Research Article

Finite-Time Stability of Uncertain Nonlinear Systems with Time-Varying Delay

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Received 7 July 2017; Accepted 13 September 2017; Published 15 October 2017

Academic Editor: J.-C. Cortés

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The problem of finite-time stability for a class of uncertain nonlinear systems with time-varying delay and external disturbances is investigated. By using the Lyapunov stability theory, sufficient conditions for the existence of finite-time state feedback controller for this class of systems are derived. The results can be applied to finite-time stability problems of linear time-delay systems with parameter uncertainties and external disturbances. Finally, two numerical examples are given to demonstrate the effectiveness of the obtained theoretical results.

1. Introduction

Time delays are frequently encountered in various engineering and physical systems such as chemical systems, networked control systems, and power systems. It is well known that time delays will deteriorate the system's performance and even make the system unstable. Therefore, it is important to study the stability issue of time-delay systems. In recent years, the stability problem of linear systems has been well addressed [1, 2]. Some results of nonlinear systems with time-varying delay can be found in [3–9] and the references therein. On the other hand, parameter uncertainties cannot be avoided in a control system due to modeling errors, external disturbances, and linearization approximations, which are the main causes of instability and poor performance of control systems. This is also the reason why some authors study stochastic differential equations [10, 11]. Many authors studied the models with parameter uncertainties [12, 13]. The stability of uncertain systems has also gained much research attention [14, 15].

It should be noted that most of the stability results in existing literature are the classical Lyapunov asymptotic stability, which is defined over an infinite time interval [16–18]. However, in practice, it is more meaningful to

investigate the finite-time stability problem. Different from Lyapunov stability, finite-time stability means that the system is Lyapunov stable and its trajectories tend to zero in a finite time. Finite-time stability leads to better robustness and disturbance rejection properties [19, 20]. In [19], the authors presented a necessary and sufficient condition for finite-time stability of continuous-time autonomous system. After that, much attention has been given to finite-time stability and stabilization. For the delay-free case, many results on finite-time stability can be found in [21–28]. The authors in [24] investigated finite-time stabilization of a class of nonlinear systems with parametric uncertainties and proposed a controller by using Lyapunov, backstepping, and input-to-state stability techniques. Reference [25] studied finite-time stability for continuous-time systems with uniqueness of solutions in forward time and obtained a necessary and sufficient condition for finite-time stability for this class of systems. Recently, huge efforts have been paid to the finite-time stability or stabilization problems for time-delay systems [29–34]. In [30], the authors provided a theoretical result on finite-time stability by using Lyapunov functional and showed how to obtain finite-time stabilization of linear systems with time delays. The authors in [32] investigated the finite-time stability and designed a finite-time H_∞ controller for a class

of nonlinear time-delay Hamiltonian systems. Meanwhile, the finite-time stability or synchronization problem of neural networks was studied in [35–42]. It is worth noting that most of the results in literature focus on the finite-time stability for systems with or without time delays. Until now, there are rare results on the finite-time stability for nonlinear systems in the presence of time delays, parameter uncertainties, and external disturbances [43]. Note that the systems studied in [24, 25] are without time delays, and those in [30, 32] are with time delays but without parameter uncertainties and external disturbances.

In this paper, motivated by the above discussions, we shall address the finite-time stability problem of nonlinear systems with time-varying delay, parameter uncertainties, and external disturbances. By using a designed delayed feedback controller, some sufficient conditions for finite-time stability for such a class of systems are established. The main results of this paper are as follows. First, finite-time stability problem for uncertain nonlinear systems with time delays and disturbances is formulated. Second, a delayed feedback controller is designed to achieve the finite-time stability for a class of uncertain nonlinear systems with time delays and external disturbances based on the finite-time Lyapunov stability theory. Sufficient conditions are derived for the finite-time stability of the studied system. Finally, two numerical examples are provided to show the effectiveness of the proposed results.

The remainder of the paper is organized as follows. In Section 2, the model of uncertain nonlinear systems with time-varying delay and external disturbances is presented. Some necessary preliminaries are also given in this section. The finite-time stability criteria are obtained in Section 3. Then, two numerical simulation examples are provided to show the effectiveness of our results in Section 4. Finally, some conclusions are drawn in Section 5.

