Research Article

On the $GI/M/1$ Queue with Vacations and Multiple Service Phases

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This paper considers a $GI/M/1$ queue with vacations and multiple service phases. Whenever the system becomes empty, the server takes a vacation, causing the system to move to vacation phase 0. If the server returns from a vacation to find no customer waiting, another vacation begins. Otherwise, the system jumps from phase 0 to some service phase i with probability q_i , $i = 1, 2, \dots, N$. Using the matrix geometric solution method and semi-Markov process, we obtain the distributions of the stationary system size at both arrival and arbitrary epochs. The distribution of the stationary waiting time of an arbitrary customer is also derived. In addition, we present some performance measures such as mean waiting time of an arbitrary customer, mean length of the type- i cycle, and mean number of customers in the system at the end of phase 0. Finally, some numerical examples are presented.

1. Introduction

Queueing system with vacations has been well studied in the past three decades and widely used to model many problems in computer networks, communication, service systems, and manufacturing systems. Levy and Yechiali [1] were first to investigate the queueing model with vacations. Later, many researchers have made an extensive study of such a model under the assumption that the server takes vacations between two successive busy periods. Excellent surveys on queueing models with vacations can be found in Doshi [2], Takagi [3], Tian and Zhang [4], Ke et al. [5], and so on.

We consider a $GI/M/1$ type vacation queue in which the interarrival times form an independent identically distributed sequence of random variables having a general distribution function. Some literatures on this type of vacation queue have emerged. In 1989, Tian et al. [6] applied the matrix analytic method to analyze a $GI/M/1$ queue with exhaustive service and multiple exponential vacations. Chatterjee and Mukherjee [7] analyzed the same model as that in Tian et al. [6]. But in [7], the vacation times are independently and identically distributed with a general distribution. Chae et al. [8] studied the $GI/M/1$ queue with single vacation and obtained the stochastic decomposition structures of queue

length and the sojourn time. Baba [9] considered a $GI/M/1$ queue with multiple working vacations. Banik et al. [10] provided a study on a finite buffer $GI/M/1$ queue with multiple working vacations. The $GI/M/1$ queue with multiple working vacations and vacation interruption was analyzed by Li et al. [11]. Recently, a $GI/M/1$ queue with single working vacation can be seen in the survey of Li and Tian [12]. Tao et al. [13] analyzed the $GI/M/1$ queue with Bernoulli-schedule-controlled vacation and vacation interruption. Ye and Liu [14] investigated the $GI/M/1$ with single working vacation and vacations.

In this paper, we extend the excellent works of Tian et al. [6] and Li and Liu [15] to a $GI/M/1$ vacation queue with multiple service phases. In [6], the service rate is a constant. In fact, in most of the literature on queueing theory, the service rate is homogeneous; that is, it is a constant. But, it is not reasonable in some cases. In real life, the service rate may change with the change of the environment. Several authors have contributed to the investigation of the queueing systems whose service rates are variable, including Yechiali and Naor [16], Neuts [17], Boxma and Kurkova [18], Cordeiro and Kharoufeh [19], and B. Kim and J. Kim [20]. Most recently, Paz and Yechiali [21] considered an $M/M/1$ queue in a multiphase random environment with disasters. In their

model, the system moves from repair phase to some service environment with a certain probability after being repaired. Jiang et al. [22, 23] extended Paz and Yechiali's work to an $M/G/1$ and $GI/M/1$ queue. In [15], Li and Liu studied a discrete-time $Geo/G/1$ queue with vacations in random environment in which the system jumps from vacation phase to a service phase with a certain probability when vacation phase ends. Inspired by [6, 15], we investigate the $GI/M/1$ vacation queue with multiple service phases. When vacation phase ends, the system switches to a service environment with a certain probability, where the service rate in each service phase may be different.

The queueing system we discuss has potential application in several situations. For example, in manufacturing systems, a flexible manufacturing facility can be considered to be a server, which is mainly used to produce customer-specified products. When there are no customer backorders, the facility keeps on taking vacations. If, on return from a vacation, there is at least one customer order waiting, the facility then resumes service normally. But the service rate may change since the operator may be different.

The rest of this paper is organized as follows. Section 2 presents the model description. In Section 3, we provide the transition analysis for an embedded Markov chain. Section 4 is devoted to derive the stationary system size distribution at arrival epochs. Section 5 gives the stationary system size distribution at arbitrary epochs. Sections 6, 7, and 8 present the waiting time distribution, mean number of customers in the system at the end of phase 0, and results for cycle analysis, respectively. Section 9 provides some numerical examples and Section 10 concludes the paper.

