Research Article

New Designs of Reduced-Order Observer-Based Controllers for Singularly Perturbed Linear Systems

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The slow and fast reduced-order observers and reduced-order observer-based controllers are designed by using the two-stage feedback design technique for slow and fast subsystems. The new designs produce an arbitrary order of accuracy, while the previously known designs produce the accuracy of $O(\epsilon)$ only where ϵ is a small singular perturbation parameter. Several cases of reduced-order observer designs are considered depending on the measured state space variables: only all slow variables are measured, only all fast variables are measured, and some combinations of the slow and fast variables are measured. Since the two-stage methods have been used to overcome the numerical ill-conditioning problem for Cases (III)–(V), they have similar procedures. The numerical ill-conditioning problem is avoided so that independent feedback controllers can be applied to each subsystem. The design allows complete time-scale separation for both the reduced-order observer and controller through the complete and exact decomposition into slow and fast time scales. This method reduces both offline and online computations.

1. Introduction

The presentation in this paper is based on the doctoral dissertation [1]. The fundamental technique used is the two-stage feedback controller design [2, 3]. The full-order observers for singularly perturbed linear systems were considered in [4–10]. The reduced-order observer for singularly perturbed discrete-time systems has been studied only in two papers [11, 12], both of them producing accuracy of $O(\epsilon)$ (an $O(\epsilon)$ is defined by $O(\epsilon) \leq k\epsilon$, where k is a bounded constant), where ϵ is a small singular perturbation parameter. There are no corresponding results reported in the literature for the reduced-order continuous time observers. Since $O(\epsilon)$ accuracy might not be acceptable, our motivation is to design reduced-order observers for this class of systems with the accuracy of $O(\epsilon^i) \leq c\epsilon^i$, $i = 1, 2, 3, \dots$, where c is a bounded constant, which is along the lines of high accuracy techniques for singularly perturbed systems [5]. Note that, for $\epsilon < 1$, $\epsilon^i \rightarrow 0$ rapidly.

Consider a singularly perturbed linear system that contains slow and fast modes [6]:

$$E\dot{x}(t) = Ax(t)$$

$$y(t) = Cx(t)$$

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & \epsilon I_{n_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad (1)$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

where ϵ is a small positive singular perturbation parameter that indicates separation of state variables $x(t) \in \mathfrak{R}^n$ into slow $x_1(t) \in \mathfrak{R}^{n_1}$ and fast $x_2(t) \in \mathfrak{R}^{n_2}$ and $n_1 + n_2 = n$, and $y(t) \in \mathfrak{R}^l$ are the system measurements. The problem matrices are constant and of appropriate dimensions. It was assumed that the matrix C has full rank equal to l . The singularly perturbed system is studied under the following standard assumption [4].

Assumption 1. A_{22} is nonsingular.

In the following we consider five cases for the reduced-order observer-based controller design for singularly perturbed linear systems depending on combinations of measured states [1].

2. Case I: Controller Design When Only All Slow Variables Are Measured

Consider the linear time invariant singularly perturbed control system [1], in which only slow variables are directly measured:

$$\begin{aligned}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \\ \dot{x}_2(t) &= \frac{1}{\epsilon}A_{21}x_1(t) + \frac{1}{\epsilon}A_{22}x_2(t) + \frac{1}{\epsilon}B_2u(t) \\ y(t) &= x_1(t),\end{aligned}\quad (2)$$

where $u(t) \in \mathfrak{R}^m$ is the control input. The reduced-order observer for (2) is given by [13]

$$\dot{\hat{e}}\hat{z}_2(t) = A_z\hat{z}_2(t) + B_zu(t) + K_z y(t), \quad (3)$$

where

$$\begin{aligned}A_z &= A_{22} - K_{11}A_{12}, \\ B_z &= B_2 - K_{11}B_1 \\ K_z &= A_{21} - K_{11}A_{11} + \frac{1}{\epsilon}A_{22}K_{11} - \frac{1}{\epsilon}K_{11}A_{12}K_{11}.\end{aligned}\quad (4)$$

The state estimation of the fast variables is obtained from

$$\hat{x}_2(t) = \hat{z}_2(t) + \frac{1}{\epsilon}K_{11}y(t) \quad (5)$$

so that

$$\hat{x}(t) = \begin{bmatrix} x_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \hat{x}_2(t) \end{bmatrix}. \quad (6)$$

The matrix K_{11} is chosen to stabilize the reduced-order observer (3); that is,

$$\lambda(A_{22} - K_{11}A_{12}) = \lambda(A_{22}^T - A_{12}^T K_{11}^T) = \lambda_{\text{robs}}^{\text{desired}}. \quad (7)$$

To design this reduced-order observer the following assumption is needed [1].

Assumption 2. The pair (A_{22}^T, A_{12}^T) is controllable, which is equivalent to the pair (A_{22}, A_{12}) being observable, which is equivalent to the requirement that the original system is observable.

In the following, the Chang transformation matrix [4] will be needed:

$$\begin{aligned}T_c &= \begin{bmatrix} I_n & \epsilon H \\ -L & I_m - \epsilon LH \end{bmatrix}, \\ T_c^{-1} &= \begin{bmatrix} I_n - \epsilon HL & -\epsilon H \\ L & I_m \end{bmatrix},\end{aligned}\quad (8)$$

where matrices L and H satisfy the algebraic equations

$$\begin{aligned}0 &= \epsilon L(A_{11} - A_{12}L) + (A_{21} - A_{22}L) \\ 0 &= \epsilon(A_{11} - A_{12}L)H + A_{12} - H(A_{22} + \epsilon LA_{12}).\end{aligned}\quad (9)$$

The solutions for L and H can be obtained using either the fixed-point iterations or Newton method or eigenvector method [5].

Using the separation principle, the observer-based controller design via the two-stage design [2] produces [1]

$$\begin{aligned}u(t) &= -F\hat{x}(t) = -\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ \hat{x}_2(t) \end{bmatrix} \\ &= -\begin{bmatrix} F_s + F_{f2}P & F_{f2} \end{bmatrix} T_c^{-1} \begin{bmatrix} y(t) \\ \hat{x}_2(t) \end{bmatrix}.\end{aligned}\quad (10)$$

The feedback gain F_s is chosen thus to place slow eigenvalues at the desired locations; that is,

$$\lambda(A_s - B_s F_s) = \lambda_s^{\text{desired}}. \quad (11)$$

The matrix P is obtained from the Sylvester algebraic equation

$$\begin{aligned}\epsilon P(A_s - B_s F_s) - B_f F_s - A_f P &= 0 \implies \\ P^{(0)} &= A_f^{-1} B_f F_s,\end{aligned}\quad (12)$$

where

$$\begin{aligned}A_s &= A_{11} - A_{12}L, \\ B_s &= B_1(I_n - \epsilon HL) - \epsilon HB_2 \\ A_f &= A_{22} + \epsilon LA_{12}, \\ B_f &= \epsilon LB_1 + B_2.\end{aligned}\quad (13)$$

The feedback gain F_{f2} is chosen thus to place the fast eigenvalues at the desired locations; that is,

$$\begin{aligned}\lambda(A_f - (B_f + \epsilon PB_s)F_{f2}) &= \lambda(A_f - B_{f\text{new}}F_{f2}) \\ &= \lambda_f^{\text{desired}}.\end{aligned}\quad (14)$$

To obtain the reduced-order feedback gains F_s and F_{f2} , the following controllability assumption is needed [14].