2. Preliminaries

Notations 1. Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ be the n -dimensional and $n \times m$ -dimensional real spaces equipped with the Euclidean norm $\|\cdot\|$, respectively. $\mathbb{Z}_+$ is the set of positive integer numbers. $\lambda_{\max}(P)$ represents the maximum eigenvalue of symmetric matrix P . $P > 0$ ($P < 0$) denotes that matrix P is a symmetric and positive definite (negative definite) matrix. Let I denote the identity matrix with appropriate dimensions. For any interval $J \subseteq \mathbb{R}$, set $S \subseteq \mathbb{R}^k$ ($1 \leq k \leq n$), $C(J, S) = \{\varphi : J \rightarrow S \text{ is continuous}\}$.

Consider the following uncertain nonlinear system with time-varying delay and external disturbances:

$$\begin{aligned} \dot{x}(t) = & (A + \Delta A)x(t) + (B + \Delta B)f(x(t - \tau(t))) \\ & + E\omega(t), \end{aligned} \quad (1)$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$ is the state vector. $E \in \mathbb{R}^{n \times m}$, $A, B \in \mathbb{R}^{n \times n}$ are constant matrices. ΔA and $\Delta B \in \mathbb{R}^{n \times n}$ are unknown matrices standing for parameter uncertainties satisfying $\Delta A^T \Delta A \leq \alpha I$, $\Delta B^T \Delta B \leq \beta I$,

and $\alpha, \beta > 0$. $\omega(t) \in \mathbb{R}^m$ is the external disturbances vector satisfying $\omega^T(t)\omega(t) \leq d$, and $d > 0$. $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \dots, f_n(x_n(t)))^T \in \mathbb{R}^n$; $\tau(t)$ is a time-varying delay, which satisfies $\dot{\tau}(t) \leq \mu < 1$. $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear function satisfying the Lipschitz condition; namely, there exists a positive constant L , for all $x, y \in \mathbb{R}^n$, such that

$$\|f(y) - f(x)\|^2 \leq L\|y - x\|^2. \quad (2)$$

The initial condition of system (1) is assumed to be

$$x(t) = \phi(t), \quad \forall t \in [-\tau, 0], \quad (3)$$

where $\phi(t) \in C([-\tau, 0], \mathbb{R}^n)$.

System (1) with a control input is given as

$$\begin{aligned} \dot{x}(t) = & (A + \Delta A)x(t) + (B + \Delta B)f(x(t - \tau(t))) \\ & + E\omega(t) + u(t), \end{aligned} \quad (4)$$

where $u(t)$ denotes the control input and is given by

$$u(t) = u_1(t) + u_2(t), \quad (5)$$

with

$$\begin{aligned} u_1(t) = & -k_1 x(t) - \lambda \text{sign}(x(t)) |x(t)|^\eta \\ & - \lambda \left(k_2 \int_{t-\tau(t)}^t x^T(s) x(s) ds \right)^{(1+\eta)/2} \frac{x(t)}{\|x(t)\|^2}, \end{aligned} \quad (6)$$

$$u_2(t) = -\frac{s_4}{2} E E^T x(t) - \frac{s_4^{-1} d}{2 \|x(t)\|^2} x(t),$$

and $|x(t)|^\eta = (|x_1(t)|^\eta, |x_2(t)|^\eta, \dots, |x_n(t)|^\eta)^T$, $\lambda > 0$, $0 < \eta < 1$, $s_4 > 0$, k_1 and k_2 are constants to be determined later, and $\text{sign}(x(t)) = \text{diag}(\text{sign}(x_1(t)), \text{sign}(x_2(t)), \dots, \text{sign}(x_n(t)))$.

For further analysis, we first give the following definition and useful lemmas.

Definition 2 (see [30]). The equilibrium $x = 0$ of system (4) is said to be finite-time stable if it is Lyapunov stable and finite-time convergent. Finite-time convergence means that, for any initial state $\varphi(s)$, there is a settling time $0 < T(\varphi) < +\infty$, such that every solution $x(t, \varphi)$ of system (4) satisfies

$$\begin{aligned} \lim_{t \rightarrow T} x(t, \varphi) &= 0, \\ x(t, \varphi) &= 0, \\ \forall t &> T(\varphi). \end{aligned} \quad (7)$$

