Research Article

Analysis and Design of Associative Memories for Memristive Neural Networks with Deviating Argument

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We investigate associative memories for memristive neural networks with deviating argument. Firstly, the existence and uniqueness of the solution for memristive neural networks with deviating argument are discussed. Next, some sufficient conditions for this class of neural networks to possess invariant manifolds are obtained. In addition, a global exponential stability criterion is presented. Then, analysis and design of autoassociative memories and heteroassociative memories for memristive neural networks with deviating argument are formulated, respectively. Finally, several numerical examples are given to demonstrate the effectiveness of the obtained results.

1. Introduction

In recent decades, analysis and design of neurodynamic systems have received much attention [1–16]. Specifically, neurodynamics of associative memories is a hot research issue [9–16]. Associative memories refer to brain-inspired computing designed to store a set of prototype patterns such that the stored patterns can be retrieved with the recalling probes containing sufficient information about the contents of patterns. In associative memories, for any given probe (e.g., noisy or corrupted version of a prototype pattern), the retrieval dynamics can converge to an ideal equilibrium point representing the prototype pattern. At present, system analysts have two main theories for explaining strange dynamical behaviors in recalling processes: autoassociative memories and heteroassociative memories [12, 13]. Two types of methods are usually used for solving problems in regard to analysis and synthesis of associative memories. The first is when neurodynamic systems are multistable, and then system states may converge to locally stable equilibrium points, and these equilibrium points are encoded as the memorized patterns. The second is when neurodynamic systems are globally monostable, and then the memorized patterns are associated with the external input patterns.

As we know, the human brain is believed to be the most powerful information processor because of its structure of

synapses. Actually, there is always a high demand for a single device to emulate artificial synaptic functions. Using a memristor, essential synaptic plasticity and learning behaviors can be mimicked. By integrating memristors into a large-scale neuromorphic circuit, memristive neural networks based neuromorphic circuits demonstrate spike-timing-dependent plasticity (a form of the Hebbian learning rule) [17–21]. From the viewpoint of theory and experiment, a memristive neural network model is a state-dependent switching system. On the other hand, this new dynamical system shows many of the characteristic features of magnetic tunnel junction [19, 20]. For these reasons, system analysis and integration for memristive neural networks will be extremely tricky. For the moment, there are two major methods for qualitative analysis of memristive neural networks. One is the differential inclusions approach [17–20], and the other is the fuzzy uncertainty approach [21]. For the differential inclusions approach, the core idea is to integrate a switched network cluster. For the fuzzy uncertainty approach, the central idea is to divide multijump network flows into a continuous subsystem and bounded subsystem. The two kinds of analysis frameworks are just the type and level of points, not the merits of good points of difference. From the perspective of cybernetics, many reported analytical skills are merely a breakthrough of system theory more than substance. How to develop more

effective analysis methods for memristive neural networks is worth studying.

For the past few years, hybrid dynamic systems have remained one of the most active fields of research in the control community [22–30]. For instance, to describe the stationary distribution of temperature along the length of a wire that is bended, the nonlinear dynamic model with deviating argument is often used. The right-hand side in nonlinear systems with deviating argument features a combination of continuous and discrete systems. Thus, nonlinear systems with deviating argument unify the advanced and retarded systems. Because of this property, related works on the control strategies for nonlinear systems with deviating argument are relatively rare. Viewed from systems involving the interplay, differential equations and difference equations are included in the nonlinear systems with deviating argument [28–30]. However, it is still obvious that many basic issues on nonlinear systems with deviating argument remain to be addressed, such as nonlinear dynamics, systems design, and analysis.

Inspired by the above discussions, the goal of this paper is to formulate the analysis and design of associative memories for a class of memristive neural networks with deviating argument. On the whole, the highlights of this paper can be outlined as follows:

- (1) Sufficient conditions are derived to ascertain the existence, uniqueness, and global exponential stability of the solution for memristive neural networks with deviating argument.
- (2) The synthetic mechanism of the analysis and design of associative memories for a general class of memristive neural networks with deviating argument is revealed.
- (3) A uniform associative memories model, which unites autoassociative memories and heteroassociative memories, is proposed for memristive neural networks with deviating argument.

In addition, it should be noted that when special and strict conditions are exerted, associative memories' performance of neural networks is usually limited. Therefore, the methods applicable to conventional stability analysis of neural networks cannot be directly employed to investigate the associative memories for memristive neural networks. Moreover, the study of dynamics for memristive neural networks with deviating argument is not an easy work, since the conditions for dynamics of neural networks cannot be simply utilized to analyze hybrid dynamic systems with deviating argument. In this paper, according to the fuzzy uncertainty approach, in combination with the theories of set-valued maps and differential inclusions, analysis and design of associative memories for memristive neural networks with deviating argument are described in detail.

The rest of the paper is arranged as follows. Design problem and preliminaries are given in Section 2. Main results are stated in Section 3. In Section 4, several numerical examples are presented. Finally, concluding remarks are given in Section 5.

2. Design Problem and Preliminaries

2.1. Problem Description. Let $\Omega = \{\beta = (\beta_1, \beta_2, \dots, \beta_n)^T \mid \beta_i = \{-p, p\}, i = 1, 2, \dots, n\}$ be the set of n -dimensional bipolar vectors, where p is a positive constant and T denotes the transpose of a vector or matrix. The problem considered is described as follows.

Given l ($l \leq 2^n$) pairs of bipolar vectors $(u^{(1)}, q^{(1)}), (u^{(2)}, q^{(2)}), \dots, (u^{(l)}, q^{(l)})$, where $u^{(r)}, q^{(r)} \in \Omega$, for $r = 1, 2, \dots, l$, design an associative memory model which satisfies the notion that the output of the model will converge to the corresponding pattern $q^{(r)}$ when $u^{(r)}$ is fed to the model input as a memory pattern.

Definition 1. The neural network is said to be an autoassociative memory if $u^{(r)} = q^{(r)}$ and a heteroassociative memory if $u^{(r)} \neq q^{(r)}$, for $r = 1, 2, \dots, l$.

In this paper, let N be the natural number set; the norm of vector $x = (x_1, x_2, \dots, x_n)^T$ is defined as $\|x\| = \sum_{i=1}^n |x_i|$. Denote $\mathcal{R}^n$ as the n -dimensional Euclidean space. Fix two real number sequences $\{\eta_k\}, \{\vartheta_k\}$, $k \in N$, satisfying $\eta_k < \eta_{k+1}$, $\eta_k \leq \vartheta_k \leq \eta_{k+1}$, and $\lim_{k \rightarrow +\infty} \eta_k = +\infty$.

2.2. Model. Consider the following neural network model:

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + B(x(t))f(x(t)) \\ &\quad + C(x(t))f(x(\gamma(t))) + Eu^{(r)}, \\ w(t) &= f(x(t)), \end{aligned} \quad (1)$$

$r = 1, 2, \dots, l$,

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathcal{R}^n$ is the neuron state, $A = \text{diag}(a_1, a_2, \dots, a_n)$, which is a constant diagonal matrix with $a_i > 0$, $i \in \{1, 2, \dots, n\}$, indicates the self-inhibition matrix, $B(x(t)) = (b_{ij}(x_j(t)))_{n \times n}$ and $C(x(t)) = (c_{ij}(x_j(t)))_{n \times n}$ are connection weight matrices at time t and $\gamma(t)$, respectively, $b_{ij}(x_j(t))$ and $c_{ij}(x_j(t))$ are defined by

$$\begin{aligned} b_{ij}(x_j(t)) &= \begin{cases} \hat{b}_{ij}, & |x_j(t)| > T_j, \\ \check{b}_{ij}, & |x_j(t)| < T_j, \end{cases} \\ c_{ij}(x_j(t)) &= \begin{cases} \hat{c}_{ij}, & |x_j(t)| > T_j, \\ \check{c}_{ij}, & |x_j(t)| < T_j, \end{cases} \end{aligned} \quad (2)$$

for $i, j = 1, 2, \dots, n$, and $b_{ij}(\pm T_j) = \hat{b}_{ij}$ or $\check{b}_{ij}$, $c_{ij}(\pm T_j) = \hat{c}_{ij}$ or $\check{c}_{ij}$, where $T_j > 0$ represents the switching jump, and $\hat{b}_{ij}, \check{b}_{ij}, \hat{c}_{ij}$, and $\check{c}_{ij}$ are all constants. $f(x(\cdot)) = (f_1(x_1(\cdot)), f_2(x_2(\cdot)), \dots, f_n(x_n(\cdot)))^T$ is the activation function. The deviating function $\gamma(t) = \vartheta_k$, when $t \in [\eta_k, \eta_{k+1})$ for any $k \in N$. $\mathbb{E}$ stands for a transform matrix of order n satisfying $\mathbb{E}u^{(r)} = q^{(r)}$ for $r = 1, 2, \dots, l$. $u^{(r)}$ and $q^{(r)}$ indicate the external input pattern and the corresponding memorized pattern, respectively. $w(t) = (w_1(t), w_2(t), \dots, w_n(t))^T$ is the output of (1) corresponding to $u^{(r)}$.