Assumption 3. The pairs (A_s, B_s) and $(A_f, B_f + \epsilon PB_s)$ are controllable.

Based on information from (3), (7), and (10), we present in Figure 1 the block diagram for the reduced-order observer-based controller when only all slow state variables are perfectly measured. In (11) and (14) we have chosen the feedback gains for the eigenvalue assignment problem. However, any feedback gains F_1 and F_2 can be used to control the system and provide corresponding design requirements.

FIGURE 1: Case I: reduced-order observer-based controller design for a singularly perturbed linear system when only all slow state variables are directly measured.

3. Case II: Controller Design When Only All Fast Variables Are Measured

Consider the linear time invariant singularly perturbed control system [1], when only all fast state variable are directly measured:

$$\begin{aligned}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \\ \dot{x}_2(t) &= \frac{1}{\epsilon}A_{21}x_1(t) + \frac{1}{\epsilon}A_{22}x_2(t) + \frac{1}{\epsilon}B_2u(t) \\ y(t) &= x_2(t).\end{aligned}\quad (15)$$

The reduced-order observer for system (15) is given by [7]

$$\dot{\hat{z}}_1(t) = A_z\hat{z}_1(t) + B_zu(t) + K_z y(t), \quad (16)$$

where

$$\begin{aligned}A_z &= A_{11} - \frac{1}{\epsilon}K_{12}A_{21}, \\ B_z &= B_2 - \frac{1}{\epsilon}K_{12}B_1 \\ K_z &= A_{12} + A_{11}K_{12} - \frac{1}{\epsilon}K_{12}A_{22} - \frac{1}{\epsilon}K_{12}A_{21}K_{12}.\end{aligned}\quad (17)$$

The slow state estimation is obtained from

$$\hat{x}_1(t) = \hat{z}_1(t) + K_{12}y(t) \quad (18)$$

so that

$$\hat{x}(t) = \begin{bmatrix} \hat{x}_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \hat{z}_1(t) \\ y(t) \end{bmatrix}. \quad (19)$$

The matrix K_{12} is chosen to stabilize the reduced-order observer (16)-(17); that is,

$$\begin{aligned}\lambda\left(A_{11} - \frac{1}{\epsilon}K_{12}A_{21}\right) &= \lambda\left(A_{11}^T - \frac{1}{\epsilon}A_{21}^TK_{12}^T\right) \\ &= \lambda_{\text{robs}}^{\text{desired}}.\end{aligned}\quad (20)$$

The reduced-order observer (16) can be designed under the following assumption [14].

Assumption 4. The pair $(A_{11}^T, (1/\epsilon)A_{21}^T)$ is controllable, which is equivalent to the pair $(A_{11}, (1/\epsilon)A_{21})$ being observable. It was shown in Appendix that this is equivalent to (A_{11}, A_{21}) being observable.

Additional matrices needed in this design can be obtained from (9) and (12)-(13). Using the separation principle, the observer-based controller can be designed via the two-stage design as

$$\begin{aligned}u(t) &= -Fx(t) = -\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ x_2(t) \end{bmatrix} \\ &= -\begin{bmatrix} F_s + F_{f2}P & F_{f2} \end{bmatrix} T_c^{-1} \begin{bmatrix} \hat{x}_1(t) \\ y(t) \end{bmatrix},\end{aligned}\quad (21)$$

where F_s and F_{f2} are obtained from (11) and (14). The feedback gains F_s and F_{f2} can be obtained under Assumption 3.

In Figure 2, the block diagram for the reduced-order observer-based controller when only all fast variables are perfectly measured is presented.

FIGURE 2: Case II: slow and fast observer-based controller design for a singularly perturbed linear system when only fast state variables are measured directly.

4. Cases (III)–(V): Controller Design When Some Combinations of Slow and Fast Variables Are Measured

In Case (III), the measurable states $x_{11}(t)$ are parts of the slow state vector $x_1(t)$ in the singularly perturbed linear system defined in (1), as

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\ \epsilon \dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) \\ y(t) &= x_{11}(t), \end{aligned} \quad (22)$$

where

$$x_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ x_{12}(t) \end{bmatrix}. \quad (23)$$

Use the following partitioning:

$$\begin{aligned} A_{11} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \\ A_{12} &= \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\ A_{21} &= [a_{31} \quad a_{32}], \\ A_{22} &= [a_{33}], \end{aligned} \quad (24)$$

where dimensions of matrices a_{11} , a_{22} and a_{33} are, respectively, $a_{11} \in R^{l \times l}$, $a_{22} \in R^{(n_1-l) \times (n_1-l)}$, and $a_{33} \in R^{n_2 \times n_2}$, with corresponding dimension of a_{ij} , $i \neq j$, matrices. System (22) with (23)-(24) can be represented as

$$\begin{aligned} \dot{x}_{11}(t) &= A_{11}^r x_{11}(t) + A_{12}^r x_2^r(t) \\ \dot{x}_2^r(t) &= A_{21}^r x_{11}(t) + A_{22}^r x_2^r(t) \\ y(t) &= x_{11}(t), \end{aligned} \quad (25)$$

where

$$\begin{aligned} x_2^r(t) &= \begin{bmatrix} x_{12}(t) \\ x_2(t) \end{bmatrix} \\ A_{11}^r &= [a_{11}], \\ A_{12}^r &= [a_{12} \quad a_{13}] \\ A_{21}^r &= \begin{bmatrix} a_{21} \\ \frac{1}{\epsilon} a_{31} \end{bmatrix}, \\ A_{22}^r &= \begin{bmatrix} a_{22} & a_{23} \\ \frac{1}{\epsilon} a_{32} & \frac{1}{\epsilon} a_{33} \end{bmatrix}. \end{aligned} \quad (26)$$

The reduced-order observer for this case is derived in [1], and it is given by

$$\dot{\hat{z}}_2^r(t) = A_z^r \hat{z}_2^r(t) + K_z^r y(t), \quad (27)$$

where

$$\begin{aligned} A_z^r &= A_{22}^r - K_2 A_{12}^r \\ &= \begin{bmatrix} a_{22} & a_{23} \\ \frac{1}{\epsilon} a_{32} & \frac{1}{\epsilon} a_{33} \end{bmatrix} - \begin{bmatrix} K_{21} \\ K_{22} \end{bmatrix} [a_{12} \ a_{13}], \\ K_z^r &= A_{21}^r - K_2 A_{11}^r + A_{22}^r K_2 - K_2 A_{12}^r K_2, \\ \hat{z}_2^r(t) &= \begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_2(t) \end{bmatrix}. \end{aligned} \quad (28)$$

For the eigenvalue assignment in A_z^r , we encounter the singularly perturbed structure, so that the two-stage method is applied for a two time-scale problem.