Lemma 3 (see [19]). Assume that $V(t)$ is a continuous positive definite function satisfying the following inequality:

$$\dot{V}(t) \leq -cV^\alpha(t), \quad \forall t \geq t_0, \quad v(t_0) \geq 0, \quad (8)$$

where $c > 0$, $0 < \alpha < 1$ are all constants. Then, for any given $t_0 \geq 0$, $V(t)$ satisfies the following inequality:

$$V^{1-\alpha}(t) \leq V^{1-\alpha}(t_0) - c(1-\alpha)(t - t_0), \quad t_0 \leq t \leq T, \quad (9)$$

and $V(t) \equiv 0$, $\forall t \geq T$, with T given by

$$T = t_0 + \frac{V^{1-\alpha}(t_0)}{c(1-\alpha)}. \quad (10)$$

Lemma 4 (see [44]). For any real matrices A , B , and Σ with appropriate dimensions and a scalar $s > 0$, if the matrix Σ satisfies $0 < \Sigma = \Sigma^T$, then the following inequality holds:

$$A^T B + B^T A \leq s A^T \Sigma A + s^{-1} B^T \Sigma^{-1} B. \quad (11)$$

Lemma 5 (see [45]). For any $a_i \in \mathbb{R}$, $i \in \mathbb{Z}_+$, and any real numbers $0 < p \leq 1$, $0 < q < 2$, we have the following assertions:

$$(1) |a_1|^q + |a_2|^q + \dots + |a_n|^q \geq (|a_1|^2 + |a_2|^2 + \dots + |a_n|^2)^{q/2}.$$

$$(2) (|a_1| + |a_2| + \dots + |a_n|)^p \leq |a_1|^p + |a_2|^p + \dots + |a_n|^p.$$

3. Main Results

In this section, we shall present some sufficient conditions for the finite-time stability of system (1) with control input $u(t)$ defined in (5).

Theorem 6. Assume that there exist constants $s_1, s_2, s_3 > 0$ and $k_1, k_2 > 0$ such that

$$(i) A^T + A - 2k_1 I + k_2 I + s_2 B B^T + s_3^{-1} I + (s_1 \alpha + s_1^{-1}) I \leq 0;$$

$$(ii) (s_2^{-1} + s_3 \beta) L - k_2 (1 - \mu) \leq 0;$$

then system (1) is finite-time stable with control input $u(t)$ in (5), and the finite time is estimated by

$$T = t_0 + \frac{V^{(1-\eta)/2}(t_0)}{\lambda(1-\eta)}. \quad (12)$$

Proof. Choose the following Lyapunov functional:

$$V(t) = x^T(t) x(t) + \int_{t-\tau(t)}^t k_2 x^T(s) x(s) ds. \quad (13)$$

Taking the derivative of $V(t)$ along the trajectory of system (4), we have

$$\begin{aligned} \dot{V}(t) &= \dot{x}^T(t) x(t) + x^T(t) \dot{x}(t) + k_2 x^T(t) x(t) \\ &\quad - k_2 x^T(t - \tau(t)) x(t - \tau(t)) (1 - \dot{\tau}(t)) \leq \dot{x}^T(t) \\ &\quad \cdot x(t) + x^T(t) \dot{x}(t) + k_2 x^T(t) x(t) - k_2 x^T(t - \tau(t)) x(t - \tau(t)) (1 - \mu) \\ &= [(A + \Delta A) x(t) + (B + \Delta B) f(x(t - \tau(t))) + E\omega(t) + u(t)]^T x(t) \\ &\quad + x^T(t) [(A + \Delta A) x(t) \end{aligned}$$

