Hindawi
Mathematical Problems in Engineering
Volume 2017, Article ID 1575828, 14 pages
https://doi.org/10.1155/2017/1575828

Research Article

The Strict-Sense Nonblocking Multirate $\log_d(N, 0, p)$ Switching Network

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Received 18 May 2016; Revised 7 November 2016; Accepted 14 November 2016; Published 7 February 2017

Academic Editor: Kyandoghere Kyamakya

This paper considers the nonblocking conditions for a multirate $\log_d(N, 0, p)$ switching network at the connection level. The necessary and sufficient conditions for the discrete bandwidth model, as well as sufficient and, in particular cases, also necessary conditions for the continuous bandwidth model, were given. The results given for $d^{\lfloor (n-1)/2 \rfloor} f_0 \geq f_1 + 1$ in the discrete bandwidth model are the same as those proposed by Hwang et al. (2005); however, in this paper, these results were extended to other values of f_0, f_1, and d. In the continuous bandwidth model for $B + b > 1$, the results given in this paper are also the same as those by Hwang et al. (2005); however, for $B + b \leq 1$, it was proved that a smaller number of vertically stacked $\log_d N$ switching networks are needed.

1. Introduction

A multirate switching network is a network in which any connection is associated with weight ω. Such a weight represents a certain bandwidth of input and output ports and interstage links connecting these input and output ports. The capacity of the links is normalized and is usually equal to 1 for interstage links. Input and output ports' capacity is in many cases lower than the interstage links' capacity and is denoted by β, where $\beta \leq 1$. Weight ω is also limited by range $[b, B]$, $b < B$, where $0 < b \leq \omega \leq B \leq \beta \leq 1$.

Depending on the possible values of ω, two models of multirate connections are considered in the literature: the discrete bandwidth model and the continuous bandwidth model. In the discrete bandwidth model, it is assumed that there are a finite number of distinct rates $\{b_1, b_2, \ldots, b_k\}$ and the smallest rate b_1 divides all other rates b_k, where $k = 2, \ldots, K$. Denote $b = b_1$ and $B = \max\{b_k : k = 1, 2, \ldots, K\}$. The smallest rate is often called a channel. In this paper, we assumed that each internal link has c_M channels and each input or output link has $c_{I/O}$ channels, where $c_M = \lfloor 1/b \rfloor$, $c_{I/O} = \lfloor \beta/b \rfloor$, and $c_{I/O} \leq c_M$. Every new connection is associated with a positive integer c, where $1 \leq c \leq C \leq c_{I/O}$ and C is the maximal number of channels that one request can demand. In the continuous bandwidth model, connections may occupy any fraction of a link's transmission capacity from interval $[b, B]$. Both models are considered in this paper.

One of the best known multistage switching networks is the 3-stage Clos network [1]. Nonblocking conditions for single-rate and multicast connections were considered by many authors [2–5]. The first upper bound of nonblocking conditions in the case of the continuous bandwidth model was proposed by Melen and Turner in [6]. This upper bound was later improved by Chung and Ross in [7]. In turn, asymmetrical switch configurations were considered in [8]. More generalized 3-stage Clos switching fabrics were considered by Liotopoulos and Chalasani [9]. The results derived in those papers were limited to $b = 0$ or $B \in (1 - b, B]$. Both sufficient and necessary nonblocking conditions for any B and $b > 0$ were proved in [10, 11] in the case of symmetrical and asymmetrical 3-stage Clos switching networks, respectively. In some papers, the blocking probability at the connection level was also considered [12–14].

Another switching network considered in the literature is the vertically stacked Banyan type switching network [15–18]. Multirate $\log_d(N, m, p)$ switching fabrics were considered in [19, 20], where necessary and sufficient conditions were given for the discrete bandwidth model when $1/b$ is an integer, and a sufficient condition, as well as a necessary condition for $B \in (1-b, \beta]$, for the continuous bandwidth model was proved. Better upper bounds, which in some cases are also lower bounds, for $\log_2(N, 0, p)$ switching networks were given in [21, 22]. Multirate $\log_2(N, 0, p)$ switching fabrics with multicast connections were considered in turn in [23]. Some architectures, which may be considered as special cases of $\log_2(N, m, p)$ networks, like extended delta and Cantor switching fabrics, were considered earlier in [6, 7, 24].

The results presented in [19, 20] have been improved by Hwang et al. in [25]. They proved both sufficient and necessary conditions for the strict-sense nonblocking operation of $\log_d(N, m, p)$ networks for the discrete bandwidth model, but only where $d^{\lfloor (n-1)/2 \rfloor} c_{I/O} \geq c_M + 1$. They also proved sufficient and necessary conditions for the continuous bandwidth model when $b + B > 1$ and sufficient conditions when $b + B \leq 1$ (the conditions are also necessary in case $b = 0$).

In this paper, we also consider a multirate $\log_d(N, m, p)$ switching network, however, only for $m = 0$. The results for $m \geq 1$ are under study. First, we extended the results given in [25] for the discrete bandwidth model to the general case. Then, we also introduce sufficient condition for the continuous bandwidth model when $b + B \leq 1$. In most cases, these sufficient conditions are also necessary.

It should be noted that nowadays multirate switching networks [6–14, 16–19, 23, 24] are getting more popular each day [26–30]. A special kind of such a multirate network is an elastic optical network [31–36] which constitutes a "hot topic" in optical networking and switching. In the future, elastic optical networks will replace the current optical networks used by network operators and will also probably be used in data centers. In the elastic optical network, an optical path may occupy a bandwidth which is a multiple of the so-called frequency slot unit. This frequency slot unit occupies 12.5 GHz of the bandwidth and c adjacent frequency slot units may be assigned to one optical path. This may be described by using the discrete bandwidth model with b denoting the frequency slot unit and c denoting the number of such units in one connection.

Multirate type of structure can be used, for example, in data center networks [37–41] or in multiprocessor systems. By using multirate structures, it is also possible to handle network traffic in telecommunication and computer networks generated by many services handled by network providers or companies. Using a new type of switching network structures in data centers or multiprocessor systems allows building more energy-efficient and cheaper architectures, where the cost could be understood as the number of cross points [3, 4] or as the number of active and passive optical switching elements [42]. The topic of energy efficiency is not considered in this paper; however, this paper could be a starting point for such an investigation.

In turn, the classification of different types of services enables the proper management of resources and appropriate performance for each of these services especially for the 4G/5G networks. Each service requires different resources, expressed very often in basic bandwidth units or in the number of channels. And for the 4G/5G networks bandwidth is a very crucial aspect. If sufficient resources are available, each service considered can be realized in such a switching network. It was assumed that one connection represents some service and each service requires a different number of channels c, where the maximal number of channels one service could demand is C, and $c \leq C$.

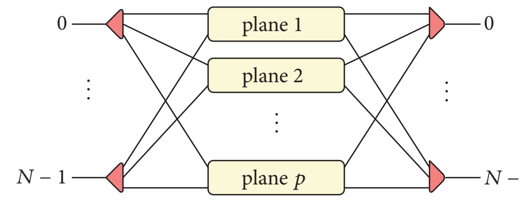

Figure 1: An example of $\log_2(16, 0, 2)$ switching network.

Motivation of this paper was to improve already known best results for the strict-sense nonblocking multirate $\log_d(N, 0, p)$ switching network [25]. In the next sections, it is described in detail how better results were achieved. The $\log_d(N, 0, p)$ switching network constitutes nowadays a quite interesting architecture which could be used in optical network nodes especially for $d = 2$, where physical implementation is much easier than that for $d \geq 3$, where d denotes the size of one switching element.

This paper is organized as follows. In Section 2, the model used in this paper is described. In the next section, the nonblocking operation of the considered network is discussed for the discrete bandwidth model. In Section 4, strict-sense nonblocking conditions for the continuous bandwidth model are considered. In the next section, a few numerical examples for both discrete and continuous bandwidth models are presented. In the same section, the results obtained in this paper are compared with the already known upper bounds. The last section includes conclusions.

2. Model Description

The $\log_2(N, 0, p)$ switching network was proposed in [16]. Such a network is constructed by vertically stacking p copies of $\log_2 N$ networks (the main idea of stacking planes is shown in Figure 1). This architecture was later extended to the $\log_2(N, m, p)$ switching network in [17]. Such an extended network is obtained by adding m extra stages to the $\log_2 N$ network and vertically stacking p copies of the $\log_2(N, m, 1)$ network. The $\log_2(N, m, p)$ switching network is a particular case of the $\log_d(N, m, p)$ switching network which consists of $d \times d$ symmetrical switching elements. The $\log_d(N, m, p)$ switching network is nonblocking in the strict sense when there are a sufficient number of vertically stacked planes p, so any connection between a free input and a free output can be realized regardless of the routing algorithm used.

The architecture of the $\log_d(N, 0, p)$ switching network (where $d = 2$, $N = 16$, and $p = 2$) is shown in Figure 2. It consists of p vertically stacked $\log_d N$ networks composed of $d \times d$ switches. These $\log_d N$ networks are called planes. At each input terminal there is a splitter $1 \times p$ and at each output terminal there is a combiner $p \times 1$. Throughout the

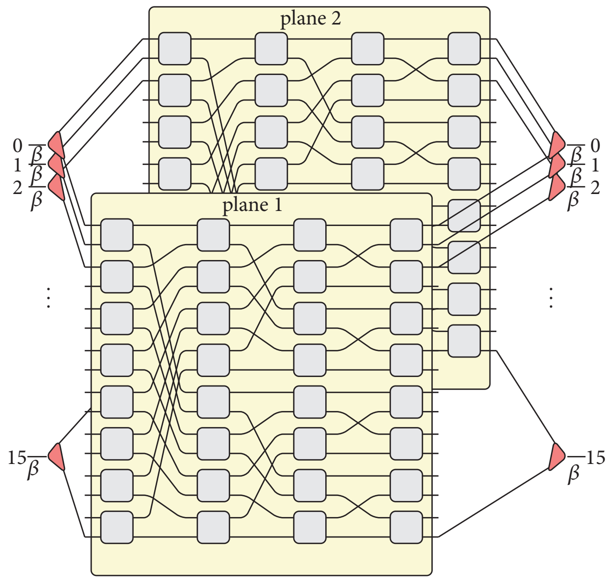

FIGURE 2: An example of $\log_2(16, 0, 2)$ switching network.

discussion in this paper, bipartite graphs [43–46] will be used to represent the topology of the switching network [16, 18]. In a bipartite graph, which represents the space-division network, two paths are not allowed to intersect at a node. If this happens, this means that one of these two paths is blocked.

