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Research Article

On Dynamical Behavior of a Friction-Induced Oscillator with 2-DOF on a Speed-Varying Traveling Belt

Jinjun Fan, Shuangshuang Li, and Ge Chen

School of Mathematical Sciences, Shandong Normal University, Jinan 250014, China

Correspondence should be addressed to Jinjun Fan; fjj18@126.com

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The dynamical behavior of a friction-induced oscillator with 2-DOF on a speed-varying belt is investigated by using the flow switchability theory of discontinuous dynamical systems. The mechanical model consists of two masses and a speed-varying traveling belt. Both of the masses on the traveling belt are connected with three linear springs and three dampers and are harmonically excited. Different domains and boundaries for such system are defined according to the friction discontinuity. Based on the above domains and boundaries, the analytical conditions of the passable motions, stick motions, and grazing motions for the friction-induced oscillator are obtained mathematically. An analytical prediction of periodic motions is performed through the mapping dynamics. With appropriate mapping structure, the simulations of the stick and nonstick motions in the two-degree friction-induced oscillator are illustrated for a better understanding of the motion complexity.

1. Introduction

In mechanical engineering, the friction contact between two surfaces of two bodies is an important connection and friction phenomenon widely exists. In recent years, much research effort in science and engineering has focussed on nonsmooth dynamical systems [1–11]. This problem can go back to the 30s of last century. In 1930, Hartog [1] investigated the nonstick periodic motion of the forced linear oscillator with Coulomb and viscous damping. In 1960, Levitan [2] proved the existence of periodic motions in a friction oscillator with the periodically driven base. In 1964, Filippov [3] investigated the motion in the Coulomb friction oscillator and presented differential equation theory with discontinuous right-hand sides. The investigations of such discontinuous differential equations were summarized in Filippov [4]. However, Filippov's theory mainly focused on the existence and uniqueness of the solutions for nonsmooth dynamical systems. Such a differential equation theory with discontinuity is difficult to apply to practical problems. In 2003, Awrejcewicz and Olejnik [5] studied a two-degree-of-freedom autonomous system with friction numerically and illustrated some interesting examples of stick-slip regular and chaotic dynamics. In 2014, Pascal [6] discussed a system composed of two masses connected by linear springs: one of the masses is in contact with a rough surface and the other is also subjected to a harmonic external force. Several periodic orbits were obtained in closed form, and symmetry in space and time had been proved for some of these periodic solutions. More discussion about discontinuous system can refer to [7–11].

However, a lot of questions caused by the discontinuity (i.e., the local singularity and the motion switching on the separation boundary) were not discussed in detail. So the further investigation on discontinuous dynamical systems should be deepened and expanded. In 2005–2012, Luo [12–17] developed a general theory to define real, imaginary, sink, and source flows and to handle the local singularity and flow switchability in discontinuous dynamical systems. By using this theory, a lot of discontinuous systems were discussed (e.g., [18–20]). Luo and Gegg [18] presented the force criteria for the stick and nonstick motions for 1-DOF (degree of freedom) oscillator moving on the belt with dry friction. In 2009, Luo and Wang [19] investigated the analytical conditions for stick and nonstick motions in 2-DOF friction induced oscillator moving on two belts. Velocity and force responses for stick and nonstick motions in such system were illustrated for a better understanding of the motion complexity. Based on this improved model, which consists of two masses moving on one speed-varying traveling belt and in which the two masses are connected with three linear springs and three dampers and are exerted by two periodic excitations, nonlinear dynamics mechanism of such a 2-DOF oscillator system will be investigated.

In this paper, a model of frictional-induced oscillator with two degrees of freedom (2-DOF) on a speed-varying belt is proposed in which multiple discontinuity boundaries exist: they are caused by the presence of friction between the mass and the belt. The model allows a simple representation of engineering applications with multiple nonsmooth characteristics as for instance friction wheels or slipping mechanisms in multiblock structures. The main goal is to study the analytical conditions of motion switching and stick motions of the oscillator on the corresponding boundaries by using the theory of discontinuous dynamical systems. Based on the discontinuity, domain partitions and boundaries will be defined and the analytical conditions of the passable motions, stick motions, and grazing motions for the friction-induced oscillator are obtained mathematically, from which it can be seen that such oscillator has more complicated and rich dynamical behaviors. An analytical condition of periodic motions is performed through the mapping dynamics. With appropriate mapping structure, the simulations of the stick and nonstick motions of the oscillator with 2-DOF are illustrated for a better understanding of the motion complexity. There are more simulations about such oscillator to be discussed in future.

2. Preliminaries

For convenience, the fundamental theory on flow switchability of discontinuous dynamical systems will be presented; that is, concepts of G-functions and the decision theorems of semipassable flow, sink flow, and grazing flow to a separation boundary are stated in the following, respectively (see [16, 17]).

Assume that Ω is a bounded simply connected domain in R^n and its boundary $\partial\Omega \subset R^{n-1}$ is a smooth surface.

Consider a dynamic system consisting of N subdynamic systems in a universal domain $\Omega \subset R^n$. The universal domain is divided into N accessible subdomains Ω_α ($\alpha \in I$) and the inaccessible domain Ω_0. The union of all the accessible subdomains is $\bigcup_{\alpha\in I}\Omega_\alpha$ and $\Omega = \bigcup_{\alpha\in I}\Omega_\alpha \cup \Omega_0$ is the universal domain. On the αth open subdomain Ω_α, there is a C^{r_α}-continuous system ($r_\alpha \geq 1$) in form of

$$\dot{\mathbf{x}}^{(\alpha)} \equiv \mathbf{F}^{(\alpha)}\left(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha\right) \in R^n,$$
$$\mathbf{x}^{(\alpha)} = \left(x_1^{(\alpha)}, x_2^{(\alpha)}, \ldots, x_n^{(\alpha)}\right)^{\mathrm{T}} \in \Omega_\alpha. \tag{1}$$

The time is t and $\dot{\mathbf{x}} = d\mathbf{x}/dt$. In an accessible subdomain Ω_α, the vector field $\mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha)$ with parameter vector $\mathbf{p}_\alpha = (p_\alpha^{(1)}, p_\alpha^{(2)}, \ldots, p_\alpha^{(l)})^{\mathrm{T}} \in R^l$ is C^{r_α}-continuous ($r_\alpha \geq 1$) in $\mathbf{x} \in \Omega_\alpha$ and for all time t.

The flow on the boundary $\partial\Omega_{ij} = \Omega_i \cap \Omega_j$ can be determined by

$$\dot{\mathbf{x}}^{(0)} \equiv \mathbf{F}^{(0)}\left(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}\right)$$
$$\text{with } \varphi_{ij}\left(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}\right) = 0, \tag{2}$$

where $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})^{\mathrm{T}}$. With specific initial conditions, one always obtains different flows on $\varphi_{ij}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) = \varphi_{ij}(\mathbf{x}_0^{(0)}, t_0, \boldsymbol{\lambda})$.

Consider a dynamic system in (1) in domain Ω_α ($\alpha \in \{i, j\}$) which has a flow $\mathbf{x}_t^{(\alpha)} = \boldsymbol{\Phi}^{(\alpha)}(t_0, \mathbf{x}_0^{(\alpha)}, \mathbf{p}_\alpha, t)$ with an initial condition $(t_0, \mathbf{x}_0^{(\alpha)})$, and on the boundary $\partial\Omega_{ij}$, there is an enough smooth flow $\mathbf{x}_t^{(0)} = \boldsymbol{\Phi}^{(0)}(t_0, \mathbf{x}_0^{(0)}, \boldsymbol{\lambda}, t)$ with an initial condition $(t_0, \mathbf{x}_0^{(0)})$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t-\varepsilon, t)$ and $(t, t+\varepsilon]$ for flow $\mathbf{x}_t^{(\alpha)}$ ($\alpha \in \{i, j\}$) and the flow $\mathbf{x}_t^{(\alpha)}$ approaches the separation boundary at time t_m, that is, $\mathbf{x}_{t_{m\pm}}^{(\alpha)} = \mathbf{x}_m = \mathbf{x}_{t_m}^{(0)}$, where $\mathbf{x}_{t_{m\pm}}^{(\alpha)} = \mathbf{x}^{(\alpha)}(t_m\pm)$, $\mathbf{x}_{t_m}^{(0)} = \mathbf{x}^{(0)}(t_m)$, and $\mathbf{x}_m \in \partial\Omega_{ij}$.

Definition 1. The G-functions $G_{\partial\Omega_{ij}}^{(\alpha)}$ of the flow $\mathbf{x}_t^{(\alpha)}$ to the flow $\mathbf{x}_t^{(0)}$ on the boundary $\partial\Omega_{ij}$ are defined as

$$G_{\partial\Omega_{ij}}^{(\alpha)}\left(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}\right) = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}}\left(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}\right) \cdot \left[\mathbf{F}^{(\alpha)}\left(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha\right) - \mathbf{F}^{(0)}\left(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}\right)\right]\Big|_{(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_{m\pm})}, \tag{3}$$

where $\mathbf{x}_m^{(0)} = \mathbf{x}^{(0)}(t_m)$, $\mathbf{x}_{m\pm}^{(\alpha)} = \mathbf{x}^{(\alpha)}(t_{m\pm})$, $t_{m\pm} \equiv t_m \pm 0$ is to represent the quantity in the domain rather than on the boundary, and $G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda})$ is a time rate of the inner product of displacement difference and the normal direction $\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}, t_m, \boldsymbol{\lambda})$.

Definition 2. The kth-order G-functions of the domain flow $\mathbf{x}_t^{(\alpha)}$ to the boundary flow $\mathbf{x}_t^{(0)}$ in the normal direction of $\partial\Omega_{ij}$ are defined as

$$G_{\partial\Omega_{ij}}^{(k,\alpha)}\left(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}\right) = \sum_{s=1}^{k+1} C_{k+1}^{s} D_0^{k+1-s} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[D_\alpha^{s-1}\mathbf{F}\left(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha\right) - D_0^{s-1}\mathbf{F}^{(0)}\left(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}\right)\right]\Big|_{(\mathbf{x}_m^{(0)}, \mathbf{x}_{m\pm}^{(\alpha)}, t_{m\pm})}, \tag{4}$$

where the total derivative operators are defined as

$$D_0(\cdot) \equiv \frac{\partial(\cdot)}{\partial\mathbf{x}^{(0)}}\dot{\mathbf{x}}^{(0)} + \frac{\partial(\cdot)}{t},$$
$$D_\alpha(\cdot) \equiv \frac{\partial(\cdot)}{\partial\mathbf{x}^{(\alpha)}}\dot{\mathbf{x}}^{(\alpha)} + \frac{\partial(\cdot)}{t}. \tag{5}$$

For $k = 0$, we have

$$G_{\partial\Omega_{ij}}^{(k,\alpha)}\left(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}\right) = G_{\partial\Omega_{ij}}^{(\alpha)}\left(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}\right). \tag{6}$$

Definition 3. For a discontinuous dynamical system in (1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_m - \varepsilon, t_m)$ and $(t_m, t_m + \varepsilon]$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$; if

$$\begin{aligned}&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\left(\mathbf{x}^{(0)}_{m-\varepsilon}\right)\cdot\left[\mathbf{x}^{(0)}_{m-\varepsilon}-\mathbf{x}^{(i)}_{m-\varepsilon}\right]>0,\\&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\left(\mathbf{x}^{(0)}_{m+\varepsilon}\right)\cdot\left[\mathbf{x}^{(j)}_{m+\varepsilon}-\mathbf{x}^{(0)}_{m+\varepsilon}\right]>0\end{aligned} \tag{7}$$

for $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$, then a resultant flow of two flows $\mathbf{x}^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) is a semipassable flow from domain Ω_i to Ω_j at point $(\mathbf{x}_m, t_m)$ to boundary $\partial\Omega_{ij}$, where $\mathbf{x}^{(0)}_{m\pm\varepsilon} = \mathbf{x}^{(0)}(t_m \pm \varepsilon)$, $\mathbf{x}^{(\alpha)}_{m\pm\varepsilon} = \mathbf{x}^{(\alpha)}(t_m \pm \varepsilon)$.

To simplify notation usage, the symbols $t_{m\pm\varepsilon}$ represent $t_m \pm \varepsilon$ in next paragraphs.

Lemma 4. *For a discontinuous dynamical system in (1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha \in \{i, j\}$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$, two flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$ and $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$-continuous for time t, respectively, and $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$ ($r_\alpha \geq 1$, $\alpha \in \{i, j\}$). The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ to the boundary $\partial\Omega_{ij}$ are semipassable from domain Ω_i to Ω_j if and only if*

either

$$\begin{aligned}&G^{(i)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}\right) > 0,\\&G^{(j)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}\right) > 0\\&\qquad\qquad for\ \mathbf{n}_{\partial\Omega_{ij}} \longrightarrow \Omega_j,\end{aligned} \tag{8}$$

or

$$\begin{aligned}&G^{(i)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}\right) < 0,\\&G^{(j)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m+}, \mathbf{p}_j, \boldsymbol{\lambda}\right) < 0\\&\qquad\qquad for\ \mathbf{n}_{\partial\Omega_{ij}} \longrightarrow \Omega_i.\end{aligned} \tag{9}$$

Lemma 5. *For a discontinuous dynamical system in (1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha \in \{i, j\}$). For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m-})$. Both flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$-continuous for time t and $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$ ($r_\alpha \geq 1$, $\alpha \in \{i, j\}$). The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ to the boundary $\partial\Omega_{ij}$ are sink flow if and only if*

either

$$\begin{aligned}&G^{(i)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}\right) > 0,\\&G^{(j)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m-}, \mathbf{p}_j, \boldsymbol{\lambda}\right) < 0\\&\qquad\qquad for\ \mathbf{n}_{\partial\Omega_{ij}} \longrightarrow \Omega_j,\end{aligned} \tag{10}$$

or

$$\begin{aligned}&G^{(i)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}\right) < 0,\\&G^{(j)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m-}, \mathbf{p}_j, \boldsymbol{\lambda}\right) > 0\\&\qquad\qquad for\ \mathbf{n}_{\partial\Omega_{ij}} \longrightarrow \Omega_i.\end{aligned} \tag{11}$$

Lemma 6. *For a discontinuous dynamical system in (1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha \in \{i, j\}$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$ and $C^{r_j}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous for time t, respectively, and $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$ ($r_\alpha \geq 2$, $\alpha \in \{i, j\}$). The sliding fragmentation bifurcation of the nonpassable flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ of the first kind at point $(\mathbf{x}_m, t_m)$ switching to the passable flow on the boundary $\widetilde{\partial\Omega_{ij}}$ occurs if and only if*

$$\begin{aligned}&G^{(j)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}\right) = 0,\\&G^{(i)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}\right) > 0 \quad for\ \mathbf{n}_{\partial\Omega_{ij}} \longrightarrow \Omega_j,\\&G^{(i)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}\right) < 0 \quad for\ \mathbf{n}_{\partial\Omega_{ij}} \longrightarrow \Omega_i,\\&G^{(1,j)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}\right) > 0 \quad for\ \mathbf{n}_{\partial\Omega_{ij}} \longrightarrow \Omega_j,\\&G^{(1,j)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}\right) < 0 \quad for\ \mathbf{n}_{\partial\Omega_{ij}} \longrightarrow \Omega_i.\end{aligned} \tag{12}$$