$$\begin{aligned} &+ (B + \Delta B) f(x(t - \tau(t))) + E\omega(t) + u(t)] \\ &+ k_2 x^T(t) x(t) - k_2 x^T(t - \tau(t)) x(t - \tau(t)) (1 - \mu) \\ &= [(A + \Delta A) x(t) + (B + \Delta B) f(x(t - \tau(t))) + E\omega(t) + u_1(t) + u_2(t)]^T x(t) + x^T(t) \\ &\quad \cdot [(A + \Delta A) x(t) + (B + \Delta B) f(x(t - \tau(t))) + E\omega(t) + u_1(t) + u_2(t)] + k_2 x^T(t) x(t) \\ &\quad - k_2 x^T(t - \tau(t)) x(t - \tau(t)) (1 - \mu) \\ &= \left[(A + \Delta A) x(t) + (B + \Delta B) f(x(t - \tau(t))) - k_1 x(t) - \lambda \text{sign}(x(t)) |x(t)|^\eta \right. \\ &\quad \left. - \lambda \left(k_2 \int_{t-\tau(t)}^t x^T(s) x(s) ds \right)^{(1+\eta)/2} \frac{x(t)}{\|x(t)\|^2} \right]^T \\ &\quad \cdot x(t) + x^T(t) \left[(A + \Delta A) x(t) + (B + \Delta B) f(x(t - \tau(t))) - k_1 x(t) \right. \\ &\quad \left. - \lambda \text{sign}(x(t)) |x(t)|^\eta - \lambda \left(k_2 \int_{t-\tau(t)}^t x^T(s) x(s) ds \right)^{(1+\eta)/2} \frac{x(t)}{\|x(t)\|^2} \right] \\ &\quad + k_2 x^T(t) x(t) - k_2 x^T(t - \tau(t)) x(t - \tau(t)) (1 - \mu) \\ &\quad + \omega^T(t) E^T x(t) + u_2^T(t) x(t) + x^T(t) E\omega(t) + x^T(t) u_2(t) \\ &= \left[x^T(t) (A + \Delta A)^T + f^T(x(t - \tau(t))) (B + \Delta B)^T - k_1 x^T(t) \right. \\ &\quad \left. - \lambda (|x(t)|^\eta)^T \text{sign}^T(x(t)) - \lambda \left(k_2 \int_{t-\tau(t)}^t x^T(s) x(s) ds \right)^{(1+\eta)/2} \frac{x^T(t)}{\|x(t)\|^2} \right] x(t) \\ &\quad + x^T(t) \left[(A + \Delta A) x(t) + (B + \Delta B) f(x(t - \tau(t))) - k_1 x(t) \right. \\ &\quad \left. - \lambda \text{sign}(x(t)) |x(t)|^\eta - \lambda \left(k_2 \int_{t-\tau(t)}^t x^T(s) x(s) ds \right)^{(1+\eta)/2} \frac{x(t)}{\|x(t)\|^2} \right] \\ &\quad + k_2 x^T(t) x(t) - k_2 x^T(t - \tau(t)) x(t - \tau(t)) (1 - \mu) + 2x^T(t) E\omega(t) + 2x^T(t) u_2(t) = x^T(t) \end{aligned}$$

$$\begin{aligned}
& \cdot \left[(A + \Delta A)^T + (A + \Delta A) - 2k_1 I + k_2 I \right] x(t) \\
& + \left[f^T(x(t - \tau(t))) (B + \Delta B)^T x(t) \right. \\
& + x^T(t) (B + \Delta B) f(x(t - \tau(t))) \left. \right] - k_2 x^T(t - \tau(t)) x(t - \tau(t)) (1 - \mu) \\
& + \left[-\lambda (|x(t)|^\eta)^T \text{sign}^T(x(t)) x(t) \right. \\
& - \lambda x^T(t) \text{sign}(x(t)) |x(t)|^\eta \left. \right] \\
& - 2\lambda \left(k_2 \int_{t-\tau(t)}^t x^T(s) x(s) ds \right)^{(1+\eta)/2} + 2x^T(t) \\
& \cdot E\omega(t) + 2x^T(t) \left[-\frac{s_4}{2} EE^T x(t) - \frac{s_4^{-1} d}{2 \|x(t)\|^2} x(t) \right]. \quad (14)
\end{aligned}$$

Based on Lemma 4, one can obtain that the following inequality holds:

$$\begin{aligned}
& x^T(t) \Delta A^T x(t) + x^T(t) \Delta A x(t) \\
& \leq s_1 x^T(t) \Delta A^T \Delta A x(t) + s_1^{-1} x^T(t) x(t) \\
& \leq s_1 x^T(t) \alpha x(t) + s_1^{-1} x^T(t) x(t) \\
& = (s_1 \alpha + s_1^{-1}) x^T(t) x(t). \quad (15)
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
& f^T(x(t - \tau(t))) (B + \Delta B)^T x(t) + x^T(t) (B + \Delta B) \\
& \cdot f(x(t - \tau(t))) = f^T(x(t - \tau(t))) B^T x(t) \\
& + f^T(x(t - \tau(t))) \Delta B^T x(t) + x^T(t) \\
& \cdot B f(x(t - \tau(t))) + x^T(t) \Delta B f(x(t - \tau(t))) \\
& = \left[f^T(x(t - \tau(t))) B^T x(t) \right. \\
& + x^T(t) B f(x(t - \tau(t))) \left. \right] \\
& + \left[f^T(x(t - \tau(t))) \Delta B^T x(t) \right. \\
& + x^T(t) \Delta B f(x(t - \tau(t))) \left. \right] \leq \left[s_2 x^T(t) B B^T x(t) \right. \\
& + s_2^{-1} f^T(x(t - \tau(t))) f(x(t - \tau(t))) \left. \right] \\
& + \left[s_3 f^T(x(t - \tau(t))) \Delta B^T \Delta B f(x(t - \tau(t))) \right. \\
& + s_3^{-1} x^T(t) x(t) \left. \right] \leq \left[s_2 x^T(t) B B^T x(t) \right. \\
& + s_2^{-1} L x^T(t - \tau(t)) x(t - \tau(t)) \left. \right] \\
& + \left[s_3 \beta L x^T(t - \tau(t)) x(t - \tau(t)) + s_3^{-1} x^T(t) x(t) \right]
\end{aligned}$$

$$\begin{aligned}
& = x^T(t) (s_2 B B^T + s_3^{-1} I) x(t) + x^T(t - \tau(t)) (s_2^{-1} L I \\
& + s_3 \beta L I) x(t - \tau(t)), \\
& 2x^T(t) E\omega(t) \leq s_4 x^T(t) E E^T x(t) + s_4^{-1} \omega^T(t) \omega(t). \quad (16)
\end{aligned}$$

Also, we can derive the following inequality:

$$\begin{aligned}
& -\lambda (|x(t)|^\eta)^T \text{sign}^T(x(t)) x(t) - \lambda x^T(t) \text{sign}(x(t)) \\
& \cdot |x(t)|^\eta = -\lambda (|x(t)|^\eta)^T |x(t)| - \lambda |x(t)|^T |x(t)|^\eta \\
& = -2\lambda |x(t)|^T |x(t)|^\eta = -2\lambda (|x_1(t)|^{1+\eta} \\
& + |x_2(t)|^{1+\eta} + \dots + |x_n(t)|^{1+\eta}) \\
& = -2\lambda (|x_1(t)|^{2((1+\eta)/2)} + |x_2(t)|^{2((1+\eta)/2)} + \dots \\
& + |x_n(t)|^{2((1+\eta)/2)}) \leq -2\lambda (|x_1(t)|^2 + |x_2(t)|^2 + \dots \\
& + |x_n(t)|^2)^{(1+\eta)/2} = -2\lambda |x^T(t) x(t)|^{(1+\eta)/2}. \quad (17)
\end{aligned}$$

Substituting (15)–(17) into (14), it yields that

$$\begin{aligned}
\dot{V}(t) & \leq x^T(t) \left[A^T + A - 2k_1 I + k_2 I + (s_1 \alpha + s_1^{-1}) I \right] \\
& \cdot x(t) + x^T(t) (s_2 B B^T + s_3^{-1} I) x(t) + x^T(t - \tau(t)) \\
& \cdot (s_2^{-1} L I + s_3 \beta L I) x(t - \tau(t)) - k_2 (1 - \mu) x^T(t - \tau(t)) \\
& \cdot x(t - \tau(t)) - 2\lambda |x(t)^T x(t)|^{(1+\eta)/2} \\
& - 2\lambda \left(k_2 \int_{t-\tau(t)}^t x^T(s) x(s) ds \right)^{(1+\eta)/2} + s_4 x^T(t) \\
& \cdot E E^T x(t) + s_4^{-1} \omega^T(t) \omega(t) + 2x^T(t) \\
& \cdot \left[-\frac{s_4}{2} E E^T x(t) - \frac{s_4^{-1} d}{2 \|x(t)\|^2} x(t) \right] \leq x^T(t) \left[A^T \right. \\
& + A - 2k_1 I + k_2 I + (s_1 \alpha + s_1^{-1}) I + s_2 B B^T + s_3^{-1} I \left. \right] \\
& \cdot x(t) + x^T(t - \tau(t)) \left[(s_2^{-1} L + s_3 \beta L) I \right. \\
& - k_2 (1 - \mu) I \left. \right] x(t - \tau(t)) - 2\lambda |x(t)^T x(t)|^{(1+\eta)/2} \\
& - 2\lambda \left(k_2 \int_{t-\tau(t)}^t x^T(s) x(s) ds \right)^{(1+\eta)/2} + s_4 x^T(t) \\
& \cdot E E^T x(t) + s_4^{-1} d + 2x^T(t) \left[-\frac{s_4}{2} E E^T x(t) \right. \\
& - \frac{s_4^{-1} d}{2 \|x(t)\|^2} x(t) \left. \right] \leq -2\lambda |x^T(t) x(t)|^{(1+\eta)/2} \\
& - 2\lambda \left(k_2 \int_{t-\tau(t)}^t x^T(s) x(s) ds \right)^{(1+\eta)/2}
\end{aligned}$$