Neural network model (1) is called an associative memory if the output $w(t)$ converges to $q^{(r)}$.

Obviously, neural network model (1) is of hybrid type: switching and mixed. When $|x_j(t)| > T_j$, $b_{ij}(x_j(t)) = \bar{b}_{ij}$, $c_{ij}(x_j(t)) = \bar{c}_{ij}$. When $|x_j(t)| < T_j$, $b_{ij}(x_j(t)) = \check{b}_{ij}$, $c_{ij}(x_j(t)) = \check{c}_{ij}$. From this perspective, (1) is a switching system. On the other hand, for any fixed interval $[\eta_k, \eta_{k+1})$, when $\gamma(t) = \vartheta_k < t < \eta_{k+1}$, neural network model (1) is a retarded system. When $\eta_k \leq t < \gamma(t) = \vartheta_k$, neural network model (1) is an advanced system. Then, from this point of view, (1) is a mixed system.

For neural network model (1), the conventional definition of solution for differential equations cannot apply here. To tackle this problem, the solution concept for differential equations with deviating argument is introduced [24–30]. According to this theory, a solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ of (1) is a consecutive function such that (1) $\dot{x}(t)$ exists on $[0, +\infty)$, except at the points η_k , $k \in N$, where a one-sided derivative exists, and (2) $x(t)$ satisfies (1) within each interval (η_k, η_{k+1}) , $k \in N$.

In this paper, the activation functions are selected as

$$f_i(\chi) = \sum_{\ell=1}^2 (-1)^\ell \frac{p}{2d} (|\chi + z_\ell^{(i)}| - |\chi - z_\ell^{(i)}|), \quad (3)$$

$$i = 1, 2, \dots, n,$$

where $p > 0$ and $d > 0$ are used to adjust the slope and corner point of activation functions and $z_1^{(i)} = T_i$, $z_2^{(i)} = T_i + d$.

It is easy to see that $f_i(\cdot)$ satisfies $f_i(\pm T_i) = 0$ and

$$|f_i(l_1) - f_i(l_2)| \leq \alpha |l_1 - l_2|, \quad i = 1, 2, \dots, n, \quad (4)$$

for any $l_1, l_2 \in (-\infty, +\infty)$, where $\alpha = p/d$.

For technical convenience, denote $\bar{b}_{ij} = \max\{|\hat{b}_{ij}|, |\check{b}_{ij}|\}$, $b_{ij}^0 = (1/2)(\hat{b}_{ij} + \check{b}_{ij})$, $b_{ij}^+ = (1/2)(\hat{b}_{ij} + \check{b}_{ij})$, $b_{ij}^- = (1/2)(\hat{b}_{ij} - \check{b}_{ij})$, $\bar{c}_{ij} = \max\{|\hat{c}_{ij}|, |\check{c}_{ij}|\}$, $c_{ij}^0 = (1/2)(\hat{c}_{ij} + \check{c}_{ij})$, $c_{ij}^+ = (1/2)(\hat{c}_{ij} + \check{c}_{ij})$, $c_{ij}^- = (1/2)(\hat{c}_{ij} - \check{c}_{ij})$, and $S_i = \max\{|\hat{b}_{ii}p - a_i d|, |\check{b}_{ii}p - a_i d|\}$, for $i, j = 1, 2, \dots, n$. Let $S_b = \max_{1 \leq i \leq n} (a_i + \sum_{j=1}^n \alpha (b_{ji}^+ + b_{ji}^-))$, $S_b^- = \min_{1 \leq i \leq n} (a_i - \sum_{j=1}^n (b_{ji}^+ + b_{ji}^-)\alpha)$, and $S_c = \max_{1 \leq i \leq n} (\sum_{j=1}^n \alpha (c_{ji}^+ + c_{ji}^-))$. $\overline{\text{co}}\{\bar{\mathcal{M}}, \check{\mathcal{M}}\}$ denotes the convex closure of a set constituted by real numbers $\bar{\mathcal{M}}$ and $\check{\mathcal{M}}$.

2.3. Autoassociative Memories. From Definition 1, matrix $\mathbb{E}$ is selected as an identity matrix when designing autoassociative memories (i.e., $\mathbb{E}u^{(r)} = u^{(r)} = q^{(r)}$ for $r = 1, 2, \dots, l$). For the convenience of analysis and formulation in autoassociative memories, neural network model (1) can be rewritten in component form:

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij}(x_j(t)) f_j(x_j(t)) + \sum_{j=1}^n c_{ij}(x_j(t)) f_j(x_j(\gamma(t))) + u_i,$$

$$w_i(t) = f_i(x_i(t)),$$

$$i = 1, 2, \dots, n. \quad (5)$$

2.4. Properties. For neural network model (1), we consider the following set-valued maps:

$$K[b_{ij}(x_j(t))] = \begin{cases} \bar{b}_{ij}, & |x_j(t)| > T_j, \\ \overline{\text{co}}\{\bar{b}_{ij}, \check{b}_{ij}\}, & |x_j(t)| = T_j, \\ \check{b}_{ij}, & |x_j(t)| < T_j, \end{cases} \quad (6)$$

$$K[c_{ij}(x_j(t))] = \begin{cases} \bar{c}_{ij}, & |x_j(t)| > T_j, \\ \overline{\text{co}}\{\bar{c}_{ij}, \check{c}_{ij}\}, & |x_j(t)| = T_j, \\ \check{c}_{ij}, & |x_j(t)| < T_j, \end{cases} \quad (7)$$

for $i, j = 1, 2, \dots, n$.

Based on the theory of differential inclusions [17–20], from (5), we can get

$$\begin{aligned} \dot{x}_i(t) \in & -a_i x_i(t) + \sum_{j=1}^n K[b_{ij}(x_j(t))] f_j(x_j(t)) \\ & + \sum_{j=1}^n K[c_{ij}(x_j(t))] f_j(x_j(\gamma(t))) + u_i, \end{aligned} \quad (8)$$

or, equivalently, there exist $b_{ij}^*(x_j(t)) \in K[b_{ij}(x_j(t))]$ and $c_{ij}^*(x_j(t)) \in K[c_{ij}(x_j(t))]$ such that

$$\begin{aligned} \dot{x}_i(t) = & -a_i x_i(t) + \sum_{j=1}^n b_{ij}^*(x_j(t)) f_j(x_j(t)) \\ & + \sum_{j=1}^n c_{ij}^*(x_j(t)) f_j(x_j(\gamma(t))) + u_i, \end{aligned} \quad (9)$$

$$i = 1, 2, \dots, n,$$

for almost all $t \in [0, +\infty)$.

Employing the fuzzy uncertainty approach [21], neural network model (5) can be expressed as follows:

$$\begin{aligned} \dot{x}_i(t) = & -a_i x_i(t) + \sum_{j=1}^n b_{ij}^0 f_j(x_j(t)) \\ & + \sum_{j=1}^n c_{ij}^0 f_j(x_j(\gamma(t))) + u_i \\ & + \sum_{j=1}^n \delta_{ij}^b(x_j(t)) b_{ij}^- f_j(x_j(t)) + \sum_{j=1}^n \delta_{ij}^c(x_j(t)) \\ & \times c_{ij}^- f_j(x_j(\gamma(t))), \end{aligned} \quad (10)$$

where

$$\begin{aligned}\delta_{ij}^b(x_j(t)) &= \begin{cases} \operatorname{sgn}(\widehat{b}_{ij} - \check{b}_{ij}), & |x_j(t)| > T_j, \\ -\operatorname{sgn}(\widehat{b}_{ij} - \check{b}_{ij}), & |x_j(t)| < T_j, \end{cases} \\ \delta_{ij}^c(x_j(t)) &= \begin{cases} \operatorname{sgn}(\widehat{c}_{ij} - \check{c}_{ij}), & |x_j(t)| > T_j, \\ -\operatorname{sgn}(\widehat{c}_{ij} - \check{c}_{ij}), & |x_j(t)| < T_j, \end{cases}\end{aligned}\quad (11)$$

and $\operatorname{sgn}(\cdot)$ is the sign function.

Definition 2. A constant vector $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ is said to be an equilibrium point of neural network model (5) if and only if

$$\begin{aligned}0 &\in -a_i x_i^* + \sum_{j=1}^n K[b_{ij}(x_j^*)] f_j(x_j^*) \\ &+ \sum_{j=1}^n K[c_{ij}(x_j^*)] f_j(x_j^*) + u_i, \quad i = 1, 2, \dots, n,\end{aligned}\quad (12)$$

or, equivalently, there exist $\tilde{b}_{ij}(x_j^*) \in K[b_{ij}(x_j^*)]$ and $\tilde{c}_{ij}(x_j^*) \in K[c_{ij}(x_j^*)]$ such that

$$\begin{aligned}-a_i x_i^* + \sum_{j=1}^n \tilde{b}_{ij}(x_j^*) f_j(x_j^*) + \sum_{j=1}^n \tilde{c}_{ij}(x_j^*) f_j(x_j^*) + u_i \\ = 0,\end{aligned}\quad (13)$$

for $i = 1, 2, \dots, n$.