2. Model Description

In this paper, we provide a detailed analysis of the $GI/M/1$ queue with vacations and multiple service phases. Our queueing system is described as follows. Whenever the system becomes empty, the server begins a vacation of random length V ; that is to say, the system moves to vacation phase 0. If the server returns from a vacation to find no customer waiting, another vacation begins. Otherwise, the system jumps from phase 0 to some service phase i with probability q_i , $i = 1, 2, \dots, N$, where $q_i > 0$ and $\sum_{i=1}^N q_i = 1$. Let τ_ℓ represent the arrival epoch of ℓ th customer with $\tau_0 = 0$. The interarrival times $\{A_\ell, \ell \geq 1\}$ are independent and identically distributed (i.i.d.) with a general distribution function, denoted by $A(t)$, a mean, denoted by $1/\lambda$, and a Laplace Stieltjes transform (LST), denoted by $\bar{A}(s)$. When the system is in phase 0, the server stops serving customers, and the vacation times are i.i.d. exponentially distributed with rate θ . When the system is in service phase i , the service times are i.i.d. exponential random variables with rate of μ_i , $i = 1, 2, \dots, N$. Note that the system can not directly move from one service phase to another service phase. That is, if the system becomes empty, it must move to phase 0 first. In addition, once the system moves to service phase i , $i = 1, 2, \dots, N$, the customers are served with the fixed rate μ_i . Finally, the interarrival times, service times, and vacation times are assumed to be mutually independent, and

customers are served according to the discipline of first-come-first-served.

3. Embedded Markov Chain

Let $L(t)$ denote the number of customers in the system at time t , and let $L_\ell = L(\tau_\ell - 0)$ represent the number of customers before the ℓ th arrival. Define

$$J_\ell = \begin{cases} 0, & \text{the } \ell\text{th arrival occurs during phase 0,} \\ i, & \text{the } \ell\text{th arrival occurs during phase } i, \end{cases} \quad (1)$$

where $i = 1, 2, \dots, N$.

Because both vacation and service times have memory-less property, the process $\{(L_\ell, J_\ell), \ell \geq 1\}$ is an embedded Markov chain with the state space

$$\Omega = \{(m, 0), m \geq 0\} \cup \{(m, i), m \geq 1, i = 1, 2, \dots, N\}. \quad (2)$$

In order to express the transition matrix of $\{(L_\ell, J_\ell), \ell \geq 1\}$, let us define

$$P_{(n,i),(m,j)} = P\{L_{\ell+1} = m, J_{\ell+1} = j \mid L_\ell = n, J_\ell = i\},$$

$$b_{k,i} = \int_0^\infty \frac{(\mu_i t)^k}{k!} e^{-\mu_i t} dA(t), \quad k = 0, 1, 2, \dots, i = 1, 2, \dots, N, \quad (3)$$

$$v_{k,i} = \int_0^\infty \left\{ \int_0^t \frac{[\mu_i(t-x)]^k}{k!} e^{-\mu_i(t-x)} \theta e^{-\theta x} dx \right\} dA(t), \quad k = 0, 1, 2, \dots, i = 1, 2, \dots, N.$$

Now, we consider the transition probabilities of (L_ℓ, J_ℓ) . First, considering the case that the system is in service phase i , $i = 1, 2, \dots, N$, the transition from (n, i) to (m, i) occurs if there are $n+1-m$ service completions during an interarrival time. Then, we have

$$P_{(n,i),(m,i)} = \begin{cases} \int_0^\infty \frac{(\mu_i t)^{n+1-m}}{(n+1-m)!} e^{-\mu_i t} dA(t) = b_{n+1-m,i}, & 1 \leq m \leq n+1, \\ 0, & m \geq n+2. \end{cases} \quad (4)$$

The transition from (n, i) to $(0, 0)$ arises if $n+1$ customers are served and then the server enters into the vacation phase. From (4), we obtain

$$P_{(n,i),(0,0)} = \int_0^\infty \left\{ \int_0^t \frac{\mu_i (\mu_i x)^n}{n!} e^{-\mu_i x} dx \right\} dA(t) = 1 - \sum_{k=0}^n b_{k,i}, \quad n = 1, 2, \dots \quad (5)$$

Second, the transition from $(n, 0)$ to $(m, 0)$ is possible, only if $m = n+1$. This transition occurs if the remaining vacation time is greater than an interarrival time. Then, we have

$$P_{(n,0),(n+1,0)} = \int_0^\infty e^{-\theta t} dA(t) = \bar{A}(\theta). \quad (6)$$