Lemma 7. *For a discontinuous dynamical system in (1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha \in \{i, j\}$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$ and $C^{r_j}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous for time t, respectively, and $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$ ($r_\alpha \geq 2$, $\alpha \in \{i, j\}$). The sliding bifurcation of the passable flow of $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ switching to the sink flow on the boundary $\overrightarrow{\partial\Omega_{ij}}$ occurs if and only if*

$$\begin{aligned}&G^{(j)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}\right) = 0,\\&G^{(i)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}\right) > 0 \quad for\ \mathbf{n}_{\partial\Omega_{ij}} \longrightarrow \Omega_j,\\&G^{(i)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m-}, \mathbf{p}_i, \boldsymbol{\lambda}\right) < 0 \quad for\ \mathbf{n}_{\partial\Omega_{ij}} \longrightarrow \Omega_i,\\&G^{(1,j)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}\right) > 0 \quad for\ \mathbf{n}_{\partial\Omega_{ij}} \longrightarrow \Omega_j,\\&G^{(1,j)}_{\partial\Omega_{ij}}\left(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_j, \boldsymbol{\lambda}\right) < 0 \quad for\ \mathbf{n}_{\partial\Omega_{ij}} \longrightarrow \Omega_i.\end{aligned} \tag{13}$$

Lemma 8. *For a discontinuous dynamical system in (1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha \in \{i, j\}$). For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$.*

The flow $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r_\alpha}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$*-continuous* ($r_\alpha \geq 2$) *for time* t, *and* $\|d^{r_\alpha+1}\mathbf{x}^{(\alpha)}/dt^{r_\alpha+1}\| < \infty$ ($\alpha \in \{i,j\}$). *A flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is tangential to the boundary* $\partial\Omega_{ij}$ *if and only if*

$$G^{(0,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) = 0 \quad \textit{for } \alpha \in \{i,j\}; \tag{14}$$

either

$$G^{(1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) < 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \longrightarrow \Omega_\beta, \tag{15}$$

or

$$G^{(1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \boldsymbol{\lambda}) > 0 \quad \textit{for } \mathbf{n}_{\partial\Omega_{\alpha\beta}} \longrightarrow \Omega_\alpha. \tag{16}$$

More detailed theory on the flow switchability such as the definitions or theorems about various flow passability in discontinuous dynamical systems can be referred to [16, 17].

3. Physical Model

Consider a friction-induced oscillator with two degrees of freedom on the speed-varying traveling belt, as shown in Figure 1. The system consists of two masses m_α ($\alpha = 1,2$), which are connected with three linear springs of stiffness k_α ($\alpha = 1,2,3$) and three dampers of coefficient r_α ($\alpha = 1,2,3$). Both of masses move on the belt with varying speed $V(t)$. Two periodic excitations $A_\alpha + B_\alpha \cos\Omega t$ ($\alpha = 1,2$) with frequency Ω, amplitudes B_α ($\alpha = 1,2$), and constant forces A_α ($\alpha = 1,2$) are exerted on the two masses, respectively.

There exist friction forces between the two masses and the belt, so the two masses can move or stay on the surface of the belt. Let $V(t)$ be the speed of the belt and

$$V(t) = V_0 \cos(\Omega t + \beta) + V_1, \tag{17}$$

where Ω and β are the oscillation frequency and primary phase of the traveling belt, respectively, V_0 is the oscillation amplitude of the traveling belt, and V_1 is constant.

Further, the friction force shown in Figure 2 is described by

$$F_f^{(\alpha)}(\dot{x}_\alpha)\begin{cases} = \mu_k F_N^{(\alpha)}, & \dot{x}_\alpha > V(t); \\ \in \left[-\mu_k F_N^{(\alpha)}, \mu_k F_N^{(\alpha)}\right], & \dot{x}_\alpha = V(t); \\ = -\mu_k F_N^{(\alpha)}, & \dot{x}_\alpha < V(t), \end{cases} \tag{18}$$

where $\dot{x}_\alpha = dx_\alpha/dt$, μ_k is the coefficient of friction between m_α and the belt, $F_N^{(\alpha)} = m_\alpha g$ ($\alpha = 1,2$), and g is the acceleration of gravity. The nonfriction force acting on the mass m_α in the x_α-direction is defined as

$$\begin{aligned} F_s^{(\alpha)} &= B_\alpha \cos\Omega t + A_\alpha - r_\alpha \dot{x}_\alpha - r_3\left(\dot{x}_\alpha - \dot{x}_\beta\right) - k_\alpha x_\alpha \\ &\quad - k_3\left(x_\alpha - x_\beta\right), \end{aligned} \tag{19}$$

where $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$. From now on, $F_f^{(\alpha)} = \mu_k \cdot F_N^{(\alpha)}$.

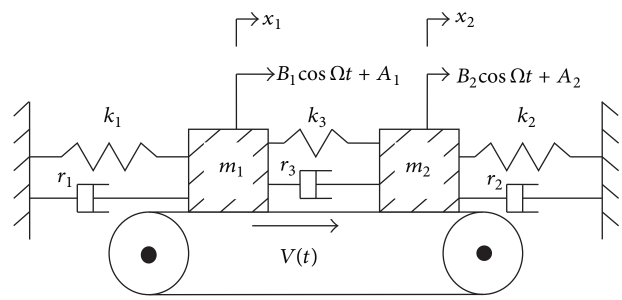

Figure 1: Physical model.

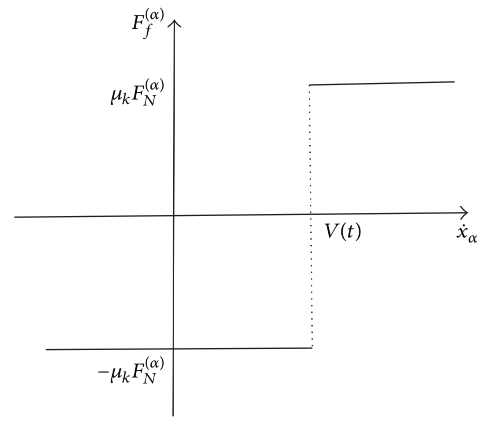

Figure 2: Force of friction.

From the previous discussion, there are four cases of motions:

Case 1 (nonstick motion ($\dot{x}_\alpha \neq V(t)$) ($\alpha = 1,2$)). When $F_s^{(\alpha)}$ can overcome the static friction force $F_f^{(\alpha)}$ (i.e., $|F_s^{(\alpha)}| > |F_f^{(\alpha)}|$, $\alpha = 1,2$), the mass m_α has relative motion to the belt, that is,

$$\dot{x}_\alpha \neq V(t), \quad (\alpha = 1,2). \tag{20}$$

For the nonstick motion of the mass m_α ($\alpha = 1,2$), the total force acting on the mass m_α is

$$\begin{aligned} F^{(\alpha)} &= F_s^{(\alpha)} - F_f^{(\alpha)} \operatorname{sgn}(\dot{x}_\alpha - V(t)) \\ &= B_\alpha \cos\Omega t + A_\alpha - r_\alpha \dot{x}_\alpha - r_3\left(\dot{x}_\alpha - \dot{x}_\beta\right) - k_\alpha x_\alpha \\ &\quad - k_3\left(x_\alpha - x_\beta\right) - F_f^{(\alpha)} \operatorname{sgn}(\dot{x}_\alpha - V(t)), \end{aligned} \tag{21}$$

and the equations of nonstick motion for the 2-DOF dry friction induced oscillator are

$$\begin{aligned} &m_\alpha \ddot{x}_\alpha + r_\alpha \dot{x}_\alpha + r_3\left(\dot{x}_\alpha - \dot{x}_\beta\right) + k_\alpha x_\alpha + k_3\left(x_\alpha - x_\beta\right) \\ &\quad = B_\alpha \cos\Omega t + A_\alpha - F_f^{(\alpha)} \operatorname{sgn}(\dot{x}_\alpha - V(t)), \end{aligned} \tag{22}$$

where $\alpha, \beta \in \{1,2\}$, $\alpha \neq \beta$.

Case 2 (single stick motion ($\dot{x}_1 = V(t)$, $\dot{x}_2 \neq V(t)$)). When $F_s^{(1)}$ cannot overcome the static friction force $F_f^{(1)}$ (i.e., $|F_s^{(1)}| \leq |F_f^{(1)}|$), mass m_1 does not have any relative motion to the belt, that is,

$$\begin{aligned} \dot{x}_1 &= V(t), \\ \ddot{x}_1 &= \dot{V}(t) = -V_0\Omega\sin(\Omega t+\beta); \end{aligned} \tag{23}$$

meanwhile, when $F_s^{(2)}$ can overcome the static friction force $F_f^{(2)}$ (i.e., $|F_s^{(2)}| > |F_f^{(2)}|$), the mass m_2 has relative motion to the belt, that is,

$$\begin{aligned} &\dot{x}_2 \neq V(t), \\ &m_2\ddot{x}_2 + r_2\dot{x}_2 + r_3(\dot{x}_2 - \dot{x}_1) + k_2x_2 + k_3(x_2 - x_1) \\ &\quad = B_2\cos\Omega t + A_2 - F_f^{(2)}\operatorname{sgn}(\dot{x}_2 - V(t)). \end{aligned} \tag{24}$$

Case 3 (single stick motion ($\dot{x}_2 = V(t)$, $\dot{x}_1 \neq V(t)$)). When $F_s^{(2)}$ cannot overcome the static friction force $F_f^{(2)}$ (i.e., $|F_s^{(2)}| \leq |F_f^{(2)}|$), mass m_2 does not have any relative motion to the belt, that is,

$$\begin{aligned} \dot{x}_2 &= V(t), \\ \ddot{x}_2 &= \dot{V}(t) = -V_0\Omega\sin(\Omega t+\beta); \end{aligned} \tag{25}$$

meanwhile, when $F_s^{(1)}$ can overcome the static friction force $F_f^{(1)}$ (i.e., $|F_s^{(1)}| > |F_f^{(1)}|$), mass m_1 has relative motion to the belt, that is,

$$\begin{aligned} &\dot{x}_1 \neq V(t), \\ &m_1\ddot{x}_1 + r_1\dot{x}_1 + r_3(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_3(x_1 - x_2) \\ &\quad = B_1\cos\Omega t + A_1 - F_f^{(1)}\operatorname{sgn}(\dot{x}_1 - V(t)). \end{aligned} \tag{26}$$

Case 4 (double stick motions ($\dot{x}_\alpha = V(t)$) ($\alpha = 1,2$)). When $F_s^{(\alpha)}$ cannot overcome the static friction force $F_f^{(\alpha)}$ (i.e., $|F_s^{(\alpha)}| \leq |F_f^{(\alpha)}|$), mass m_α does not have any relative motion to the belt, that is,

$$\begin{aligned} \dot{x}_\alpha &= V(t), \\ \ddot{x}_\alpha &= \dot{V}(t) = -V_0\Omega\sin(\Omega t+\beta). \end{aligned} \tag{27}$$

Integrating (17) leads to the displacement of the belt:

$$\begin{aligned} x(t) = {}&\frac{V_0}{\Omega}\left[\sin(\Omega t+\beta) - \sin(\Omega t_i+\beta)\right] + V_1(t-t_i) \\ &+ x_{t_i}, \end{aligned} \tag{28}$$

where $t > t_i$ and $x_{t_i} = x(t_i)$.

4. Domains and Boundaries

Due to frictions between the mass m_α ($\alpha = 1,2$) and the traveling belt, the motions become discontinuous and more complicated. The phase space of the discontinuous dynamical system is divided into four 4-dimensional domains.

The state variables and vector fields are introduced by

$$\begin{aligned} \mathbf{x} &= (x_1, \dot{x}_1, x_2, \dot{x}_2)^{\mathrm{T}} = (x_1, y_1, x_2, y_2)^{\mathrm{T}}, \\ \mathbf{F} &= (y_1, F_1, y_2, F_2)^{\mathrm{T}}. \end{aligned} \tag{29}$$

By the state variables, the domains are defined as

$$\begin{aligned} \Omega_1 &= \{(x_1, y_1, x_2, y_2) \mid y_1 > V(t),\ y_2 > V(t)\}, \\ \Omega_2 &= \{(x_1, y_1, x_2, y_2) \mid y_1 > V(t),\ y_2 < V(t)\}, \\ \Omega_3 &= \{(x_1, y_1, x_2, y_2) \mid y_1 < V(t),\ y_2 < V(t)\}, \\ \Omega_4 &= \{(x_1, y_1, x_2, y_2) \mid y_1 < V(t),\ y_2 > V(t)\} \end{aligned} \tag{30}$$

and the corresponding boundaries are defined as

$$\begin{aligned} \partial\Omega_{12} = \partial\Omega_{21} = \{&(x_1, y_1, x_2, y_2) \mid \varphi_{12} = \varphi_{21} = y_2 \\ &- V(t) = 0,\ y_1 \geq V(t)\}, \\ \partial\Omega_{23} = \partial\Omega_{32} = \{&(x_1, y_1, x_2, y_2) \mid \varphi_{23} = \varphi_{32} = y_1 \\ &- V(t) = 0,\ y_2 \leq V(t)\}, \\ \partial\Omega_{34} = \partial\Omega_{43} = \{&(x_1, y_1, x_2, y_2) \mid \varphi_{34} = \varphi_{43} = y_2 \\ &- V(t) = 0,\ y_1 \leq V(t)\}, \\ \partial\Omega_{14} = \partial\Omega_{41} = \{&(x_1, y_1, x_2, y_2) \mid \varphi_{12} = \varphi_{21} = y_1 \\ &- V(t) = 0,\ y_2 \geq V(t)\}. \end{aligned} \tag{31}$$

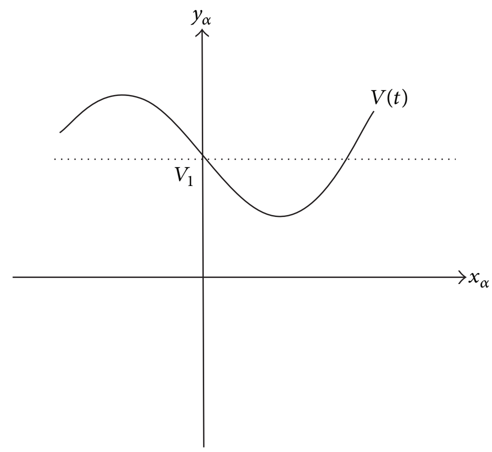

FIGURE 3: Phase plane of m_α.