Definition 3. The equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ of neural network model (5) is globally exponentially stable if there exist constants $M > 0$ and $\epsilon > 0$ such that

$$\|x(t) - x^*\| \leq M \|x^0 - x^*\| \exp\{-\epsilon t\}, \quad (14)$$

for any $t \geq 0$, where $x(t)$ is the state vector of neural network model (5) with initial condition x^0 .

Lemma 4. For neural network model (5), one has

$$\begin{aligned}&|K[b_{ij}(x_j)] f_j(x_j) - K[b_{ij}(y_j)] f_j(y_j)| \\ &\leq \bar{b}_{ij} \alpha |x_j - y_j|, \\ &|K[c_{ij}(x_j)] f_j(x_j) - K[c_{ij}(y_j)] f_j(y_j)| \\ &\leq \bar{c}_{ij} \alpha |x_j - y_j|,\end{aligned}\quad (15)$$

for $i, j = 1, 2, \dots, n$, $x_j, y_j \in (-\infty, +\infty)$, where $K[b_{ij}(\cdot)]$ and $K[c_{ij}(\cdot)]$ are defined as those in (6) and (7), respectively.

Lemma 4 is a basic conclusion. To get more specific, please see [Neural Networks, vol. 51, pp. 1–8, 2014], or [Information Sciences, vol. 279, pp. 358–373, 2014], or others.

In the following, we end this section with some basic assumptions:

(A1) There exists a positive constant η satisfying

$$\eta_{k+1} - \eta_k < \eta, \quad k \in N. \quad (16)$$

(A2) $\eta(S_b + 2S_c) \exp\{S_b \eta\} < 1$.

(A3) $\eta[S_c + S_b(1 + S_c \eta) \exp\{S_b \eta\}] < 1$.

3. Main Results

In this section, we first present the conditions ensuring the existence and uniqueness of solution for neural network model (5) and then analyze its global exponential stability; finally, we discuss how to design the procedures of autoassociative memories and heteroassociative memories.

3.1. Existence and Uniqueness of Solution

Theorem 5. Let (A1) and (A2) hold. Then, there exists a unique solution $x(t) = x(t, t_0, x^0)$, $t \geq t_0$, of (5) for every $(t_0, x^0) \in (\mathfrak{R}^+, \mathfrak{R}^n)$.

Proof. The proof of the assertion is divided into two steps.

Firstly, we discuss the existence of solution.

For a fixed $k \in N$, assume that $\eta_k \leq t_0 \leq \vartheta_k = \gamma(t) < t < \eta_{k+1}$ and the other case $\eta_k \leq t_0 < t < \vartheta_k = \gamma(t) < \eta_{k+1}$ can be discussed in a similar way.

Let $v(t) = x(t, t_0, x^0)$, and consider the following equivalent equation:

$$\begin{aligned}v_i(t) &= x_i^0 + \int_{t_0}^t \left[-a_i v_i(s) + \sum_{j=1}^n b_{ij}^0 f_j(v_j(s)) + \sum_{j=1}^n c_{ij}^0 \right. \\ &\times f_j(v_j(\vartheta_k)) + u_i + \sum_{j=1}^n \delta_{ij}^b(v_j(t)) \tilde{b}_{ij}^- f_j(v_j(s)) \\ &\left. + \sum_{j=1}^n \delta_{ij}^c(v_j(t)) \tilde{c}_{ij}^- f_j(v_j(\vartheta_k)) \right] ds, \\ &\quad i = 1, 2, \dots, n.\end{aligned}\quad (17)$$

Construct the following sequence $\{v_i^m(t)\}$, $m \in N$, $i = 1, 2, \dots, n$:

$$\begin{aligned}v_i^{m+1}(t) &= x_i^0 + \int_{t_0}^t \left[-a_i v_i^m(s) + \sum_{j=1}^n b_{ij}^0 f_j(v_j^m(s)) \right. \\ &+ \sum_{j=1}^n c_{ij}^0 \times f_j(v_j^m(\vartheta_k)) + u_i \\ &\left. + \sum_{j=1}^n \delta_{ij}^b(v_j^m(t)) \tilde{b}_{ij}^- f_j(v_j^m(s)) \right. \\ &\left. + \sum_{j=1}^n \delta_{ij}^c(v_j^m(t)) \tilde{c}_{ij}^- f_j(v_j^m(s)) \right] ds,\end{aligned}$$

$$\left. + \sum_{j=1}^n \delta_{ij}^c (v_j^m(t)) c_{ij}^- f_j(v_j^m(\vartheta_k)) \right] ds, \\ i = 1, 2, \dots, n, \quad (18)$$

and $v_i^0(t) = x_i^0$.

Define a norm $\|v(t)\|_M = \max_{[t_0, t]} \|v(t)\|$, and we can get

$$\|v^{m+1}(t) - v^m(t)\|_M \leq [(S_b + S_c)\eta]^m \varsigma, \quad (19)$$

where $\varsigma = \eta[\sum_{i=1}^n a_i |x_i^0| + \sum_{i=1}^n \sum_{j=1}^n (b_{ij}^+ + b_{ij}^- + c_{ij}^+ + c_{ij}^-)p + \sum_{i=1}^n |u_i|]$.

From (A2), we have $(S_b + S_c)\eta < 1$, and hence

$$\lim_{m \rightarrow +\infty} \|v^{m+1}(t) - v^m(t)\|_M = 0. \quad (20)$$

Therefore, there exists a solution $v(t) = x(t, t_0, x^0)$ on the interval $[t_0, \vartheta_k]$ for (17). From (4), the solution can be consecutive to η_{k+1} . Utilizing a similar method, $x(t)$ can be consecutive to ϑ_{k+1} and then to η_{k+2} . Applying the mathematical induction, the proof of existence of solution for (5) is completed.

Next, we analyze the uniqueness of solution.

Fix $k \in N$ and choose $t_0 \in [\eta_k, \eta_{k+1}]$, $x^1 \in \mathfrak{R}^n$, $x^2 \in \mathfrak{R}^n$. Denote $x^1(t) = x(t, t_0, x^1)$ and $x^2(t) = x(t, t_0, x^2)$ as two solutions of (5) with different initial conditions (t_0, x^1) and (t_0, x^2) , respectively.

Then, we get

$$\begin{aligned} & \|x^1(t) - x^2(t)\| \\ & \leq (\|x^1 - x^2\| + S_c \eta \|x^1(\vartheta_k) - x^2(\vartheta_k)\|) \\ & \quad + \int_{t_0}^t S_b \|x^1(s) - x^2(s)\| ds. \end{aligned} \quad (21)$$

Based on the Grönwall-Bellman inequality,

$$\begin{aligned} & \|x^1(t) - x^2(t)\| \\ & \leq (\|x^1 - x^2\| + S_c \eta \|x^1(\vartheta_k) - x^2(\vartheta_k)\|) \\ & \quad \times \exp\{S_b \eta\}, \end{aligned} \quad (22)$$

and, particularly,

$$\begin{aligned} & \|x^1(\vartheta_k) - x^2(\vartheta_k)\| \\ & \leq (\|x^1 - x^2\| + S_c \eta \|x^1(\vartheta_k) - x^2(\vartheta_k)\|) \\ & \quad \times \exp\{S_b \eta\}. \end{aligned} \quad (23)$$

Hence,

$$\|x^1(t) - x^2(t)\| \leq \left(\frac{\exp\{S_b \eta\}}{1 - S_c \eta \exp\{S_b \eta\}} \right) \|x^1 - x^2\|. \quad (24)$$

Suppose that there exists some $\bar{t} \in [\eta_k, \eta_{k+1}]$ satisfying $x^1(\bar{t}) = x^2(\bar{t})$; then,

$$\begin{aligned} & \|x^1 - x^2\| \leq S_c \eta \|x^1(\vartheta_k) - x^2(\vartheta_k)\| \\ & \quad + \int_{t_0}^{\bar{t}} S_b \|x^1(s) - x^2(s)\| ds. \end{aligned} \quad (25)$$

From (A2), we get

$$0 < \eta \exp\{S_b \eta\} (S_b + S_c) < 1 - \eta S_c \exp\{S_b \eta\}; \quad (26)$$

that is,

$$0 < \frac{\eta \exp\{S_b \eta\} (S_b + S_c)}{1 - \eta S_c \exp\{S_b \eta\}} < 1. \quad (27)$$

Substituting (24) into (25), from (27), we obtain

$$\begin{aligned} & \|x^1 - x^2\| \leq S_c \eta \frac{\exp\{S_b \eta\}}{1 - S_c \eta \exp\{S_b \eta\}} \|x^1 - x^2\| \\ & \quad + S_b \eta \frac{\exp\{S_b \eta\}}{1 - S_c \eta \exp\{S_b \eta\}} \|x^1 - x^2\| \\ & \leq \frac{\eta \exp\{S_b \eta\} (S_b + S_c)}{1 - \eta S_c \exp\{S_b \eta\}} \|x^1 - x^2\| \\ & < \|x^1 - x^2\|. \end{aligned} \quad (28)$$

This poses a contradiction. Therefore, the uniqueness of solution is true. Taken together, the proof of the existence and uniqueness of solution for (5) is completed. $\square$

3.2. Invariant Manifolds

Lemma 6. The solution $x(t)$ of neural network model (5) will fall into $\prod_{i=1}^n [d + T_i, +\infty)$ ultimately if, for $i = 1, 2, \dots, n$, the following conditions hold:

$$u_i > S_i + a_i T_i + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p + \sum_{j=1}^n \bar{c}_{ij} p. \quad (29)$$

Proof. We distinguish four cases to prove the lemma.