Before we move on to the next part of this paper, we will determine the number of paths A that may intersect with a given path in one node. Let us assume that we want to set up connection $\langle i, o\rangle$ between input terminal i and output terminal o. This connection will be blocked in one plane if another connection is already set up through one of the nodes belonging to path $\langle i, o\rangle$. These nodes can belong to the part of the switching network that is reachable from input terminals or to the part of the switching network that is reachable from output terminals. These two parts of the switching network will be denoted as set $\mathbb{I}$. These parts are equal and both of them have $(n-1)/2$ or $(n-2)/2$ stages for n odd or n even, respectively, where $n = \log_d N$. Let us determine

$$x = (n-1) \bmod 2. \tag{1}$$

When n is odd, we have $x = 0$, or $x = 1$ when n is even. So, the above stage numbers can be superseded to $(n-1-x)/2$. The number of paths reachable at the node of stage $(n-1-x)/2$ is equal to $d^{(n-1-x)/2}$. However, one of these paths is the path which belongs to the considered connection $\langle i, o\rangle$, so the number of paths that intersect with the given path $\langle i, o\rangle$ is equal to

$$A_{\mathbb{I}} = d^{(n-1-x)/2} - 1. \tag{2}$$

When n is even, there is also the central stage $n/2$ and there are $A_{\mathbb{M}}$ additional paths that intersect with the considered path $\langle i, o\rangle$, where $\mathbb{M}$ denotes a set of these additional paths. The number of these additional paths is equal to

$$A_{\mathbb{M}} = (d-1) \cdot d^{(n-1-x)/2}. \tag{3}$$

According to formulas (2) and (3), it should also be noted that the number of all paths that can intersect with the considered path $\langle i, o\rangle$ is denoted by A and is given by the following equation:

$$A = A_{\mathbb{I}} + A_{\mathbb{M}}. \tag{4}$$

3. Discrete Bandwidth Model

In the discrete bandwidth model, the capacity of the link is divided into bandwidth units also called channels. A connection may occupy c such channels, where c is usually limited by C, which denotes the maximum number of channels one connection may occupy, and $1 \le c \le C \le c_{I/O} \le c_M$. In the general case, $c_{I/O}$ denotes the number of channels at each input and output link, while c_M is the number of such channels in any interstage link. A connection between input i and output o which requests c channels will be denoted by $\langle i, o, c\rangle$. This connection can be realized in input i and output o when $c_{I/O} - \sum_k c_k \ge c$, where $\sum_k c_k$ denotes the number of channels already occupied by the existing connections at input i (output o). Similarly, an interstage link can be used by the connection when $c_M - \sum_j c_j \ge c$, where $\sum_j c_j$ denotes the number of channels occupied by connections already set up through this interstage link. This means that an interstage link which has

$$z(c) \ge c_M - c + 1 \tag{5}$$

channels occupied is inaccessible for connection $\langle i, o, c\rangle$. We will also use the following equations to make the final formulas more transparent:

$$X(i; c) = \left\lfloor \frac{i \cdot c_{I/O}}{z(c)} \right\rfloor, \tag{6}$$

$$Y(i; c) = i - \left\lfloor \frac{i}{z(c)} \right\rfloor \cdot z(c), \tag{7}$$

$$Z(c) = Y(A \cdot c_{I/O}; c) + c_{I/O} - c. \tag{8}$$

Theorem 1. *The $\log_d(N, 0, p)$ switching network is strict-sense nonblocking for the discrete bandwidth model, and $1 \le c \le C \le c_{I/O} \le c_M$ if and only if*

$$p \ge 2 \cdot X(A_{\mathbb{I}}; C) + x \cdot X(A_{\mathbb{M}}; C) + \left\lfloor \frac{Z(C)}{z(C)} \right\rfloor + \left\lfloor \frac{x \cdot Y(A_{\mathbb{M}} \cdot c_{I/O}; C) + Z(C)}{z(C)} \right\rfloor + 1. \tag{9}$$

Proof. Sufficient and necessary conditions will be proved by showing the worst state in the switching network. Let a new connection be $\langle i, o, c\rangle$. This connection may be blocked if any node of stages 1 to $n-1$ carries connections that already occupy $z(c)$ channels. These blocking connections can be set up from $A_{\mathbb{I}}$ inputs, and they block $\langle i, o, c\rangle$ in nodes of stages 1 to $\lfloor (n-1)/2 \rfloor$, or to $A_{\mathbb{I}}$ outputs (they will block the new connection in stages $\lceil (n+1)/2 \rceil$ to $n-1$). When n is even, $\langle i, o, c\rangle$ may also be blocked in the node of stage $n/2$ by connections from $A_{\mathbb{M}}$ inputs. Since there are $c_{I/O}$ channels at each input or output and $z(c)$ channels block the connection in one plane, the number of planes that may be inaccessible by $\langle i, o, c\rangle$ is

$$\begin{aligned} p_1 &= 2 \cdot \left\lfloor \frac{A_{\mathbb{I}} \cdot c_{I/O}}{z(c)} \right\rfloor + x \cdot \left\lfloor \frac{A_{\mathbb{M}} \cdot c_{I/O}}{z(c)} \right\rfloor \\ &= 2 \cdot X\left(A_{\mathbb{I}}; c\right) + x \cdot X\left(A_{\mathbb{M}}; c\right). \end{aligned} \tag{10}$$

In the set of $A_{\mathbb{I}}$ inputs, we may still have

$$A_{\mathbb{I}} \cdot c_{I/O} - \left\lfloor \frac{A_{\mathbb{I}} \cdot c_{I/O}}{z(c)} \right\rfloor \cdot z(c) = Y\left(A_{\mathbb{I}} \cdot c_{I/O}; c\right) \tag{11}$$

free channels and $c_{I/O} - c$ channels at input i and output o. Therefore, we may have

$$Z(c) = Y\left(A_{\mathbb{I}} \cdot c_{I/O}; c\right) + c_{I/O} - c \tag{12}$$

channels free in input and output ports. Connections in these channels may occupy

$$p_2 = \left\lfloor \frac{Z(c)}{z(c)} \right\rfloor \tag{13}$$

additional planes. Similarly, connections to the set of $A_{\mathbb{I}}$ outputs may occupy additional planes, but when n is even, we also have to consider the center stage. It must only be counted once, since connections from inputs of the set of $\mathbb{M}$ inputs have to be set up to the set of $\mathbb{M}$ outputs, to intersect with the path of the new connection in the node of stage $n/2$. So, we may have

$$p_3 = \left\lfloor \frac{Z(c) + x \cdot Y\left(A_{\mathbb{M}} \cdot c_{I/O}; c\right)}{z(c)} \right\rfloor \tag{14}$$

additional planes inaccessible by the new connection. In the worst-case scenario, one more plane is needed to set up connection $\langle i, o, c\rangle$. Thus, finally,

$$\begin{aligned} p &\geq p_1 + p_2 + p_3 + 1 \\ &= 2 \cdot \left\lfloor \frac{A_{\mathbb{I}} \cdot c_{I/O}}{z(c)} \right\rfloor + x \cdot \left\lfloor \frac{A_{\mathbb{M}} \cdot c_{I/O}}{z(c)} \right\rfloor + \left\lfloor \frac{Z(c)}{z(c)} \right\rfloor \\ &\quad + \left\lfloor \frac{Z(c) + x \cdot Y\left(A_{\mathbb{M}} \cdot c_{I/O}; c\right)}{z(c)} \right\rfloor + 1 \end{aligned} \tag{15}$$

planes are needed.

The number of planes p must be maximized through all c and it reaches a maximum at $c = C$; then, by putting C in respective formulas ((10), (13), and (14)), we obtain conditions given in Theorem 1 (see formula (9)). □

In the case when $C = c_{I/O} = c_M = 1$, we have the space-division switching network and we may simplify the previous formulas. So, this time, the interstage link is inaccessible by considered connection $\langle i, o, C\rangle$ if it has

$$z(C) = 1 - 1 + 1 = 1 \tag{16}$$

channels occupied, and expressions (6), (7), and (8) have form as follows, respectively:

$$X(i; C) = \left\lfloor \frac{i}{1} \right\rfloor = i, \tag{17}$$

$$Y(i; C) = i - i = 0, \tag{18}$$

$$Z(C) = 0 + 1 - 1 = 0. \tag{19}$$

As a result, Theorem 1 in this case is

$$\begin{aligned} p &\geq 2 \cdot A_{\mathbb{I}} + x \cdot A_{\mathbb{M}} + \left\lfloor \frac{0}{1} \right\rfloor + \left\lfloor \frac{0+0}{1} \right\rfloor + 1 \\ &= 2 \cdot A_{\mathbb{I}} + x \cdot A_{\mathbb{M}} + 1. \end{aligned} \tag{20}$$

This result corresponds to the results given for space-division switching networks by Lea [16].