The phase plane of m_α is shown in Figure 3.

The 2-dimensional edges of the 3-dimensional boundaries are defined by

$$\angle\Omega_{\alpha_1\alpha_2\alpha_3} = \partial\Omega_{\alpha_1\alpha_2} \cap \partial\Omega_{\alpha_2\alpha_3} = \bigcap_{i=1}^{3}\Omega_{\alpha_i}, \tag{32}$$

where $\alpha_i \in \{1,2,3,4\}$, $i = 1,2,3$, and $\alpha_i \neq \alpha_j$ $(i \neq j)$, $i, j \in \{1,2,3\}$. The intersection of four 2-dimensional edges is

$$\begin{aligned} \angle\Omega_{1234} &= \bigcap \angle\Omega_{\alpha_1\alpha_2\alpha_3} = \{(x_1, y_1, x_2, y_2) \mid \varphi_{12} = \varphi_{34} \\ &= y_2 - V(t) = 0,\ \varphi_{23} = \varphi_{14} = y_1 - V(t) = 0\}. \end{aligned} \tag{33}$$

From the above discussion, the motion equations of the oscillator described in Section 3 in absolute coordinates are

$$\begin{aligned} \dot{\mathbf{x}}^{(\alpha)} &= \mathbf{F}^{(\alpha)}\left(\mathbf{x}^{(\alpha)}, t\right) \quad \text{in } \Omega_\alpha, \\ \dot{\mathbf{x}}^{(\alpha_1\alpha_2)} &= \mathbf{F}^{(\alpha_1\alpha_2)}\left(\mathbf{x}^{(\alpha_1\alpha_2)}, t\right) \quad \text{on } \partial\Omega_{\alpha_1\alpha_2}, \\ \dot{\mathbf{x}}^{(\alpha_1\alpha_2\alpha_3)} &= \mathbf{F}^{(\alpha_1\alpha_2\alpha_3)}\left(\mathbf{x}^{(\alpha_1\alpha_2\alpha_3)}, t\right) \quad \text{on } \partial\Omega_{\alpha_1\alpha_2\alpha_3}, \\ \mathbf{x}^{(\alpha)} &= \mathbf{x}^{(\alpha_1\alpha_2)} = \mathbf{x}^{(\alpha_1\alpha_2\alpha_3)} = \left(x_1, y_1, x_2, y_2\right)^{\mathrm{T}}, \\ \mathbf{F}^{(\alpha)} &= \left(y_1, F_1^{(\alpha)}, y_2, F_2^{(\alpha)}\right)^{\mathrm{T}}, \\ \mathbf{F}^{(\alpha_1\alpha_2)} &= \left(y_1, F_1^{(\alpha_1\alpha_2)}, y_2, F_2^{(\alpha_1\alpha_2)}\right)^{\mathrm{T}}, \\ \mathbf{F}^{(\alpha_1\alpha_2\alpha_3)} &= \left(y_1, F_1^{(\alpha_1\alpha_2\alpha_3)}, y_2, F_2^{(\alpha_1\alpha_2\alpha_3)}\right)^{\mathrm{T}}, \end{aligned} \tag{34}$$

where the forces of per unit mass for the 2-DOF friction induced oscillator in the domain Ω_α $(\alpha \in \{1,2,3,4\})$ are

$$\begin{aligned} F_1^{(1)} &= F_1^{(2)} \\ &= b_1 \cos\Omega t + a_1 - c_1 y_1 - p_1\left(y_1 - y_2\right) - d_1 x_1 \\ &\quad - q_1\left(x_1 - x_2\right) - f_1, \\ F_1^{(3)} &= F_1^{(4)} \\ &= b_1 \cos\Omega t + a_1 - c_1 y_1 - p_1\left(y_1 - y_2\right) - d_1 x_1 \\ &\quad - q_1\left(x_1 - x_2\right) + f_1, \\ F_2^{(1)} &= F_2^{(4)} \\ &= b_2 \cos\Omega t + a_2 - c_2 y_2 - p_2\left(y_2 - y_1\right) - d_2 x_2 \\ &\quad - q_2\left(x_2 - x_1\right) - f_2, \\ F_2^{(2)} &= F_2^{(3)} \\ &= b_2 \cos\Omega t + a_2 - c_2 y_2 - p_2\left(y_2 - y_1\right) - d_2 x_2 \\ &\quad - q_2\left(x_2 - x_1\right) + f_2; \end{aligned} \tag{35}$$

here

$$\begin{aligned} a_\alpha &= \frac{A_\alpha}{m_\alpha}, \\ b_\alpha &= \frac{B_\alpha}{m_\alpha}, \\ c_\alpha &= \frac{r_\alpha}{m_\alpha}, \\ d_\alpha &= \frac{k_\alpha}{m_\alpha}, \\ p_\alpha &= \frac{r_3}{m_\alpha}, \\ q_\alpha &= \frac{k_3}{m_\alpha}, \\ f_\alpha &= \frac{F_f^{(\alpha)}}{m_\alpha}, \\ &\alpha \in \{1,2\}, \end{aligned} \tag{36}$$

and the forces of per unit mass of the oscillator on the boundary $\partial\Omega_{\alpha_1\alpha_2}$ $(\alpha_1, \alpha_2 \in \{1,2,3,4\}, \alpha_1 \neq \alpha_2)$ are

$$\begin{aligned} F_1^{(12)} &\equiv b_1 \cos\Omega t + a_1 - c_1 y_1 - p_1\left(y_1 - y_2\right) - d_1 x_1 \\ &\quad - q_1\left(x_1 - x_2\right) - f_1, \\ F_2^{(12)} &= 0 \quad \text{for stick on } \partial\Omega_{12}, \\ F_2^{(12)} &\in \left[F_2^{(1)}, F_2^{(2)}\right] \quad \text{for nonstick on } \partial\Omega_{12}; \\ F_2^{(23)} &\equiv b_2 \cos\Omega t + a_2 - c_2 y_2 - p_2\left(y_2 - y_1\right) - d_2 x_2 \\ &\quad - q_2\left(x_2 - x_1\right) + f_2, \\ F_1^{(23)} &= 0 \quad \text{for stick on } \partial\Omega_{23}, \\ F_1^{(23)} &\in \left[F_1^{(2)}, F_1^{(3)}\right] \quad \text{for nonstick on } \partial\Omega_{23}; \\ F_1^{(34)} &\equiv b_1 \cos\Omega t + a_1 - c_1 y_1 - p_1\left(y_1 - y_2\right) - d_1 x_1 \\ &\quad - q_1\left(x_1 - x_2\right) + f_1, \\ F_2^{(34)} &= 0 \quad \text{for stick on } \partial\Omega_{34}, \\ F_2^{(34)} &\in \left[F_2^{(4)}, F_2^{(3)}\right] \quad \text{for nonstick on } \partial\Omega_{34}; \\ F_2^{(14)} &\equiv b_2 \cos\Omega t + a_2 - c_2 y_2 - p_2\left(y_2 - y_1\right) - d_2 x_2 \\ &\quad - q_2\left(x_2 - x_1\right) - f_2, \\ F_1^{(14)} &= 0 \quad \text{for stick on } \partial\Omega_{14}, \\ F_1^{(14)} &\in \left[F_1^{(1)}, F_1^{(4)}\right] \quad \text{for nonstick on } \partial\Omega_{14}. \end{aligned} \tag{37}$$

The forces of per unit mass of the oscillator on the boundary $\partial\Omega_{\alpha_1\alpha_2\alpha_3}$ $(\alpha_i \in \{1,2,3,4\},\ i = 1,2,3;\ \alpha_1, \alpha_2, \alpha_3$ are not equal to each other without repeating) are

$$\begin{aligned} F_\alpha^{(\alpha_1\alpha_2\alpha_3)} &\in \left(F_1^{(\alpha_1\alpha_2)}, F_2^{(\alpha_2\alpha_3)}\right), \alpha \in \{1,2\} \\ &\qquad \text{for nonstick on } \partial\Omega_{\alpha_1\alpha_2\alpha_3}; \\ F_\alpha^{(\alpha_1\alpha_2\alpha_3)} &= 0, \alpha \in \{1,2\} \quad \text{for full stick on } \partial\Omega_{\alpha_1\alpha_2\alpha_3}. \end{aligned} \tag{38}$$

For simplicity, the relative displacement, velocity, and acceleration between the mass m_α $(\alpha = 1, 2)$ and the traveling belt are defined as

$$\begin{aligned} z_\alpha &= x_\alpha - x(t), \\ v_\alpha &= \dot{x}_\alpha - V(t), \\ \ddot{z}_\alpha &= \ddot{x}_\alpha - \dot{V}(t). \end{aligned} \tag{39}$$

The domains and boundaries in relative coordinates are defined as

$$\begin{aligned} \Omega_1 &= \{(z_1, v_1, z_2, v_2) \mid v_1 > 0,\ v_2 > 0\}, \\ \Omega_2 &= \{(z_1, v_1, z_2, v_2) \mid v_1 > 0,\ v_2 < 0\}, \\ \Omega_3 &= \{(z_1, v_1, z_2, v_2) \mid v_1 < 0,\ v_2 < 0\}, \\ \Omega_4 &= \{(z_1, v_1, z_2, v_2) \mid v_1 < 0,\ v_2 > 0\}, \end{aligned} \tag{40}$$

$$\begin{aligned} \partial\Omega_{12} &= \partial\Omega_{21} \\ &= \{(z_1, v_1, z_2, v_2) \mid \varphi_{12} = \varphi_{21} = v_2 = 0,\ v_1 \ge 0\}, \\ \partial\Omega_{23} &= \partial\Omega_{32} \\ &= \{(z_1, v_1, z_2, v_2) \mid \varphi_{23} = \varphi_{32} = v_1 = 0,\ v_2 \le 0\}, \\ \partial\Omega_{34} &= \partial\Omega_{43} \\ &= \{(z_1, v_1, z_2, v_2) \mid \varphi_{34} = \varphi_{43} = v_2 = 0,\ v_1 \le 0\}, \\ \partial\Omega_{14} &= \partial\Omega_{41} \\ &= \{(z_1, v_1, z_2, v_2) \mid \varphi_{14} = \varphi_{41} = v_1 = 0,\ v_2 \ge 0\}, \end{aligned} \tag{41}$$

$$\angle\Omega_{\alpha_1\alpha_2\alpha_3} = \partial\Omega_{\alpha_1\alpha_2} \cap \partial\Omega_{\alpha_2\alpha_3} = \bigcap_{i=1}^{3} \Omega_{\alpha_i}, \tag{42}$$

where $\alpha_i \in \{1, 2, 3, 4\}$, $i = 1, 2, 3$, and $\alpha_i \neq \alpha_j$ $(i \neq j)$, $i, j \in \{1, 2, 3\}$. The intersection of four 2-dimensional edges is

$$\begin{aligned} \angle\Omega_{1234} = \bigcap \angle\Omega_{\alpha_1\alpha_2\alpha_3} &= \{(z_1, v_1, z_2, v_2) \mid \varphi_{12} = \varphi_{34} \\ &= v_2 = 0,\ \varphi_{23} = \varphi_{14} = v_1 = 0\}. \end{aligned} \tag{43}$$

The domain partitions and boundaries in relative coordinates are shown in Figure 4.

From the foregoing equations, the motion equations in relative coordinates are as follows:

$$\begin{aligned} \dot{\mathbf{z}}^{(\alpha)} &= \mathbf{g}^{(\alpha)}\left(\mathbf{z}^{(\alpha)}, \mathbf{x}^{(\alpha)}, t\right) \quad \text{in } \Omega_\alpha, \\ \dot{\mathbf{z}}^{(\alpha_1\alpha_2)} &= \mathbf{g}^{(\alpha_1\alpha_2)}\left(\mathbf{z}^{(\alpha_1\alpha_2)}, \mathbf{x}^{(\alpha_1\alpha_2)}, t\right) \quad \text{on } \partial\Omega_{\alpha_1\alpha_2}, \\ \dot{\mathbf{z}}^{(\alpha_1\alpha_2\alpha_3)} &= \mathbf{g}^{(\alpha_1\alpha_2\alpha_3)}\left(\mathbf{z}^{(\alpha_1\alpha_2\alpha_3)}, \mathbf{x}^{(\alpha_1\alpha_2\alpha_3)}, t\right) \\ &\qquad \text{on } \partial\Omega_{\alpha_1\alpha_2\alpha_3}, \end{aligned} \tag{44}$$

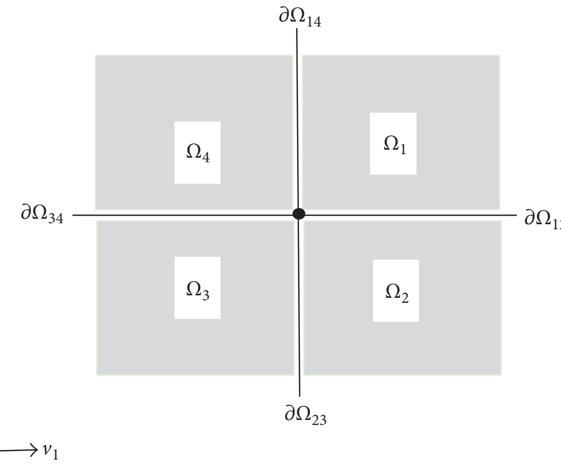

Figure 4: Relative domains and boundaries.

where

$$\begin{aligned} \mathbf{z}^{(\alpha)} &= \mathbf{z}^{(\alpha_1\alpha_2)} = \mathbf{z}^{(\alpha_1\alpha_2\alpha_3)} = \left(z_1, \dot{z}_1, z_2, \dot{z}_2\right)^{\mathrm{T}} \\ &= \left(z_1, v_1, z_2, v_2\right)^{\mathrm{T}}, \\ \mathbf{g}^{(\alpha)} &= \left(\dot{z}_1, g_1^{(\alpha)}, \dot{z}_2, g_2^{(\alpha)}\right)^{\mathrm{T}} = \left(v_1, g_1^{(\alpha)}, v_2, g_2^{(\alpha)}\right)^{\mathrm{T}}, \\ \mathbf{g}^{(\alpha_1\alpha_2)} &= \left(\dot{z}_1, g_1^{(\alpha_1\alpha_2)}, \dot{z}_2, g_2^{(\alpha_1\alpha_2)}\right)^{\mathrm{T}} \\ &= \left(v_1, g_1^{(\alpha_1\alpha_2)}, v_2, g_2^{(\alpha_1\alpha_2)}\right)^{\mathrm{T}}, \\ \mathbf{g}^{(\alpha_1\alpha_2\alpha_3)} &= \left(\dot{z}_1, g_1^{(\alpha_1\alpha_2\alpha_3)}, \dot{z}_2, g_2^{(\alpha_1\alpha_2\alpha_3)}\right)^{\mathrm{T}} \\ &= \left(v_1, g_1^{(\alpha_1\alpha_2\alpha_3)}, v_2, g_2^{(\alpha_1\alpha_2\alpha_3)}\right)^{\mathrm{T}}. \end{aligned} \tag{45}$$