The reduced-order sequential observer configuration obtained using the two-stage method [1] is given by

$$\begin{aligned} &\begin{bmatrix} \dot{\hat{z}}_s(t) \\ \epsilon \dot{\hat{z}}_{fnew}(t) \end{bmatrix} \\ &= \begin{bmatrix} A_{sr} - K_{sr} C_{sr} & 0 \\ -\epsilon K_{f2r} C_{sr} & A_{fr} - K_{f2r} C_{fnewr} \end{bmatrix} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix} \\ &+ \begin{bmatrix} K_{sr}^* \\ K_{f2r}^* \end{bmatrix} y(t), \end{aligned} \quad (29)$$

where K_{sr}^* and K_{f2r}^* are obtained from $T_{4r}^{-1} K_z^r$, with T_{4r} defined by

$$\begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_2(t) \end{bmatrix} = T_{1r} T_{cr} T_{2r} = T_{4r} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix}, \quad (30)$$

where

$$\begin{aligned} T_{1r} &= \begin{bmatrix} I_{(n_1-l)} & 0 \\ 0 & \frac{1}{\epsilon} I_{n_2} \end{bmatrix}, \\ T_{2r} &= \begin{bmatrix} I_{(n_1-l)} & P_{or}^T \\ 0 & I_{n_2} \end{bmatrix}, \\ T_{cr}^T &= \begin{bmatrix} I_{(n_1-l)} & -\epsilon L_r^T \\ H_r^T & I_{n_2} - \epsilon H_r^T L_r^T \end{bmatrix}, \end{aligned} \quad (31)$$

where L_r^T and H_r^T are matrices that satisfy

$$\begin{aligned} 0 &= \epsilon (a_{22}^T - L_r^T a_{23}^T) L_r^T + (a_{32}^T - L_r^T a_{33}^T) \\ 0 &= \epsilon H_r^T (a_{22}^T - L_r^T a_{23}^T) + a_{23}^T - (a_{33}^T + \epsilon a_{23}^T L_r^T) H_r^T \end{aligned} \quad (32)$$

with a_{ij} matrices defined in (26). The reduced-order observer (29) has a *sequential structure*. It can be block diagonalized and used in a *parallel structure* as follows:

$$\begin{aligned} \dot{\hat{z}}_s(t) &= (A_{sr} - K_{sr} C_{sr}) \hat{z}_s(t) + K_{sr}^* y(t) \\ \epsilon \dot{\hat{z}}_{fnew2}(t) &= (A_{fr} - K_{f2r} C_{fnewr}) \hat{z}_{fnew2}(t) \\ &+ K_{f3r} y(t), \end{aligned} \quad (33)$$

where

$$\begin{aligned} K_{f3r} &= \epsilon (P_{o2r} K_{sr}^* + K_{f2r}^*), \\ \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew2}(t) \end{bmatrix} &= \begin{bmatrix} I_{(n_1-l)} & 0 \\ P_{o2r} & I_{n_2} \end{bmatrix} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix} \\ &= T_{3r} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew}(t) \end{bmatrix}. \end{aligned} \quad (34)$$

The original coordinates $\hat{z}_{12}(t)$ and $\hat{z}_2(t)$ and the coordinates $\hat{z}_s(t)$ and $\hat{z}_{fnew2}(t)$ are related via

$$\begin{aligned} \begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_2(t) \end{bmatrix} &= T_{4r} T_{3r}^{-1} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew2}(t) \end{bmatrix} = T_r \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew2}(t) \end{bmatrix} \\ &= \begin{bmatrix} I_{(n_1-l)} - P_{or}^T P_{o2r} - H_r P_{o2r} & P_{or}^T + H_r \\ -L_r + L_r P_{or}^T P_{o2r} - \frac{1}{\epsilon} (I_{n_2} - \epsilon L_r H_r) P_{o2r} & -L_r P_{or}^T + \frac{1}{\epsilon} (I_{n_2} - \epsilon L_r H_r) \end{bmatrix} \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew2}(t) \end{bmatrix}, \end{aligned} \quad (35)$$

where P_{o2r} satisfies the algebraic Sylvester equation represented by

$$\begin{aligned} &\epsilon P_{o2r} (A_{sr} - K_{sr} C_{sr}) - \epsilon K_{f2r} C_{sr} \\ &- (A_{fr} - K_{f2r} C_{fnewr}) P_{o2r} = 0 \implies \end{aligned}$$

$$P_{o2r}^0 = O(\epsilon).$$

(36)

The steps used in the sequential and parallel observer design structures are summarized in Figure 3.

FIGURE 3: Case III: design outlines for the sequential and parallel reduced-order observer.

We use the parallel observers structure (33) and consider the reduced-order observer-based controller design for singularly perturbed linear systems. Observer (33) is now driven by the system measurements and *control inputs*; that is,

$$\begin{aligned}\dot{\hat{z}}_s(t) &= (A_{sr} - K_{sr}C_{sr})\hat{z}_s(t) + B_{sr2}u(t) \\ &\quad + K_{sr}^*y(t) \\ \epsilon\dot{\hat{z}}_{fnew2}(t) &= (A_{fq} - K_{f2}C_{fnew})\hat{z}_{fnew2}(t) \\ &\quad + B_{f2r}u(t) + K_{f3}y(t),\end{aligned}\quad (37)$$

where B_{sr2} and B_{f2r} can be obtained from $T_r^{-1}B$ as

$$\begin{aligned}B_{sr2} &= (I_{(n_1-l)} - \epsilon H_r L_r)B_{z1} - \epsilon P_{or}^T L_r B_{z1} - H_r B_{z2} \\ &\quad - P_{or}^T B_{z2}, \\ B_{f2r} &= \epsilon P_{or2r} (I_{(n_1-l)} - \epsilon H_r L_r)B_{z1} - \epsilon^2 P_{or2r} P_{or}^T L_r B_{z1} \\ &\quad + \epsilon^2 L_r B_{z1} - \epsilon P_{or2r} H_r B_{z2} - \epsilon P_{or2r} P_{or}^T B_{z2} \\ &\quad + \epsilon B_{z2}.\end{aligned}\quad (38)$$

The control input in the $\hat{z}_s$ - $\hat{z}_{fnew2}$ coordinates is given by

$$\begin{aligned}u(t) &= -F_r \hat{x}(t) = -[F_{12} \ F_2] \begin{bmatrix} \hat{x}_{12}(t) \\ \hat{x}_2(t) \end{bmatrix} \\ &= -\left([F_{12} \ F_2] \begin{bmatrix} K_{21} \\ K_{22} \end{bmatrix} y(t) \right. \\ &\quad \left. + [F_{12} \ F_2] T_r \begin{bmatrix} \hat{z}_s(t) \\ \hat{z}_{fnew2}(t) \end{bmatrix} \right) \\ &= -F_r K_2 y(t) - F_{sr2} \hat{z}_s(t) - F_{fr2} \hat{z}_{fnew2}(t), \\ [F_{11} \ F_{12} \ | \ F_2] &= [F_1 \ | \ F_2] \\ &= [F_s + F_{f2}P \ | \ F_{f2}] T_c^{-1} \\ \text{so that } [F_{11} \ | \ F_{12} \ F_2] &= [F_{11} \ | \ F_r]\end{aligned}\quad (39)$$

with

$$\begin{aligned}F_{sr2} &= F_{12} (I_{(n_1-l)} - P_{or}^T P_{or2r} - H_r P_{or2r}) \\ &\quad - F_2 L (I_n - P_{or}^T P_{or2r}) \\ &\quad - \frac{1}{\epsilon} F_2 (I_{n_2} - \epsilon L_r H_r) P_{or2r} \\ F_{fr2} &= F_{12} (P_{or}^T + H_r) - F_2 L_r P_{or}^T \\ &\quad + \frac{1}{\epsilon} F_2 (I_{n_2} - \epsilon L_r H_r).\end{aligned}\quad (40)$$

The corresponding block diagram for the observer driven controller is presented in Figure 4. This block diagram clearly indicates full parallelism of the slow controller driven by the slow observer and the fast controller driven by the fast observer.