When $C = c_{I/O} = 1$ and $c_M = \nu$, we obtain the so-called ν-dilated switching network presented in Figure 3. ν interstage links are inaccessible by a new connection $\langle i, o, C\rangle$ when they have

$$z(C) = \nu - 1 + 1 = \nu \tag{21}$$

channels occupied and expressions (6), (7), and (8) have form as follows, respectively:

$$X(i; C) = \left\lfloor \frac{i}{\nu} \right\rfloor, \tag{22}$$

$$Y(i; C) = i - \left\lfloor \frac{i}{\nu} \right\rfloor \cdot \nu, \tag{23}$$

$$Z(C) = Y\left(A_{\mathbb{I}}; C\right). \tag{24}$$

Theorem 1 in this case is

$$\begin{aligned} p &\geq 2 \cdot \left\lfloor \frac{A_{\mathbb{I}}}{\nu} \right\rfloor + x \cdot \left\lfloor \frac{A_{\mathbb{M}}}{\nu} \right\rfloor + \left\lfloor \frac{Y\left(A_{\mathbb{I}}; C\right)}{\nu} \right\rfloor \\ &\quad + \left\lfloor \frac{x \cdot Y\left(A_{\mathbb{M}}; C\right) + Y\left(A_{\mathbb{I}}; C\right)}{\nu} \right\rfloor + 1. \end{aligned} \tag{25}$$

The discrete model could be adopted in all systems where channels are used. Thus, for example, this model will work for the UMTS system, the 4G system, the 5G system, and so on. It could be assumed that each kind of channel used, for example, in the UMTS/4G/5G could be divided into basic bandwidth units (or in fact into the smallest channels or slots). One UMTS/4G/5G channel then will use maximum C small channels/slot.

4. Continuous Bandwidth Model

In the continuous bandwidth model, a connection may occupy any fraction of the link capacity. This fraction is called the weight of a connection and is denoted by ω. A connection from input i to output o of weight ω will be denoted by $\langle i, o, \omega\rangle$. Weight ω is limited by interval $[b, B]$, $b < B$, and $0 < b \leq \omega \leq B \leq \beta \leq 1$, where β denotes the normalized bandwidth of input and output links. Input i (output o) is accessible by the new connection $\langle i, o, \omega\rangle$ if $\beta - \sum_k \omega_k \leq \omega$, where $\sum_k \omega_k$ denotes the total weight of all connections already set up at input i (output o). An interstage link is accessible by this new connection when $1 - \sum_j \omega_j \leq \omega$, where $\sum_j \omega_j$ denotes the total weight of all connections already set up through this interstage link. Similarly as in [3, 22], the following functions will be used:

$$\zeta(i;j) = \begin{cases} \left\lfloor \dfrac{i}{j} \right\rfloor, & \dfrac{i}{j} \text{ is not an integer or } \left\lfloor \dfrac{i}{j} \right\rfloor = 0; \\ 0, & \text{for } j = 0; \\ \left\lfloor \dfrac{i}{j} \right\rfloor - 1, & \dfrac{i}{j} \text{ is an integer and } \dfrac{i}{j} > 0; \end{cases} \tag{26}$$

$$\phi_1(i;j) = \begin{cases} i - j \cdot \zeta(i;j), & \text{for } \zeta(i;j) \neq 0,\ i - j \cdot \zeta(j;j) > b; \\ 0, & \text{for } \zeta(i;j) \neq 0,\ i - j \cdot \zeta(i;j) \leq b; \\ i, & \text{for } \zeta(i;j) = 0; \end{cases} \tag{27}$$

$$\phi_2(i;j) = \begin{cases} \left\lfloor \dfrac{j}{\phi_1(i;j)} \right\rfloor, & \text{for } \phi_1(i;j) \geq b; \\ 0, & \text{for } \phi_1(i;j) < b; \end{cases} \tag{28}$$

$$\phi_3(i) = \begin{cases} i, & \text{for } i \geq b; \\ 0, & \text{for } i < b. \end{cases} \tag{29}$$

In these equations, $\zeta(\beta; 1-\omega)$ denotes the number of connections that can be set up in one input (output) link of the smallest weight which will block a connection of weight ω in one interstage link. In turn, $\phi_1(\beta; 1-\omega)$ denotes a bandwidth in an input (output) link which remains after setting up $\zeta(\beta; 1-\omega)$ connections of weight $1-\omega+\epsilon$, where $\epsilon \to 0$, in this link. $\phi_2(\beta; 1-\omega)$ denotes the number of connections of weight $\phi_1(\beta; 1-\omega)$, so that $\phi_2(\beta; 1-\omega) + 1$ such connections may block one interstage link for a new connection of weight ω. $\phi_3(i)$ indicates whether weight i can be used by the next connection or not.

We will also use the following equations to make our final formulas more transparent:

$$\alpha(i;\beta;B) = \left[i - (\phi_2(\beta;1-B)+1) \left\lfloor \frac{i}{\phi_2(\beta;1-B)+1} \right\rfloor \right] \cdot \phi_1(\beta;1-B); \tag{30}$$

$$\gamma(i;\beta;B) = i \cdot \phi_1(\beta;1-B); \tag{31}$$

$$\delta = \begin{cases} \left\lceil \dfrac{1-B}{b} \right\rceil, & \dfrac{1-B}{b} \text{ is not an integer;} \\ 1 + \left\lceil \dfrac{1-B}{b} \right\rceil, & \dfrac{1-B}{b} \text{ is an integer.} \end{cases} \tag{32}$$

Theorem 2. *The* $\log_d(N,0,p)$ *switching network is strict-sense nonblocking for the continuous bandwidth case, and* $0 < b < B \leq \beta \leq 1$ *if*

$$p \geq 2\psi(A_{\mathbb{I}}) + x\psi(A_{\mathbb{M}}) + 1, \tag{33}$$

where $\psi(i)$ *is given in the following formula:*

$$\psi(i) = \begin{cases} i \left\lfloor \dfrac{\beta}{b} \right\rfloor, & \textit{for } 1-b < B; \\ \left\lfloor \dfrac{i\lfloor \beta/b \rfloor + \lfloor A_{\mathbb{I}}/i \rfloor \cdot \lfloor (\beta-B)/b \rfloor}{2} \right\rfloor, & \textit{for } 1-2b < B \leq \dfrac{1}{2},\ \dfrac{1}{4} < b < \dfrac{1}{2} \\ i \cdot \zeta(\beta;1-B), & \textit{for other } B\textit{'s},\ \phi_1(\beta;1-B) < b; \\ i \cdot \zeta(\beta;1-B) + \left\lfloor \dfrac{i}{\phi_2(\beta;1-B)+1} \right\rfloor & \\ \quad + \zeta\left(\alpha(i;\beta;B) + \left\lfloor \dfrac{A_{\mathbb{I}}}{i} \right\rfloor \phi_3(\beta-B); 1-B \right) & \\ \quad + \left\lfloor \dfrac{i}{A_{\mathbb{M}}} \right\rfloor [\zeta(\phi_1(\alpha(A_{\mathbb{I}};\beta;B) + \phi_3(\beta-B);1-B) & \\ \qquad + \phi_1(\alpha(A_{\mathbb{M}};\beta;B);1-B);1-B)], & \textit{for other } B\textit{'s},\ b \leq \phi_1(\beta;1-B) < 2b \\ & \textit{or } \phi_1(\beta;1-B) = 2b,\ \zeta(\beta;1-B) > 0; \\ \left\lfloor \dfrac{i\lfloor \beta/b \rfloor + \lfloor A_{\mathbb{I}}/i \rfloor \cdot \lfloor (\beta-B)/b \rfloor}{\delta} \right\rfloor, & \textit{for other } B\textit{'s},\ \phi_1(\beta;1-B) = 2b, \\ & \zeta(\beta;1-B) = 0; \\ i \cdot \zeta(\beta;1-B) & \\ \quad + \zeta\left(\gamma(i;\beta;B) + \left\lfloor \dfrac{A_{\mathbb{I}}}{i} \right\rfloor \cdot \phi_3(\beta-B); 1-B \right) & \\ \quad + \left\lfloor \dfrac{i}{A_{\mathbb{M}}} \right\rfloor \cdot [\zeta(\phi_1(\gamma(A_{\mathbb{I}};\beta;B) + \phi_3(\beta-B);1-B) & \\ \qquad + \phi_1(\gamma(A_{\mathbb{M}};\beta;B);1-B);1-B)], & \textit{for other } B\textit{'s},\ \phi_1(\beta;1-B) > 2b. \end{cases} \tag{34}$$

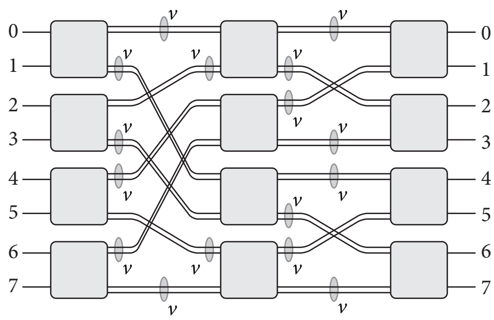

FIGURE 3: The ν-dilated $\log_2(8, 0, 1)$ switching network.