The forces of per unit mass for the 2-DOF friction induced oscillator in the domain Ω_α $(\alpha \in \{1, 2, 3, 4\})$ in relative coordinates are

$$\begin{aligned} g_1^{(1)} &= g_1^{(2)} \\ &= b_1 \cos\Omega t + a_1 - c_1 v_1 - p_1(v_1 - v_2) - d_1 z_1 \\ &\quad - q_1(z_1 - z_2) - c_1 V(t) - d_1 x(t) - f_1 - \dot{V}(t), \\ g_1^{(3)} &= g_1^{(4)} \\ &= b_1 \cos\Omega t + a_1 - c_1 v_1 - p_1(v_1 - v_2) - d_1 z_1 \\ &\quad - q_1(z_1 - z_2) - c_1 V(t) - d_1 x(t) + f_1 - \dot{V}(t); \\ g_2^{(1)} &= g_2^{(4)} \\ &= b_2 \cos\Omega t + a_2 - c_2 v_2 - p_2(v_2 - v_1) - d_2 z_2 \\ &\quad - q_2(z_2 - z_1) - c_2 V(t) - d_2 x(t) - f_2 - \dot{V}(t), \end{aligned}$$

$$\begin{aligned} g_2^{(2)} &= g_2^{(3)} \\ &= b_2\cos\Omega t + a_2 - c_2 v_2 - p_2(v_2 - v_1) - d_2 z_2 \\ &\quad - q_2(z_2 - z_1) - c_2 V(t) - d_2 x(t) + f_2 - \dot{V}(t). \end{aligned} \tag{46}$$

The forces of per unit mass of the friction induced oscillator on the boundary $\partial\Omega_{\alpha_1\alpha_2}$ in relative coordinates are

$$\begin{aligned} g_1^{(12)} &\equiv b_1\cos\Omega t + a_1 - c_1 v_1 - p_1(v_1 - v_2) - d_1 z_1 \\ &\quad - q_1(z_1 - z_2) - c_1 V(t) - d_1 x(t) - f_1 \\ &\quad - \dot{V}(t), \\ g_2^{(12)} &= 0 \quad \text{for stick on } \partial\Omega_{12}, \\ g_2^{(12)} &\in \left[g_2^{(1)}, g_2^{(2)}\right] \quad \text{for nonstick on } \partial\Omega_{12}; \\ g_2^{(23)} &\equiv b_2\cos\Omega t + a_2 - c_2 v_2 - p_2(v_2 - v_1) - d_2 z_2 \\ &\quad - q_2(z_2 - z_1) - c_2 V(t) - d_2 x(t) + f_2 \\ &\quad - \dot{V}(t), \\ g_1^{(23)} &= 0 \quad \text{for stick on } \partial\Omega_{23}, \\ g_1^{(23)} &\in \left[g_1^{(2)}, g_1^{(3)}\right] \quad \text{for nonstick on } \partial\Omega_{23}; \\ g_1^{(34)} &\equiv b_1\cos\Omega t + a_1 - c_1 v_1 - p_1(v_1 - v_2) - d_1 z_1 \\ &\quad - q_1(z_1 - z_2) - c_1 V(t) - d_1 x(t) + f_1 \\ &\quad - \dot{V}(t), \\ g_2^{(34)} &= 0 \quad \text{for stick on } \partial\Omega_{34}, \\ g_2^{(34)} &\in \left[g_2^{(4)}, g_2^{(3)}\right] \quad \text{for nonstick on } \partial\Omega_{34}; \\ g_2^{(14)} &\equiv b_2\cos\Omega t + a_2 - c_2 v_2 - p_2(v_2 - v_1) - d_2 z_2 \\ &\quad - q_2(z_2 - z_1) - c_2 V(t) - d_2 x(t) - f_2 \\ &\quad - \dot{V}(t), \\ g_1^{(14)} &= 0 \quad \text{for stick on } \partial\Omega_{14}, \\ g_1^{(14)} &\in \left[g_1^{(1)}, g_1^{(4)}\right] \quad \text{for nonstick on } \partial\Omega_{14}. \end{aligned} \tag{47}$$

The forces of per unit mass of the oscillator on the boundary $\partial\Omega_{\alpha_1\alpha_2\alpha_3}$ ($\alpha_i \in \{1,2,3,4\}$, $i = 1,2,3$, and $\alpha_i \neq \alpha_j$ $(i \neq j)$, $i, j \in \{1,2,3\}$) are

$$\begin{aligned} g_\alpha^{(\alpha_1\alpha_2\alpha_3)} &\in \left(g_1^{(\alpha_1\alpha_2)}, g_2^{(\alpha_2\alpha_3)}\right), \alpha \in \{1,2\} \\ &\qquad \text{for no full stick on } \partial\Omega_{\alpha_1\alpha_2\alpha_3}; \end{aligned} \tag{48}$$

$$g_\alpha^{(\alpha_1\alpha_2\alpha_3)} = 0, \alpha \in \{1,2\} \quad \text{for full stick on } \partial\Omega_{\alpha_1\alpha_2\alpha_3}.$$

In other words, we have the following formulas:

$$\begin{aligned} & g_1^{(123)}, g_2^{(123)} \in \left(g_1^{(12)}, g_2^{(23)}\right) \\ &\qquad \text{for no full stick on } \angle\Omega_{123}, \\ &\quad g_1^{(123)} = 0, \\ &\quad g_2^{(123)} = 0 \\ &\qquad \text{for full stick on } \angle\Omega_{123}, \\ & g_1^{(234)}, g_2^{(234)} \in \left(g_1^{(23)}, g_2^{(34)}\right) \\ &\qquad \text{for no full stick on } \angle\Omega_{234}, \\ &\quad g_1^{(234)} = 0, \\ &\quad g_2^{(234)} = 0 \\ &\qquad \text{for full stick on } \angle\Omega_{234}, \\ & g_1^{(341)}, g_2^{(341)} \in \left(g_1^{(34)}, g_2^{(41)}\right) \\ &\qquad \text{for no full stick on } \angle\Omega_{341}, \\ &\quad g_1^{(341)} = 0, \\ &\quad g_2^{(341)} = 0 \\ &\qquad \text{for full stick on } \angle\Omega_{341}, \\ & g_1^{(412)}, g_2^{(412)} \in \left(g_1^{(41)}, g_2^{(12)}\right) \\ &\qquad \text{for no full stick on } \angle\Omega_{412}, \\ &\quad g_1^{(412)} = 0, \\ &\quad g_2^{(412)} = 0 \\ &\qquad \text{for full stick on } \angle\Omega_{412}. \end{aligned} \tag{49}$$

5. Analytical Conditions

Using the absolute coordinates, it is very difficult to develop the analytical conditions for the complex motions of the oscillator described in Section 3 because the boundaries are dependent on time; thus the relative coordinates are needed herein for simplicity.

From (3) and (4) in Section 2, we have

$$\begin{aligned} & G^{(0,\alpha_1)}\left(\mathbf{z}^{(\alpha)}, \mathbf{x}^{(\alpha)}, t_{m\pm}\right) \\ &\quad = \mathbf{n}_{\partial\Omega_{\alpha_1\alpha_2}}^{\mathrm{T}} \cdot \mathbf{g}^{(\alpha_1)}\left(\mathbf{z}^{(\alpha)}, \mathbf{x}^{(\alpha)}, t_{m\pm}\right), \end{aligned} \tag{50}$$

$$\begin{aligned} & G^{(1,\alpha_1)}\left(\mathbf{z}^{(\alpha)}, \mathbf{x}^{(\alpha)}, t_{m\pm}\right) \\ &\quad = 2D\mathbf{n}_{\partial\Omega_{\alpha_1\alpha_2}}^{\mathrm{T}} \cdot \left[\mathbf{g}^{(\alpha_1)}(t_{m\pm}) - \mathbf{g}^{(\alpha_1\alpha_2)}(t_m)\right] + \mathbf{n}_{\partial\Omega_{\alpha_1\alpha_2}}^{\mathrm{T}} \\ &\qquad \cdot \left[D\mathbf{g}^{(\alpha_1)}(t_{m\pm}) - D\mathbf{g}^{(\alpha_1\alpha_2)}(t_m)\right]. \end{aligned} \tag{51}$$

In relative coordinates, the boundary $\partial\Omega_{\alpha_1\alpha_2}$ is independent on t, so $D\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} = 0$. Because of

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1\alpha_2)} = 0, \quad (52)$$

therefore

$$D\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1\alpha_2)} + \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot D\mathbf{g}^{(\alpha_1\alpha_2)} = 0; \quad (53)$$

thus

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot D\mathbf{g}^{(\alpha_1\alpha_2)} = 0. \quad (54)$$

Equation (51) is simplified as

$$G^{(1,\alpha_1)}\left(\mathbf{z}_\alpha, \mathbf{x}_\alpha, t_{m\pm}\right) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot D\mathbf{g}^{(\alpha_1)}\left(t_{m\pm}\right). \quad (55)$$

t_m represents the time for the motion on the velocity boundary and $t_{m\pm} = t_m \pm 0$ reflects the responses in the domain rather than on the boundary.

From the previous descriptions for the system, the normal vector of the boundary $\partial\Omega_{\alpha_1\alpha_2}$ in the relative coordinates is

$$\mathbf{n}_{\partial\Omega_{\alpha_1\alpha_2}} = \left(\frac{\partial\varphi_{\alpha_1\alpha_2}}{\partial z_1}, \frac{\partial\varphi_{\alpha_1\alpha_2}}{\partial v_1}, \frac{\partial\varphi_{\alpha_1\alpha_2}}{\partial z_2}, \frac{\partial\varphi_{\alpha_1\alpha_2}}{\partial v_2}\right)^{\mathrm{T}}. \quad (56)$$

With (41) and (56), we have

$$\begin{aligned} \mathbf{n}_{\partial\Omega_{23}} &= \mathbf{n}_{\partial\Omega_{14}} = (0,1,0,0)^{\mathrm{T}}, \\ \mathbf{n}_{\partial\Omega_{12}} &= \mathbf{n}_{\partial\Omega_{34}} = (0,0,0,1)^{\mathrm{T}}. \end{aligned} \quad (57)$$

Theorem 9. *For the 2-DOF friction induced oscillator described in Section 3, the nonstick motion (or called passable motion to boundary) on $\mathbf{x}_m \in \partial\Omega_{\alpha_1\alpha_2}$ at time t_m appears if and only if*

(a) $\alpha_1 = 2$, $\alpha_2 = 1$:

$$\begin{aligned} &g_2^{(2)}\left(t_{m-}\right) > 0, \\ &g_2^{(1)}\left(t_{m+}\right) > 0 \\ &\qquad \text{from } \Omega_2 \longrightarrow \Omega_1; \end{aligned} \quad (58)$$

(b) $\alpha_1 = 1$, $\alpha_2 = 2$:

$$\begin{aligned} &g_2^{(1)}\left(t_{m-}\right) < 0, \\ &g_2^{(2)}\left(t_{m+}\right) < 0 \\ &\qquad \text{from } \Omega_1 \longrightarrow \Omega_2; \end{aligned} \quad (59)$$

(c) $\alpha_1 = 3$, $\alpha_2 = 4$:

$$\begin{aligned} &g_2^{(3)}\left(t_{m-}\right) > 0, \\ &g_2^{(4)}\left(t_{m+}\right) > 0 \\ &\qquad \text{from } \Omega_3 \longrightarrow \Omega_4; \end{aligned} \quad (60)$$

(d) $\alpha_1 = 4$, $\alpha_2 = 3$:

$$\begin{aligned} &g_2^{(4)}\left(t_{m-}\right) < 0, \\ &g_2^{(3)}\left(t_{m+}\right) < 0 \\ &\qquad \text{from } \Omega_4 \longrightarrow \Omega_3; \end{aligned} \quad (61)$$

(e) $\alpha_1 = 2$, $\alpha_2 = 3$:

$$\begin{aligned} &g_1^{(2)}\left(t_{m-}\right) < 0, \\ &g_1^{(3)}\left(t_{m+}\right) < 0 \\ &\qquad \text{from } \Omega_2 \longrightarrow \Omega_3; \end{aligned} \quad (62)$$

(f) $\alpha_1 = 3$, $\alpha_2 = 2$:

$$\begin{aligned} &g_1^{(3)}\left(t_{m-}\right) > 0, \\ &g_1^{(2)}\left(t_{m+}\right) > 0 \\ &\qquad \text{from } \Omega_3 \longrightarrow \Omega_2; \end{aligned} \quad (63)$$

(g) $\alpha_1 = 4$, $\alpha_2 = 1$:

$$\begin{aligned} &g_1^{(4)}\left(t_{m-}\right) > 0, \\ &g_1^{(1)}\left(t_{m+}\right) > 0 \\ &\qquad \text{from } \Omega_4 \longrightarrow \Omega_1; \end{aligned} \quad (64)$$

(h) $\alpha_1 = 1$, $\alpha_2 = 4$:

$$\begin{aligned} &g_1^{(1)}\left(t_{m-}\right) < 0, \\ &g_1^{(4)}\left(t_{m+}\right) < 0 \\ &\qquad \text{from } \Omega_1 \longrightarrow \Omega_4. \end{aligned} \quad (65)$$

Proof. By Lemma 4, the passable motion for a flow from domain Ω_{α_1} to Ω_{α_2} on the boundary $\partial\Omega_{\alpha_1\alpha_2}$ at time t_m appears if and only if for $\mathbf{n}_{\partial\Omega_{\alpha_1\alpha_2}} \rightarrow \Omega_{\alpha_1}$

$$\begin{aligned} G^{(0,\alpha_1)}\left(t_{m-}\right) &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1)}\left(t_{m-}\right) < 0, \\ G^{(0,\alpha_2)}\left(t_{m+}\right) &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_2)}\left(t_{m+}\right) < 0. \end{aligned} \quad (66)$$

From (57) and $\mathbf{g}^{(\alpha)} = (v_1, g_1^{(\alpha)}, v_2, g_2^{(\alpha)})$, we have

$$\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(\alpha)}\left(t_{m\pm}\right) &= g_2^{(\alpha)}\left(t_{m\pm}\right) \quad (\alpha = 1,2), \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{34}} \cdot \mathbf{g}^{(\alpha)}\left(t_{m\pm}\right) &= g_2^{(\alpha)}\left(t_{m\pm}\right) \quad (\alpha = 3,4), \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{23}} \cdot \mathbf{g}^{(\alpha)}\left(t_{m\pm}\right) &= g_1^{(\alpha)}\left(t_{m\pm}\right) \quad (\alpha = 2,3), \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{14}} \cdot \mathbf{g}^{(\alpha)}\left(t_{m\pm}\right) &= g_1^{(\alpha)}\left(t_{m\pm}\right) \quad (\alpha = 1,4). \end{aligned} \quad (67)$$