Third, the transition from $(n, 0)$ to (m, i) occurs if there are $n + 1 - m$ service completions between the end of a vacation and a customer arriving. Therefore we have

$$P_{(n,0),(m,i)} = q_i \int_0^\infty \left\{ \int_0^t \frac{[\mu_i(t-x)]^{n+1-m}}{(n+1-m)!} e^{-\mu_i(t-x)} \theta e^{-\theta x} dx \right\} dA(t) \quad (7)$$

$$= q_i v_{n+1-m,i},$$

for $m = 1, 2, \dots, n+1$, $i = 1, 2, \dots, N$. For $m = n+2, n+3, \dots$, $i = 1, 2, \dots, N$, we have

$$P_{(n,0),(m,i)} = 0. \quad (8)$$

From (6)–(8), we get

$$P_{(n,0),(0,0)} = 1 - \bar{A}(\theta) - \sum_{i=1}^N q_i \sum_{k=0}^n v_{k,i}. \quad (9)$$

If we sequence the states in the lexicographic order, $(0, 0), ((k, 0), (k, 1), \dots, (k, N))$, $k = 1, 2, \dots$, the transition probability matrix of $\{(L_\ell, J_\ell), \ell \geq 1\}$ can be written as the Block-Jacobi matrix

$$P = \begin{pmatrix} B_{00} & A_{01} & 0 & 0 & \cdots \\ B_{10} & A_1 & A_0 & 0 & \cdots \\ B_{20} & A_2 & A_1 & A_0 & \cdots \\ B_{30} & A_3 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (10)$$

where

$$B_{00} = 1 - \bar{A}(\theta) - \sum_{i=1}^N q_i v_{0,i},$$

$$A_{01} = (\bar{A}(\theta), q_1 v_{0,1}, q_2 v_{0,2}, \dots, q_N v_{0,N}),$$

$$B_{k0} = \left(1 - \bar{A}(\theta) - \sum_{i=1}^N q_i \sum_{h=0}^k v_{h,i}, 1 - \sum_{h=0}^k b_{h,1}, 1 - \sum_{h=0}^k b_{h,2}, \dots, 1 - \sum_{h=0}^k b_{h,N} \right)^T,$$

$$A_0 = \begin{pmatrix} \bar{A}(\theta) & q_1 v_{0,1} & q_2 v_{0,2} & \cdots & q_N v_{0,N} \\ 0 & b_{0,1} & 0 & 0 & 0 \\ 0 & 0 & b_{0,2} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & b_{0,N} \end{pmatrix},$$

$$A_k = \begin{pmatrix} 0 & q_1 v_{k,1} & q_2 v_{k,2} & \cdots & q_N v_{k,N} \\ 0 & b_{k,1} & 0 & 0 & 0 \\ 0 & 0 & b_{k,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & b_{k,N} \end{pmatrix}, \quad k \geq 1. \quad (11)$$

It is clear that P is a stochastic matrix and the process $\{(L_\ell, J_\ell), \ell \geq 1\}$ is irreducible and aperiodic.

4. Stationary Distribution at Arrival Epochs

In this section, we will derive the stationary distribution of $\{(L_\ell, J_\ell), \ell \geq 1\}$ at the arrival epochs by using the matrix geometric approach. First, we introduce the following lemmas.

Lemma 1. *The Markov chain $\{(L_\ell, J_\ell), \ell \geq 1\}$ is stable if and only if $\rho_i = \lambda/\mu_i < 1$ and $\theta > 0$, $i = 1, 2, \dots, N$.*

Proof. It is easy to see that our system acts as the classical GI/M/1 queue with arrival rate λ and service rate μ_i during service phase i , $i = 1, 2, \dots, N$. So as long as $\rho_i = \lambda/\mu_i < 1$ and $\theta > 0$, $i = 1, 2, \dots, N$, the Markov chain $\{(L_\ell, J_\ell), \ell \geq 1\}$ is stable. That is, $\rho_i = \lambda/\mu_i < 1$ and $\theta > 0$, $i = 1, 2, \dots, N$, are a sufficient condition for stability of $\{(L_\ell, J_\ell), \ell \geq 1\}$. On the other hand, obviously, if $\theta = 0$, the Markov chain $\{(L_\ell, J_\ell), \ell \geq 1\}$ will not be stable, and if there is an i , $\rho_i = \lambda/\mu_i \geq 1$, $i = 1, 2, \dots, N$, since $q_i > 0$, then when the system moves to service phase i , the Markov chain $\{(L_\ell, J_\ell), \ell \geq 1\}$ will not be stable, either. Thus, $\rho_i = \lambda/\mu_i < 1$ and $\theta > 0$, $i = 1, 2, \dots, N$, are necessary for stability of $\{(L_\ell, J_\ell), \ell \geq 1\}$. $\square$