The remaining matrices defined in (29) are given by

$$\begin{aligned}A_{sr}^T &= a_{22}^T - L^T a_{23}^T, \\ A_{fr}^T &= a_{33}^T + \epsilon a_{23}^T L^T \\ C_{sr}^T &= a_{12}^T - L^T a_{13}^T, \\ C_{fr}^T &= \epsilon H^T a_{12}^T + (I_{n_2} - \epsilon H^T L^T) a_{13}^T \\ C_{fnewr} &= C_{fr} + \epsilon C_{sr} P_{or}^T.\end{aligned}\quad (41)$$

K_{sr} , K_{f2r} , and P_{or} can be obtained from the following formulas:

$$\begin{aligned}\lambda (A_{sr}^T - C_{sr}^T K_{sr}^T) &= \lambda (A_{sr} - K_{sr} C_{sr}) = \lambda_s^{\text{desired}} \\ \lambda (A_{fr} - K_{f2r} C_{fnewr}) &= \lambda_f^{\text{desired}} \\ \epsilon P_{or} (A_{sr}^T - C_{sr}^T K_{sr}^T) - C_{fr}^T K_{sr}^T - A_{fr}^T P_{or} &= 0 \implies \\ P_{or} &= O(\epsilon).\end{aligned}\quad (42)$$

FIGURE 4: Case III: slow and fast observer-based controller designs for singularly perturbed linear systems with the system feedback gains obtained in (39) when parts of slow variables are measured.

In Case (IV), the measurable states $x_{21}(t)$ are parts of the fast state vector $x_2(t)$ in the singularly perturbed linear system defined in (1):

$$\begin{aligned}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) \\ \epsilon \dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) \\ y(t) &= I_l x_{22}(t),\end{aligned}\quad (43)$$

where

$$x_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \begin{bmatrix} x_{21}(t) \\ y(t) \end{bmatrix}, \quad (44)$$

using the following partitioning:

$$\begin{aligned}A_{11} &= [a_{11}], \\ A_{12} &= [a_{12} \ a_{13}] \\ A_{21} &= \begin{bmatrix} a_{21} \\ a_{31} \end{bmatrix}, \\ A_{22} &= \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix},\end{aligned}\quad (45)$$

where dimensions of matrices a_{11} , a_{22} , and a_{33} are, respectively, $a_{11} \in R^{n_1 \times n_1}$, $a_{22} \in R^{(n_2-l) \times (n_2-l)}$, and $a_{33} \in R^{l \times l}$, with the corresponding dimensions of a_{ij} , $i \neq j$, matrices. System (43) with (44)-(45) can be represented as

$$\begin{aligned}\dot{x}_1^r(t) &= A_{11}^r x_1^r(t) + A_{12}^r x_{22}(t) \\ \dot{x}_{22}(t) &= A_{21}^r x_1^r(t) + A_{22}^r x_{22}(t) \\ y(t) &= I_l x_{22}(t),\end{aligned}\quad (46)$$

where

$$\begin{aligned}x_1^r(t) &= \begin{bmatrix} x_1(t) \\ x_{21}(t) \end{bmatrix} \\ A_{11}^r &= \begin{bmatrix} a_{11} & a_{12} \\ \frac{1}{\epsilon} a_{21} & \frac{1}{\epsilon} a_{22} \end{bmatrix}, \\ A_{12}^r &= \begin{bmatrix} a_{13} \\ \frac{1}{\epsilon} a_{23} \end{bmatrix} \\ A_{21}^r &= \begin{bmatrix} \frac{1}{\epsilon} a_{31} & \frac{1}{\epsilon} a_{32} \end{bmatrix}, \\ A_{22}^r &= \begin{bmatrix} \frac{1}{\epsilon} a_{33} \end{bmatrix}.\end{aligned}\quad (47)$$

The corresponding reduced-order observer structure is given by [1]

$$\dot{\hat{z}}_1^r(t) = A_{z,2}^r \hat{z}_1^r(t) + B_{z,2}^r u(t) + K_{z,2}^r y(t), \quad (48)$$

where

$$\hat{z}_1^r(t) = \begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_{21}(t) \end{bmatrix} \quad (49)$$

The observer gain K_3 and matrices $B_{z,2}^r$ and $K_{z,2}^r$ [7] are obtained from

$$\begin{aligned}A_{z,2}^r &= A_{11}^r - K_3 A_{21}^r \\ &= \begin{bmatrix} a_{11} & a_{12} \\ \frac{1}{\epsilon} a_{21} & \frac{1}{\epsilon} a_{22} \end{bmatrix} - \begin{bmatrix} K_{31} \\ K_{32} \end{bmatrix} \begin{bmatrix} \frac{1}{\epsilon} a_{31} & \frac{1}{\epsilon} a_{32} \end{bmatrix},\end{aligned}$$

$$\begin{aligned} B_{z,2}^r &= B_2 - K_3 B_1 = \begin{bmatrix} B_{z1} \\ B_{z2} \end{bmatrix}, \\ K_{z,2}^r &= A_{12}^r - K_3 A_{22}^r + A_{11}^r K_3 - K_3 A_{21}^r K_3. \end{aligned} \quad (50)$$

For the eigenvalue assignment in $A_{z,2}^r$, we encounter singularly perturbed structure so that the two-stage design is applied to the slow and fast subsystems.

The two-stage method provides the parallel observer [1]. Using the parallel observer we can form the reduced-order observer-based controller given as

$$\begin{aligned} &\begin{bmatrix} \dot{\hat{z}}_{s,2}(t) \\ \epsilon \dot{\hat{z}}_{f\text{new},2}(t) \end{bmatrix} \\ &= \begin{bmatrix} A_{sr,2} - K_{sr,2} C_{sr,2} & 0 \\ 0 & A_{fr,2} - K_{f2r,2} C_{f\text{new},r,2} \end{bmatrix} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{f\text{new},2}(t) \end{bmatrix} \\ &+ \begin{bmatrix} B_{s2r,2} \\ B_{f2r,2} \end{bmatrix} u(t) + \begin{bmatrix} K_{sr,2}^* \\ K_{f3r,2} \end{bmatrix} y(t), \end{aligned} \quad (51)$$

where

$$\begin{aligned} A_{sr,2}^T &= a_{11}^T - L_{r,2}^T a_{12}^T, \\ A_{fr,2}^T &= a_{22}^T + \epsilon a_{12}^T L_{r,2}^T, \\ B_{s2r,2} &= (I_{n_1} - \epsilon H_{r,2} L_{r,2}) B_{z1} - \epsilon P_{or,2}^T L_{r,2} B_{z1} \end{aligned}$$

$$\begin{aligned} &\begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_{21}(t) \end{bmatrix} = T_{r,2} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{f\text{new},2}(t) \end{bmatrix} = T_{4r,2} T_{3r,2}^{-1} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{f\text{new},2}(t) \end{bmatrix} \\ &= \begin{bmatrix} I_{n_1} - P_{or,2}^T P_{o2r,2} - H_{r,2} P_{o2r,2} & P_{or,2}^T + H_{r,2} \\ -L_{r,2} + L_{r,2} P_{or,2}^T P_{o2r,2} - \frac{1}{\epsilon} (I_{(n_2-l)} - \epsilon L_{r,2} H_{r,2}) P_{o2r,2} & -L_{r,2} P_{or,2}^T + \frac{1}{\epsilon} (I_{(n_2-l)} - \epsilon L_{r,2} H_{r,2}) \end{bmatrix} \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{f\text{new},2}(t) \end{bmatrix}, \end{aligned} \quad (54)$$

where

$$\begin{aligned} T_{4r,2} &= \begin{bmatrix} I_{n_1} & P_{or,2}^T + H_{r,2} \\ -L_{r,2} & -L_{r,2} P_{or,2}^T + \frac{1}{\epsilon} (I_{(n_2-l)} - \epsilon L_{r,2} H_{r,2}) \end{bmatrix}, \\ T_{3r,2}^{-1} &= \begin{bmatrix} I_{n_1} & 0 \\ -P_{o2r,2} & I_{(n_2-l)} \end{bmatrix}. \end{aligned} \quad (55)$$