Substituting the first formula of (67) into (66), we have

$$\begin{aligned}
&G^{(0,1)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(1)}(t_{m-}) = g_2^{(1)}(t_{m-}) < 0,\\
&G^{(0,2)}(t_{m+}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(2)}(t_{m+}) = g_2^{(2)}(t_{m+}) < 0\\
&\qquad\qquad\qquad \text{from } \Omega_1 \longrightarrow \Omega_2,\\
&G^{(0,2)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(2)}(t_{m-}) = g_2^{(2)}(t_{m-}) > 0,\\
&G^{(0,1)}(t_{m+}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(1)}(t_{m+}) = g_2^{(1)}(t_{m+}) > 0\\
&\qquad\qquad\qquad \text{from } \Omega_2 \longrightarrow \Omega_1.
\end{aligned} \tag{68}$$

So, (a) and (b) hold. Similarly, (c)–(h) can be proved. □

Theorem 10. *For the 2-DOF friction induced oscillator described in Section 3, the stick motion in physics (or called the sliding motion in mathematics) to the boundary $\partial\Omega_{\alpha_1\alpha_2}$ is guaranteed if and only if*

$$\begin{aligned}
&g_2^{(2)}(t_{m-}) > 0,\\
&g_2^{(1)}(t_{m-}) < 0\\
&\qquad \textit{on } \partial\Omega_{12};\\
&g_2^{(3)}(t_{m-}) > 0,\\
&g_2^{(4)}(t_{m-}) < 0\\
&\qquad \textit{on } \partial\Omega_{34};\\
&g_1^{(4)}(t_{m-}) > 0,\\
&g_1^{(1)}(t_{m-}) < 0\\
&\qquad \textit{on } \partial\Omega_{14};\\
&g_1^{(3)}(t_{m-}) > 0,\\
&g_1^{(2)}(t_{m-}) < 0\\
&\qquad \textit{on } \partial\Omega_{23}.
\end{aligned} \tag{69}$$

Proof. By Lemma 5 and (50), the necessary and sufficient conditions of the sliding motion on the boundary $\partial\Omega_{\alpha_1\alpha_2}$ are

$$\begin{aligned}
&G^{(0,\alpha_1)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1)}(t_{m-}) < 0,\\
&G^{(0,\alpha_2)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_2)}(t_{m-}) > 0\\
&\qquad\qquad \text{for } \mathbf{n}_{\partial\Omega_{\alpha_1\alpha_2}} \longrightarrow \Omega_{\alpha_1}.
\end{aligned} \tag{70}$$

Substitute the first formula of (67) into (70); we have

$$\begin{aligned}
&G^{(0,1)}(t_{m-}) = \mathbf{n}^{\mathbf{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(1)}(t_{m-}) = g_2^{(1)}(t_{m-}) < 0,\\
&G^{(0,2)}(t_{m-}) = \mathbf{n}^{\mathbf{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(2)}(t_{m-}) = g_2^{(2)}(t_{m-}) > 0\\
&\qquad\qquad \text{for } \mathbf{n}_{\partial\Omega_{12}} \longrightarrow \Omega_1.
\end{aligned} \tag{71}$$

So the conclusion on $\partial\Omega_{12}$ is proved. Similarly, the other formulas in (69) can also be proved. □

Theorem 11. *For the 2-DOF friction induced oscillator described in Section 3, the stick motion on the boundary $\partial\Omega_{\alpha_1\alpha_2}$ appears if and only if*

$$\begin{aligned}
&g_2^{(2)}(t_{m-}) > 0,\\
&g_2^{(1)}(t_{m\pm}) = 0,\\
&Dg_2^{(1)}(t_{m\pm}) < 0\\
&\qquad \textit{from } \Omega_1 \textit{ to } \partial\Omega_{12},
\end{aligned} \tag{72}$$

$$\begin{aligned}
&g_2^{(2)}(t_{m\pm}) = 0,\\
&g_2^{(1)}(t_{m-}) < 0,\\
&Dg_2^{(2)}(t_{m\pm}) > 0\\
&\qquad \textit{from } \Omega_2 \textit{ to } \partial\Omega_{21};
\end{aligned} \tag{73}$$

$$\begin{aligned}
&g_1^{(3)}(t_{m-}) > 0,\\
&g_1^{(2)}(t_{m\pm}) = 0,\\
&Dg_1^{(2)}(t_{m\pm}) < 0\\
&\qquad \textit{from } \Omega_2 \textit{ to } \partial\Omega_{23},
\end{aligned} \tag{74}$$

$$\begin{aligned}
&g_1^{(3)}(t_{m\pm}) = 0,\\
&g_1^{(2)}(t_{m-}) < 0,\\
&Dg_1^{(3)}(t_{m\pm}) > 0\\
&\qquad \textit{from } \Omega_3 \textit{ to } \partial\Omega_{32};
\end{aligned} \tag{75}$$

$$\begin{aligned}
&g_2^{(3)}(t_{m-}) > 0,\\
&g_2^{(4)}(t_{m\pm}) = 0,\\
&Dg_2^{(4)}(t_{m\pm}) < 0\\
&\qquad \textit{from } \Omega_4 \textit{ to } \partial\Omega_{43},
\end{aligned} \tag{76}$$

$$\begin{aligned}
&g_2^{(3)}(t_{m\pm}) = 0,\\
&g_2^{(4)}(t_{m-}) < 0,\\
&Dg_2^{(3)}(t_{m\pm}) > 0\\
&\qquad \textit{from } \Omega_3 \textit{ to } \partial\Omega_{34};
\end{aligned} \tag{77}$$

$$\begin{aligned}
&g_1^{(4)}(t_{m-}) > 0,\\
&g_1^{(1)}(t_{m\pm}) = 0\\
&Dg_1^{(1)}(t_{m\pm}) < 0\\
&\qquad \textit{from } \Omega_1 \textit{ to } \partial\Omega_{14},
\end{aligned} \tag{78}$$

$$\begin{aligned} &g_1^{(4)}(t_{m\pm}) = 0, \\ &g_1^{(1)}(t_{m-}) < 0, \\ &Dg_1^{(4)}(t_{m\pm}) > 0 \\ &\qquad \textit{from } \Omega_4 \textit{ to } \partial\Omega_{41}. \end{aligned} \tag{79}$$

Proof. By Lemma 7 and (50) and (55), if the normal direction on $\partial\Omega_{\alpha_1\alpha_2}$ is for $\mathbf{n}_{\partial\Omega_{\alpha_1\alpha_2}} \to \Omega_{\alpha_1}$, the analytical conditions for the appearance of the stick motion are

$$\begin{aligned} &G^{(0,\alpha_1)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1)}(t_{m-}) < 0, \\ &G^{(0,\alpha_2)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_2)}(t_{m\pm}) = 0, \\ &G^{(1,\alpha_2)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot D\mathbf{g}^{(\alpha_2)}(t_{m\pm}) > 0 \\ &\qquad \text{from } \Omega_{\alpha_2} \longrightarrow \partial\Omega_{\alpha_1\alpha_2}. \end{aligned} \tag{80}$$

By (57), we have

$$\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot D\mathbf{g}^{(\alpha)}(t_{m\pm}) = Dg_2^{(\alpha)}(t_{m\pm}), \quad (\alpha = 1, 2) \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_3\alpha_4}} \cdot D\mathbf{g}^{(\alpha)}(t_{m\pm}) = Dg_2^{(\alpha)}(t_{m\pm}), \quad (\alpha = 3, 4) \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_2\alpha_3}} \cdot D\mathbf{g}^{(\alpha)}(t_{m\pm}) = Dg_1^{(\alpha)}(t_{m\pm}), \quad (\alpha = 2, 3) \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_4}} \cdot D\mathbf{g}^{(\alpha)}(t_{m\pm}) = Dg_1^{(\alpha)}(t_{m\pm}), \quad (\alpha = 1, 4), \end{aligned} \tag{81}$$

where

$$\begin{aligned} Dg_2^{(1)}(t_{m\pm}) &= Dg_2^{(2)}(t_{m\pm}) = Dg_2^{(3)}(t_{m\pm}) \\ &= Dg_2^{(4)}(t_{m\pm}) \\ &= -b_2\Omega\sin\Omega t - c_2\dot{v}_2 - p_2(\dot{v}_2 - \dot{v}_1) \\ &\quad - d_2 v_2 - q_2(v_2 - v_1) - c_2\dot{V}(t) \\ &\quad - d_2 V(t) - \ddot{V}(t), \\ Dg_1^{(1)}(t_{m\pm}) &= Dg_1^{(2)}(t_{m\pm}) = Dg_1^{(3)}(t_{m\pm}) \\ &= Dg_1^{(4)}(t_{m\pm}) \\ &= -b_1\Omega\sin\Omega t - c_1\dot{v}_1 - p_1(\dot{v}_1 - \dot{v}_2) \\ &\quad - d_1 v_1 - q_1(v_1 - v_2) - c_1\dot{V}(t) \\ &\quad - d_1 V(t) - \ddot{V}(t). \end{aligned} \tag{82}$$

Substitute the first formula of (67) and (81) into (80); we have

$$\begin{aligned} &G^{(0,1)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(1)}(t_{m-}) = g_2^{(1)}(t_{m-}) < 0, \\ &G^{(0,2)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(2)}(t_{m\pm}) = g_2^{(2)}(t_{m\pm}) = 0, \\ &G^{(1,2)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot D\mathbf{g}^{(2)}(t_{m\pm}) = Dg_2^{(2)}(t_{m\pm}) > 0 \\ &\qquad \text{from } \Omega_2 \text{ to } \partial\Omega_{21}; \\ &G^{(0,1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(1)}(t_{m\pm}) = g_2^{(1)}(t_{m\pm}) = 0, \\ &G^{(0,2)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(2)}(t_{m-}) = g_2^{(2)}(t_{m-}) > 0, \\ &G^{(1,1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot D\mathbf{g}^{(1)}(t_{m\pm}) = Dg_2^{(1)}(t_{m\pm}) < 0 \\ &\qquad \text{from } \Omega_1 \text{ to } \partial\Omega_{12}. \end{aligned} \tag{83}$$

So the cases on $\partial\Omega_{12}$ and $\partial\Omega_{21}$ are proved. Similarly, the other equations in (74) to (79) can also be proved. □

Theorem 12. *For the 2-DOF friction induced oscillator described in Section 3, the analytical conditions for vanishing of the stick motion from $\partial\Omega_{\alpha_1\alpha_2}$ and entering domain Ω_{α_1} are*

$$\begin{aligned} &g_2^{(2)}(t_{m-}) > 0, \\ &g_2^{(1)}(t_{m\pm}) = 0, \\ &Dg_2^{(1)}(t_{m\pm}) > 0 \\ &\qquad \textit{from } \partial\Omega_{12} \longrightarrow \Omega_1, \end{aligned} \tag{84}$$

$$\begin{aligned} &g_2^{(2)}(t_{m\pm}) = 0, \\ &g_2^{(1)}(t_{m-}) < 0, \\ &Dg_2^{(2)}(t_{m\pm}) < 0 \\ &\qquad \textit{from } \partial\Omega_{12} \longrightarrow \Omega_2; \end{aligned} \tag{85}$$

$$\begin{aligned} &g_2^{(3)}(t_{m-}) > 0, \\ &g_2^{(4)}(t_{m\pm}) = 0, \\ &Dg_2^{(4)}(t_{m\pm}) > 0 \\ &\qquad \textit{from } \partial\Omega_{34} \longrightarrow \Omega_4, \end{aligned} \tag{86}$$

$$\begin{aligned} &g_2^{(3)}(t_{m\pm}) = 0, \\ &g_2^{(4)}(t_{m-}) < 0, \\ &Dg_2^{(3)}(t_{m\pm}) < 0 \\ &\qquad \textit{from } \partial\Omega_{34} \longrightarrow \Omega_3; \end{aligned} \tag{87}$$

$$\begin{aligned} &g_1^{(4)}(t_{m-}) > 0, \\ &g_1^{(1)}(t_{m\pm}) = 0, \\ &Dg_1^{(1)}(t_{m\pm}) > 0 \\ &\qquad \textit{from } \partial\Omega_{14} \longrightarrow \Omega_1, \end{aligned} \tag{88}$$

$$\begin{aligned} &g_1^{(4)}(t_{m\pm}) = 0, \\ &g_1^{(1)}(t_{m-}) < 0, \\ &Dg_1^{(4)}(t_{m\pm}) < 0 \\ &\qquad \textit{from } \partial\Omega_{14} \longrightarrow \Omega_4; \end{aligned} \tag{89}$$

$$\begin{aligned} &g_1^{(3)}(t_{m-}) > 0, \\ &g_1^{(2)}(t_{m\pm}) = 0, \\ &Dg_1^{(2)}(t_{m\pm}) > 0 \\ &\qquad \text{from } \partial\Omega_{23} \longrightarrow \Omega_2, \end{aligned} \tag{90}$$

$$\begin{aligned} &g_1^{(3)}(t_{m\pm}) = 0, \\ &g_1^{(2)}(t_{m-}) < 0, \\ &Dg_1^{(3)}(t_{m\pm}) < 0 \\ &\qquad \text{from } \partial\Omega_{23} \longrightarrow \Omega_3. \end{aligned} \tag{91}$$

Proof. By Lemma 6 and (50) and (55), if the normal direction on $\partial\Omega_{\alpha_1\alpha_2}$ is for $\mathbf{n}_{\partial\Omega_{\alpha_1\alpha_2}} \to \Omega_{\alpha_1}$, the necessary and sufficient conditions for the oscillator sliding motion switched into passible motion are

$$\begin{aligned} &G^{(0,\alpha_1)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1)}(t_{m-}) < 0, \\ &G^{(0,\alpha_2)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_2)}(t_{m\pm}) = 0, \\ &G^{(1,\alpha_2)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot D\mathbf{g}^{(\alpha_2)}(t_{m\pm}) < 0 \\ &\qquad \text{from } \partial\Omega_{\alpha_1\alpha_2} \longrightarrow \Omega_{\alpha_2}, \\ &G^{(0,\alpha_1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1)}(t_{m\pm}) = 0, \\ &G^{(0,\alpha_2)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_2)}(t_{m-}) > 0, \\ &G^{(1,\alpha_1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot D\mathbf{g}^{(\alpha_1)}(t_{m\pm}) > 0 \\ &\qquad \text{from } \partial\Omega_{\alpha_1\alpha_2} \longrightarrow \Omega_{\alpha_1}. \end{aligned} \tag{92}$$