$$\leq -2\lambda \left[x^T(t) x(t) + k_2 \int_{t-\tau(t)}^t x^T(s) x(s) ds \right]^{(1+\eta)/2} = -2\lambda V^{(1+\eta)/2}(t). \quad (18)$$

It then follows from Lemma 3 that

$$V^{1-(1+\eta)/2}(t) \leq V^{1-(1+\eta)/2}(t_0) - 2\lambda \left(1 - \frac{1+\eta}{2} \right) (t - t_0), \quad (19)$$

$$t_0 \leq t \leq T,$$

and $V(t) \equiv 0$, $\forall t \geq T$, with T given by

$$T = t_0 + \frac{V^{1-(1+\eta)/2}(t_0)}{2\lambda(1 - (1+\eta)/2)} = t_0 + \frac{V^{(1-\eta)/2}(t_0)}{\lambda(1-\eta)}. \quad (20)$$

Therefore, we obtain that

$$\|x(t)\| \equiv 0, \quad t \geq T. \quad (21)$$

Then system (1) is finite-time stable with control input $u(t)$ in (5), and the finite time is estimated by

$$T = t_0 + \frac{V^{(1-\eta)/2}(t_0)}{\lambda(1-\eta)}. \quad (22)$$

The proof is completed. $\square$

Remark 7. In the paper, the finite-time stability problem of nonlinear systems with time-varying delays, parameter uncertainties, and external disturbances is addressed. Although the finite-time stability problem of some kinds of systems with time delays is studied in [30, 32], the parameter uncertainties and external disturbances are not taken into account in these papers. So our methods are more general compared with the existing results.

Particularly, system (1) is linear if we let $f(x) = x$. For such a linear system, we can obtain the following corollary by choosing $L = 1$ in condition (ii) of Theorem 6.

Corollary 8. Assume that there exist constants $s_1, s_2, s_3 > 0$ and $k_1, k_2 > 0$ such that

- (i) $A^T + A - 2k_1 I + k_2 I + s_2 BB^T + s_3^{-1} I + (s_1 \alpha + s_1^{-1}) I \leq 0$;
- (ii) $s_2^{-1} + s_3 \beta - k_2(1 - \mu) \leq 0$;

then the linear system is finite-time stable with control input $u(t)$ in (5), and the finite time is estimated by

$$T = t_0 + \frac{V^{(1-\eta)/2}(t_0)}{\lambda(1-\eta)}. \quad (23)$$

Based on Theorem 6, one can easily get the following corollary.

Corollary 9. Assume that there exist constants $s_1, s_2, s_3 > 0$ such that

$$\begin{aligned} k_1 &\geq \frac{\lambda_{\max}(A + A^T) + s_2 \lambda_{\max}(BB^T) + s_3^{-1} + s_1 \alpha + s_1^{-1} + k_2}{2}, \\ k_2 &\geq \frac{(s_2^{-1} + s_3 \beta) L}{1 - \mu}; \end{aligned} \quad (24)$$

then system (1) is finite-time stable with control input $u(t)$ in (5), and the finite time is estimated by

$$T = t_0 + \frac{V^{(1-\eta)/2}(t_0)}{\lambda(1-\eta)}. \quad (25)$$

4. Simulation Examples

In this section, two examples are provided to verify the effectiveness of the results obtained in the previous section.

Example 1. Consider a two-dimensional uncertain nonlinear system without external disturbances:

$$\dot{x}(t) = (A + \Delta A) x(t) + (B + \Delta B) f(x(t - \tau(t))), \quad (26)$$

where $x(t) = (x_1(t), x_2(t))^T$, $\tau(t) = 0.4 + 0.1 \cos t$,

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \Delta A &= \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \\ B &= \begin{pmatrix} -2.5 & 0.2 \\ 0.1 & -1.5 \end{pmatrix}, \\ \Delta B &= \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \end{aligned} \quad (27)$$

and $f(x(t)) = \tanh(x(t))$.