Proof. Sufficient conditions will be proved by showing the worst state in the switching network. Suppose we want to add the new connection denoted by $\langle i, o, \omega \rangle$, where $0 < b \leq \omega \leq B \leq \beta \leq 1$. The connection path from input terminal i will be inaccessible for the new connection of weight ω if there is a node in the connecting path $\langle i, o \rangle$ which already carries connections of the total weight greater than $1 - \omega$. In the worst-case scenario, this sum of weights should be as small as possible, say $1 - \omega + \epsilon$, where ϵ is close to but greater than 0. However, in the worst-case scenario, when $1 - \omega < b$, the path is inaccessible if a connection of weight b is set up through at least one of its nodes. When $1 - \omega \geq B$, it is not possible that one connection will occupy weight $1 - \omega + \epsilon$. In this case, at least two connections must be set up. When $1 - \omega < 2b$, the total weight of the blocking connections is equal to $2b$. In other cases, it is always possible to set up one or more connections of the total weight $1 - \omega + \epsilon$. Therefore, three cases can be distinguished:

(i) Case 1: for $1 - \omega < b$

(ii) Case 2: for $B \leq 1 - \omega < 2b$

(iii) Case 3: for other values of $1 - \omega$

Case 1 ($1 - \omega < b$). A plane is inaccessible for a new connection $\langle i, o, \omega \rangle$ if there is a node on the connecting path $\langle i, o \rangle$ which already carries a connection of weight b. In the worst-case scenario, each of such connections can be set up through a different plane. In one input link, $\lfloor \beta / b \rfloor$ connections of weight b can be set up. If $\lfloor \beta / b \rfloor$ is not an integer, there is some free bandwidth at this input terminal, but its weight is lower than b and it cannot be used to set up the next connection in this link. The connecting path of connection $\langle i, o, \omega \rangle$ may be blocked by another connecting path from input or output terminals. Connection $\langle i, o, \omega \rangle$ may be blocked in the node of stage k, where $1 \leq k \leq \lfloor (n - 1 - x)/2 \rfloor$, so there are $A_{\mathbb{I}}$ such paths from the input terminals. Connection $\langle i, o, \omega \rangle$ may also be blocked in the node of stage l, where $\lfloor (n + 1 + x)/2 \rfloor \leq l \leq n - 1$, so there are $A_{\mathbb{I}}$ such paths to the output terminals, too. Thus, there are $2 \cdot A_{\mathbb{I}}$ available inputs and outputs, and connections of weight b from these inputs or to these outputs may block

$$p_1 = 2 \cdot A_{\mathbb{I}} \cdot \left\lfloor \frac{\beta}{b} \right\rfloor \tag{35}$$

planes. When n is even, there is also the central stage $n/2$ and $x = 1$ (when n is odd, $x = 0$). So, there may be additional

$$p_2 = x \cdot A_{\mathbb{M}} \cdot \left\lfloor \frac{\beta}{b} \right\rfloor \tag{36}$$

planes blocked. In the worst-case scenario, one more plane is needed to set up connection $\langle i, o, \omega \rangle$. Therefore, finally,

$$p \geq p_1 + p_2 + 1 = 2 \cdot A_{\mathbb{I}} \left\lfloor \frac{\beta}{b} \right\rfloor + x \cdot A_{\mathbb{M}} \left\lfloor \frac{\beta}{b} \right\rfloor + 1 \tag{37}$$

planes are needed.

Case 2 ($B \leq 1 - \omega < 2b$). A plane is inaccessible for a new connection $\langle i, o, \omega \rangle$, if there is a node on the connecting path $\langle i, o \rangle$ which already carries two connections of weight b. Similar to the previous case, there may be $\lfloor \beta / b \rfloor$ connections of weight b at each input terminal other than i and $\lfloor (\beta - \omega)/b \rfloor$ connections of weight b at input terminal i. Thus, there are $A_{\mathbb{I}} \cdot \lfloor \beta / b \rfloor + \lfloor (\beta - \omega)/b \rfloor$ connections, and 2 connections make a node inaccessible for the new connection $\langle i, o, \omega \rangle$. The same situation occurs at every output terminal. Therefore,

$$p_1 = 2 \cdot \left\lfloor \frac{A_{\mathbb{I}} \cdot \lfloor \beta / b \rfloor + \lfloor (\beta - \omega) / b \rfloor}{2} \right\rfloor \tag{38}$$

planes are needed. When n is even, then there is also the central stage. Connections passing through the node which belongs to this stage block

$$p_2 = \left\lfloor \frac{A_{\mathbb{M}} \cdot \lfloor \beta / b \rfloor}{2} \right\rfloor \tag{39}$$

planes. It should be noted that since $A_{\mathbb{M}} = (d - 1) \cdot d^{(n-1-x)/2}$, $A_{\mathbb{M}}$ is always even and there will be no bandwidth of weight b available for the next connection from inputs counted in $A_{\mathbb{M}}$. There may only be a free bandwidth of weight b at inputs counted in $A_{\mathbb{I}}$ or in input i; however, one connection of such weight will not block the new connection. So, in the worst-case scenario, one more plane is needed to set up connection $\langle i, o, \omega \rangle$. Therefore, finally,

$$\begin{aligned} p &\geq p_1 + p_2 + 1 \\ &= 2 \cdot \left\lfloor \frac{A_{\mathbb{I}} \cdot \lfloor \beta / b \rfloor + \lfloor (\beta - \omega) / b \rfloor}{2} \right\rfloor + x \\ &\quad \cdot \left\lfloor \frac{A_{\mathbb{M}} \cdot \lfloor \beta / b \rfloor}{2} \right\rfloor + 1 \end{aligned} \tag{40}$$

planes are needed.

Case 3 (other values of $1 - \omega$). A plane is inaccessible for a new connection $\langle i, o, \omega \rangle$, if there is a node on its connecting path that already carries connections of the total weight greater than $1 - \omega$. When $1 - \omega < B$, only one connection of such weight may be set up. In the other case, at least two connections must be set up and their total weight should be greater than $1-\omega$. At each input or output terminal, $\zeta(\beta;1-\omega)$ such connections may be set up and

$$p_1 = A_{\mathbb{I}} \cdot \zeta(\beta; 1-\omega) \quad (41)$$

planes may be blocked for the new connection. However, there is still a free bandwidth of weight $\beta-\omega$ at input terminal i and at output terminal o, too. When $\beta - \omega \le 1 - \omega$, a connection of weight greater than $1-\omega$ at these terminals cannot be set up. So, there is still a free bandwidth of weight $\phi_1(\beta; 1-\omega)$ at each input and output terminal, except i and o. It may be used by subsequent connections, provided that $\phi_1(\beta;1-\omega) \ge b$. If $b \le \phi_1(\beta;1-\omega) < 2b$ or $\phi_1(\beta;1-\omega) = 2b$ and $\zeta(\beta;1-\omega) > 0$, several connections of such weight, which pass through one node, may make the plane inaccessible by the new connection. The minimum number of these connections is denoted by $\phi_2(\beta;1-\omega)+1$. Therefore, the next

$$p_2 = \left\lfloor \frac{A_{\mathbb{I}}}{\phi_2(\beta; 1-\omega) + 1} \right\rfloor \quad (42)$$

planes will be inaccessible by the new connection. However, there can still be some free bandwidth of weight $\phi_3(\beta-\omega)$ at input terminal i and free bandwidth of weight

$$\begin{aligned} &\alpha(A_{\mathbb{I}}; \beta; \omega) \\ &= \left[A_{\mathbb{I}} - (\phi_2(\beta;1-\omega)+1) \cdot \left\lfloor \frac{A_{\mathbb{I}}}{\phi_2(\beta;1-\omega)+1} \right\rfloor \right] \\ &\quad \cdot \phi_1(\beta;1-\omega) \end{aligned} \quad (43)$$

at input terminals not counted in p_2. The same situation occurs at the output terminals. When this weight is greater than $1-\omega$, it may block

$$p_3 = \zeta(\alpha(A_{\mathbb{I}};\beta;\omega) + \phi_3(\beta-\omega); 1-\omega) \quad (44)$$

additional planes. We still have a free bandwidth of weight $\phi_1(\alpha(A_{\mathbb{I}};\beta;\omega)+\phi_3(\beta-\omega);1-\omega)$ but this bandwidth is lower than or equal to $1-\omega$ and cannot block a plane for the new connection. When $\phi_1(\beta;1-\omega) > 2b$, the total weight of the free bandwidth at the input terminals is equal to

$$\gamma(A_{\mathbb{I}};\beta;\omega) = A_{\mathbb{I}} \cdot \phi_1(\beta;1-\omega). \quad (45)$$

Connections using this bandwidth and the free bandwidth of weight $\phi_3(\beta-\omega)$ at input terminal i will occupy not more than

$$p_4 = \zeta(\gamma(A_{\mathbb{I}};\beta;\omega) + \phi_3(\beta-\omega); 1-\omega) \quad (46)$$

planes. The weight of all the remaining bandwidth is now $\phi_1(\gamma(A_{\mathbb{I}};\beta;\omega)+\phi_3(\beta-\omega);1-\omega)$. The same situation occurs at the output terminals, including o. When n is even, similar considerations may be used to receive the number of planes $\psi(A_{\mathbb{M}})$ blocked by the connections from inputs counted in $A_{\mathbb{M}}$ (it should be noted that, in $A_{\mathbb{M}}$, we have no input i and output o, so $\phi_3(\beta-\omega)$ cannot be considered). Thus, there are

$$\begin{aligned} p_1' &= A_{\mathbb{M}} \cdot \zeta(\beta;1-\omega), \\ &\qquad \text{for } \phi_1(\beta,1-\omega) < b; \\ p_2' &= \left\lfloor \frac{A_{\mathbb{M}}}{\phi_2(\beta;1-\omega)+1} \right\rfloor, \\ &\qquad \text{for } b \le \phi_1(\beta;1-\omega) < 2b; \\ p_3' &= \zeta(\alpha(A_{\mathbb{M}};\beta;\omega);1-\omega), \\ &\qquad \text{for } b \le \phi_1(\beta;1-\omega) < 2b \\ &\text{or } \phi_1(\beta;1-\omega) = 2\text{b}, \ \zeta(\beta;1-\omega) > 0; \\ p_4' &= \zeta(\gamma(A_{\mathbb{M}};\beta;\omega);1-\omega), \\ &\qquad \text{for } \phi_1(\beta;1-\omega) > 2b \end{aligned} \quad (47)$$

planes.