Substitute the first formula of (67) and (81) into (92); we have

$$\begin{aligned} &G^{(0,1)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(1)}(t_{m-}) = g_2^{(1)}(t_{m-}) < 0, \\ &G^{(0,2)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(2)}(t_{m\pm}) = g_2^{(2)}(t_{m\pm}) = 0, \\ &G^{(1,2)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot D\mathbf{g}^{(2)}(t_{m\pm}) = Dg_2^{(2)}(t_{\pm}) < 0 \\ &\qquad \text{from } \partial\Omega_{12} \longrightarrow \Omega_2; \\ &G^{(0,1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(1)}(t_{m\pm}) = g_2^{(1)}(t_{m\pm}) = 0, \\ &G^{(0,2)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(2)}(t_{m-}) = g_2^{(2)}(t_{m-}) > 0, \\ &G^{(1,1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot D\mathbf{g}^{(1)}(t_{m\pm}) = Dg_2^{(1)}(t_{\pm}) > 0 \\ &\qquad \text{from } \partial\Omega_{12} \longrightarrow \Omega_1. \end{aligned} \tag{93}$$

So the cases on $\partial\Omega_{12}$ are proved. Similarly, the other cases in (86) to (91) can also be proved. □

Theorem 13. *For the 2-DOF friction induced oscillator described in Section 3, the grazing motion on the boundary $\partial\Omega_{\alpha_1\alpha_2}$ is guaranteed if and only if*

$$\begin{aligned} &g_2^{(1)}(t_{m\pm}) = 0, \\ &Dg_2^{(1)}(t_{m\pm}) > 0 \\ &\qquad \textit{on } \partial\Omega_{12} \textit{ in } \Omega_1, \end{aligned} \tag{94}$$

$$\begin{aligned} &g_2^{(2)}(t_{m\pm}) = 0, \\ &Dg_2^{(2)}(t_{m\pm}) < 0 \\ &\qquad \textit{on } \partial\Omega_{12} \textit{ in } \Omega_2; \end{aligned} \tag{95}$$

$$\begin{aligned} &g_2^{(4)}(t_{m\pm}) = 0, \\ &Dg_2^{(4)}(t_{m\pm}) > 0 \\ &\qquad \textit{on } \partial\Omega_{34} \textit{ in } \Omega_4, \end{aligned} \tag{96}$$

$$\begin{aligned} &g_2^{(3)}(t_{m\pm}) = 0, \\ &Dg_2^{(3)}(t_{m\pm}) < 0 \\ &\qquad \textit{on } \partial\Omega_{34} \textit{ in } \Omega_3; \end{aligned} \tag{97}$$

$$\begin{aligned} &g_1^{(1)}(t_{m\pm}) = 0, \\ &Dg_1^{(1)}(t_{m\pm}) > 0 \\ &\qquad \textit{on } \partial\Omega_{14} \textit{ in } \Omega_1, \end{aligned} \tag{98}$$

$$\begin{aligned} &g_1^{(4)}(t_{m\pm}) = 0, \\ &Dg_1^{(4)}(t_{m\pm}) < 0 \\ &\qquad \textit{on } \partial\Omega_{14} \textit{ in } \Omega_4; \end{aligned} \tag{99}$$

$$\begin{aligned} &g_1^{(2)}(t_{m\pm}) = 0, \\ &Dg_1^{(2)}(t_{m\pm}) > 0 \\ &\qquad \textit{on } \partial\Omega_{23} \textit{ in } \Omega_2, \end{aligned} \tag{100}$$

$$\begin{aligned} &g_1^{(3)}(t_{m\pm}) = 0, \\ &Dg_1^{(3)}(t_{m\pm}) < 0 \\ &\qquad \textit{on } \partial\Omega_{23} \textit{ in } \Omega_3. \end{aligned} \tag{101}$$

Proof. By Lemma 8 and (50) and (55), the conditions for the grazing motion in domain Ω_{α_1} to the boundary $\partial\Omega_{\alpha_1\alpha_2}$ are

$$\begin{aligned} &G^{(0,\alpha_1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1)}(t_{m\pm}) = 0, \\ &G^{(1,\alpha_1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot D\mathbf{g}^{(\alpha_1)}(t_{m\pm}) > 0 \\ &\qquad \text{for } \mathbf{n}_{\partial\Omega_{\alpha_1\alpha_2}} \longrightarrow \Omega_{\alpha_1}; \end{aligned}$$

$$G^{(0,\alpha_1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot \mathbf{g}^{(\alpha_1)}(t_{m\pm}) = 0,$$
$$G^{(1,\alpha_1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha_1\alpha_2}} \cdot D\mathbf{g}^{(\alpha_1)}(t_{m\pm}) < 0$$
$$\text{for } \mathbf{n}_{\partial\Omega_{\alpha_1\alpha_2}} \longrightarrow \Omega_{\alpha_2}. \quad (102)$$

By (67) and (81), we have

$$G^{(0,1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(1)}(t_{m\pm}) = g_2^{(1)}(t_{m\pm}) = 0,$$
$$G^{(1,1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot D\mathbf{g}^{(1)}(t_{m\pm}) = Dg_2^{(1)}(t_{m\pm}) > 0$$
$$\text{on } \partial\Omega_{12} \text{ in } \Omega_1; \quad (103)$$
$$G^{(0,1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{g}^{(2)}(t_{m\pm}) = g_2^{(2)}(t_{m\pm}) = 0,$$
$$G^{(1,1)}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot D\mathbf{g}^{(2)}(t_{m\pm}) = Dg_2^{(2)}(t_{m\pm}) < 0$$
$$\text{on } \partial\Omega_{12} \text{ in } \Omega_2.$$

So the cases on $\partial\Omega_{12}$ are proved. Similarly, the other formulas in (96)–(101) can also be proved. □

6. Mapping Structures and Periodic Motions

From the boundary $\partial\Omega_{\alpha_1\alpha_2}$ in (41), the switching sets are

$$\Sigma_1^+ = \{(z_{1(i)}, v_{1(i)}, z_{2(i)}, v_{2(i)}, \Omega t_i) \mid v_{1(i)} = 0^+,\ i \in N\},$$
$$\Sigma_1^0 = \{(z_{1(i)}, v_{1(i)}, z_{2(i)}, v_{2(i)}, \Omega t_i) \mid v_{1(i)} = 0,\ i \in N\},$$
$$\Sigma_1^- = \{(z_{1(i)}, v_{1(i)}, z_{2(i)}, v_{2(i)}, \Omega t_i) \mid v_{1(i)} = 0^-,\ i \in N\}, \quad (104)$$
$$\Sigma_2^+ = \{(z_{1(i)}, v_{1(i)}, z_{2(i)}, v_{2(i)}, \Omega t_i) \mid v_{2(i)} = 0^+,\ i \in N\},$$
$$\Sigma_2^0 = \{(z_{1(i)}, v_{1(i)}, z_{2(i)}, v_{2(i)}, \Omega t_i) \mid v_{2(i)} = 0,\ i \in N\},$$
$$\Sigma_2^- = \{(z_{1(i)}, v_{1(i)}, z_{2(i)}, v_{2(i)}, \Omega t_i) \mid v_{2(i)} = 0^-,\ i \in N\},$$

where $0^\pm = \lim_{\varepsilon\to 0^+}(0 \pm \varepsilon)$ and the switching set on the edge $\angle\Omega_{\alpha_1\alpha_2\alpha_3}$ is defined by

$$\Sigma_0^0 = \{(z_{1(i)}, v_{1(i)}, z_{2(i)}, v_{2(i)}, \Omega t_i) \mid v_{1(i)} = 0,\ v_{2(i)} = 0,\ i \in N\}. \quad (105)$$

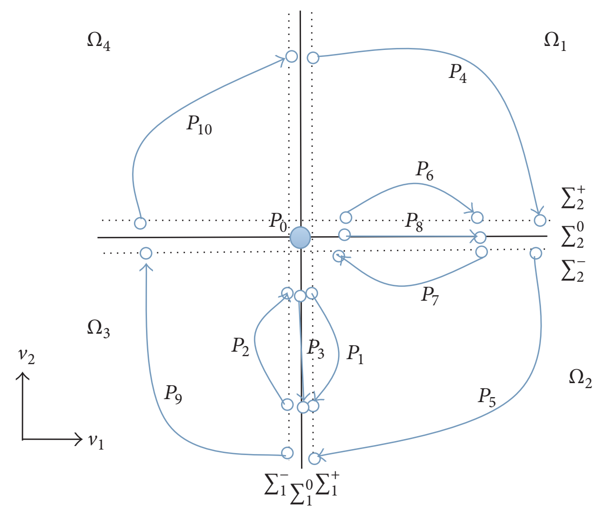

FIGURE 5: Switching sets and mappings.

Therefore, eleven basic mappings will be defined as

$$\begin{aligned} P_1 &: \Sigma_1^+ \longrightarrow \Sigma_1^+,\\ P_2 &: \Sigma_1^- \longrightarrow \Sigma_1^-,\\ P_3 &: \Sigma_1^0 \longrightarrow \Sigma_1^0,\\ P_6 &: \Sigma_2^+ \longrightarrow \Sigma_2^+,\\ P_7 &: \Sigma_2^- \longrightarrow \Sigma_2^-,\\ P_8 &: \Sigma_2^0 \longrightarrow \Sigma_2^0,\\ P_4 &: \Sigma_1^+ \longrightarrow \Sigma_2^+,\\ P_5 &: \Sigma_2^- \longrightarrow \Sigma_1^+,\\ P_9 &: \Sigma_1^- \longrightarrow \Sigma_2^-,\\ P_{10} &: \Sigma_2^+ \longrightarrow \Sigma_1^-,\\ P_0 &: \Sigma_0^0 \longrightarrow \Sigma_0^0. \end{aligned} \quad (106)$$

Because the switching set Σ_0^0 is the special case of the switching sets Σ_1^0 and Σ_2^0, the mappings P_3 and P_8 can apply to Σ_0^0, that is,

$$\begin{aligned} P_3 &: \Sigma_0^0 \longrightarrow \Sigma_1^0,\\ P_3 &: \Sigma_1^0 \longrightarrow \Sigma_0^0;\\ P_8 &: \Sigma_0^0 \longrightarrow \Sigma_2^0,\\ P_8 &: \Sigma_2^0 \longrightarrow \Sigma_0^0. \end{aligned} \quad (107)$$

The switching sets and mappings are shown in Figure 5.

In all eleven mappings, P_n are the local mappings when $n = 0, 1, 2, 3, 6, 7, 8$ and P_n are the global mappings when $n = 4, 5, 9, 10$. From the previous defined mappings, for each

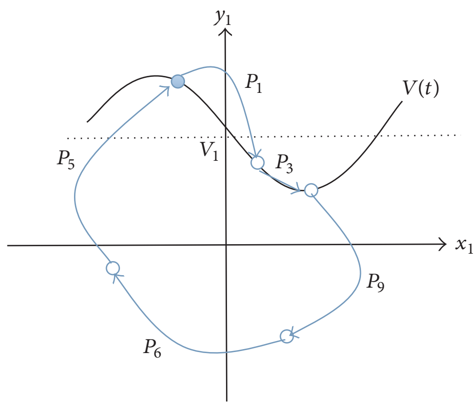

FIGURE 6: A mapping structure in absolute space of m_1.

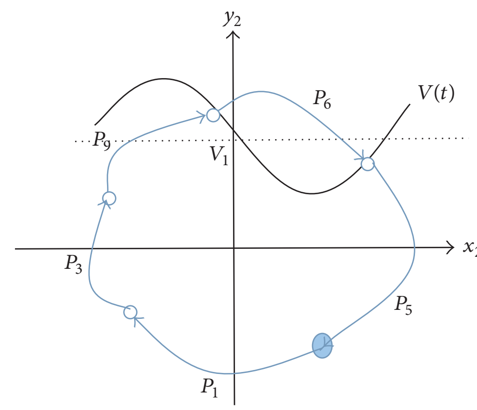

FIGURE 8: A mapping structure in absolute space of m_2.

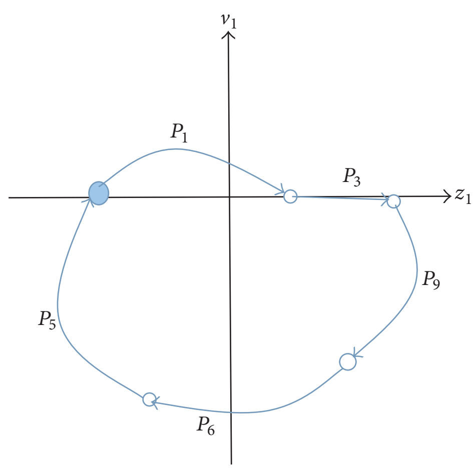

FIGURE 7: A mapping structure in relative space of m_1.

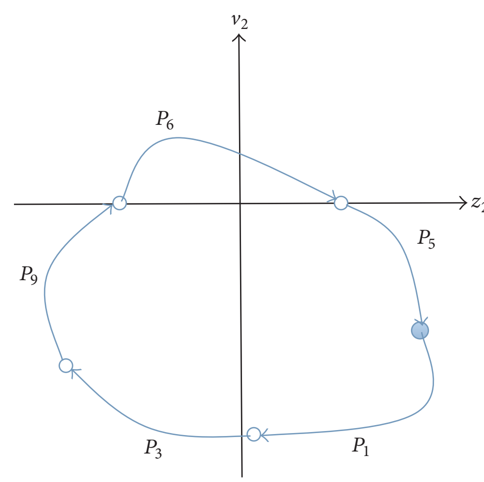

FIGURE 9: A mapping structure in relative space of m_2.

mapping P_n $(n = 0, 1, \ldots, 10)$, one obtains a set of nonlinear algebraic equations

$$\mathbf{f}^{(n)}\left(\mathbf{z}_i, t_i, \mathbf{z}_{i+1}, t_{i+1}\right) = \mathbf{0}, \tag{108}$$

where

$$\begin{aligned}\mathbf{z}_i &= \left(z_{1(i)}, v_{1(i)}, z_{2(i)}, v_{2(i)}\right)^{\mathrm{T}},\\ \mathbf{f}^{(n)} &= \left(f_1^{(n)}, f_2^{(n)}, f_3^{(n)}, f_4^{(n)}\right)^{\mathrm{T}}\end{aligned} \tag{109}$$

with the constraints for $\mathbf{z}_i$ and $\mathbf{z}_{i+1}$ from the boundaries.

The system in Section 3 has complicated motions, while any possible physical motion can be generated by the combination of the above eleven mappings in this section. The periodic motion with stick should be specially discussed. Consider the mappings of periodic motion with stick of m_1 in absolute space and in relative space, which are shown in Figures 6 and 7, respectively. And the corresponding mappings of periodic motion without stick of m_2 in absolute space and in relative space are shown in Figures 8 and 9, respectively.