(1) If, for some t , $x_i(t) \in [T_i, d + T_i]$, $i \in \{1, 2, \dots, n\}$, then

$$\begin{aligned} \dot{x}_i(t) & = -a_i x_i(t) + \sum_{j=1}^n b_{ij}(x_j(t)) f_j(x_j(t)) \\ & \quad + \sum_{j=1}^n c_{ij}(x_j(t)) \times f_j(x_j(\gamma(t))) + u_i \end{aligned}$$

$$\begin{aligned}
&\geq -a_i x_i(t) + b_{ii}(x_i(t)) \frac{p}{d} [x_i(t) - T_i] \\
&\quad - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p - \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
&\geq -a_i [x_i(t) - T_i] + b_{ii}(x_i(t)) \frac{p}{d} [x_i(t) - T_i] \\
&\quad - a_i T_i - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p - \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
&\geq \left[-a_i + b_{ii}(x_i(t)) \frac{p}{d} \right] [x_i(t) - T_i] - a_i T_i \\
&\quad - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p - \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
&\geq - \left| -a_i + b_{ii}(x_i(t)) \frac{p}{d} \right| d - a_i T_i - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
&\quad - \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
&\geq - |b_{ii}(x_i(t)) p - a_i d| - a_i T_i - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
&\quad - \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
&\geq -S_i - a_i T_i - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p - \sum_{j=1}^n \bar{c}_{ij} p + u_i > 0.
\end{aligned} \tag{30}$$

That is to say, $\dot{x}_i(t) > 0$, when $x_i(t) \in [T_i, d + T_i]$.

(2) If, for some t , $x_i(t) \in (-T_i, T_i)$, $i \in \{1, 2, \dots, n\}$, then

$$\begin{aligned}
\dot{x}_i(t) &\geq -a_i x_i(t) + b_{ii}(x_i(t)) f_i(x_i(t)) - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
&\quad - \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
&\geq -a_i T_i - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p - \sum_{j=1}^n \bar{c}_{ij} p + u_i > S_i > 0.
\end{aligned} \tag{31}$$

That is to say, $\dot{x}_i(t) > 0$, when $x_i(t) \in (-T_i, T_i)$.

(3) If, for some t , $x_i(t) \in (-T_i - d, -T_i)$, $i \in \{1, 2, \dots, n\}$, then

$$\begin{aligned}
\dot{x}_i(t) &\geq -a_i x_i(t) + b_{ii}(x_i(t)) f_i(x_i(t)) - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
&\quad - \sum_{j=1}^n \bar{c}_{ij} p + u_i
\end{aligned}$$

$$\begin{aligned}
&= -a_i [x_i(t) + T_i] + b_{ii}(x_i(t)) \frac{p}{d} [x_i(t) + T_i] \\
&\quad - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p - \sum_{j=1}^n \bar{c}_{ij} p + u_i + a_i T_i \\
&= \left[-a_i + b_{ii}(x_i(t)) \frac{p}{d} \right] [x_i(t) + T_i] - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
&\quad - \sum_{j=1}^n \bar{c}_{ij} p + u_i + a_i T_i \\
&\geq -|b_{ii}(x_i(t)) p - a_i d| + a_i T_i - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
&\quad - \sum_{j=1}^n \bar{c}_{ij} p + u_i > 2a_i T_i > 0.
\end{aligned} \tag{32}$$

That is to say, $\dot{x}_i(t) > 0$, when $x_i(t) \in (-T_i - d, -T_i)$.

(4) If, for some t , $x_i(t) \in (-\infty, -T_i - d)$, $i \in \{1, 2, \dots, n\}$, then

$$\begin{aligned}
\dot{x}_i(t) &\geq -a_i x_i(t) + b_{ii}(x_i(t)) f_i(x_i(t)) - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
&\quad - \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
&\geq a_i (T_i + d) - b_{ii}(x_i(t)) p - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
&\quad - \sum_{j=1}^n \bar{c}_{ij} p + u_i
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\geq -|b_{ii}(x_i(t)) p - a_i d| + a_i T_i - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
&\quad - \sum_{j=1}^n \bar{c}_{ij} p + u_i > 2a_i T_i > 0.
\end{aligned}$$

That is to say, $\dot{x}_i(t) > 0$, when $x_i(t) \in (-\infty, -T_i - d)$.

From the above analysis, we can get that any component $x_i(t)$ ($i = 1, 2, \dots, n$) of the solution $x(t)$ of neural network model (5) will fall into $[d + T_i, +\infty)$ ultimately. Hence, the proof is completed. $\square$

Lemma 7. The solution $x(t)$ of neural network model (5) will fall into $\prod_{i=1}^n (-\infty, -d - T_i]$ ultimately if, for $i = 1, 2, \dots, n$, the following conditions hold:

$$u_i < -S_i - a_i T_i - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p - \sum_{j=1}^n \bar{c}_{ij} p. \tag{34}$$

Proof. We distinguish four cases to prove the lemma.

(1) If, for some t , $x_i(t) \in [d + T_i, +\infty)$, $i \in \{1, 2, \dots, n\}$, then

$$\begin{aligned}
 \dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^n b_{ij}(x_j(t)) f_j(x_j(t)) \\
 &\quad + \sum_{j=1}^n c_{ij}(x_j(t)) f_j(x_j(\gamma(t))) + u_i \\
 &\leq -a_i x_i(t) + b_{ii}(x_i(t)) f_i(x_i(t)) + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
 &\quad + \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
 &\leq -a_i(d + T_i) + b_{ii}(x_i(t)) p + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
 &\quad + \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
 &\leq |a_i d - b_{ii}(x_i(t)) p| - a_i T_i + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
 &\quad + \sum_{j=1}^n |\bar{c}_{ij}| p + u_i \\
 &\leq S_i - a_i T_i + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p + \sum_{j=1}^n \bar{c}_{ij} p + u_i < -2a_i T_i \\
 &< 0.
 \end{aligned} \tag{35}$$

That is to say, $\dot{x}_i(t) < 0$, when $x_i(t) \in [d + T_i, +\infty)$.

(2) If, for some t , $x_i(t) \in (T_i, d + T_i)$, $i \in \{1, 2, \dots, n\}$, then

$$\begin{aligned}
 \dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^n b_{ij}(x_j(t)) f_j(x_j(t)) \\
 &\quad + \sum_{j=1}^n c_{ij}(x_j(t)) f_j(x_j(\gamma(t))) + u_i \\
 &\leq -a_i[x_i(t) - T_i] + b_{ii}(x_i(t)) \frac{p}{d} [x_i(t) - T_i] \\
 &\quad - a_i T_i + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p + \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
 &\leq \left[-a_i + b_{ii}(x_i(t)) \frac{p}{d} \right] [x_i(t) - T_i] - a_i T_i \\
 &\quad + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p + \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
 &\leq \left[-a_i + b_{ii}(x_i(t)) \frac{p}{d} \right] d - a_i T_i + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
 &\leq |b_{ii}(x_i(t)) p - a_i d| - a_i T_i + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
 &\quad + \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
 &\leq S_i - a_i T_i + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p + \sum_{j=1}^n \bar{c}_{ij} p + u_i < -2a_i T_i \\
 &< 0.
 \end{aligned} \tag{36}$$

That is to say, $\dot{x}_i(t) < 0$, when $x_i(t) \in (T_i, d + T_i)$.

(3) If, for some t , $x_i(t) \in (-T_i, T_i)$, $i \in \{1, 2, \dots, n\}$, then

$$\begin{aligned}
 \dot{x}_i(t) &\leq -a_i x_i(t) + b_{ii}(x_i(t)) f_i(x_i(t)) + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
 &\quad + \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
 &\leq a_i T_i + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p + \sum_{j=1}^n \bar{c}_{ij} p + u_i < -S_i < 0.
 \end{aligned} \tag{37}$$

That is to say, $\dot{x}_i(t) < 0$, when $x_i(t) \in (-T_i, T_i)$.