Nevertheless, at each input terminal counted in the central stage, there may be a free bandwidth of weight $\phi_1(\alpha(A_{\mathbb{M}};\beta;\omega);1-\omega)$ or $\phi_1(\gamma(A_{\mathbb{M}};\beta;\omega);1-\omega)$, depending on $\phi_1(\beta;1-\omega)$ being higher or lower than $2b$. Therefore, we may have additional

$$\begin{aligned} p_5 = \zeta(&\phi_1(\alpha(A_{\mathbb{I}};\beta;\omega)+\phi_3(\beta-\omega);1-\omega) \\ &+ \phi_1(\alpha(A_{\mathbb{M}};\beta;\omega);1-\omega);1-\omega) \end{aligned} \quad (48)$$

planes inaccessible when $b \le \phi_1(\beta;1-\omega) < 2b$ or $\phi_1(\beta;1-\omega) = 2b$ and $\zeta(\beta;1-\omega) > 0$, or there may be

$$\begin{aligned} p_6 = \zeta(&\phi_1(\gamma(A_{\mathbb{I}};\beta;\omega)+\phi_3(\beta-\omega);1-\omega) \\ &+ \phi_1(\gamma(A_{\mathbb{M}};\beta;\omega);1-\omega);1-\omega) \end{aligned} \quad (49)$$

planes inaccessible when $\phi_1(\beta;1-\omega) > 2b$. It is also possible that $\phi_1(\beta;1-\omega) = 2b$ and $\zeta(\beta;1-\omega) = 0$. In this case, $\phi_1(\beta;1-\omega)$ is always equal to β, so in the worst state there can be only $\lfloor \beta/b \rfloor = 2$ connections at each input (output) terminal, and each of them has weight b, or one connection of weight $b \le \omega \le B$. Therefore, in the worst-case scenario, only connections of weight b may be considered. There are $A_{\mathbb{I}}$ such input (output) terminals. At input i and output o, we have a free bandwidth of weight $\beta-\omega$ which may be used by $\lfloor(\beta-\omega)/b\rfloor$ connections of weight b. At one interstage link, there are Z connections of weight b which make a plane inaccessible for connection $\langle i, o, \omega \rangle$. So,

$$p_7 = 2 \cdot \left\lfloor \frac{A_{\mathbb{I}} \lfloor \beta/b \rfloor + \lfloor (\beta-\omega)/b \rfloor}{\delta} \right\rfloor \quad (50)$$

planes are needed. Similarly, in the central stage, there are

$$p_7' = \left\lfloor \frac{A_{\mathbb{M}} \lfloor \beta/b \rfloor}{\delta} \right\rfloor \quad (51)$$

inaccessible planes. In a particular case, $\delta = 2$ and we have the same formula as these from Case 2. In the worst state, one more plane is needed to set up connection $\langle i, o, \omega \rangle$. So, finally, the number of planes needed for other values of $1-\omega$ is

$$p \geq \begin{cases} 2 \cdot p_1 + x \cdot p_1' + 1, & \text{for } \phi_1(\beta, 1-\omega) < b; \\ 2 \cdot (p_1 + p_2 + p_3) + x \cdot \left(p_1' + p_2' + p_3' + p_5\right) + 1, & \text{for } b \leq \phi_1(\beta; 1-\omega) < 2b \\ & \text{or } \phi_1(\beta; 1-\omega) = 2b,\ \zeta(\beta; 1-\omega) > 0; \\ 2 \cdot (p_1 + p_4) + x \cdot \left(p_1' + p_4' + p_6\right) + 1, & \text{for } \phi_1(\beta; 1-\omega) > 2b; \\ 2 \cdot p_7 + x \cdot p_7' + 1 & \text{for } \phi_1(\beta; 1-\omega) = 2b,\ \zeta(\beta; 1-\omega) = 0. \end{cases} \tag{52}$$

Combining all cases together and using $\psi(i)$, we can write

$$p \geq \max_{b \leq \omega \leq B} \left\{2 \cdot \psi(A_{\parallel}) + x \cdot \psi(A_{\mathbb{M}}) + 1\right\}. \tag{53}$$

In Case 2, we have

$$\psi(A_{\mathbb{M}}) = \left\lfloor \frac{A_{\mathbb{M}} \cdot \lfloor \beta/b \rfloor + \lfloor A_{\parallel}/A_{\mathbb{M}} \rfloor \cdot \lfloor (\beta - B)/b \rfloor}{2} \right\rfloor, \tag{54}$$

but since $A_{\mathbb{M}} > A_{\parallel}$, we have $\lfloor A_{\parallel}/A_{\mathbb{M}} \rfloor = 0$. It should be noted that, for Cases 1 and 2, the given conditions are also necessary. In Case 3 and for $\phi_1(\beta; 1-\omega) > 2b$, our conditions are also sufficient. For $\phi_1(\beta; 1-\omega) \leq 2b$, the conditions given in Theorem 2 are also necessary.

The number of planes p must be maximized through all ω and it reaches a maximum at $\omega = B$, so putting B in the respective formulas, we obtain the conditions given in Theorem 2. □

5. Comparison and Results

5.1. Examples for Discrete Bandwidth Model. In the following few examples, the worst-case scenarios in switching fabrics with different parameters are shown for the discrete bandwidth model.

Example 1. Let us consider a switching network with $d = 2$, $n = 4$, $c_{I/O} = 3$, and $c_M = 6$, and let the maximum number of channels occupied by one connection be $C = 2$. In this example, the switching fabric requires $p = 3$ planes to be strict-sense nonblocking. The worst state in this switching network is shown in Figure 4. The plane will be inaccessible for connection $\langle i, o, 2 \rangle$ if there is a node on the connecting path $\langle i, o \rangle$ which already carries connections of the total number of occupied channels greater than or equal to $z(C) = 5$. In this example, there is also a central stage, so $p_1 = 1$ plane is inaccessible by $\langle i, o, 2 \rangle$ (in Figure 4, it is plane 1). In the set of $A_{\parallel}$ inputs, there are still $Y(A_{\parallel} \cdot c_{I/O}; A_{\mathbb{M}}) = 3$ free channels and 1 free channel at input i, but these $Z(C) = 4$ free channels cannot block a plane for $\langle i, o, 2 \rangle$; therefore, $p_2 = 0$. There is also a central stage and $Y(A_{\mathbb{M}} \cdot c_{I/O}; C) = 1$ free channel. So, $p_3 = 1$ plane is inaccessible by the new connection (in Figure 4, it is plane 2). Similarly, at the output terminals, there are also $Z(C) = 4$ free channels, but it is still possible to set up connection $\langle i, o, 2 \rangle$ and we need only one additional plane (in Figure 4, it is plane 3). Finally, $p \geq p_1 + p_2 + p_3 + 1 = 3$ planes are needed.

Example 2. This time, let us consider a switching network with $d = 2$, $n = 4$, $c_{I/O} = 4$, and $c_M = 9$, and let the maximum number of channels occupied by one connection be $C = 3$. The switching network, according to Theorem 1, requires $p \geq 2$ planes. The worst state in the switching network is shown in Figure 5. The plane will be inaccessible for connection $\langle i, o, 3 \rangle$ if there is a node on the connecting path $\langle i, o \rangle$ which already carries a connection of the total number of occupied channels greater than or equal to $z(C) = 7$. We also have a central stage, so $p_1 = 1$ plane is needed (in Figure 5, it is plane 1). In the set of $A_{\parallel}$, we have $Y(A_{\parallel} \cdot c_{I/O}; C) = 4$ free channels and 1 free channel at input i. But these $Z(C) = 5$ channels cannot block a plane for $\langle i, o, 3 \rangle$ and $p_2 = 0$. In the central stage, there is $Y(A_{\mathbb{M}} \cdot c_{I/O}; C) = 1$ free channel, but, together with $Z(C) = 5$ channels, the 6 mentioned channels (5 + 1) cannot block a plane for connection $\langle i, o, 3 \rangle$; therefore, $p_3 = 0$. Similarly, at the output terminals, there are $Z(C) = 5$ free channels and it is still possible to set up a new connection in the same plane with them. So, we only need one additional plane (in Figure 5, it is plane 2). Finally, $p \geq p_1 + p_2 + p_3 + 1 = 2$ planes are needed.

Example 3. Let us consider a switching network with parameters $d = 2$, $n = 6$, $c_{I/O} = 3$, and $c_M = 12$, and let the maximum number of channels occupied by one connection be $C = 3$. In this example, the switching fabric contains 3 planes. The worst state in the switching network is shown in Figure 6. The plane will be again inaccessible for connection $\langle i, o, 3 \rangle$ if there is a node on the connecting path $\langle i, o \rangle$ which already carries a connection of the total number of occupied channels greater than or equal to $z(C) = 10$. There is also a central stage and $p_1 = 1$ plane is inaccessible by $\langle i, o, 3 \rangle$ (in Figure 6, it is plane 1). We have the set of $A_{\parallel}$ inputs and there are still $Y(A_{\parallel} \cdot c_{I/O}; C) = 9$ free channels. In this example, there are no free channels at input i and output o, so $Z(C)$ is also equal to 9 and these free channels cannot block a plane for $\langle i, o, 3 \rangle$; therefore, $p_2 = 0$. In the central stage, there are $Y(A_{\parallel} \cdot c_{I/O}; C) = 2$ free channels. So, $p_3 = 1$ plane is inaccessible by the new connection (in Figure 6, it is plane 2). At the input terminal, there is now only one free channel and it will not block a plane for the next connection. At the output terminals, there are $Z(C) = 9$ free channels, but it is still possible to set up connection $\langle i, o, 3 \rangle$ and we only

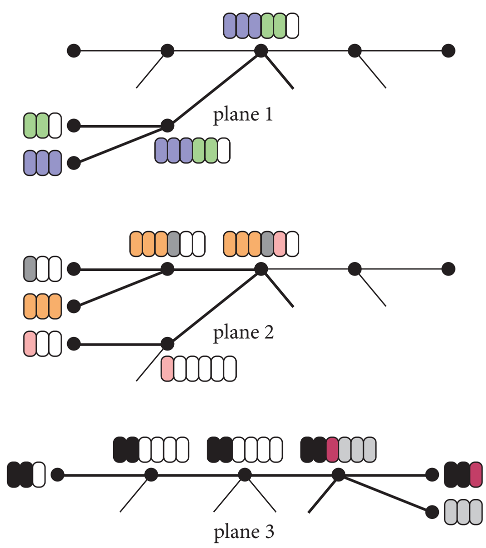

FIGURE 4: The worst state in the switching network with $d = 2, n = 4, c_{I/O} = 3, c_M = 6, C = 2$, and $p = 3$; filled cells denote occupied channels; empty cells denote free channels.