For $P_{56931} = P_5 \circ P_6 \circ P_9 \circ P_3 \circ P_1$, the corresponding mapping relations are

$$\begin{aligned}
&P_1 : \left(t_i, z_{1(i)}, 0, \mathbf{z}_{2(i)}\right) \longrightarrow \left(t_{i+1}, z_{1(i+1)}, 0, \mathbf{z}_{2(i+1)}\right),\\
&P_3 : \left(t_{i+1}, z_{1(i+1)}, 0, \mathbf{z}_{2(i+1)}\right)\\
&\quad \longrightarrow \left(t_{i+2}, z_{1(i+2)}, 0, \mathbf{z}_{2(i+2)}\right),\\
&P_9 : \left(t_{i+2}, z_{1(i+2)}, 0, \mathbf{z}_{2(i+2)}\right)\\
&\quad \longrightarrow \left(t_{i+3}, \mathbf{z}_{1(i+3)}, z_{2(i+3)}, 0\right),\\
&P_6 : \left(t_{i+3}, \mathbf{z}_{1(i+3)}, z_{2(i+3)}, 0\right)\\
&\quad \longrightarrow \left(t_{i+4}, \mathbf{z}_{1(i+4)}, z_{2(i+4)}, 0\right),\\
&P_5 : \left(t_{i+4}, \mathbf{z}_{1(i+4)}, z_{2(i+4)}, 0\right)\\
&\quad \longrightarrow \left(t_{i+5}, z_{1(i+5)}, 0, \mathbf{z}_{2(i+5)}\right).
\end{aligned} \tag{110}$$

Such mapping relations provide the nonlinear algebraic equations, that is,

$$\begin{aligned}
f_1^{(n)}(\mathbf{z}_i, t_i, \mathbf{z}_{i+1}, t_{i+1}) &= 0, \\
f_3^{(n)}(\mathbf{z}_{i+1}, t_{i+1}, \mathbf{z}_{i+2}, t_{i+2}) &= 0, \\
f_9^{(n)}(\mathbf{z}_{i+2}, t_{i+2}, \mathbf{z}_{i+3}, t_{i+3}) &= 0, \\
f_6^{(n)}(\mathbf{z}_{i+3}, t_{i+3}, \mathbf{z}_{i+4}, t_{i+4}) &= 0, \\
f_5^{(n)}(\mathbf{z}_{i+4}, t_{i+4}, \mathbf{z}_{i+5}, t_{i+5}) &= 0,
\end{aligned} \tag{111}$$

where $\mathbf{z}_{1(i+5)} = \mathbf{z}_{1(i)}$ and $\mathbf{z}_{2(i+5)} = \mathbf{z}_{2(i)}$ and $t_{i+5} = t_i + NT$ (T is a period; $N = 1, 2, \ldots$), and give the switching points for the periodic solutions. Similarly, the other mapping structures can be developed to analytically predict the switching points for periodic motions in the 2-DOF friction induced oscillator.

7. Numerical Simulations

To illustrate the analytical conditions of stick and nonstick motions, the motions of the 2-DOF oscillator will be demonstrated through the time histories of displacement and velocity, the corresponding trajectory of the oscillator in phase space. The starting points of motions are represented by blue-solid circular symbols; the switching points at which the oscillator contacts on the moving boundaries are depicted by red-solid circular symbols. The moving boundaries, that is, the velocity curves of the traveling belt, are represented by blue curves, and the displacement, the velocity or the forces of per unit mass, and the corresponding trajectories of the oscillator in phase space are shown by the green curves, red curves, and dark curves, respectively.

Consider a set of system parameters for numerical illustration:

$$\begin{aligned}
m_1 &= 2\,\text{kg}, \\
r_1 &= 0.1\,\text{N·s/m}, \\
k_1 &= 1\,\text{N/m}, \\
A_1 &= 0.1\,\text{N}, \\
B_1 &= -5\,\text{N}, \\
m_2 &= 2\,\text{kg}, \\
r_2 &= 0.5\,\text{N·s/m}, \\
k_2 &= 2\,\text{N/m}, \\
A_2 &= -0.5\,\text{N}, \\
B_2 &= 5\,\text{N}, \\
r_3 &= 0.05\,\text{N·s/m}, \\
k_3 &= 0.5\,\text{N/m}, \\
\Omega &= 2\,\text{rad/s}, \\
\mu_k &= 0.4, \\
g &= 10\,\text{m/s}^2,
\end{aligned}$$

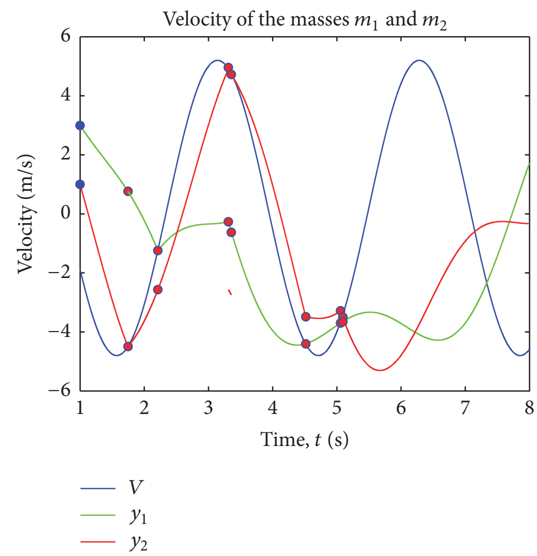

FIGURE 10: Velocity-time history of the masses m_1 and m_2 and the traveling belt ($m_1 = 2$, $r_1 = 0.1$, $k_1 = 1$, $A_1 = 0.1$, $B_1 = -5$, $m_2 = 2$, $r_2 = 0.5$, $k_2 = 2$, $A_2 = -0.5$, $B_2 = 5$, $r_3 = 0.05$, $k_3 = 0.5$, $\Omega = 2$, $\mu_k = 0.4$, $g = 10$, $\beta = 0$, $V_0 = 5$, $V_1 = 0.2$). The initial condition is $t_0 = 1$, $x_{10} = 1$, $y_{10} = 3$, $x_{20} = 2$, $y_{20} = 1$.

$$\begin{aligned}
\beta &= 0, \\
V_0 &= 5\,\text{m/s}, \\
V_1 &= 0.2\,\text{m/s}.
\end{aligned}$$

For the above system parameters, the nonstick motions and stick motions of mass m_1 and mass m_2 are presented in Figures 10, 11, and 12 with the initial conditions of $t_0 = 1\,\text{s}$, $x_{10} = 1\,\text{m}$, $y_{10} = 3\,\text{m/s}$, $x_{20} = 2\,\text{m}$, $y_{20} = 1\,\text{m/s}$. Consider the complex mappings $P = P_{10} \circ P_1 \circ P_{10} \circ P_8 \circ P_9 \circ P_5 \circ P_6$. The time histories of velocities of the traveling belt and the masses m_1 and m_2 are depicted in Figure 10. The time histories of the velocities, displacements, trajectories of the oscillators in phase space, and the corresponding forces of per unit mass of mass m_1 and mass m_2 are shown in Figures 11(a), 11(b), 11(c), and 11(d) and 12(a), 12(b), 12(c), and 12(d), respectively. The time history of the force of per unit mass of m_2 is shown in Figure 12(e) when the sliding motion of m_2 occurs.

When $t \in [1, 1.7450)$, m_1 and m_2 move freely in domain Ω_1, which satisfy $y_1 > V$ and $y_2 > V$, as shown in Figure 10. In this time interval the time histories of velocities of m_1 and m_2 are shown in Figures 11(a) and 12(a), respectively. The displacements of m_1 and m_2 are shown in Figures 11(b) and 12(b), respectively. At the time $t_1 = 1.7450$, the velocity of m_2 reached the speed boundary $\partial\Omega_{12}$ (i.e., $y_2 = V$). Since the forces $F_{2-}^{(1)}$ and $F_{2+}^{(2)}$ of per unit mass satisfy the conditions of $F_{2-}^{(1)} < \dot{V}$ and $F_{2+}^{(2)} < \dot{V}$ (as shown in Figure 12(d)), the analytical condition (59) of the passable motion on the boundary $\partial\Omega_{12}$ is satisfied in Theorem 9 (b). At such a point the motion enters into the domain Ω_2 relative to $y_2 < V$, as shown in Figure 10. Due to the movement in the area Ω_1 at the time $t_0 = 1$ and the movement that reached the boundary $\partial\Omega_{12}$ at the time $t_1 = 1.7450$, the mapping for this process is P_6. When $t \in (1.7450, 2.2120)$, m_1 and m_2 move freely in

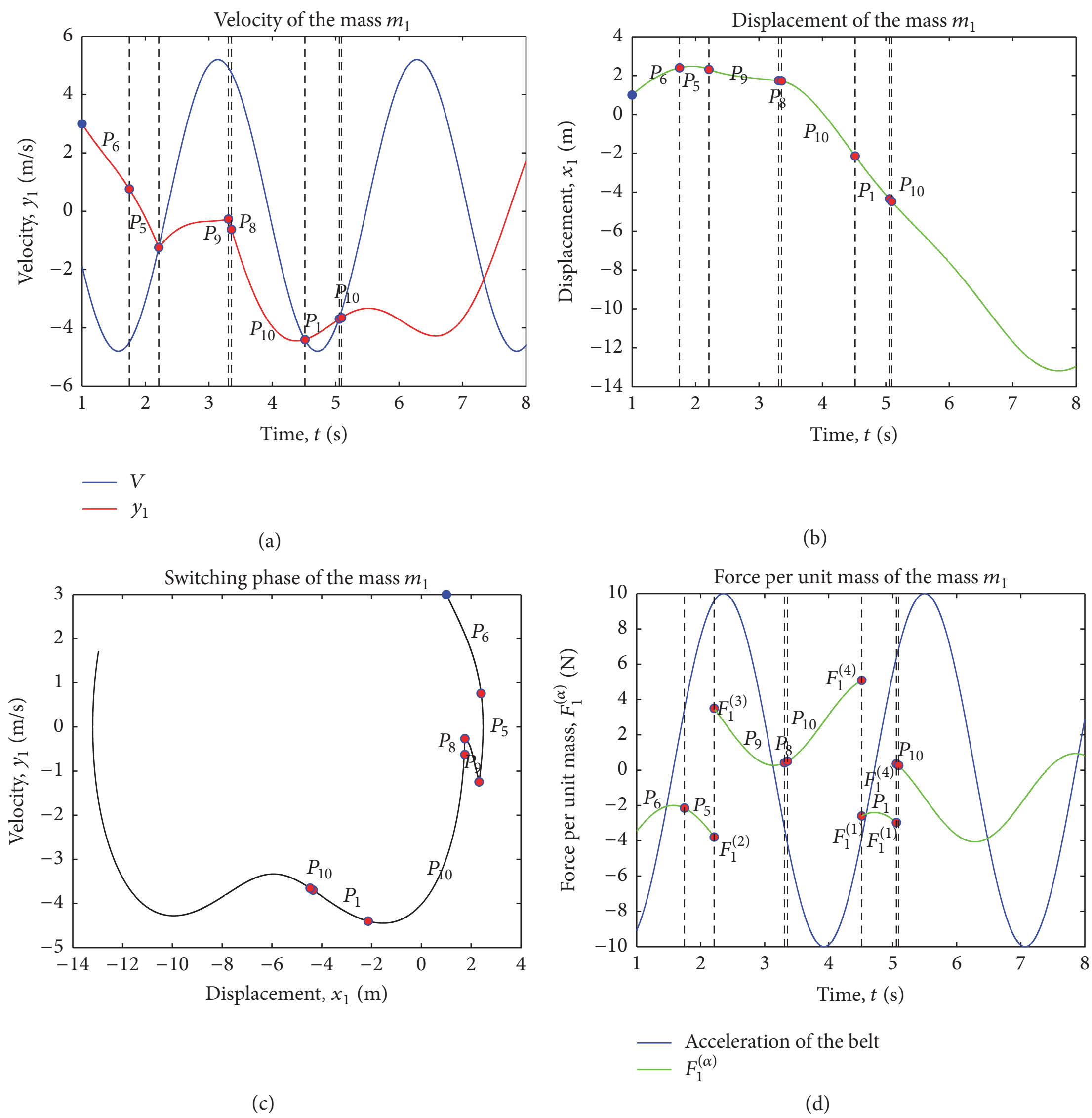

Figure 11: Numerical simulation of a motion of m_1 relative to mapping $P = P_{10} \circ P_1 \circ P_{10} \circ P_8 \circ P_9 \circ P_5 \circ P_6$: (a) velocity-time history, (b) displacement-time history, (c) phase trajectory, and (d) force-time history ($m_1 = 2$, $r_1 = 0.1$, $k_1 = 1$, $A_1 = 0.1$, $B_1 = -5$, $m_2 = 2$, $r_2 = 0.5$, $k_2 = 2$, $A_2 = -0.5$, $B_2 = 5$, $r_3 = 0.05$, $k_3 = 0.5$, $\Omega = 2$, $\mu_k = 0.4$, $g = 10$, $\beta = 0$, $V_0 = 5$, $V_1 = 0.2$). The initial condition is $t_0 = 1$, $x_{10} = 1$, $y_{10} = 3$, $x_{20} = 2$, $y_{20} = 1$.

domain Ω_2, which satisfy $y_1 > V$ and $y_2 < V$, as shown in Figure 10. The displacements of m_1 and m_2 are shown in Figures 11(b) and 12(b), respectively. At the time $t_2 = 2.2120$, the velocity of m_1 reached the speed boundary $\partial\Omega_{23}$ (i.e., $y_1 = V$). Since the forces $F_{1-}^{(2)}$ and $F_{1+}^{(3)}$ of per unit mass satisfy the conditions of $F_{1-}^{(2)} < \dot{V}$ and $F_{1+}^{(3)} < \dot{V}$ (as shown in Figure 11(d)), the analytical condition (62) of the passable motion on the boundary $\partial\Omega_{23}$ is satisfied in Theorem 9 (e). At such a point the motion enters into the domain Ω_3 relative to $y_1 < V$, as shown in Figure 10. Due to the movement on the boundary $\partial\Omega_{12}$ at the time $t_1 = 1.7450$ and the movement that reached the boundary $\partial\Omega_{23}$ at the time $t_2 = 2.2120$, the mapping for this process is P_5. When $t \in (2.2120, 3.3090)$, m_1 and m_2 move freely in domain Ω_3, which satisfy $y_1 < V$ and $y_2 < V$, as shown in Figure 10. The displacements of m_1 and m_2 are shown in Figures 11(b) and 12(b), respectively. At the time $t_3 = 3.3090$, the velocity of m_2 reached the speed boundary $\partial\Omega_{34}$ (i.e., $y_2 = V$). Since the forces $F_{2-}^{(3)}$ and $F_{2-}^{(4)}$ of per unit mass satisfy the conditions of $F_{2-}^{(3)} > \dot{V}$ and $F_{2-}^{(4)} < \dot{V}$ (as shown in Figure 12(d)), the analytical condition (69) of the stick motion on the boundary $\partial\Omega_{34}$ is satisfied in Theorem 10. At such a point the sliding motion of m_2 occurs on the boundary $\partial\Omega_{34}$ and keeps to $t_4 = 3.3570$. Due to the movement on the boundary $\partial\Omega_{23}$ at the time $t_2 = 2.2120$ and the movement that reached the boundary $\partial\Omega_{34}$ at the time $t_3 = 3.3090$, the mapping for this process is P_9.