Lemma 2. *If $\rho_i = \lambda/\mu_i < 1$, then the equation $z = \bar{A}(\mu_i(1-z))$ has a unique root $r_{i,i}$ in the range $0 < z < 1$, $i = 1, 2, \dots, N$.*

Proof. We set $\varphi(z) = z - \bar{A}(\mu_i(1-z))$. Evidently, $\varphi(0) = -\bar{A}(\mu_i) < 0$, and $\varphi(1) = 1 - \bar{A}(0) = 0$. Further, for $0 < z < 1$, we have

$$\varphi''(z) = -\mu_i^2 \int_0^\infty t^2 e^{-\mu_i(1-z)t} dA(t) < 0. \quad (12)$$

Since $\varphi'(z) = 1 - \mu_i \int_0^\infty t e^{-\mu_i(1-z)t} dA(t)$, $\rho_i = \lambda/\mu_i < 1$, then $\varphi'(1) = 1 - 1/\rho_i < 0$. Thus, the equation $z = \bar{A}(\mu_i(1-z))$ has a unique root in the range $0 < z < 1$, denoted by $r_{i,i}$. $\square$

Lemma 3. *If $\rho_i = \lambda/\mu_i < 1$, $i = 1, 2, \dots, N$, then the matrix equation $R = \sum_{k=0}^\infty R^k A_k$ has the minimal nonnegative solution*

$$R = \begin{pmatrix} r_{0,0} & r_{0,1} & r_{0,2} & \cdots & r_{0,N} \\ 0 & r_{1,1} & 0 & 0 & 0 \\ 0 & 0 & r_{2,2} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{N,N} \end{pmatrix}, \quad (13)$$

where $r_{0,0} = \tilde{A}(\theta)$, $r_{i,i}$ is the unique root of equation $z = \tilde{A}(\mu_i(1-z))$ in the range $0 < z < 1$, $i = 1, 2, \dots, N$, and

$$r_{0,i} = \frac{q_i \theta}{\theta - \mu_i (1 - \tilde{A}(\theta))} (r_{i,i} - \tilde{A}(\theta)), \quad (14)$$

$$i = 1, 2, \dots, N.$$

Proof. Since each A_k is a special upper diagonal matrix, if there exists the solution R of the matrix equation $R =$

$\sum_{k=0}^{\infty} R^k A_k$, then R can be assumed having the same upper diagonal structure as

$$R = \begin{pmatrix} r_{0,0} & r_{0,1} & r_{0,2} & \cdots & r_{0,N} \\ 0 & r_{1,1} & 0 & 0 & 0 \\ 0 & 0 & r_{2,2} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{N,N} \end{pmatrix}. \quad (15)$$

Then

$$R^k = \begin{pmatrix} r_{0,0}^k & r_{0,1} \sum_{j=0}^{k-1} r_{0,0}^j r_{1,1}^{k-1-j} & r_{0,2} \sum_{j=0}^{k-1} r_{0,0}^j r_{2,2}^{k-1-j} & \cdots & r_{0,N} \sum_{j=0}^{k-1} r_{0,0}^j r_{N,N}^{k-1-j} \\ 0 & r_{1,1}^k & 0 & 0 & 0 \\ 0 & 0 & r_{2,2}^k & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{N,N}^k \end{pmatrix}, \quad k \geq 1. \quad (16)$$

Substituting R^k into the matrix equation, we have

$$r_{0,0} = \tilde{A}(\theta),$$

$$r_{i,i} = \sum_{k=0}^{\infty} r_{i,i}^k b_{k,i} = \tilde{A}(\mu_i(1 - r_{i,i})), \quad i \geq 1, \quad (17)$$

$$r_{0,i} = q_i \sum_{k=0}^{\infty} r_{0,0}^k v_{k,i} + r_{0,i} \sum_{k=1}^{\infty} b_{k,i} \sum_{j=0}^{k-1} r_{i,i}^j r_{0,0}^{k-1-j},$$

$$i = 1, 2, \dots, N.$$

Thus, $r_{0,0} = \tilde{A}(\theta)$, and from Lemma 2, $r_{i,i}$ is the unique root of equation $z = \tilde{A}(\mu_i(1-z))$ in the range $0 < z < 1$, $i = 1, 2, \dots, N$. Substituting $r_{0,0} = \tilde{A}(\theta)$ in the third equation of (17), we have

$$q_i \sum_{k=0}^{\infty} r_{0,0}^k v_{k,i} = q_i \sum_{k=0}^{\infty} [\tilde{A}(\theta)]^k v_{k,i}$$