Choose $\alpha = 0.04$, $\beta = 1$, and $\mu = 0.5$. It is obvious that $\Delta A^T \Delta A \leq \alpha I$, $\Delta B^T \Delta B \leq \beta I$, and $\dot{\tau}(t) \leq \mu < 1$. We can select $L = 1$. Under the initial condition $x_1(s) = 0.5$ and $x_2(s) = -0.5$, $\forall s \in [-1, 0]$, the state trajectories of $x_1(t)$ and $x_2(t)$ of system (26) are shown in Figure 1.

Now, we consider system (26) with control input; the resulting system is given by

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A) x(t) + (B + \Delta B) f(x(t - \tau(t))) \\ &\quad + u(t). \end{aligned} \quad (28)$$

For numerical simulation, we can select $\lambda = 0.1$, $\eta = 0.8$, $s_1 = 1$, $s_2 = 1$, and $s_3 = 1$. Based on Corollary 9, we have

FIGURE 1: State trajectories of system (26) without control input under the initial conditions $x_1(s) = 0.5$ and $x_2(s) = -0.5$, $\forall s \in [-1, 0)$.

FIGURE 2: State trajectories of system (26) with control input (29) under the initial conditions $x_1(s) = 0.5$ and $x_2(s) = -0.5$, $\forall s \in [-1, 0)$.

$k_2 \geq 4$ and $k_1 \geq 7.6819$. We choose $k_2 = 4$ and $k_1 = 8$. So the controller $u(t) = u_1(t) = (u_{11}(t), u_{12}(t))^T$ is designed as

$$\begin{aligned} u_{11}(t) &= -8x_1(t) - 0.1 \operatorname{sign}(x_1(t)) |x_1(t)|^{0.8} - 0.1 \\ &\quad \cdot 4^{0.9} \left(\int_{t-(0.4+0.1 \cos t)}^t (x_1^2(s) + x_2^2(s)) ds \right)^{0.9} \\ &\quad \cdot \frac{x_1(t)}{x_1^2(t) + x_2^2(t)}, \\ u_{12}(t) &= -8x_2(t) - 0.1 \operatorname{sign}(x_2(t)) |x_2(t)|^{0.8} - 0.1 \\ &\quad \cdot 4^{0.9} \left(\int_{t-(0.4+0.1 \cos t)}^t (x_1^2(s) + x_2^2(s)) ds \right)^{0.9} \\ &\quad \cdot \frac{x_2(t)}{x_1^2(t) + x_2^2(t)}. \end{aligned} \quad (29)$$

Under the initial condition $x_1(s) = 0.5$ and $x_2(s) = -0.5$, $\forall s \in [-1, 0)$, the state trajectories of $x_1(t)$ and $x_2(t)$ of system (28) are shown in Figure 2. Hence, system (26) is finite-time stable with control input $u(t)$ defined in (29).

Example 2. Consider a three-dimensional uncertain nonlinear system with external disturbances:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (B + \Delta B)f(x(t - \tau(t))) \\ &\quad + E\omega(t), \end{aligned} \quad (30)$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T$, $\tau(t) = 0.2$, $E = (0, 0, 1)^T$, $\omega(t) = \sin t$,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{aligned} \Delta A &= \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}, \\ B &= \begin{pmatrix} 1 & 0.5 & -0.8 \\ -0.6 & 1.5 & 0.5 \\ 0.4 & 0.6 & 1 \end{pmatrix}, \\ \Delta B &= \begin{pmatrix} 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 \end{pmatrix}, \end{aligned} \quad (31)$$

and $f(x(t)) = \tanh(x(t))$.

Choose $\alpha = 0.04$, $\beta = 0.5$, $\mu = 0.5$, and $d = 1$. It is obvious that $\Delta A^T \Delta A \leq \alpha I$, $\Delta B^T \Delta B \leq \beta I$, $\omega^T(t)\omega(t) \leq d$, and $\dot{\tau}(t) \leq \mu < 1$. We can select $L = 1$. Under the initial condition $x_1(s) = -0.6$, $x_2(s) = 0.5$, and $x_3(s) = -1.5$, $\forall s \in [-1, 0)$, the state trajectories of variables $x_1(t)$, $x_2(t)$, and $x_3(t)$ of system (30) are shown in Figure 3.