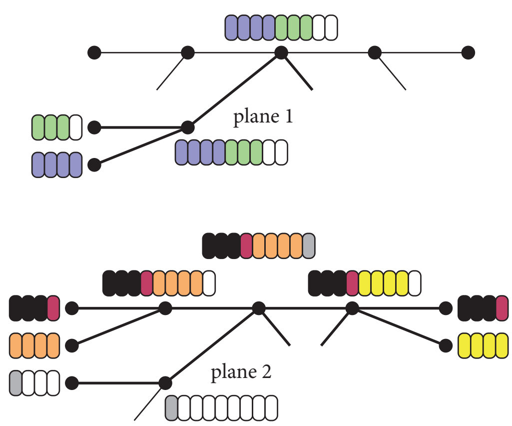

FIGURE 5: The worst state in the switching network with $d = 2, n = 4, c_{I/O} = 4, c_M = 9, C = 3$, and $p = 2$; filled cells denote occupied channels; empty cells denote free channels.

need one additional plane (in Figure 6, it is plane 3). Finally, $p \geq p_1 + p_2 + p_3 + 1 = 3$ planes are needed.

5.2. Examples for Continuous Bandwidth Model. In the following few examples, the worst-case scenarios in switching fabrics with different parameters are shown for the continuous bandwidth model.

Example 1. Let us consider a switching network with $d = 2$, $n = 6$, and $\beta = 0.5$, and let connection weights be between $b = 0.4$ and $B = 0.5$. In this example, the nonblocking conditions are determined by Case 2, because $1 - 2b < B \leq 1/2$ and $1/4 < b < 1/2$. This switching network contains 5 planes. The worst state in the switching network is shown in Figure 7. A plane will be inaccessible for connection $\langle i, o, 0.5\rangle$ if there is a node on the connecting path $\langle i, o\rangle$ which already carries

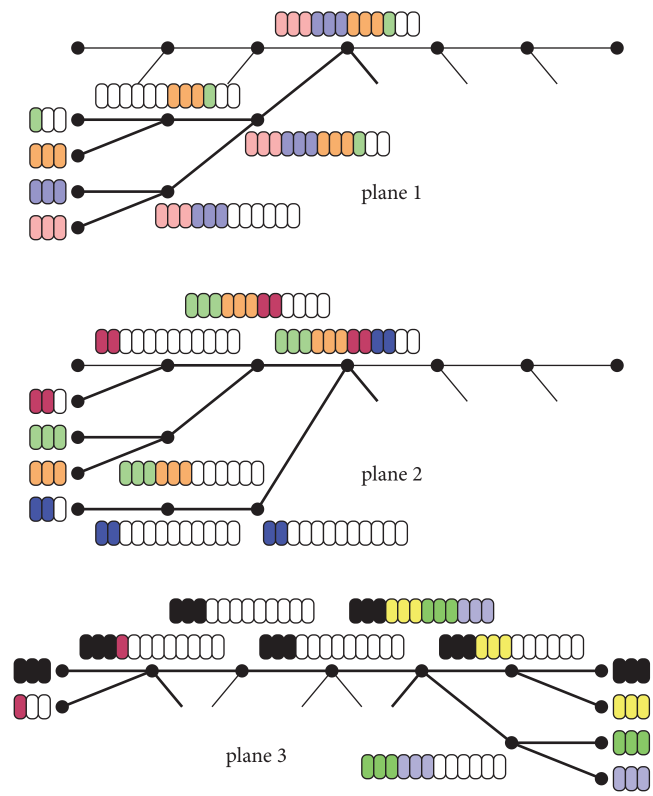

FIGURE 6: The worst state in the switching network with $d = 2, n = 6, c_{I/O} = 3, c_M = 12, C = 3$, and $p = 3$; filled cells denote occupied channels; empty cells denote free channels.

connections of the total weight higher than 0.5. Two such connections can be set up in two input terminals. In both of these input terminals, there is still a free bandwidth of weight lower than 0.4 and it cannot be used by the next connection in this link. In both input terminal i and output terminal o, there is no free bandwidth, because connection $\langle i, o, 0.5\rangle$ occupies the whole link capacity. Therefore, there are $p_1 = 2$ planes inaccessible by connection $\langle i, o, 0.5\rangle$ (in Figure 7, these are planes 1 and 2). In this case, there is also a central stage. Similarly, there are $p_2 = 2$ additional inaccessible planes (in Figure 7, these are planes 3 and 4). But in the worst state, one more plane is needed to set up connection $\langle i, o, 0.5\rangle$ (in Figure 7, this is plane 5). So, finally, we need $p \geq p_1 + p_2 + 1 = 5$ planes.

Example 2. Let us this time consider a switching network with $d = 2$, $n = 5$, and $\beta = 0.75$, and let connection weights be between $b = 0.2$ and $B = 0.4$. In this example, the nonblocking conditions are determined by Case 3, because B has a different value and $\phi_1(\beta; 1 - B) < b$. The switching network, according to Theorem 2, contains $p \geq 7$ planes. The worst state in the switching network is shown in Figure 8. The plane will be inaccessible for connection $\langle i, o, 0.4\rangle$ if there is a node on the connecting path $\langle i, o\rangle$ which already carries a set of connections of the weight higher than 0.6. One such set of connections can be set up in one input terminal. In this input terminal, there is free bandwidth of weight lower than

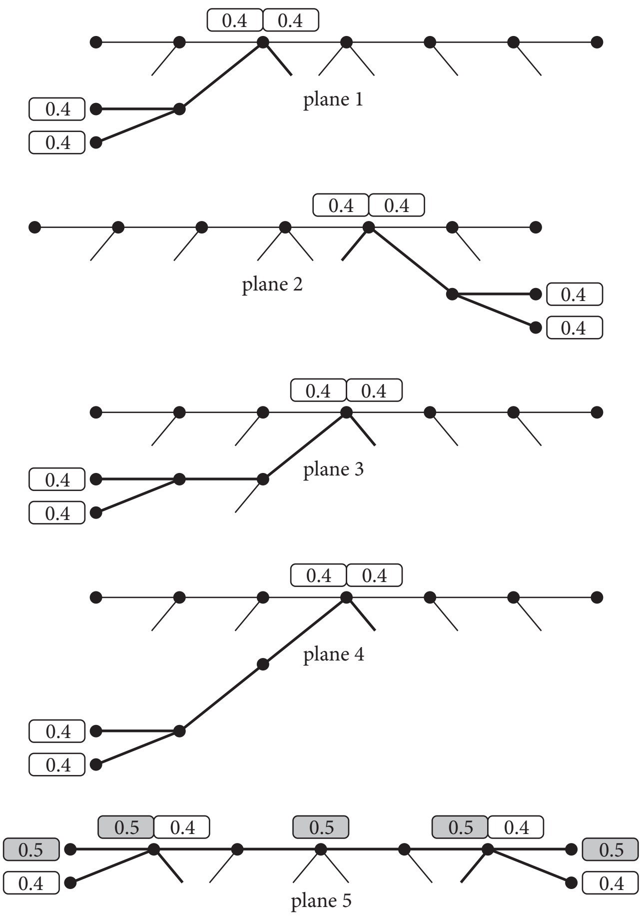

Figure 7: The worst state in the switching network with $d = 2, n = 6$, $\beta = 0.5$, $B = 0.5$, $b = 0.4$, and $p = 5$.

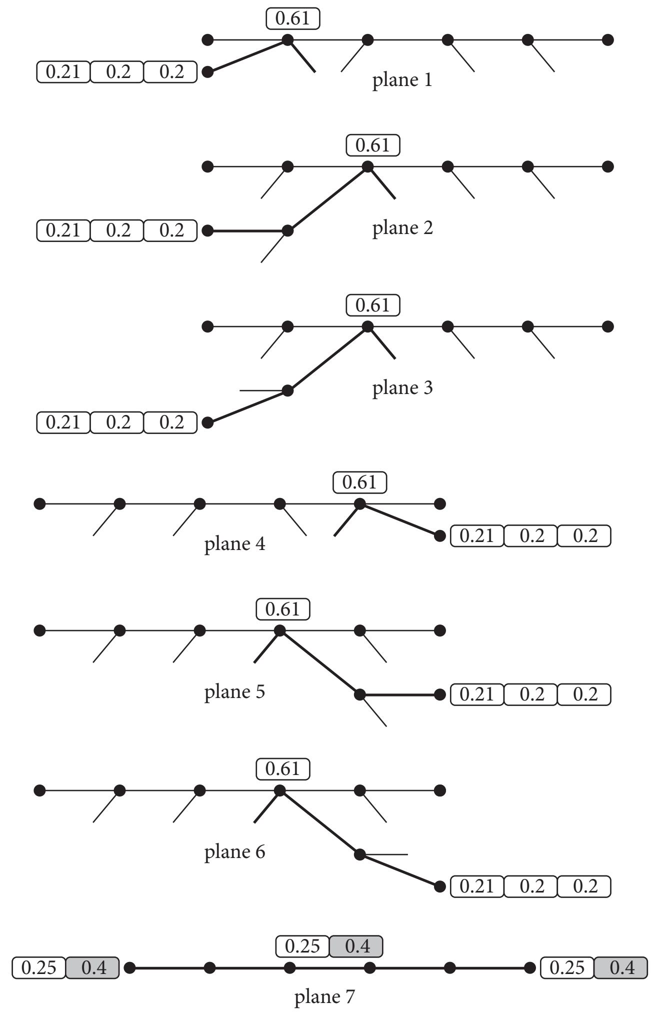

Figure 8: The worst state in the switching network with $d = 2, n = 5$, $\beta = 0.75$, $B = 0.4$, $b = 0.2$, and $p = 7$.