$$= q_i \int_0^{\infty} \left\{ \int_0^t \sum_{k=0}^{\infty} [\tilde{A}(\theta)]^k \frac{[\mu_i(t-x)]^k}{k!} e^{-\mu_i(t-x)} \theta e^{-\theta x} dx \right\} dA(t)$$

$$= q_i \int_0^{\infty} \left\{ \int_0^t e^{-\mu_i(t-x)[1-\tilde{A}(\theta)]-\theta x} \theta dx \right\} dA(t)$$

$$= \frac{q_i \theta}{\theta - \mu_i (1 - \tilde{A}(\theta))} [\tilde{A}(\mu_i(1 - \tilde{A}(\theta))) - \tilde{A}(\theta)],$$

$$1 - \sum_{k=1}^{\infty} b_{k,i} \sum_{j=0}^{k-1} r_{i,i}^j [\tilde{A}(\theta)]^{k-1-j}$$

$$= 1 - \frac{1}{r_{i,i} - \tilde{A}(\theta)} \sum_{k=1}^{\infty} b_{k,i} (r_{i,i}^k - (\tilde{A}(\theta))^k)$$

$$= 1 - \frac{1}{r_{i,i} - \tilde{A}(\theta)} [r_{i,i} - \tilde{A}(\mu_i(1 - \tilde{A}(\theta)))]$$

$$= \frac{1}{r_{i,i} - \tilde{A}(\theta)} [\tilde{A}(\mu_i(1 - \tilde{A}(\theta))) - \tilde{A}(\theta)]. \quad (18)$$

By substituting (18) and in the third equation of (17), we can obtain

$$r_{0,i} = \frac{q_i \theta}{\theta - \mu_i (1 - \tilde{A}(\theta))} (r_{i,i} - \tilde{A}(\theta)), \quad (19)$$

$$i = 1, 2, \dots, N.$$

Then the proof is completed. $\square$

Let $\pi = (\pi_0, \pi_1, \pi_2, \dots, \pi_k, \dots)$ be the stationary probability vector of $\{(L_\ell, J_\ell), \ell \geq 1\}$, where

$$\pi_0 = \pi_{0,0} = \lim_{\ell \rightarrow \infty} P\{L_\ell = 0, J_\ell = 0\},$$

$$\pi_k = (\pi_{k,0}, \pi_{k,1}, \dots, \pi_{k,N}), \quad k \geq 1,$$

$$\pi_{k,i} = \lim_{\ell \rightarrow \infty} P\{L_\ell = k, J_\ell = i\}, \quad (20)$$

$$k \geq 1, \quad i = 0, 1, 2, \dots, N.$$

Theorem 4. If $\rho_i = \lambda/\mu_i < 1$ and $\theta > 0$, $i = 1, 2, \dots, N$, the stationary probability vector $(\pi_{0,0}, \pi_{1,0}, \pi_{1,1}, \dots, \pi_{1,N})$ exists and is given by

$$\begin{aligned} & (\pi_{0,0}, \pi_{1,0}, \pi_{1,1}, \dots, \pi_{1,N}) B[R] \\ &= (\pi_{0,0}, \pi_{1,0}, \pi_{1,1}, \dots, \pi_{1,N}), \end{aligned} \quad (21)$$

$$\pi_{0,0} + (\pi_{1,0}, \pi_{1,1}, \dots, \pi_{1,N}) (\mathbf{I} - R)^{-1} \mathbf{e} = 1,$$

and for $k \geq 2$, π_k can be obtained by

$$\pi_k = \pi_1 R^{k-1}, \quad (22)$$

where $\mathbf{e}$ is a column vector with all its components equal to one, $\mathbf{I}$ is the identity matrix, and

$$B[R] = \begin{bmatrix} \Delta_0 & \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_N \\ \gamma_0 & 0 & \delta_1 & \delta_2 & \cdots & \delta_N \\ \gamma_1 & 0 & \Delta_1 & 0 & \cdots & 0 \\ \gamma_2 & 0 & 0 & \Delta_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_N & 0 & 0 & 0 & \cdots & \Delta_N \end{bmatrix} \quad (23)$$