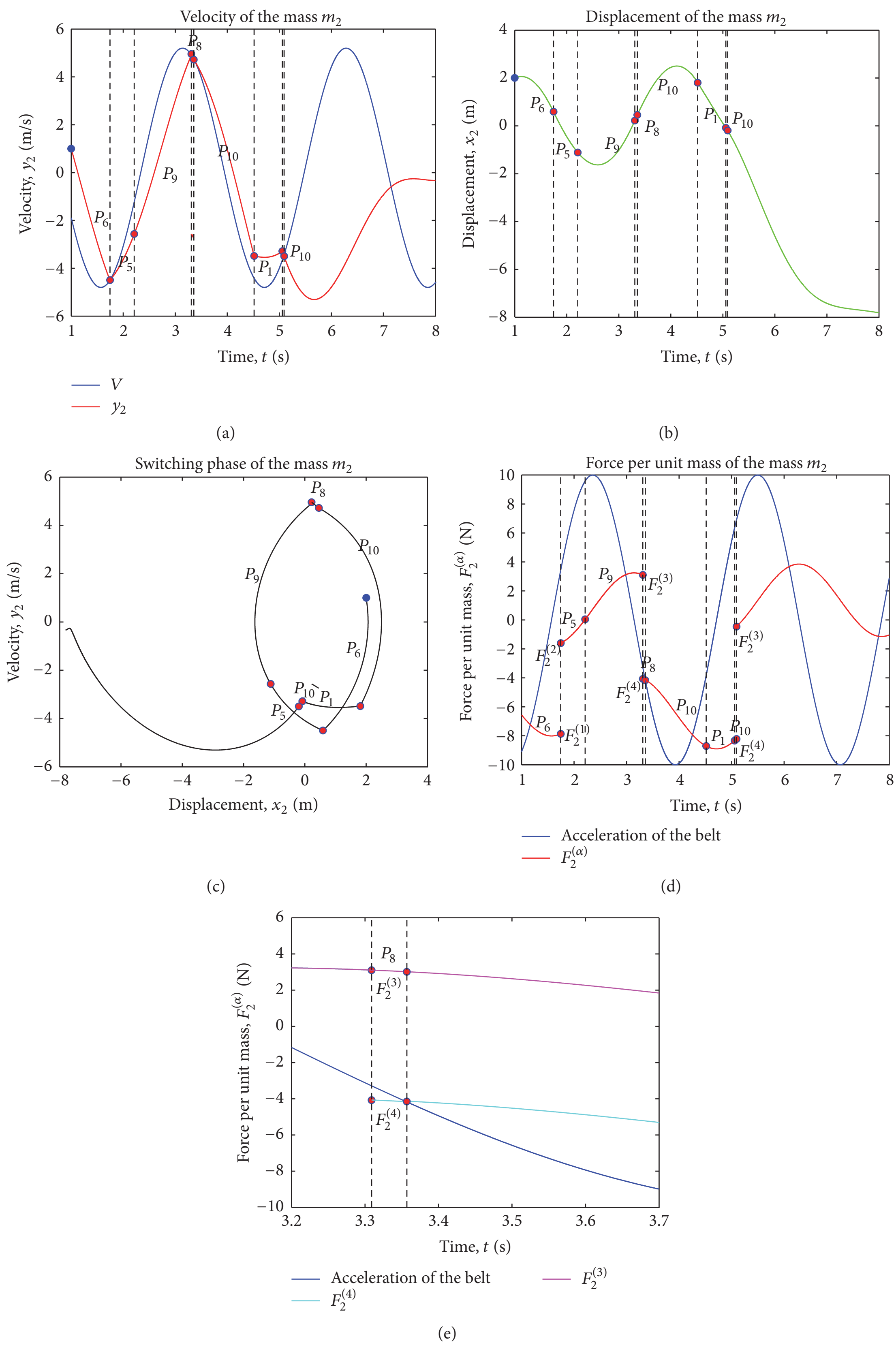

FIGURE 12: Numerical simulation of a motion of m_2 relative to mapping $P = P_{10} \circ P_1 \circ P_{10} \circ P_8 \circ P_9 \circ P_5 \circ P_6$: (a) velocity-time history, (b) displacement-time history, (c) phase trajectory, (d) force-time history, (e) and force of per unit mass of m_2 relative to mapping P_8 when sliding motion ($m_1 = 2$, $r_1 = 0.1$, $k_1 = 1$, $A_1 = 0.1$, $B_1 = -5$, $m_2 = 2$, $r_2 = 0.5$, $k_2 = 2$, $A_2 = -0.5$, $B_2 = 5$, $r_3 = 0.05$, $k_3 = 0.5$, $\Omega = 2$, $\mu_k = 0.4$, $g = 10$, $\beta = 0$, $V_0 = 5$, $V_1 = 0.2$). The initial condition is $t_0 = 1$, $x_{10} = 1$, $y_{10} = 3$, $x_{20} = 2$, $y_{20} = 1$.

When $t \in (3.3090, 3.3570)$, m_1 moves freely in domain Ω_3, satisfying $y_1 < V$, as shown in Figure 10. The displacement of m_1 is shown in Figure 11(b), correspondingly. However, at such time interval, m_2 maintains sliding motion, and the time history of force per unit mass of m_2 is shown in Figure 12(e). In this time period, the force product satisfies $F_2^{(3)} \cdot F_2^{(4)} < 0$ relative to $\dot{V}$, where $F_2^{(3)}$ is represented by pink curves and $F_2^{(4)}$ is represented by light cyan curves. At the time $t_4 = 3.3570$, the forces $F_{2-}^{(3)}$ and $F_2^{(4)}$ of per unit mass satisfy the conditions of $F_{2-}^{(3)} > \dot{V}$, $F_2^{(4)} = \dot{V}$ and $DF_{2\pm}^{(4)} > \ddot{V}$, so the analytical condition (86) of the vanishing of stick motion on the boundary $\partial\Omega_{34}$ is satisfied in Theorem 12 and the sliding motion of m_2 vanishes and the motion of m_2 enters the domain Ω_4. Due to the movement on the boundary $\partial\Omega_{34}$ from the time $t_3 = 3.3090$ to the time $t_4 = 3.3570$, the mapping for this process is P_8.

When $t \in (3.3570, 4.5150)$, m_1 and m_2 move freely again in domain Ω_4, which satisfy $y_1 < V$ and $y_2 > V$, as shown in Figure 10. The displacements of m_1 and m_2 are shown in Figures 11(b) and 12(b), respectively. At the time $t_5 = 4.5150$, the velocity of m_1 reached the speed boundary $\partial\Omega_{14}$ (i.e., $y_1 = V$). Since the forces $F_{1-}^{(4)}$ and $F_{1+}^{(1)}$ of per unit mass satisfy the conditions of $F_{1-}^{(4)} > \dot{V}$ and $F_{1+}^{(1)} > \dot{V}$ (as shown in Figure 11(d)), the analytical condition (64) of the passable motion on the boundary $\partial\Omega_{41}$ is satisfied in Theorem 9 (g). At such a point the motion enters into the domain Ω_1 relative to $y_1 > V$ as shown in Figure 10. Due to the movement on the boundary $\partial\Omega_{34}$ at the time $t_4 = 3.3570$ and the movement that reached the boundary $\partial\Omega_{41}$ at the time $t_5 = 4.5150$, the mapping for this process is P_{10}. When $t \in (4.5150, 5.0570)$, m_1 and m_2 move freely in domain Ω_1, which satisfy $y_1 > V$ and $y_2 > V$, as shown in Figure 10. The displacements of m_1 and m_2 are shown in Figures 11(b) and 12(b), respectively. At the time $t_6 = 5.0570$, the velocity of m_1 reached the speed boundary $\partial\Omega_{41}$ (i.e., $y_1 = V$). Since the forces $F_{1-}^{(1)}$ and $F_{1+}^{(4)}$ of per unit mass satisfy the conditions of $F_{1-}^{(1)} < \dot{V}$ and $F_{1+}^{(4)} < \dot{V}$ (as shown in Figure 11(d)), the analytical condition (65) of the passable motion on the boundary $\partial\Omega_{41}$ is satisfied in Theorem 9 (h). At such a point the motion enters into the domain Ω_4 relative to $y_1 < V$ as shown in Figure 10. Due to the movement on the boundary $\partial\Omega_{41}$ at the time $t_5 = 4.5150$ and the movement that reached the boundary $\partial\Omega_{41}$ at the time $t_6 = 5.0570$, the mapping for this process is P_1. When $t \in (5.0570, 5.0910)$, m_1 and m_2 move freely in domain Ω_4, which satisfy $y_1 < V$ and $y_2 > V$, as shown in Figure 10. The displacements of m_1 and m_2 are shown in Figures 11(b) and 12(b), respectively. At the time $t_7 = 5.0910$, the velocity of m_2 reached the speed boundary $\partial\Omega_{34}$ (i.e., $y_2 = V$). Since the forces $F_{2-}^{(4)}$ and $F_{2+}^{(3)}$ of per unit mass satisfy the conditions of $F_{2-}^{(4)} < \dot{V}$ and $F_{2+}^{(3)} < \dot{V}$, the analytical condition (61) of the passable motion on the boundary $\partial\Omega_{34}$ is satisfied in Theorem 9 (d). At such a point the motion enters into the domain Ω_3 relative to $y_2 < V$ as shown in Figure 10. Due to the movement on the boundary $\partial\Omega_{41}$ at the time $t_6 = 5.0570$ and the movement that reached the boundary $\partial\Omega_{34}$ at the time $t_7 = 5.0910$, the mapping for this process is P_{10}.

When $t > 5.0910$, the movement will continue, but, here, the later motion will not be described. In the whole process, the phase trajectories of m_1 and m_2 are shown in Figures 11(c) and 12(c), respectively.

8. Conclusion

The model of frictional-induced oscillator with two degrees of freedom on a speed-varying traveling belt was proposed. The dynamics of such oscillator with two harmonically external excitations on a speed-varying traveling belt were investigated by using the theory of flow switchability for discontinuous dynamical systems. The dynamics of this system are of interest because it is a simple representation of mechanical systems with multiple nonsmooth characteristics. Different domains and boundaries for such system were defined according to the friction discontinuity. Based on the above domains and boundaries, the analytical conditions for the passable motions and the onset or vanishing of stick motions and grazing motions were presented. The basic mappings were introduced to describe motions in such an oscillator. Analytical conditions of periodic motions were developed by the mapping dynamics. Numerical simulations were carried out to illustrate stick and nonstick motions for a better understanding of complicated dynamics of such mechanical model. Through the velocity and force responses of such motions, it is possible to validate analytical conditions for the motion switching in such a discontinuous system. There are more simulations about such an oscillator to be discussed in future.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

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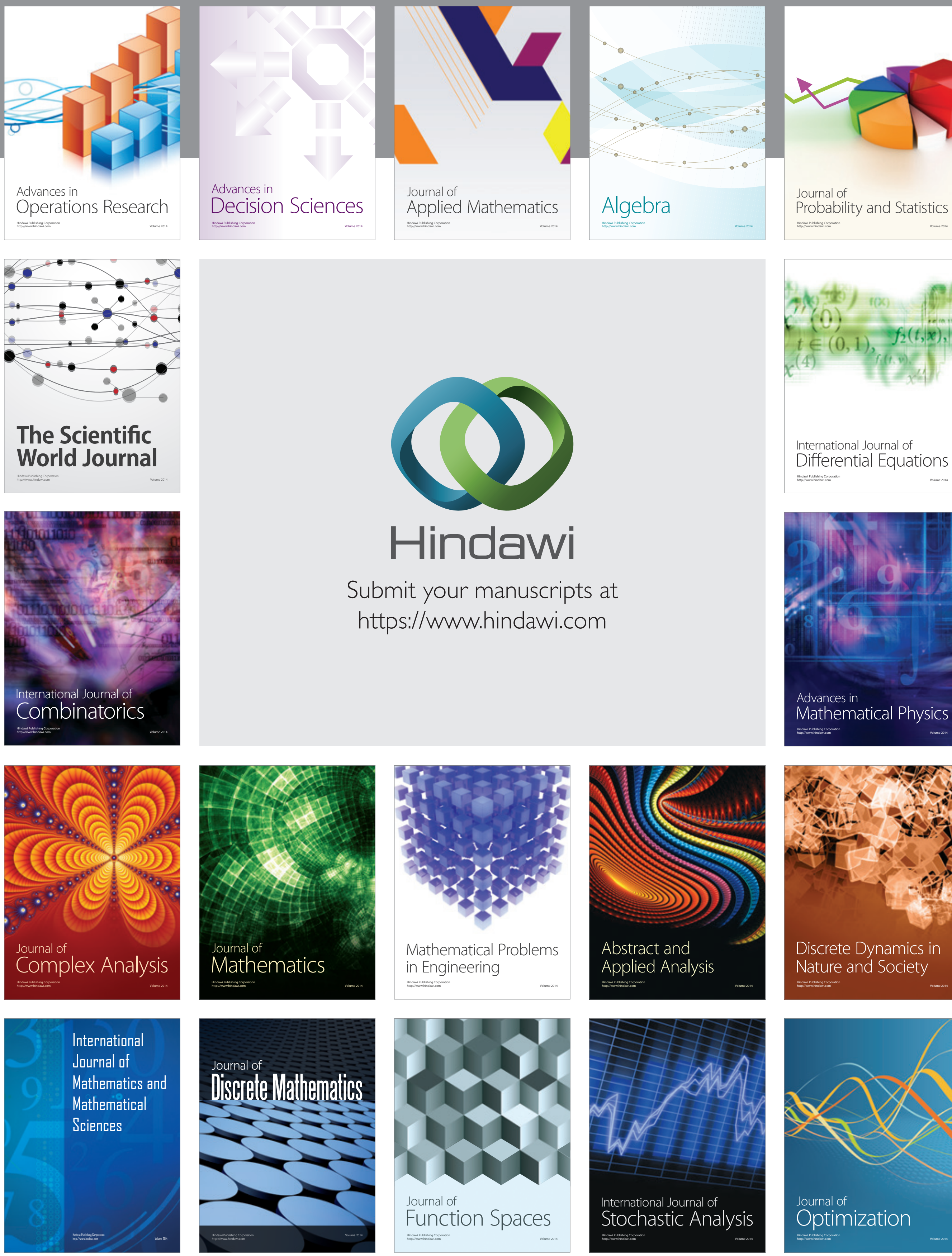
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