0.2 (because $\phi_1(0.75; 0.6) = 0$) and it cannot be used by a new connection in this link. Therefore, $p_1 = 3$ planes will be inaccessible by connection $\langle i, o, 0.4\rangle$ (in Figure 8, these are planes 1, 2, and 3). The same situation occurs at the output terminals (in Figure 8, these are planes 4, 5, and 6). In this example, there is no central stage and $\phi_1(\beta; 1 - B) < b$, so there are no other unconsidered terminals, except terminals i and o. In both of them, it is possible to set up a connection of weight 0.2, but the plane will be still accessible for connection $\langle i, o, 0.4\rangle$. Thus, one additional plane is needed (in Figure 8, this is plane 7). Finally, there are $p \ge 2p_1 + 1 = 7$ planes.

Example 3. In this example, we consider a switching network with $d = 2$, $n = 5$, and $\beta = 1.00$, and the connection weights are between $b = 0.2$ and $B = 0.75$. The nonblocking conditions are determined by Case 3, because B has different values and $b \le \phi_1(\beta; 1 - B) < 2b$. This switching network contains 23 planes. The worst state is shown in Figure 9. The plane will be inaccessible for connection $\langle i, o, 0.75\rangle$ if there is a node on the connecting path $\langle i, o\rangle$ which already carries a set of connections of the total weight greater than 0.25. Three such sets of connections can be set up in one input terminal, so $p_1 = 9$ (in Figure 9, these are planes from 1 to 9). In one of these input terminals, there is a free bandwidth of weight higher than 0.2 (because $\phi_1(1.00; 0.25) = 0.25$) and it can be used by the next connection in this link. However, this bandwidth from one link cannot block connection $\langle i, o, 0.75\rangle$, but $\phi_2(\beta; 1 - B) + 1 = 2$ such connections can make a plane inaccessible to a new connection. So, there is another $p_2 = 1$ inaccessible plane (in Figure 9, this is plane 10). But there is free bandwidth $\alpha(A_{\parallel}; \beta; B) = 0.25$ at the input terminals not considered in p_2. We also have some free bandwidth $\phi_3(\beta - B) = 0.25$ at input terminal i, so $p_3 = 1$ plane will be inaccessible for connection $\langle i, o, 0.75\rangle$ (in Figure 9, this is plane 11). The same situation occurs at the output terminals (in Figure 9, these are planes from 12 to 22). In this example, there is no central stage, so, in the worst state, one more plane is needed (in Figure 9, this is plane 23). So, finally, there are $p \ge 2(p_1 + p_2 + p_3) + 1 = 23$ planes.

5.3. Comparison and Results for Continuous Bandwidth Model. In Tables 1–3, we compared our results with the sufficient conditions proposed in [25], where $T2$ means Theorem 2. In these tables, when n grows, the number of the

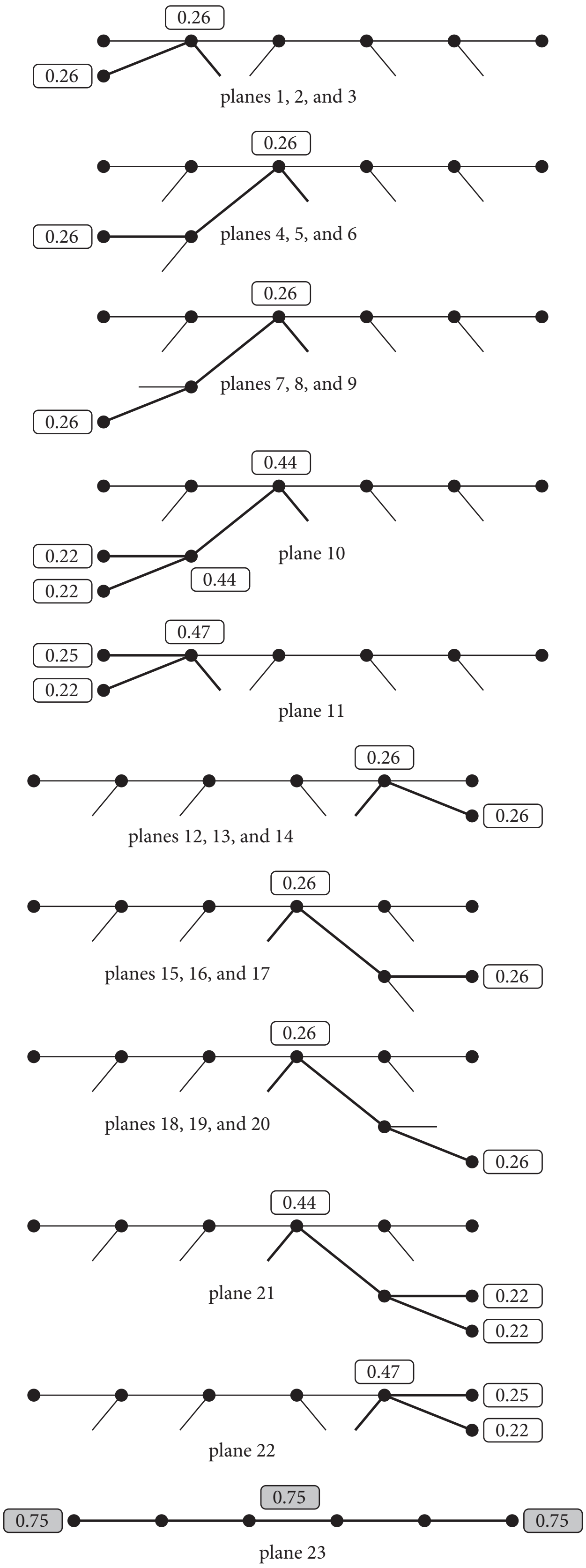

Figure 9: The worst state in the switching network with $d = 2, n = 5$, $\beta = 1.00$, $B = 0.3$, $b = 0.2$, and $p = 23$.

Table 1: Number of planes p for $b = 0.2$, $B = 0.4$, and $\beta = 0.75$ (Case 3 for $\phi_1(\beta; 1 - B) < b$).

n	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
	[25]	$T2$	[25]	$T2$	[25]	$T2$	[25]	$T2$
3	3	3	7	5	9	7	11	9
4	6	5	14	11	24	19	36	29
5	9	7	21	17	39	31	61	49
6	14	11	44	35	99	79	186	149
7	19	15	67	53	159	127	311	249
8	29	23	134	107	399	319	936	749

Table 2: Number of planes p for $b = 0.2$, $B = 0.75$, and $\beta = 1.00$ (Case 3 for $b \leq \phi_1(\beta; 1 - B) < 2b$).

n	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
	[25]	$T2$	[25]	$T2$	[25]	$T2$	[25]	$T2$
3	9	9	17	15	25	23	33	29
4	17	16	41	36	73	65	113	99
5	25	23	65	57	121	107	193	169
6	41	37	137	120	313	275	593	519
7	57	51	209	183	505	443	993	869
8	89	79	425	372	1273	1115	2993	2619

Table 3: Number of planes p for $b = 0.2$, $B = 0.5$, and $\beta = 1.00$ (Case 3 for $\phi_1(\beta; 1 - B) > 2b$).

n	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
	[25]	$T2$	[25]	$T2$	[25]	$T2$	[25]	$T2$
3	5	5	9	9	13	13	17	17
4	9	9	21	21	37	37	57	57
5	13	13	33	33	61	61	97	97
6	21	21	69	69	157	157	297	297
7	29	29	105	105	253	253	497	497
8	45	45	213	213	637	637	1497	1497

required planes also grows, but, in the cases for $d = 2$, our results are better than or equal to those in [25]. For any case, except for Case 3 when $\phi_1(\beta; 1 - B) > 2b$ and $d \geq 3$, our results show that fewer planes are required than in [25].

In Table 4, the number of planes for different d is compared when $n = 5$, $b = 0.2$, and $\beta = 1.00$. In this table, we have Cases 1 and 3. As it can be seen from this table, for Case 3 for $\phi_1(\beta; 1 - B) < b$, we always get a better result than the ones given in [25]. For Case 3, for $b \leq \phi_1(\beta; 1 - B) < 2b$ and for $\phi_1(\beta; 1 - B) = 2b$ and $\zeta(\beta; 1 - B) > 0$ when $d > 2$, we always get better results too. In Case 3, for $\phi_1(\beta; 1 - B) > 2b$, our results are in the upper bound and they are the same as those proposed in [25].

In Tables 5 and 6, switching networks for Case 2 are compared. As it can be seen from Table 5, for Case 2, our results are better than the ones given in [25]. In Table 6, for Cases 2 and 3, for $\phi_1(\beta; 1 - B) = 2b$ and $\zeta(\beta; 1 - B) > 0$, we always get better results than in [25]. In other cases in these tables (it is always Case 3 for $\phi_1(\beta; 1 - B) > 2b$), our results

TABLE 4: Number of planes p for $n = 5$, $b = 0.2$, and $\beta = 1.00$ (c1: Case 1; c3a: Case 3 for $\phi_1(\beta; 1-B) < b$; c3b: Case 3 for $b \le \phi_1(\beta; 1-B) < 2b$ or for $\phi_1(\beta; 1-B) = 2b$ and $\zeta(\beta; 1-B) > 0$; c3c: Case 3 for $\phi_1(\beta; 1-B) > 2b$).