with

$$\begin{aligned} \Delta_0 &= 1 - r_{0,0} - \sum_{i=1}^N \frac{q_i \theta}{\theta - \mu_i} (\bar{A}(\mu_i) - r_{0,0}), \\ \beta_0 &= r_{0,0}, \\ \beta_i &= \frac{q_i \theta}{\theta - \mu_i} (\bar{A}(\mu_i) - r_{0,0}), \quad i = 1, 2, \dots, N, \\ \delta_i &= \frac{q_i}{r_{0,0}} \left\{ \frac{\theta}{\theta - \mu_i (1 - r_{0,0})} [\bar{A}(\mu_i (1 - r_{0,0})) - r_{0,0}] \right. \\ &\quad \left. - \frac{\theta}{\theta - \mu_i} [\bar{A}(\mu_i) - r_{0,0}] \right\} \\ &\quad + \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left\{ \frac{1}{r_{0,0}} [\bar{A}(\mu_i (1 - r_{0,0})) - \bar{A}(\mu_i)] \right. \\ &\quad \left. - \frac{1}{r_{i,i}} [r_{i,i} - \bar{A}(\mu_i)] \right\}, \quad i = 1, 2, \dots, N, \\ \gamma_0 &= 1 - \sum_{i=1}^N \delta_i, \end{aligned}$$

$$\gamma_i = \frac{\bar{A}(\mu_i)}{r_{i,i}}, \quad i = 1, 2, \dots, N,$$

$$\Delta_i = 1 - \frac{\bar{A}(\mu_i)}{r_{i,i}} = 1 - \gamma_i, \quad i = 1, 2, \dots, N. \quad (24)$$

Proof. Based on Theorem 1.5.1 of Neuts [24], if the Markov chain $\{(L_\ell, J_\ell), \ell \geq 1\}$ is positive recurrent, then the matrix

$$B[R] = \begin{pmatrix} B_{00} & A_{01} \\ \sum_{k=1}^{\infty} R^{k-1} B_{k0} & \sum_{k=1}^{\infty} R^{k-1} A_k \end{pmatrix} \quad (25)$$

has positive left invariant vector, and $(\pi_{0,0}, \pi_{1,0}, \pi_{1,1}, \dots, \pi_{1,N})$ is given by the positive left invariant vector and the normalizing condition

$$\pi_{0,0} + (\pi_{1,0}, \pi_{1,1}, \dots, \pi_{1,N}) (\mathbf{I} - R)^{-1} \mathbf{e} = 1, \quad (26)$$

where $\mathbf{e}$ is a column vector with all its components equal to one and $\mathbf{I}$ is the identity matrix. From Lemma 1, if $\rho_i = \lambda/\mu_i < 1$ and $\theta > 0$, $i = 1, 2, \dots, N$, then $\{(L_\ell, J_\ell), \ell \geq 1\}$ is positive recurrent; thus we have (21) and (22).

For $k \geq 2$, using equation (1.5.4) of Neuts [24], we obtain

$$\pi_k = \pi_1 R^{k-1}. \quad (27)$$

Now, we are ready to derive the expression of $B[R]$ in the following.

Substituting the expressions for B_{00} , A_{01} , R , A_k , and B_{k0} in (25), we have

$$B[R] = \begin{bmatrix} \Delta_0 & \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_N \\ \gamma_0 & 0 & \delta_1 & \delta_2 & \cdots & \delta_N \\ \gamma_1 & 0 & \Delta_1 & 0 & \cdots & 0 \\ \gamma_2 & 0 & 0 & \Delta_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_N & 0 & 0 & 0 & \cdots & \Delta_N \end{bmatrix}, \quad (28)$$

where

$$\Delta_0 = B_{00} = 1 - \bar{A}(\theta) - \sum_{i=1}^N q_i v_{0,i},$$

$$(\beta_0, \beta_1, \dots, \beta_N) = A_{01} = (\bar{A}(\theta), q_1 v_{0,1}, q_2 v_{0,2}, \dots, q_N v_{0,N}),$$

$$\begin{aligned}
(\gamma_0, \gamma_1, \dots, \gamma_N)^T &= \sum_{k=1}^{\infty} R^{k-1} B_{k0} = \begin{pmatrix} \sum_{k=1}^{\infty} \left[r_{0,0}^{k-1} \left(1 - \widetilde{A}(\theta) - \sum_{i=1}^N q_i \sum_{h=0}^k v_{h,i} \right) + \sum_{i=1}^N r_{0,i} \sum_{j=0}^{k-2} r_{0,0}^j r_{i,i}^{k-2-j} \left(1 - \sum_{h=0}^k b_{h,i} \right) \right] \\ \sum_{k=1}^{\infty} r_{1,1}^{k-1} \left(1 - \sum_{h=0}^k b_{h,1} \right) \\ \vdots \\ \sum_{k=1}^{\infty} r_{N,N}^{k-1} \left(1 - \sum_{h=0}^k b_{h,N} \right) \end{pmatrix}, \\
\begin{bmatrix} 0 & \delta_1 & \cdots & \delta_N \\ 0 & \Delta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_N \end{bmatrix} &= \sum_{k=1}^{\infty} R^{k-1} A_k = \sum_{k=1}^{\infty} \begin{bmatrix} 0 & q_1 r_{0,0}^{k-1} v_{k,1} + r_{0,1} b_{k,1} \sum_{j=0}^{k-2} r_{0,0}^j r_{1,1}^{k-2-j} & \cdots & q_N r_{0,0}^{k-1} v_{k,N} + r_{0,N} b_{k,N} \sum_{j=0}^{k-2} r_{0,0}^j r_{N,N}^{k-2-j} \\ 0 & r_{1,1}^{k-1} b_{k,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{N,N}^{k-1} b_{k,N} \end{bmatrix}.
\end{aligned} \tag{29}$$