(4) If, for some t , $x_i(t) \in (-T_i - d, -T_i)$, $i \in \{1, 2, \dots, n\}$, then

$$\begin{aligned}
 \dot{x}_i(t) &\leq -a_i x_i(t) + b_{ii}(x_i(t)) f_i(x_i(t)) + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
 &\quad + \sum_{j=1}^n \bar{c}_{ij} p + u_i \\
 &= -a_i[x_i(t) + T_i] + b_{ii}(x_i(t)) \frac{p}{d} [x_i(t) + T_i] \\
 &\quad + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p + \sum_{j=1}^n \bar{c}_{ij} p + u_i + a_i T_i \\
 &= \left[-a_i + b_{ii}(x_i(t)) \frac{p}{d} \right] [x_i(t) + T_i] + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
 &\quad + \sum_{j=1}^n \bar{c}_{ij} p + u_i + a_i T_i \\
 &\leq |b_{ii}(x_i(t)) p - a_i d| + a_i T_i + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \\
 &\quad + \sum_{j=1}^n \bar{c}_{ij} p + u_i < 0.
 \end{aligned} \tag{38}$$

That is to say, $\dot{x}_i(t) < 0$, when $x_i(t) \in (-T_i - d, -T_i)$.

From the above analysis, we can get that any component $x_i(t)$ ($i = 1, 2, \dots, n$) of the solution $x(t)$ of neural network model (5) will fall into $(-\infty, -d - T_i]$ ultimately. Hence, the proof is completed. $\square$

3.3. Global Exponential Stability Analysis. Denote

$$\begin{aligned}\Delta_1 &= \left\{ i \mid i \in \{1, 2, \dots, n\}, u_i > S_i + a_i T_i + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \right. \\ &\quad \left. + \sum_{j=1}^n \bar{c}_{ij} p \right\}, \\ \Delta_2 &= \left\{ i \mid i \in \{1, 2, \dots, n\}, u_i > S_i + a_i T_i + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p \right. \\ &\quad \left. + \sum_{j=1}^n \bar{c}_{ij} p \right\}.\end{aligned}\quad (39)$$

In order to guarantee the existence of equilibrium point for neural network model (5), the following assumption is needed:

(A4) For any $i \in \{1, 2, \dots, n\}$, $i \in \Delta_1 \cup \Delta_2$.

Theorem 8. Let (A4) hold. Then, neural network model (5) has at least one equilibrium point $x^* \in \Theta$, where

$$\begin{aligned}\Theta &= \{x \in \mathfrak{R}^n \mid x_i \leq -T_i - d \text{ or } x_i \geq d + T_i, i \\ &\in \{1, 2, \dots, n\}\}.\end{aligned}\quad (40)$$

Proof. Define a mapping

$$F(x) = (F_1(x), F_2(x), \dots, F_n(x))^T, \quad (41)$$

where

$$\begin{aligned}F_i(x) &= \frac{1}{a_i} \left[\sum_{j=1}^n K[b_{ij}(x_j)] f_j(x_j) \right. \\ &\quad \left. + \sum_{j=1}^n K[c_{ij}(x_j)] f_j(x_j) + u_i \right], \quad i = 1, 2, \dots, n.\end{aligned}\quad (42)$$

It is obvious that $F(x)$ is equicontinuous.

When $i \in \Delta_1$, there exists a small enough positive number ε_1 satisfying

$$u_i > S_i + a_i T_i + \sum_{j=1, j \neq i}^n \bar{b}_{ij} p + \sum_{j=1}^n \bar{c}_{ij} p + a_i \varepsilon_1, \quad (43)$$

and when $i \in \Delta_2$, there exists a small enough positive number ε_2 satisfying

$$u_i < -S_i - a_i T_i - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p - \sum_{j=1}^n \bar{c}_{ij} p - a_i \varepsilon_2. \quad (44)$$

Take $0 < \varepsilon < \min(\varepsilon_1, \varepsilon_2)$ and denote

$$\begin{aligned}\bar{\Theta} &= \left\{ x \in \mathfrak{R}^n \mid x_i \in \left[d + T_i + \varepsilon, \frac{(d + T_i)}{\varepsilon} \right], i \right. \\ &\quad \left. \in \Delta_1, \text{ or } x_i \in \left[\frac{-(d + T_i)}{\varepsilon}, -d - T_i - \varepsilon \right], i \in \Delta_2 \right\},\end{aligned}\quad (45)$$

which is a bounded and closed set.

When $x_i \in [d + T_i + \varepsilon, (d + T_i)/\varepsilon]$, $i \in \Delta_1$,

$$\begin{aligned}F_i(x) &= \frac{1}{a_i} \left[\sum_{j=1}^n K[b_{ij}(x_j)] f_j(x_j) \right. \\ &\quad \left. + \sum_{j=1}^n K[c_{ij}(x_j)] f_j(x_j) + u_i \right] \\ &\geq \frac{1}{a_i} \left[K[b_{ii}(x_i)] p - \sum_{j=1, j \neq i}^n \bar{b}_{ij} p - \sum_{j=1}^n \bar{c}_{ij} p + u_i \right] \\ &> \frac{1}{a_i} [(K[b_{ii}(x_i)] p - a_i d) + S_i + a_i T_i + a_i d + a_i \varepsilon] \\ &\geq T_i + d + \varepsilon.\end{aligned}\quad (46)$$

Similarly, we can obtain

$$F_i(x) < -T_i - d - \varepsilon \quad (47)$$

for $x_i \in [-(d + T_i)/\varepsilon, -d - T_i - \varepsilon]$, $i \in \Delta_2$.

From the above discussion, we can get $F(x) \in \Theta$ for $x \in \bar{\Theta}$. Based on the generalized Brouwer's fixed point theorem, there exists at least one equilibrium $x^* \in \Theta$. This completes the proof. $\square$

Recalling (10), for $i = 1, 2, \dots, n$, then

$$\begin{aligned}\dot{x}_i(t) &\in -a_i x_i(t) + \sum_{j=1}^n b_{ij}^0 f_j(x_j(t)) \\ &\quad + \sum_{j=1}^n c_{ij}^0 f_j(x_j(\gamma(t))) + u_i \\ &\quad + \sum_{j=1}^n K[\delta_{ij}^b(x_j(t))] \bar{b}_{ij} f_j(x_j(t)) \\ &\quad + \sum_{j=1}^n K[\delta_{ij}^c(x_j(t))] \bar{c}_{ij} f_j(x_j(\gamma(t))).\end{aligned}\quad (48)$$

Let $y_i(t) = x_i(t) - x_i^*$, $i = 1, 2, \dots, n$,

$$\begin{aligned}\dot{y}_i(t) &\in -a_i y_i(t) + \sum_{j=1}^n b_{ij}^0 g_j(y_j(t)) \\ &\quad + \sum_{j=1}^n c_{ij}^0 g_j(y_j(\gamma(t))) + \sum_{j=1}^n A_{ij}^b(y_j(t)) \\ &\quad + \sum_{j=1}^n A_{ij}^c(y_j(\gamma(t))), \quad i = 1, 2, \dots, n,\end{aligned}\quad (49)$$

where $g_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*)$, $A_{ij}^b(y_j(t)) = b_{ij}^-(K[\delta_{ij}^b(y_j(t) + x_j^*)]f_j(y_j(t) + x_j^*) - K[\delta_{ij}^b(x_j^*)]f_j(x_j^*))$, or $A_{ij}^c(y_j(t)) = c_{ij}^-(K[\delta_{ij}^c(y_j(t) + x_j^*)]f_j(y_j(t) + x_j^*) - K[\delta_{ij}^c(x_j^*)]f_j(x_j^*))$, or there exists $\phi_{ij}^b(y_j(t)) \in A_{ij}^b(y_j(t))$, $\varphi_{ij}^c(y_j(t)) \in A_{ij}^c(y_j(t))$ for $i, j = 1, 2, \dots, n$, such that

$$\begin{aligned} \dot{y}_i(t) = & -a_i y_i(t) + \sum_{j=1}^n b_{ij}^0 g_j(y_j(t)) \\ & + \sum_{j=1}^n c_{ij}^0 g_j(y_j(\gamma(t))) + \sum_{j=1}^n \phi_{ij}^b(y_j(t)) \\ & + \sum_{j=1}^n \varphi_{ij}^c(y_j(\gamma(t))), \quad i = 1, 2, \dots, n. \end{aligned} \quad (50)$$

According to Lemma 4,

$$\begin{aligned} |A_{ij}^b(y_j(t))| & \leq b_{ij}^- \alpha |y_j(t)|, \\ |A_{ij}^c(y_j(\gamma(t)))| & \leq c_{ij}^- \alpha |y_j(\gamma(t))|, \end{aligned} \quad (51)$$

for $i, j = 1, 2, \dots, n$.