$K_{sr,2}^*$ and $(1/\epsilon)K_{f2r,2}^*$ are obtained from $T_{4r,2}^{-1}K_{z,2}^r$. The slow feedback gain $K_{sr,2}$ can be obtained using the eigenvalue assignment; that is,

$$\begin{aligned} \lambda(A_{sr,2}^T - C_{sr,2}^T K_{sr,2}^T) &= \lambda(A_{sr,2} - K_{sr,2} C_{sr,2}) \\ &= \lambda_s^{\text{desired}} \end{aligned} \quad (56)$$

requiring that the following assumption is satisfied [14].

$$\begin{aligned} &-H_{r,2} B_{z2} - P_{or,2}^T B_{z2}, \\ B_{f2r,2} &= \epsilon P_{o2r,2} (I_{n_1} - \epsilon H_{r,2} L_{r,2}) B_{z1} \\ &- \epsilon^2 P_{o2r,2} P_{or,2}^T L_{r,2} B_{z1} + \epsilon^2 L_{r,2} B_{z1} \\ &- \epsilon P_{o2r,2} H_{r,2} B_{z2} - \epsilon P_{o2r,2} P_{or,2}^T B_{z2} \\ &+ \epsilon B_{z2} \end{aligned}$$

$$C_{sr,2}^T = \frac{1}{\epsilon} a_{31}^T - \frac{1}{\epsilon} L_{r,2}^T a_{32}^T,$$

$$c_{fr,2}^T = H_{r,2}^T a_{31}^T + \frac{1}{\epsilon} (I_{(n_2-l)} - \epsilon H_{r,2}^T L_{r,2}^T) a_{32}^T,$$

$$C_{f\text{new},r,2} = (C_{fr,2} + \epsilon C_{sr,2} P_{or,2}^T) \quad (52)$$

and $L_{r,2}^T$ and $H_{r,2}^T$ are the solution given as

$$\begin{aligned} 0 &= \epsilon (a_{22}^T - L_{r,2}^T a_{23}^T) L_{r,2}^T + (a_{32}^T - L_{r,2}^T a_{33}^T) \\ 0 &= \epsilon H_{r,2}^T (a_{22}^T - L_{r,2}^T a_{23}^T) + a_{23}^T \\ &- (a_{33}^T + \epsilon a_{23}^T L_{r,2}^T) H_{r,2}^T. \end{aligned} \quad (53)$$

The coordinates of the original observer (48) and the parallel observer (51) are related via a transformation as

Assumption 5. The pair $(A_{sr,2}, C_{sr,2})$ is observable.

The fast feedback gain $K_{f2r,2}$ is obtained from the eigenvalue assignment problem; that is,

$$\lambda(A_{fr,2} - K_{f2r,2} C_{f\text{new},r,2}) = \lambda_f^{\text{desired}}. \quad (57)$$

The following observability assumption is needed [14].

Assumption 6. The pair $(A_{fr,2}, C_{f\text{new},r,2})$ is observable.

The control input in $\hat{z}_{s,2}$ - $\hat{z}_{f\text{new},2}$ coordinates is given by

$$\begin{aligned} u(t) &= -F_r \hat{x}(t) = -[F_1 \ F_{21}] \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_{21}(t) \end{bmatrix} \\ &= -F_{r,2} K_3 y(t) - [F_{sr,2} \ F_{fr,2}] \begin{bmatrix} \hat{z}_{s,2}(t) \\ \hat{z}_{f\text{new},2}(t) \end{bmatrix} \end{aligned}$$

FIGURE 5: Case IV: slow and fast observer-based controller design for a singularly perturbed linear system measurable variables being part of the fast variables only.

$$\begin{aligned} [F_1 \mid F_{21} \mid F_{22}] &= [F_1 \mid F_2] = [F_s + F_{f2}P \mid F_{f2}] T_c^{-1} \\ \text{so that } [F_1 \mid F_{21} \mid F_{22}] &= [F_r \mid F_{22}] \end{aligned} \quad (58)$$

with

$$\begin{aligned} F_{sr,2,2} &= F_1 \left(I_{n_1} - P_{or,2}^T P_{o2r,2} - H_{r,2} P_{o2r,2} \right) \\ &\quad - F_{21} L \left(I_{(n_2-l)} - P_{or,2}^T P_{o2r,2} \right) \\ &\quad - \frac{1}{\epsilon} F_2 \left(I_{(n_2-l)} - \epsilon L_{r,2} H_{r,2} \right) P_{o2r,2} \quad (59) \\ F_{fr,2,2} &= F_1 \left(P_{or,2}^T + H_{r,2} \right) - F_{21} L_{r,2} P_{or,2}^T \\ &\quad + \frac{1}{\epsilon} F_{21} \left(I_{(n_2-l)} - \epsilon L_{r,2} H_{r,2} \right). \end{aligned}$$

The remaining matrices $P_{or,2}$ and $P_{o2r,2}$ satisfy the algebraic Sylvester equation represented by

$$\begin{aligned} \epsilon P_{or,2} \left(A_{sr,2}^T - C_{sr,2}^T K_{sr,2}^T \right) - C_{fr,2}^T K_{sr,2}^T - A_{fr,2}^T P_{or,2} \\ = 0 \implies \\ P_{or,2} = O(\epsilon) \\ \epsilon P_{o2r,2} \left(A_{sr,2} - K_{sr,2} C_{sr,2} \right) - \epsilon K_{f2r,2} C_{sr,2} \\ - \left(A_{fr,2} - K_{f2r,2} C_{fnew,2} \right) P_{o2r,2} = 0 \implies \\ P_{o2r,2}^0 = O(\epsilon). \end{aligned} \quad (60)$$

The corresponding block diagram for the observer driven controller is presented in Figure 5.

In Case (V), the measurable states $x_{11}(t)$ and $x_{21}(t)$ are parts of the slow state vector $x_1(t)$ and the fast state $x_2(t)$ in the singularly perturbed linear system defined as