For Δ_0 , we first have

$$\begin{aligned}
v_{0,i} &= \int_0^{\infty} \left[\int_0^t e^{-\mu_i(t-x)} \theta e^{-\theta x} dx \right] dA(t) \\
&= \frac{\theta}{\theta - \mu_i} [\widetilde{A}(\mu_i) - \widetilde{A}(\theta)], \quad i = 1, 2, \dots, N.
\end{aligned} \tag{30}$$

From $r_{0,0} = \widetilde{A}(\theta)$ (see Lemma 3), then Δ_0 can be obtained by

$$\begin{aligned}
\Delta_0 &= 1 - \widetilde{A}(\theta) - \sum_{i=1}^N q_i v_{0,i} \\
&= 1 - r_{0,0} - \sum_{i=1}^N \frac{q_i \theta}{\theta - \mu_i} (\widetilde{A}(\mu_i) - r_{0,0}).
\end{aligned} \tag{31}$$

Obviously, we have $\beta_0 = \widetilde{A}(\theta) = r_{0,0}$, and from (30), we obtain

$$\beta_i = q_i v_{0,i} = \frac{q_i \theta}{\theta - \mu_i} (\widetilde{A}(\mu_i) - r_{0,0}), \quad i = 1, 2, \dots, N. \tag{32}$$

For δ_i , $i = 1, 2, \dots, N$, we have

$$\begin{aligned}
\delta_i &= \sum_{k=1}^{\infty} \left(q_i r_{0,0}^{k-1} v_{k,i} + r_{0,i} b_{k,i} \sum_{j=0}^{k-2} r_{0,0}^j r_{i,i}^{k-2-j} \right) \\
&= q_i \sum_{k=1}^{\infty} r_{0,0}^{k-1} \int_0^{\infty} \left[\int_0^t \frac{(\mu_i(t-x))^k}{k!} e^{-\mu_i(t-x)} \theta e^{-\theta x} dx \right] dA(t) \\
&\quad + r_{0,i} \sum_{k=1}^{\infty} b_{k,i} \frac{r_{0,0}^{k-1} - r_{i,i}^{k-1}}{r_{0,0} - r_{i,i}} = \frac{q_i}{r_{0,0}}
\end{aligned}$$

$$\begin{aligned}
&\cdot \int_0^{\infty} \left[\int_0^t (e^{\mu_i r_{0,0}(t-x)} - 1) e^{-\mu_i(t-x)} \theta e^{-\theta x} dx \right] dA(t) \\
&\quad + \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left\{ \int_0^{\infty} \left[\sum_{k=1}^{\infty} \frac{(\mu_i r_{0,0} t)^k}{r_{0,0} k!} \right] e^{-\mu_i t} dA(t) \right. \\
&\quad \left. - \int_0^{\infty} \left[\sum_{k=1}^{\infty} \frac{(\mu_i r_{i,i} t)^k}{r_{i,i} k!} \right] e^{-\mu_i t} dA(t) \right\} \\
&= \frac{q_i}{r_{0,0}} \left\{ \frac{\theta}{\theta - \mu_i (1 - r_{0,0})} [\widetilde{A}(\mu_i (1 - r_{0,0})) - \widetilde{A}(\theta)] \right. \\
&\quad \left. - \frac{\theta}{\theta - \mu_i} [\widetilde{A}(\mu_i) - \widetilde{A}(\theta)] \right\} \\
&\quad + \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left\{ \frac{1}{r_{0,0}} [\widetilde{A}(\mu_i (1 - r_{0,0})) - \widetilde{A}(\mu_i)] \right. \\
&\quad \left. - \frac{1}{r_{i,i}} [r_{i,i} - \widetilde{A}(\mu_i)] \right\}.
\end{aligned} \tag{33}$$