Lemma 9. Let (A1)–(A3) hold and $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$ be a solution of (49) or (50). Then,

$$\|y(\gamma(t))\| \leq \mu \|y(t)\|, \quad (52)$$

for $t \geq 0$, where $\mu = (1 - \eta[S_c + S_b(1 + S_c\eta)\exp\{S_b\eta\}])^{-1}$.

Proof. For any $t > 0$, there exists a unique $k \in N$, such that $t \in [\eta_k, \eta_{k+1})$; then,

$$\begin{aligned} y_i(t) = & y_i(\vartheta_k) + \int_{\vartheta_k}^t \left[-a_i y_i(s) + \sum_{j=1}^n b_{ij}^0 g_j(y_j(s)) \right. \\ & + \sum_{j=1}^n c_{ij}^0 g_j(y_j(\vartheta_k)) + \sum_{j=1}^n \phi_{ij}^b(y_j(s)) \\ & \left. + \sum_{j=1}^n \varphi_{ij}^c(y_j(\vartheta_k)) \right] ds, \quad i = 1, 2, \dots, n. \end{aligned} \quad (53)$$

Taking the absolute value on both sides of (53),

$$\begin{aligned} |y_i(t)| \leq & |y_i(\vartheta_k)| + \int_{\vartheta_k}^t \left[a_i |y_i(s)| \right. \\ & + \sum_{j=1}^n b_{ij}^+ |g_j(y_j(s))| + \sum_{j=1}^n c_{ij}^+ |g_j(y_j(\vartheta_k))| \\ & \left. + \sum_{j=1}^n |\phi_{ij}^b(y_j(s))| + \sum_{j=1}^n |\varphi_{ij}^c(y_j(\vartheta_k))| \right] ds, \\ & i = 1, 2, \dots, n, \end{aligned} \quad (54)$$

and then

$$\begin{aligned} \sum_{i=1}^n |y_i(t)| \leq & \sum_{i=1}^n |y_i(\vartheta_k)| + \int_{\vartheta_k}^t \left[\sum_{i=1}^n a_i |y_i(s)| \right. \\ & + \sum_{i=1}^n \sum_{j=1}^n b_{ij}^+ |g_j(y_j(s))| + \sum_{i=1}^n \sum_{j=1}^n c_{ij}^+ |g_j(y_j(\vartheta_k))| \\ & \left. + \sum_{i=1}^n \sum_{j=1}^n |\phi_{ij}^b(y_j(s))| + \sum_{i=1}^n \sum_{j=1}^n |\varphi_{ij}^c(y_j(\vartheta_k))| \right] ds. \end{aligned} \quad (55)$$

Applying Lemma 4,

$$\begin{aligned} \sum_{i=1}^n |y_i(t)| \leq & \sum_{i=1}^n |y_i(\vartheta_k)| + \int_{\vartheta_k}^t \left[\sum_{i=1}^n a_i |y_i(s)| \right. \\ & + \sum_{i=1}^n \sum_{j=1}^n b_{ij}^+ \alpha |y_j(s)| + \sum_{i=1}^n \sum_{j=1}^n c_{ij}^+ \alpha |y_j(\vartheta_k)| \\ & + \sum_{i=1}^n \sum_{j=1}^n b_{ij}^- \alpha |y_j(s)| + \sum_{i=1}^n \sum_{j=1}^n c_{ij}^- \alpha |y_j(\vartheta_k)| \left. \right] ds \\ = & \sum_{i=1}^n |y_i(\vartheta_k)| + \int_{\vartheta_k}^t \left[\sum_{i=1}^n a_i |y_i(s)| \right. \\ & + \sum_{i=1}^n \sum_{j=1}^n b_{ji}^+ \alpha |y_i(s)| + \sum_{i=1}^n \sum_{j=1}^n c_{ji}^+ \alpha |y_i(\vartheta_k)| \\ & \left. + \sum_{i=1}^n \sum_{j=1}^n b_{ji}^- \alpha |y_i(s)| + \sum_{i=1}^n \sum_{j=1}^n c_{ji}^- \alpha |y_i(\vartheta_k)| \right] ds; \end{aligned} \quad (56)$$

that is

$$\begin{aligned} \|y(t)\| \leq & \sum_{i=1}^n |y_i(\vartheta_k)| \\ & + \int_{\vartheta_k}^t \left[\sum_{i=1}^n \left(a_i + \sum_{j=1}^n \alpha (b_{ji}^+ + b_{ji}^-) \right) |y_i(s)| \right. \\ & \left. + \sum_{i=1}^n \left(\sum_{j=1}^n \alpha (c_{ji}^+ + c_{ji}^-) \right) |y_i(\vartheta_k)| \right] ds; \end{aligned} \quad (57)$$

then

$$\begin{aligned} \|y(t)\| \leq & \|y(\vartheta_k)\| + \int_{\vartheta_k}^t [S_b \|y(s)\| + S_c \|y(\vartheta_k)\|] ds \\ \leq & (1 + S_c \eta) \|y(\vartheta_k)\| + \int_{\vartheta_k}^t S_b \|y(s)\| ds. \end{aligned} \quad (58)$$

According to the Gronwall-Bellman inequality,

$$\begin{aligned} \|y(t)\| \leq & (1 + S_c \eta) \|y(\vartheta_k)\| \exp \left\{ \int_{\vartheta_k}^t S_b ds \right\} \\ \leq & (1 + S_c \eta) \exp \{S_b \eta\} \|y(\vartheta_k)\|. \end{aligned} \quad (59)$$

Exchanging the location of $y_i(t)$ and $y_i(\vartheta_k)$ in (53), we get

$$\begin{aligned} \|y(\vartheta_k)\| &\leq \|y(t)\| + \int_{\vartheta_k}^t [S_b \|y(s)\| + S_c \|y_{\vartheta_k}\|] ds \\ &\leq \|y(t)\| + \int_{\vartheta_k}^t [S_b ((1 + S_c \eta) \exp \{S_b \eta\} \|y(\vartheta_k)\|) \\ &\quad + S_c \|y_{\vartheta_k}\|] ds \leq \|y(t)\| + \eta [S_c + S_b (1 \\ &\quad + S_c \eta)] \|y(\vartheta_k)\|, \end{aligned} \quad (60)$$

and then

$$\|y(\vartheta_k)\| \leq \mu \|y(t)\| \quad (61)$$

holds for $t \in [\eta_k, \eta_{k+1})$, so $\|y(\gamma(t))\| \leq \mu \|y(t)\|$, where $\mu = (1 - \eta[S_c + S_b(1 + S_c \eta) \exp \{S_b \eta\}])^{-1}$. This completes the proof. $\square$

Remark 10. Neural network model (49) is also of a hybrid type. The difficulty in investigating this class of neural networks lies in the existence of a switching jump and deviating argument. For the switching jump, we introduce the differential inclusions and fuzzy uncertainty approach to compensate state-jump uncertainty. For the deviating argument, by the aid of effective computational analysis, we estimate the norm of deviating state $y(\gamma(t))$ by the corresponding state $y(t)$.

In the following, the criterion to guarantee the global exponential stability of neural network model (5) based on the Lyapunov method is established.

Theorem 11. Let (A1)–(A4) hold. The origin of neural network model (49) is globally exponentially stable, which implies that the equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ of neural network model (5) is globally exponentially stable, if the following condition is satisfied:

$$\mu S_c - S_b^- < 0. \quad (62)$$

Proof. Denote

$$z_i(t) = \exp \{\lambda t\} |y_i(t)|, \quad i = 1, 2, \dots, n, \quad (63)$$

where $\lambda = -(1/2)(\mu S_c - S_b^-) > 0$ is a positive constant satisfying

$$\lambda + \mu S_c - S_b^- = \frac{1}{2} (\mu S_c - S_b^-) < 0. \quad (64)$$

Then, for $t \neq \eta_k, k \in N$, calculate the derivative of $z_i(t)$ along the trajectory of (49) or (50):

$$\begin{aligned} \dot{z}_i(t) &= \lambda \exp \{\lambda t\} |y_i(t)| + \exp \{\lambda t\} \operatorname{sgn}(y_i(t)) \dot{y}_i(t) \\ &= \lambda \exp \{\lambda t\} |y_i(t)| + \exp \{\lambda t\} \operatorname{sgn}(y_i(t)) \end{aligned}$$

$$\begin{aligned} &\cdot \left(-a_i y_i(t) + \sum_{j=1}^n b_{ij}^0 g_j(y_j(t)) \right. \\ &\quad + \sum_{j=1}^n c_{ij}^0 g_j(y_j(\gamma(t))) + \sum_{j=1}^n \phi_{ij}^b(y_j(t)) \\ &\quad \left. + \sum_{j=1}^n \phi_{ij}^c(y_j(\gamma(t))) \right) \leq \lambda \exp \{\lambda t\} |y_i(t)| \\ &\quad + \exp \{\lambda t\} \left(-a_i |y_i(t)| + \sum_{j=1}^n b_{ij}^+ |g_j(y_j(t))| \right. \\ &\quad + \sum_{j=1}^n c_{ij}^+ |g_j(y_j(\gamma(t)))| + \sum_{j=1}^n |\phi_{ij}^b(y_j(t))| \\ &\quad \left. + \sum_{j=1}^n |\phi_{ij}^c(y_j(\gamma(t)))| \right), \quad i = 1, 2, \dots, n. \end{aligned} \quad (65)$$