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t)$$

$$\epsilon \dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t)$$

$$\begin{aligned} y(t) &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \\ x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix} \quad (61) \\ &= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} x_1(t) &= \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}, \\ x_1(t) &= \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ x_{12}(t) \end{bmatrix}, \quad (62) \\ x_2(t) &= \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \begin{bmatrix} x_{21}(t) \\ y_2(t) \end{bmatrix}, \end{aligned}$$

using the following partitioning:

$$\begin{aligned} A_{11} &= \begin{bmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{bmatrix}, \\ A_{12} &= \begin{bmatrix} a_{13}^* & a_{14}^* \\ a_{23}^* & a_{24}^* \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A_{21} &= \begin{bmatrix} a_{31}^* & a_{32}^* \\ a_{41}^* & a_{42}^* \end{bmatrix}, \\ A_{22} &= \begin{bmatrix} a_{33}^* & a_{34}^* \\ a_{43}^* & a_{44}^* \end{bmatrix}, \end{aligned} \quad (63)$$

where dimensions of matrices a_{11}^* , a_{22}^* , and a_{33}^* are, respectively, $a_{11}^* \in R^{l_1 \times l_1}$, $a_{22}^* \in R^{(n_1-l_1) \times (n_1-l_1)}$, and $a_{33}^* \in R^{l_2 \times l_2}$, with the corresponding dimensions of a_{ij} , $i \neq j$, matrices. System (61) with (62)-(63) can be represented as

$$\begin{aligned} \dot{x}_m(t) &= A_1^r x_m(t) + A_2^r x_u(t) \\ \dot{x}_u(t) &= A_3^r x_m(t) + A_4^r x_u(t) \\ y(t) &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \\ x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix} \\ &= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \end{aligned} \quad (64)$$

where

$$\begin{aligned} x_m(t) &= \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix}, \\ x_u(t) &= \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix} \\ A_1^r &= \begin{bmatrix} a_{11}^* & a_{13}^* \\ \frac{1}{\epsilon} a_{31}^* & \frac{1}{\epsilon} a_{33}^* \end{bmatrix}, \\ A_2^r &= \begin{bmatrix} a_{12}^* & a_{14}^* \\ \frac{1}{\epsilon} a_{32}^* & \frac{1}{\epsilon} a_{34}^* \end{bmatrix}, \\ A_3^r &= \begin{bmatrix} a_{21}^* & a_{23}^* \\ \frac{1}{\epsilon} a_{41}^* & \frac{1}{\epsilon} a_{43}^* \end{bmatrix}, \\ A_4^r &= \begin{bmatrix} a_{22}^* & a_{24}^* \\ \frac{1}{\epsilon} a_{42}^* & \frac{1}{\epsilon} a_{44}^* \end{bmatrix}. \end{aligned} \quad (65)$$

$x_m(t)$ are the measurable states and $x_u(t)$ are the unmeasurable states. A_1^r and A_3^r are elements in (62) relevant to the measurable states and A_2^r and A_4^r are elements in (62) relevant to the unmeasurable states. The reduced-order observer is given by

$$\dot{\hat{z}}_1^r(t) = A_{z,3}^r \hat{z}_1^r(t) + B_{z,3}^r u(t) + K_{z,3}^r y(t). \quad (66)$$

The observer gains are obtained from

$$\begin{aligned} A_{z,3}^r &= A_4^r - K_4 A_2^r \\ &= \begin{bmatrix} a_{22}^* & a_{24}^* \\ \frac{1}{\epsilon} a_{42}^* & \frac{1}{\epsilon} a_{44}^* \end{bmatrix} - \begin{bmatrix} K_{41} \\ K_{42} \end{bmatrix} \begin{bmatrix} a_{12}^* & a_{14}^* \\ \frac{1}{\epsilon} a_{32}^* & \frac{1}{\epsilon} a_{34}^* \end{bmatrix}, \end{aligned} \quad (67)$$

$$B_{z,3}^r = B_2 - K_4 B_1 = \begin{bmatrix} B_{z1} \\ B_{z2} \end{bmatrix},$$

$$K_{z,3}^r = A_3^r - K_4 A_1^r + A_4^r K_4 - K_4 A_2^r K_4.$$

For the eigenvalue assignment in $A_{z,3}^r$, we encounter again the singularly perturbed structure.

The two-stage method provides the parallel observer; we can form the reduced-order observer-based controller given as [1]

$$\begin{aligned} &\begin{bmatrix} \dot{\hat{z}}_{s,3}(t) \\ \epsilon \dot{\hat{z}}_{fnew2,3}(t) \end{bmatrix} \\ &= \begin{bmatrix} A_{sr,3} - K_{sr,3} C_{sr,3} & 0 \\ 0 & A_{fr,3} - K_{fr,3} C_{fnewr,3} \end{bmatrix} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix} \\ &+ \begin{bmatrix} B_{sr,3} \\ B_{fr,3} \end{bmatrix} u(t) + \begin{bmatrix} K_{sr,3}^* \\ K_{fr,3} \end{bmatrix} y(t), \end{aligned} \quad (68)$$

where

$$\begin{aligned} A_{sr,3}^T &= a_{22}^{*T} - L_{r,3}^T a_{24}^{*T}, \\ A_{fr,3}^T &= a_{44}^{*T} + \epsilon a_{24}^{*T} L_{r,3}^T \\ B_{sr,3} &= (I_{(n_1-l_1)} - \epsilon H_{r,3} L_{r,3}) B_{z1} - \epsilon P_{or,3}^T L_{r,3} B_{z1} \\ &\quad - H_{r,3} B_{z2} - P_{or,3}^T B_{z2}, \\ B_{fr,3} &= \epsilon P_{o2r,3} (I_{(n_1-l_1)} - \epsilon H_{r,3} L_{r,3}) B_{z1} \\ &\quad - \epsilon^2 P_{o2r,3} P_{or,3}^T L_{r,3} B_{z1} + \epsilon^2 L_{r,3} B_{z1} \\ &\quad - \epsilon P_{o2r,3} H_{r,3} B_{z2} - \epsilon P_{o2r,3} P_{or,3}^T B_{z2} \\ &\quad + \epsilon B_{z2} P_{o2r,2} P_{or,2}^T L_{r,2} B_{z1} + \epsilon^2 L_{r,2} B_{z1} \\ &\quad - \epsilon P_{o2r,2} H_{r,2} B_{z2} - \epsilon P_{o2r,2} P_{or,2}^T B_{z2} \\ &\quad + \epsilon B_{z2} \\ C_{sr,31}^T &= a_{12}^{*T} - L_{r,3}^T a_{14}^{*T}, \end{aligned}$$

$$C_{sr,32}^T = \frac{1}{\epsilon} a_{32}^{*T} - L_{r,3}^T \frac{1}{\epsilon} a_{34}^{*T}$$

$$\begin{aligned}
C_{fr,31}^T &= \epsilon H_{r,3}^T a_{12}^{*T} + (I_{(n_2-l_2)} - \epsilon H_{r,3}^T L_{r,3}^T) a_{14}^{*T}, \\
C_{fr,32}^T &= \epsilon H_{r,3}^T \frac{1}{\epsilon} a_{32}^{*T} + (I_{(n_2-l_2)} - \epsilon H_{r,3}^T L_{r,3}^T) \frac{1}{\epsilon} a_{34}^{*T} \\
C_{sr,3}^T &= [C_{sr,31}^T \quad C_{sr,32}^T], \\
C_{fr,3}^T &= [C_{fr,31}^T \quad C_{fr,32}^T] \\
C_{fnewr,3} &= (C_{fr,3} + \epsilon C_{sr,3} P_{or,3}^T) \\
&\quad - (a_{44}^{*T} + \epsilon a_{24}^{*T} L_{r,3}^T) H_{r,3}^T.
\end{aligned} \tag{69}$$

and $L_{r,3}^T$ and $H_{r,3}^T$ are the solution given as

$$\begin{aligned}
0 &= \epsilon (a_{22}^{*T} - L_{r,3}^T a_{24}^{*T}) L_{r,3}^T + (a_{42}^{*T} - L_{r,3}^T a_{44}^{*T}) \\
0 &= \epsilon H_{r,3}^T (a_{22}^{*T} - L_{r,3}^T a_{24}^{*T}) + a_{24}^{*T}
\end{aligned}$$