System (30) with control input is given as

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (B + \Delta B)f(x(t - \tau(t))) \\ &\quad + E\omega(t) + u(t). \end{aligned} \quad (32)$$

For numerical simulation, we can select $\lambda = 0.5$, $\eta = 0.6$, $s_1 = 1$, $s_2 = 1$, $s_3 = 1$, and $s_4 = 1$. Based on Corollary 8, we

FIGURE 3: State trajectories of system (30) without control input under the initial conditions $x_1(s) = -0.6$, $x_2(s) = 0.5$, and $x_3(s) = -1.5$, $\forall s \in [-1, 0)$.

have $k_2 \geq 3$ and $k_1 \geq 5.3059$. We choose $k_2 = 3$ and $k_1 = 6$. So the controller $u(t) = u_1(t) + u_2(t)$ is designed as

$$\begin{aligned} u_1(t) &= (u_{11}(t), u_{12}(t), u_{13}(t))^T, \\ u_2(t) &= (u_{21}(t), u_{22}(t), u_{23}(t))^T, \end{aligned} \quad (33)$$

where

$$\begin{aligned} u_{11}(t) &= -6x_1(t) - 0.5 \operatorname{sign}(x_1(t)) |x_1(t)|^{0.6} - 0.5 \\ &\quad \cdot 3^{0.8} \left(\int_{t-0.2}^t (x_1^2(s) + x_2^2(s) + x_3^2(s)) ds \right)^{0.8} \\ &\quad \cdot \frac{x_1(t)}{x_1^2(t) + x_2^2(t) + x_3^2(t)}, \\ u_{12}(t) &= -6x_2(t) - 0.5 \operatorname{sign}(x_2(t)) |x_2(t)|^{0.6} - 0.5 \\ &\quad \cdot 3^{0.8} \left(\int_{t-0.2}^t (x_1^2(s) + x_2^2(s) + x_3^2(s)) ds \right)^{0.8} \\ &\quad \cdot \frac{x_2(t)}{x_1^2(t) + x_2^2(t) + x_3^2(t)}, \\ u_{13}(t) &= -6x_3(t) - 0.5 \operatorname{sign}(x_3(t)) |x_3(t)|^{0.6} - 0.5 \\ &\quad \cdot 3^{0.8} \left(\int_{t-0.2}^t (x_1^2(s) + x_2^2(s) + x_3^2(s)) ds \right)^{0.8} \\ &\quad \cdot \frac{x_3(t)}{x_1^2(t) + x_2^2(t) + x_3^2(t)}, \end{aligned}$$

FIGURE 4: State trajectories of system (30) with control input (34) under the initial conditions $x_1(s) = -0.6$, $x_2(s) = 0.5$, and $x_3(s) = -1.5$, $\forall s \in [-1, 0)$.

$$\begin{aligned} u_{21}(t) &= -\frac{x_1(t)}{2(x_1^2(t) + x_2^2(t) + x_3^2(t))}, \\ u_{22}(t) &= -\frac{x_2(t)}{2(x_1^2(t) + x_2^2(t) + x_3^2(t))}, \\ u_{23}(t) &= -\frac{x_3(t)}{2} - \frac{x_3(t)}{2(x_1^2(t) + x_2^2(t) + x_3^2(t))}. \end{aligned} \quad (34)$$

Under the initial condition $x_1(s) = -0.6$, $x_2(s) = 0.5$, and $x_3(s) = -1.5$, $\forall s \in [-1, 0)$, the state trajectories of $x_1(t)$, $x_2(t)$, and $x_3(t)$ of system (32) are shown in Figure 4. Hence, system (30) is finite-time stable with control input $u(t)$ given in (34).

5. Conclusion

In this paper, we have investigated the finite-time stability problem for a class of uncertain nonlinear systems with time-varying delay and external disturbances. By using the Lyapunov stability theory, sufficient conditions guaranteeing the finite-time stability of the considered systems have been developed. Two simulation examples have been provided to validate the effectiveness and correctness of the proposed theoretical results. Future work will focus on the finite-time stability for uncertain nonlinear systems with distributed delay.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (11301308 and 61673247), the Research Fund for Distinguished Young Scholars and Excellent Young Scholars of Shandong Province (ZR201702100145 and JQ201719), and Shandong Province Key Laboratory of Medical Physics and Image Processing Technology.

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