B	$d = 2$			$d = 3$			$d = 4$			$d = 5$		
	[25]	$T2$		[25]	$T2$		[25]	$T2$		[25]	$T2$	
0.25	9	9	c3b	23	21	c3b	41	39	c3b	65	61	c3b
0.30	11	9	c3b	25	23	c3b	45	41	c3b	71	65	c3b
0.35	11	11	c3b	27	25	c3b	49	47	c3b	75	73	c3b
0.40	11	11	c3b	29	25	c3b	51	47	c3b	81	73	c3b
0.45	13	13	c3c	31	31	c3c	57	57	c3c	89	89	c3c
0.50	13	13	c3c	33	33	c3c	61	61	c3c	97	97	c3c
0.55	15	13	c3a	37	33	c3a	69	61	c3a	109	97	c3a
0.60	17	13	c3a	41	33	c3a	77	61	c3a	121	97	c3a
0.65	19	17	c3b	47	41	c3b	87	77	c3b	139	121	c3b
0.70	21	19	c3a	55	49	c3a	101	91	c3a	161	145	c3a
0.75	25	23	c3b	65	57	c3b	121	107	c3b	193	169	c3b
0.80	33	31	c3a	83	81	c3a	153	151	c3a	243	241	c3a
0.85	31	31	c1	81	81	c1	151	151	c1	241	241	c1
0.90	31	31	c1	81	81	c1	151	151	c1	241	241	c1
0.95	31	31	c1	81	81	c1	151	151	c1	241	241	c1
1.00	31	31	c1	81	81	c1	151	151	c1	241	241	c1

TABLE 5: Number of planes p for $n = 6$, $B = 0.50$, and $\beta = 0.50$ (c2: Case 2 for $1-2b \le B < 1/2$ and $1/4 < b < 1/2$ or for $\phi_1(\beta; 1-B) = 2b$ and $\zeta(\beta; 1-B) = 0$; c3c: Case 3 for $\phi_1(\beta; 1-B) > 2b$).

b	$d = 2$			$d = 3$			$d = 4$			$d = 5$		
	[25]	$T2$		[25]	$T2$		[25]	$T2$		[25]	$T2$	
0.05	9	9	c3c	33	33	c3c	77	77	c3c	147	147	c3c
0.10	9	9	c3c	33	33	c3c	77	77	c3c	147	147	c3c
0.15	9	9	c3c	33	33	c3c	77	77	c3c	147	147	c3c
0.20	9	9	c3c	33	33	c3c	77	77	c3c	147	147	c3c
0.25	9	7	c2	33	23	c2	77	53	c2	147	99	c2
0.30	9	5	c2	33	18	c2	77	39	c2	147	75	c2
0.35	9	5	c2	33	18	c2	77	39	c2	147	75	c2
0.40	9	5	c2	33	18	c2	77	39	c2	147	75	c2
0.45	9	5	c2	33	18	c2	77	39	c2	147	75	c2

are in the upper bound; however, they are not worse than the ones proposed in [25].

6. Conclusions

In this paper, we investigated the nonblocking behavior of multirate $\log_d(N, 0, p)$ switching networks. In the discrete bandwidth model, we extended the conditions proposed in [25] and we showed numerical examples for this model too. In the continuous bandwidth model, we have proved that, for $B + b \le 1$, fewer planes are needed for the multirate $\log_d(N, 0, p)$ switching network to be nonblocking in the strict sense than it was previously known. The results that we obtained have been compared in Tables 1–6. We also showed a few numerical examples for the continuous bandwidth case. For $1 - b < B \le \beta$, our results confirm those known earlier.

TABLE 6: Number of planes p for $n = 6$, $B = 0.50$, and $\beta = 1.00$ (c2: Case 2; c3b: Case 3 for $\phi_1(\beta; 1-B) = 2b$ and $\zeta(\beta; 1-B) > 0$; c3c: Case 3 for $\phi_1(\beta; 1-B) > 2b$).

b	$d = 2$			$d = 3$			$d = 4$			$d = 5$		
	[25]	$T2$		[25]	$T2$		[25]	$T2$		[25]	$T2$	
0.05	21	21	c3c	69	69	c3c	157	157	c3c	297	297	c3c
0.10	21	21	c3c	69	69	c3c	157	157	c3c	297	297	c3c
0.15	21	21	c3c	69	69	c3c	157	157	c3c	297	297	c3c
0.20	21	21	c3c	69	69	c3c	157	157	c3c	297	297	c3c
0.25	21	17	c3b	69	52	c3b	157	119	c3b	297	223	c3b
0.30	21	17	c2	69	52	c2	157	119	c2	297	223	c2
0.35	21	17	c2	69	52	c2	157	119	c2	297	223	c2
0.40	21	17	c2	69	52	c2	157	119	c2	297	223	c2
0.45	21	17	c2	69	52	c2	157	119	c2	297	223	c2

Although, in the proof of Theorem 2, only the sufficient conditions were considered, it should be noted that, for Cases 1, 2, and 3 with $\phi_1(\beta, 1-\omega) \le 2b$, they are also necessary conditions. This means that we can show the blocking state in the switching network composed of fewer planes than it was given in Theorem 2. For Case 3 with $\phi_1(\beta, 1-\omega) > 2b$, our result constitutes the upper bound.

Presented multirate switching networks can be used, for example, in data centers or in 4G/5G networks where bandwidth is a crucial aspect. The discrete model allows describing the behavior of a multiservice system where each service could require different basic bandwidth units (channels). In case where bandwidth could have any fraction of an available bandwidth, this means that some services could use any size of a bandwidth and the continuous model is then used.

The nonblocking operation of $\log_d(N, m, p)$ switching networks is currently under study.

Notations

- $\langle i, o, c\rangle$: A new connection from input i to output o which occupies c channels
- A: Total number of paths which may intersect with a considered path in a bipartite graph
- $A_{\mathbb{I}}$: Number of paths accessible from the input side of the switching network which may intersect with a considered path in a bipartite graph
- $A_{\mathbb{M}}$: Number of paths accessible from the output side of the switching network which may intersect with a considered path in a bipartite graph
- β: Capacity of input/output port
- b: Minimum capacity of a single connection
- B: Maximum capacity of a single connection
- d: Number of inputs/outputs in a single switching element
- c: Number of channels demanded by a new connection

C: Maximum number of channels demanded by a new connection
$c_{I/O}$: Number of channels available in the input/output link
c_M: Number of channels available in the inter-stage link
n: Number of stages in the $\log_d(N, 0, p)$ switching network
N: Capacity of the $\log_d(N, 0, p)$ switching network
p: Number of vertically stacked planes in a switching network
ω: Weight of a single connection
x: Stages' parity indicator.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this article.

Acknowledgments

The work described in this paper was financed from the funds of the Ministry of Science and Higher Education for the year 2016 (Grant no. 08/82/DSMK/8218).

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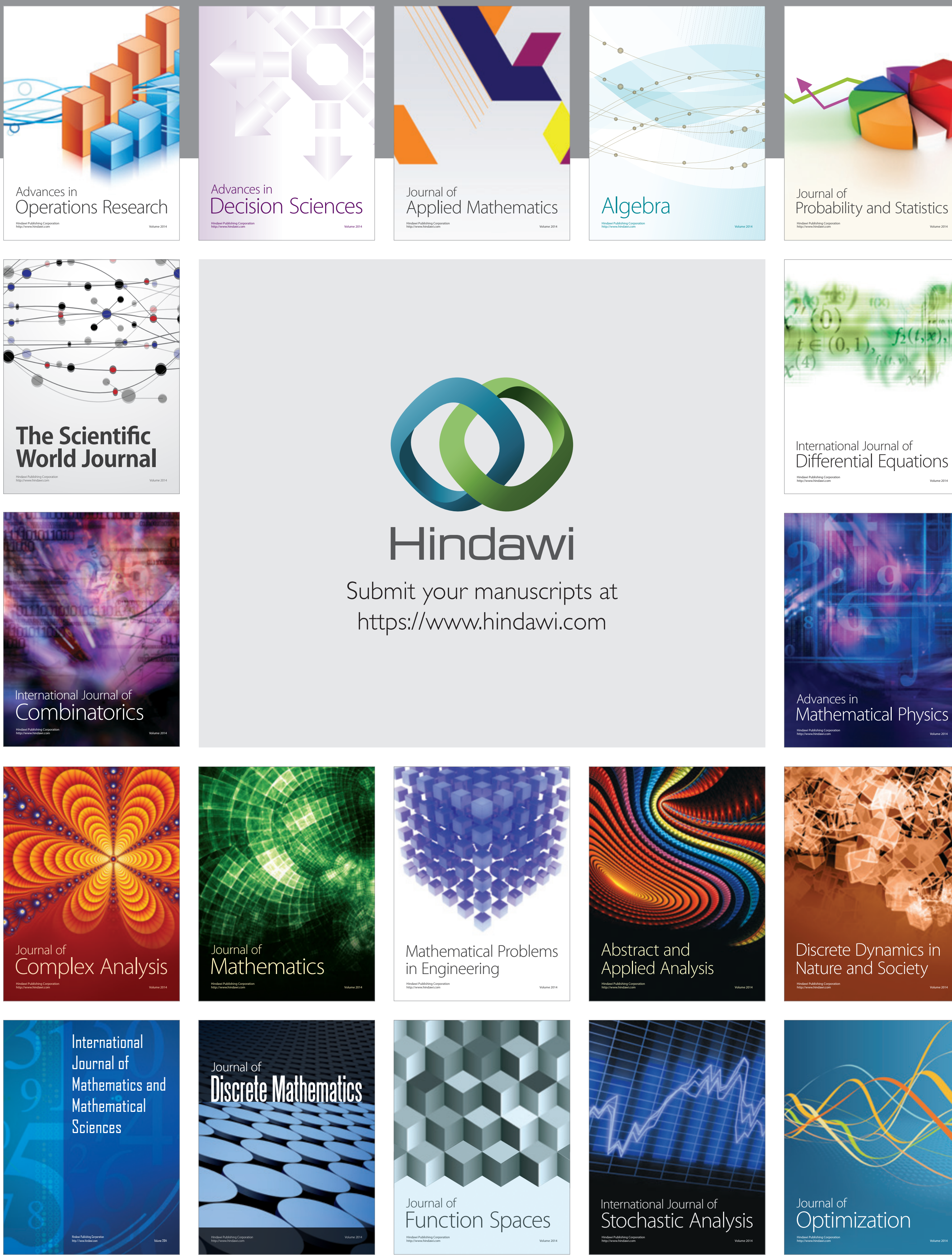

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