The matrices $P_{or,3}$ and $P_{o2r,3}$ satisfy the algebraic Sylvester equation represented by

$$\begin{aligned}
\epsilon P_{or,3} (A_{sr,3}^T - C_{sr,3}^T K_{sr,3}^T) - C_{fr,3}^T K_{sr,3}^T - A_{fr,3}^T P_{or,3} &= 0 \implies \\
P_{or,3} &= O(\epsilon) \\
\epsilon P_{o2r,3} (A_{sr,3} - K_{sr,3} C_{sr,3}) - \epsilon K_{f2r,3} C_{sr,3} \\
&\quad - (A_{fr,3} - K_{f2r,3} C_{fnewr,3}) P_{o2r,3} = 0 \implies \\
P_{o2r,3}^0 &= O(\epsilon).
\end{aligned} \tag{71}$$

The transformation is needed to relate both coordinates in which original observer (48) and parallel observer (68) are located; that is,

$$\begin{aligned}
\begin{bmatrix} \hat{z}_{12}(t) \\ \hat{z}_{22}(t) \end{bmatrix} &= T_{r,3} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix} = T_{4r,3} T_{3r,3}^{-1} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix} \\
&= \begin{bmatrix} I_{(n_1-l_1)} - P_{or,3}^T P_{o2r,3} - H_{r,3} P_{o2r,3} & P_{or,3}^T + H_{r,3} \\ -L_{r,3} + L_{r,3} P_{or,3}^T P_{o2r,3} - \frac{1}{\epsilon} (I_{(n_2-l_2)} - \epsilon L_{r,3} H_{r,3}) P_{o2r,3} & -L_{r,3} P_{or,3}^T + \frac{1}{\epsilon} (I_{(n_2-l_2)} - \epsilon L_{r,3} H_{r,3}) \end{bmatrix} \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix},
\end{aligned} \tag{72}$$

where

$$\begin{aligned}
T_{4r,3} &= \begin{bmatrix} I_{(n_1-l_1)} & P_{or,3}^T + H_{r,3} \\ -L_{r,3} & -L_{r,3} P_{or,3}^T + \frac{1}{\epsilon} (I_{(n_2-l_2)} - \epsilon L_{r,3} H_{r,3}) \end{bmatrix}, \\
T_{3r,3}^{-1} &= \begin{bmatrix} I_{(n_1-l_1)} & 0 \\ -P_{o2r,3} & I_{(n_2-l_2)} \end{bmatrix}.
\end{aligned} \tag{73}$$

Matrices $K_{sr,3}^*$ and $(1/\epsilon)K_{f2r,3}^*$ are obtained from $T_{4r,3}^{-1} K_{z,3}^{r^*}$. The slow feedback gain $K_{sr,3}$ can be obtained from the eigenvalue assignment problem; that is,

$$\begin{aligned}
\lambda (A_{sr,3}^T - C_{sr,3}^T K_{sr,3}^T) &= \lambda (A_{sr,3} - K_{sr,3} C_{sr,3}) \\
&= \lambda_s^{\text{desired}}
\end{aligned} \tag{74}$$

assuming that the following assumption is satisfied [14].

Assumption 7. The pair $(A_{sr,3}, C_{sr,3})$ is observable.

The fast feedback gain $K_{f2r,3}$ can be obtained from the eigenvalue assignment problem as

$$\lambda (A_{fr,3} - K_{f2r,3} C_{fnewr,3}) = \lambda_f^{\text{desired}}. \tag{75}$$

The following observability assumption is needed [14].

Assumption 8. The pair $(A_{fr,3}, C_{fnewr,3})$ is observable.

The control input in the $\hat{z}_{s,3}$ - $\hat{z}_{fnew2,3}$ coordinates is given by

$$\begin{aligned}
u(t) &= -F_{r,3} \hat{x}(t) = -[F_{12} \quad F_{21}] \begin{bmatrix} \hat{x}_{12}(t) \\ \hat{x}_{21}(t) \end{bmatrix} \\
&= -F_{r,3} K_4 y(t) - [F_{sr,3} \quad F_{fr,3}] \begin{bmatrix} \hat{z}_{s,3}(t) \\ \hat{z}_{fnew2,3}(t) \end{bmatrix},
\end{aligned} \tag{76}$$

$$[F_{11} \quad F_{12} \mid F_{21} \quad F_{22}] = [F_1 \mid F_2]$$

$$= [F_s + F_{f2} P \mid F_{f2}] T_c^{-1}$$

$$\text{so that } [F_{11} \mid F_{12} \mid F_{21} \mid F_{22}] = [F_{11} \mid F_{r,3} \mid F_{22}]$$

with

$$\begin{aligned}
F_{sr,3} &= F_{12} (I_{(n_1-l_1)} - P_{or,3}^T P_{o2r,3} - H_{r,3} P_{o2r,3}) \\
&\quad - F_{21} L (I_{(n_2-l_2)} - P_{or,3}^T P_{o2r,3}) \\
&\quad - \frac{1}{\epsilon} F_{21} (I_{(n_2-l_2)} - \epsilon L_{r,3} H_{r,3}) P_{o2r,3} \\
F_{fr,3} &= F_{12} (P_{or,3}^T + H_{r,3}) - F_{21} L_{r,3} P_{or,3}^T \\
&\quad + \frac{1}{\epsilon} F_{21} (I_{(n_2-l_2)} - \epsilon L_{r,3} H_{r,3}).
\end{aligned} \tag{77}$$

FIGURE 6: Case V: slow and fast observer-based controller design for a singularly perturbed linear system when some parts of the slow variables and some parts of the fast variables are measured.

The corresponding block diagram for the observer driven controller is presented in Figure 6.

5. Numerical Example for Case I

Consider a 4th-order system with the system matrices A , B , and C defined in [1]. The controllability matrix has full rank and therefore the pair (A, B) is controllable. The results obtained using MATLAB are given below. We locate the feedback system slow eigenvalues at $\lambda_{cs}^{\text{desired}} = (-2, -3)$ and the feedback system fast eigenvalues at $\lambda_{cf}^{\text{desired}} = (-7, -8)$ and the reduced-order observer eigenvalues at $\lambda_{robs}^{\text{desired}} = (-50, -60)$, given in the previous numerical example. Following the design procedure, the observer matrices A_z , K_z , K_{12} , F_1 , and F_2 are given as

$$\begin{aligned} A_z &= \begin{bmatrix} -108.9999 & 0.2620 \\ -11034.3511 & -1.0000 \end{bmatrix}, \\ K_{12} &= \begin{bmatrix} 0 & 314.5942 \\ 0 & 31983.6265 \end{bmatrix}, \\ K_z &= \begin{bmatrix} 0 & -259111.1037 \\ 0 & -35033265.2948 \end{bmatrix}, \end{aligned} \quad (78)$$

and the feedback gains are obtained as

$$\begin{aligned} F_1 &= [92.9306 \quad 37.8637], \\ F_2 &= [2.4356 \quad 0.5349]. \end{aligned} \quad (79)$$

6. Numerical Example for Case II

Consider a 4th-order system with the system matrices A , B , and C defined in Section 5. We locate the feedback system slow eigenvalues at $\lambda_{cs}^{\text{desired}} = (-2, -3)$ and the feedback

system fast eigenvalues at $\lambda_{cf}^{\text{desired}} = (-7, -8)$ and the reduced-order observer eigenvalues at $\lambda_{robs}^{\text{desired}} = (-50, -70)$, given in the previous numerical example. Following the design procedure, the observer matrices A_z , K_z , K_{12} , F_1 , and F_2 are given as

$$\begin{aligned} A_z &= \begin{bmatrix} -49.9999 & 0.0000 \\ -0.0000 & -70.0000 \end{bmatrix}, \\ K_{12} &= \begin{bmatrix} 11.0734 & 10.3042 \\ -6.4660 & 6.4112 \end{bmatrix}, \\ K_z &= \begin{bmatrix} -537.2325 & -556.1992 \\ 400.7840 & -440.7330 \end{bmatrix}, \end{aligned} \quad (80)$$

and the feedback gains are

$$\begin{aligned} F_1 &= [92.9306 \quad 37.8637], \\ F_2 &= [2.4356 \quad 0.5349]. \end{aligned} \quad (81)$$