Applying Lemma 4,

$$\begin{aligned} \dot{z}_i(t) &\leq \exp \{\lambda t\} \left((\lambda - a_i) |y_i(t)| + \sum_{j=1}^n b_{ij}^+ \alpha |y_j(t)| \right. \\ &\quad + \sum_{j=1}^n c_{ij}^+ \alpha |y_j(\gamma(t))| + \sum_{j=1}^n b_{ij}^- \alpha |y_j(t)| \\ &\quad \left. + \sum_{j=1}^n c_{ij}^- \alpha |y_j(\gamma(t))| \right), \quad i = 1, 2, \dots, n. \end{aligned} \quad (66)$$

Let

$$V(t) = \sum_{i=1}^n z_i(t); \quad (67)$$

then, for $t \neq \eta_k, k \in N$,

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^n \dot{z}_i(t) \leq \exp \{\lambda t\} \left(\sum_{i=1}^n (\lambda - a_i) |y_i(t)| \right. \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n b_{ij}^+ \alpha |y_j(t)| + \sum_{i=1}^n \sum_{j=1}^n c_{ij}^+ \alpha |y_j(\gamma(t))| \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n b_{ij}^- \alpha |y_j(t)| + \sum_{i=1}^n \sum_{j=1}^n c_{ij}^- \alpha |y_j(\gamma(t))| \Big) \\ &= \exp \{\lambda t\} \left(\sum_{i=1}^n (\lambda - a_i) |y_i(t)| + \sum_{i=1}^n \sum_{j=1}^n b_{ji}^+ \alpha |y_i(t)| \right. \\ &\quad \left. + \sum_{i=1}^n \sum_{j=1}^n c_{ji}^+ \alpha |y_i(\gamma(t))| + \sum_{i=1}^n \sum_{j=1}^n b_{ji}^- \alpha |y_i(t)| \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^n \sum_{j=1}^n c_{ji}^- \alpha |y_i(\gamma(t))| \Big) = \exp\{\lambda t\} \\
& \cdot \left[\sum_{i=1}^n \left(\lambda - a_i + \sum_{j=1}^n \alpha (b_{ji}^+ + b_{ji}^-) \right) |y_i(t)| \right. \\
& + \sum_{i=1}^n \left(\sum_{j=1}^n \alpha (c_{ji}^+ + c_{ji}^-) \right) |y_i(\gamma(t))| \Big] \leq \exp\{\lambda t\} \\
& \cdot [(\lambda - S_b^-) \|y(t)\| + S_c \|y(\gamma(t))\|].
\end{aligned} \tag{68}$$

Based on Lemma 9,

$$\dot{V}(t) \leq \exp\{\lambda t\} (\lambda - S_b^- + \mu S_c) \|y(t)\|. \tag{69}$$

From (64),

$$\lambda - S_b^- + \mu S_c < 0; \tag{70}$$

then,

$$\dot{V}(t) \leq \exp\{\lambda t\} (\lambda - S_b^- + \mu S_c) \|y(t)\| \leq 0. \tag{71}$$

It is easy to see that

$$\begin{aligned}
V(t) &= \exp\{\lambda t\} \sum_{i=1}^n |y_i(t)| = \exp\{\lambda t\} \|y(t)\| \leq V(0) \\
&= \|y(0)\|;
\end{aligned} \tag{72}$$

hence

$$\|y(t)\| \leq \|y(0)\| \exp\{-\lambda t\}; \tag{73}$$

that is

$$\|x(t) - x^*\| \leq \|x(0) - x^*\| \exp\{-\lambda t\}, \tag{74}$$

for $t \geq 0$. This completes the proof. $\square$

3.4. Design Procedure of Autoassociative Memories. Given l ($l \leq 2^n$) vectors $u^{(1)}, u^{(2)}, \dots, u^{(l)}$ to be memorized, where $u^{(k)} \in \Omega$, for $k = 1, 2, \dots, l$, the design procedure of autoassociative memories is stated as follows:

- (1) Select matrix $\mathbb{E}$ as an identity matrix.
- (2) According to (29) and (34), choose appropriate S_i , a_i , T_i , $\bar{b}_{ij}$, and $\bar{c}_{ij}$.
- (3) Based on $S_i = \max(|\bar{b}_{ii}p - a_i d|, |\check{b}_{ii}p - a_i d|)$, $\bar{b}_{ij} = \max(|\hat{b}_{ij}|, |\check{b}_{ij}|)$, and $\bar{c}_{ij} = \max(|\hat{c}_{ij}|, |\check{c}_{ij}|)$, the values of $\hat{b}_{ij}$, $\check{b}_{ij}$, $\hat{c}_{ij}$, and $\check{c}_{ij}$ can be selected.
- (4) Calculate S_b , S_b^- , and S_c . Adjust the values of $\hat{b}_{ij}$, $\check{b}_{ij}$, $\hat{c}_{ij}$, and $\check{c}_{ij}$ to make sure that $S_c - S_b^- < 0$.
- (5) Solving the following inequalities: $\eta(S_b + 2S_c) \exp\{S_b \eta\} < 1$, $\eta(S_c + S_b(1 + S_c \eta) \exp\{S_b \eta\}) < 1$, we can get $0 < \eta < \bar{\eta}$.
- (6) Take a proper $\eta \in (0, \bar{\eta})$ such that $\mu S_c - S_b^- < 0$.

The output $w(t)$ will converge to the related memory pattern when matrices A , B , C , and $\mathbb{E}$ and scalar η are chosen as above.

3.5. Design Procedure of Heteroassociative Memories. Given l ($l \leq 2^n$) vectors to be memorized as $q^{(1)}, q^{(2)}, \dots, q^{(l)}$, which correspond to the external input patterns $u^{(1)}, u^{(2)}, \dots, u^{(l)}$, where $q^{(k)} \in \Omega$, $u^{(k)} \in \Omega$, $k = 1, 2, \dots, l$, set $\mathbb{E}U = Q$, where $U = [u^{(1)}, u^{(2)}, \dots, u^{(l)}]$, $Q = [q^{(1)}, q^{(2)}, \dots, q^{(l)}]$. It is clear that heteroassociative memories can be treated as autoassociative memories when transform matrix $\mathbb{E}$ is obtained. Matrices A , B , and C and scalar η can be selected as the method used in autoassociative memories. The transform matrix $\mathbb{E}$ is obtained by the following steps:

- (1) When $l = n$, that is, Q and U are square matrices, utilizing Matlab Toolbox, we can get inverse matrix of U as U^{-1} , and $\mathbb{E}$ can be obtained as $\mathbb{E} = QU^{-1}$.
- (2) When $l < n$, add $n - l$ proper column vectors to U and Q , which constructs new matrices $\bar{U}$ and $\bar{Q}$, respectively, such that the ranks of matrices $\bar{U}$ and $\bar{Q}$ are full. And the transform matrix $\mathbb{E}$ can be obtained by $\mathbb{E} = \bar{Q}\bar{U}^{-1}$.
- (3) When $n < l$, add $l - n$ proper row vectors to U and Q , which constructs new matrices $\tilde{U}$ and $\tilde{Q}$, such that the ranks of matrices $\tilde{U}$ and $\tilde{Q}$ are full. And the transform matrix $\mathbb{E}$ can be obtained by $\mathbb{E} = \tilde{Q}\tilde{U}^{-1}$. Note in particular, in this circumstance, that the dimensionality of each input and output pattern increases from n to l . And the front n values of each external input pattern carry out associative memories.

The output $w(t)$ will converge to the related memory pattern when matrices A , B , C , and $\mathbb{E}$ and scalar η are chosen as above.

4. Experimental Verification

In this section, in order to verify the effectiveness of the proposed results, several numerical examples are provided.

Example 12. The first example is state response of autoassociative memory.

Consider neural network model (5) with $n = 12$, where $p = d = 1$, $\eta = 2/9$, $T_i = 0.1$, $a_i = 0.4$, and

$$\begin{aligned}
\hat{b}_{ij} = \hat{c}_{ij} &= \begin{cases} 0.012, & i = j, \\ 0.01, & i \neq j, \end{cases} \\
\check{b}_{ij} = \check{c}_{ij} &= \begin{cases} 0.008, & i = j, \\ 0.006, & i \neq j, \end{cases}
\end{aligned} \tag{75}$$

with the activation functions

$$\begin{aligned}
f_i(x_i) &= -\frac{1}{2} (|x_i + 0.1| - |x_i - 0.1|) \\
&\quad + \frac{1}{2} (|x_i + 1.1| - |x_i - 1.1|)
\end{aligned} \tag{76}$$

FIGURE 1: State response in neural network model (5) with external input u .

for $i, j = 1, 2, \dots, 12$. Denote $u = q = (1, 1, 1, 1, -1, -1, 1, -1, -1, 1, 1, 1)^T$ as the external input vector and the memorized vector, respectively.