7. Numerical Example for Case III

Consider a 4th-order magnetic tape system from Section 5. We locate the feedback system slow eigenvalues at $\lambda_{cs}^{\text{desired}} = (-2, -3)$ and the feedback system fast eigenvalues at $\lambda_{cf}^{\text{desired}} = (-7, -8)$ and the slow observer eigenvalues at $\lambda_{os}^{\text{desired}} = -50$ and the fast observer eigenvalues at $\lambda_{of}^{\text{desired}} = (-200, -300)$, given in the previous numerical example. Following the design procedure, the completely decoupled slow and fast observers in the z_s - z_{fnew2} coordinates, driven by the system measurements and control inputs, are

$$\begin{aligned} \dot{\hat{z}}_s(t) &= [-50.0000] \hat{z}_s(t) \\ &\quad + [-2.323033338614758] u(t) \\ &\quad + [-6196.4741] y(t). \end{aligned}$$

$$\begin{aligned}
\dot{\hat{z}}_{fnew2}(t) &= \begin{bmatrix} -1693.6295 & 113.2762 \\ -18376.0229 & 1193.6295 \end{bmatrix} \hat{z}_{fnew2}(t) \\
&+ \begin{bmatrix} -0.2499 \\ 0.4666 \end{bmatrix} u(t) \\
&+ \begin{bmatrix} -2859901.4120 \\ -33059040.2923 \end{bmatrix} y(t) \\
u(t) &= -[1761.6414] \hat{z}_s(t) \\
&- [26.9824 \quad 5.1780] \hat{z}_{fnew2}(t)
\end{aligned} \tag{82}$$

The slow and fast controller gains F_{sr2} and F_{fr2} are obtained as

$$\begin{aligned}
F_{sr2} &= [1761.6414], \\
F_{fr2} &= [26.9824 \quad 5.1780].
\end{aligned} \tag{83}$$

8. Numerical Example for Case IV

Consider a 4th-order system with the system matrices A , B , and C defined in Section 5. We locate the feedback system slow eigenvalues at $\lambda_{cs}^{\text{desired}} = (-2, -3)$ and the feedback system fast eigenvalues at $\lambda_{cf}^{\text{desired}} = (-7, -8)$ and the slow observer eigenvalues at $\lambda_{os}^{\text{desired}} = (-50, -60)$ and the fast observer eigenvalues at $\lambda_{of}^{\text{desired}} = -300$, given in the previous numerical example. Following the design procedure from Section 4, the completely decoupled slow and fast observer in the z_s - z_{fnew2} coordinates, driven by the system measurements and control inputs, are

$$\begin{aligned}
\dot{\hat{z}}_{s,2}(t) &= \begin{bmatrix} -0.0000 & -4127.7367 \\ 0.7267 & -110.0000 \end{bmatrix} \hat{z}_{s,2}(t) \\
&+ \begin{bmatrix} 4.7409 \\ -3.9951 \end{bmatrix} u(t) \\
&+ \begin{bmatrix} 113729.6251 \\ 2281.0457 \end{bmatrix} y(t) \\
\dot{\hat{z}}_{fnew2,2}(t) &= [-300.0000] \hat{z}_{fnew2,2}(t) \\
&+ [-779.5149] u(t) \\
&+ [437703.1601] y(t) \\
u(t) &= -[-6530.3242 \quad 91.6868] \hat{z}_s(t) \\
&- [332.8512 \quad -10.9008] \hat{z}_{fnew2}(t).
\end{aligned} \tag{84}$$

The slow and fast controller gains $F_{sr2,2}$ and $F_{fr2,2}$ are obtained as

$$\begin{aligned}
F_{sr2,2} &= [-6530.3242 \quad 91.6868], \\
F_{fr2,2} &= [332.8512 \quad -10.9008].
\end{aligned} \tag{85}$$

9. Numerical Example for Case V

Consider a 4th-order system with the system matrices A , B , and C defined in Section 5. We locate the feedback system slow eigenvalues at $\lambda_{cs}^{\text{desired}} = (-2, -3)$ and the feedback system fast eigenvalues at $\lambda_{cf}^{\text{desired}} = (-7, -8)$ and the slow observer eigenvalues at $\lambda_{os}^{\text{desired}} = -50$ and the fast observer eigenvalues at $\lambda_{of}^{\text{desired}} = (-200, -300)$, given in the previous numerical example. Following the design procedure, the completely decoupled slow and fast observers in the $z_{s,3}$ - $z_{fnew2,3}$ coordinates, driven by the system measurements and control inputs, are

$$\begin{aligned}
\dot{\hat{z}}_{s,3}(t) &= [-50.0000] \hat{z}_s(t) \\
&+ [-2.323033338614758] u(t) \\
&+ [-6196.4741] y(t) \\
\dot{\hat{z}}_{fnew2,3}(t) &= \begin{bmatrix} -1693.6295 & 113.2762 \\ -18376.0229 & 1193.6295 \end{bmatrix} \hat{z}_{fnew2}(t) \\
&+ \begin{bmatrix} -0.2499 \\ 0.4666 \end{bmatrix} u(t) \\
&+ \begin{bmatrix} -2859901.4120 \\ -33059040.2923 \end{bmatrix} y(t) \\
u(t) &= -[-6530.3242 \quad 91.6868] \hat{z}_s(t) \\
&- [332.8512 \quad -10.9008] \hat{z}_{fnew2}(t).
\end{aligned} \tag{86}$$

The slow and fast controller gains $F_{sr2,3}$ and $F_{fr2,3}$ are obtained as

$$\begin{aligned}
F_{sr2,3} &= [-6530.3242 \quad 91.6868], \\
F_{fr2,3} &= [332.8512 \quad -10.9008].
\end{aligned} \tag{87}$$

10. Conclusion

We have designed with high accuracy the reduced-order observer-based controllers for singularly perturbed linear systems. The numerical ill-conditioning of the original problem is removed. We have demonstrated that the full-order singularly perturbed system can be successfully controlled with the state feedback coming from the reduced-order observer-based controllers fully designed on the subsystem levels. The two-stage method is successfully implemented for both the observer and controller designs.

Appendix

It is well known that the rank condition after scalar multiplication is unchanged

$$\text{rank}(\alpha A) = \text{rank}(A). \tag{A.1}$$

In Case (II), the pair $(A_{11}, (1/\epsilon)A_{21})$ is observable, which implies

$$\text{rank} \begin{bmatrix} \frac{1}{\epsilon} A_{21} \\ \frac{1}{\epsilon} A_{21} A_{11} \\ \frac{1}{\epsilon} A_{21} A_{11}^2 \\ \vdots \\ \frac{1}{\epsilon} A_{21} A_{11}^{n-1} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{21} \\ A_{21} A_{11} \\ A_{21} A_{11}^2 \\ \vdots \\ A_{21} A_{11}^{n-1} \end{bmatrix}. \quad (\text{A.2})$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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