It follows from Theorem 11 that neural network model (5) has a unique equilibrium point, which is globally exponentially stable. The simulation result is shown in Figure 1.

It can be calculated easily that the equilibrium point in Figure 1 is

$$x^* = (2.71, 2.71, 2.71, 2.71, -2.31, -2.31, 2.71, -2.31, -2.31, 2.71, 2.71, 2.71)^T. \quad (77)$$

Clearly, the output pattern is

$$w^* = (1, 1, 1, 1, -1, -1, 1, -1, -1, 1, 1, 1)^T, \quad (78)$$

which is equivalent to the memorized pattern q . Hence, neural network model (5) can be implemented effectively as an autoassociative memory.

Remark 13. Because the equilibrium point is globally exponentially stable, the initial values can be selected randomly, and there is no influence on the performance of associative memories.

Example 14. The second example includes influences of a_i , d , and p on the location of equilibrium points.

The influences are divided into two cases to discuss.

Case 1. Let $p = 1$ be a constant and denote $u = q = (1, 1, 1, 1, -1, -1, 1, -1, -1, 1, 1, 1)^T$ as the external input vector and the memorized vector, respectively. Consider a_i and d

TABLE 1: The influences of a_i and d on the location of equilibrium points when $p = 1$ is a constant.

d	1.2	1.4	1.6	2.0	2.4	3.0
a_i						
0.3	v	v	v	v	v	—
0.4	v	v	v	v	—	—
0.5	v	v	v	—	—	—
0.6	v	v	—	—	—	—
0.7	v	—	—	—	—	—

“v” (“—”) denotes the equilibrium points located (not located) in the saturation regions.

TABLE 2: The influences of a_i and $\alpha = p/d$ on the location of equilibrium points when $d = 0.25$ is a constant.

$\alpha = \frac{p}{d}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	1	1.2	1.5
a_i						
0.40	—	v	v	v	v	v
0.45	—	—	v	v	v	v
0.60	—	—	—	v	v	v
0.70	—	—	—	—	v	v
0.90	—	—	—	—	—	v

“v” (“—”) denotes the equilibrium points located (not located) in the saturation region.

as variables in neural network model (5), taking $\eta = 1/9$, and the rest of the parameters are the same as those in Example 12. The results are shown in Table 1, from which we can see that the smaller the value of a_i , the bigger the value range d can take for the sake of memorized pattern, and vice versa.

Case 2. Let $d = 0.25$ be a constant and denote $u = q = (p, p, p, p, -p, -p, p, -p, -p, p, p, p)^T$ as the external input vector and the memorized vector, respectively, where $p > 0$. Consider a_i and p (or $\alpha = p/d$) as variables in neural network model (5), taking $\eta = 1/9$, and the rest of the parameters are the same as those in Example 12. The results are shown in Table 2, from which we can see that α (or p) should increase when a_i increases for the sake of memorized pattern, and vice versa.

Remark 15. From Tables 1 and 2, it can be concluded that the associative memories performance becomes better with the increase of $\alpha = p/d$ when a_i is fixed, but not vice versa.

Example 16. The third example includes the influences of a_i and perturbed u on the recall success rate of memorized patterns.

Let $u = (1, 1, 1, 1, -1, -1, 1, -1, -1, 1, 1, 1)^T$, $\eta = 1/9$, and consider a_i ($i = 1, 2, \dots, 12$) as variables in neural network model (5). The other parameters are the same as those in Example 12. In this example, the influences of a_i and perturbed u are analyzed on the recall success rate of memorized vectors. Experimental results are illustrated in Table 3, which makes it clear that the robustness of neural network model (5) for perturbed input u will become stronger as a_i is small enough.

TABLE 3: The influences of a_i and perturbed u on the recall success rate.

a_i	0.3	0.4	0.5	0.6	0.7
$0.9u$	100%	100%	100%	100%	100%
$0.84u$	100%	100%	100%	100%	66.67%
$0.63u$	100%	100%	100%	66.67%	0
$0.52u$	100%	100%	66.67%	0	
$0.41u$	100%	66.67%	0		
$0.25u$	66.67%	0			
$0.24u$	0

FIGURE 2: Typical evolutions of output pattern with input vector $u = q$ under three random initial values.

Remark 17. Table 3 illustrates that the robustness of associative memories on perturbed u becomes stronger with the decrease of a_i .

Example 18. The fourth example includes the application of autoassociative memory without input perturbation.

According to neural network model (5), design an autoassociative memory to memorize pattern “T” which is represented by a 4×3 -pixel image (black pixel = 1; white pixel = -1) and the corresponding input pattern is $u = (1, 1, 1, -1, 1, -1, -1, 1, -1, -1, 1, -1)^T$. The parameters are the same as those in Example 12.

Simulation results with three random initial values are depicted in Figure 2, where the dark grey block indicates that the value of the related component of output $w(t)$ is in $(0, 1)$ and the light grey block indicates that the value of the related component of output $w(t)$ is in $(-1, 0)$. Obviously, the output pattern $w(t)$ converges to the memorized pattern correctly.

Example 19. The fifth example includes the application of autoassociative memory with input perturbation.

The aim of this experiment is to discuss the influence of input pattern perturbation on the output pattern. The parameters are the same as those in Example 12. Choose pattern “T” to be memorized, and the perturbed vector is selected as $u = k * q$, where $q = (1, 1, 1, -1, 1, -1, -1, 1, -1, -1, 1, -1)^T$ is the corresponding memorized vector and k is a perturbation coefficient.

FIGURE 3: Typical evolutions of output pattern with input vector $u = 0.52 * q$ under three random initial values. FIGURE 4: Typical evolutions of output pattern with input vector $u = 0.41 * q$ under three random initial values.

In what follows, the perturbed vector u is imposed on neural network model (5) as input vector with $k = 0.52$ and $k = 0.41$, respectively. Simulation results are shown in Figures 3 and 4. In Figure 3, the output pattern is exactly the same as the memorized pattern, and in Figure 4 the output pattern is different from the memorized pattern although they can be distinguished from the outlines.

Remark 20. From the analysis above, we can see that the output pattern will converge to the memorized pattern when the perturbation coefficient is not small enough. In fact, neural network model (5) is implemented well as an associative memory if the coefficient satisfying $k \geq 0.52$ and the other parameters are the same as those in Example 19.

Example 21. The sixth example includes the application of heteroassociative memory.

Consider neural network model (1), and the parameters are the same as those in Example 12 except the transform matrix $\mathbb{E}$, which will be given later. The aim of this experiment is to design a heteroassociative memory to memorize pattern “U” when the input pattern is “C.” It is clear that the input vector is $u = (1, 1, 1, 1, -1, -1, 1, -1, -1, 1, 1, 1)^T$ and the corresponding output vector is $q = (1, -1, 1, 1, -1, 1, 1, -1, 1, 1, 1, 1)^T$.

According to the heteroassociative memories design procedure, the matrices $\bar{U}$ and $\bar{Q}$ are constructed as

$$\bar{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (79)$$

$$\bar{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively.

Using the Matlab Toolbox, we can have

$$\mathbb{E} = \bar{Q}\bar{U}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (80)$$

FIGURE 5: Typical evolutions of pattern “U” with input pattern “C” under three random initial values.

The simulation results are depicted in Figure 5, which shows that the output pattern “U” is memorized when the input pattern is “C” and the transform matrix $\mathbb{E}$ is selected as above. This indicates that neural network model (1) can perform well as a heteroassociative memory under a proper transform matrix $\mathbb{E}$, which can be easily obtained by using the Matlab Toolbox.

Remark 22. Simulation results in Figure 5 show that the associative memories model presented in this paper can be implemented well as autoassociative memories and heteroassociative memories. This model possesses the resistance capacity against external input perturbations and can be regarded as an extension and complement to the relevant results [10, 11, 16].

5. Concluding Remarks

Analysis and design of associative memories for recurrent neural networks have drawn considerable attention, whereas analysis and design of associative memories for memristive neural networks with deviating argument have not been well investigated. This paper presents a general class of memristive neural networks with deviating argument and discusses the analysis and design of autoassociative memories and heteroassociative memories. Meanwhile, the robustness of this class of neural networks is demonstrated when external input patterns are seriously perturbed. To the best of the author’s knowledge, this is the first theoretical result on associative memories for memristive neural networks with deviating argument. Future works may aim at (1) studying associative memories based on multistability of memristive neural networks with deviating argument; (2) analyzing associative memories based on discrete-time memristive neural networks with deviating argument; (3) exploring high-capacity associative memories for memristive neural networks with deviating argument.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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