It is easy to obtain the expressions for Δ_i , $i = 1, 2, \dots, N$,

$$\begin{aligned}
\Delta_i &= \sum_{k=1}^{\infty} r_{i,i}^{k-1} b_{k,i} = \sum_{k=1}^{\infty} r_{i,i}^{k-1} \int_0^{\infty} \frac{(\mu_i t)^k}{k!} e^{-\mu_i t} dA(t) \\
&= \frac{1}{r_{i,i}} [\widetilde{A}(\mu_i (1 - r_{i,i})) - \widetilde{A}(\mu_i)] \\
&= \frac{1}{r_{i,i}} (r_{i,i} - \widetilde{A}(\mu_i)) = 1 - \frac{\widetilde{A}(\mu_i)}{r_{i,i}}.
\end{aligned} \tag{34}$$

To compute the expression for δ_0 , we first have

$$\begin{aligned}
& \sum_{k=1}^{\infty} \left[r_{0,0}^{k-1} \left(1 - \widetilde{A}(\theta) - \sum_{i=1}^N q_i \sum_{h=0}^k v_{h,i} \right) \right] = \frac{1}{1-r_{0,0}} - \frac{\widetilde{A}(\theta)}{1-r_{0,0}} \\
& \quad - \sum_{i=1}^N q_i \sum_{h=0}^{\infty} \sum_{k=h}^{\infty} r_{0,0}^{k-1} \int_0^{\infty} \left\{ \int_0^t \frac{[\mu_i(t-x)]^h}{h!} e^{-\mu_i(t-x)} \theta e^{-\theta x} dx \right\} dA(t) \\
& = \frac{1}{1-r_{0,0}} - \frac{\widetilde{A}(\theta)}{1-r_{0,0}} \\
& \quad - \sum_{i=1}^N \frac{q_i}{(1-r_{0,0}) r_{0,0}} \int_0^{\infty} e^{-\mu_i(1-r_{0,0})t} \left\{ \int_0^t \theta e^{-[\theta-\mu_i(1-r_{0,0})]x} dx \right\} dA(t) \\
& \quad + \sum_{i=1}^N \frac{q_i}{(1-r_{0,0}) r_{0,0}} \int_0^{\infty} e^{-\mu_i t} \left\{ \int_0^t \theta e^{-(\theta-\mu_i)x} dx \right\} dA(t) \\
& \quad - \sum_{i=1}^N \frac{q_i}{(1-r_{0,0})} \int_0^{\infty} e^{-\mu_i t} \left\{ \int_0^t \theta e^{-(\theta-\mu_i)x} dx \right\} dA(t) \\
& = \frac{1}{1-r_{0,0}} - \frac{\widetilde{A}(\theta)}{1-r_{0,0}} \\
& \quad - \sum_{i=1}^N \frac{q_i \theta}{(1-r_{0,0}) r_{0,0} [\theta - \mu_i(1-r_{0,0})]} [\widetilde{A}(\mu_i(1-r_{0,0})) - \widetilde{A}(\theta)] \\
& \quad + \sum_{i=1}^N \frac{q_i \theta}{r_{0,0} (\theta - \mu_i)} [\widetilde{A}(\mu_i) - \widetilde{A}(\theta)], \\
& \sum_{k=1}^{\infty} \left[\sum_{i=1}^N r_{0,i} \sum_{j=0}^{k-2} r_{0,0}^j r_{i,i}^{k-2-j} \left(1 - \sum_{h=0}^k b_{h,i} \right) \right] = \sum_{k=1}^{\infty} \sum_{i=1}^N r_{0,i} \frac{r_{0,0}^{k-1} - r_{i,i}^{k-1}}{r_{0,0} - r_{i,i}} \left(1 - \sum_{h=0}^k b_{h,i} \right) \\
& = \sum_{i=1}^N \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left(\frac{1}{1-r_{0,0}} - \frac{1}{1-r_{i,i}} - \sum_{k=1}^{\infty} r_{0,0}^{k-1} \sum_{h=0}^k b_{h,i} + \sum_{k=1}^{\infty} r_{i,i}^{k-1} \sum_{h=0}^k b_{h,i} \right) \\
& = \sum_{i=1}^N \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left(\frac{1}{1-r_{0,0}} - \frac{1}{1-r_{i,i}} \right) \\
& \quad - \sum_{i=1}^N \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left\{ \frac{1}{1-r_{0,0}} \int_0^{\infty} \left[\sum_{h=1}^{\infty} \frac{(\mu_i r_{0,0} t)^h}{r_{0,0} h!} + 1 \right] e^{-\mu_i t} dA(t) \right\} \\
& \quad + \sum_{i=1}^N \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left\{ \frac{1}{1-r_{i,i}} \int_0^{\infty} \left[\sum_{h=1}^{\infty} \frac{(\mu_i r_{i,i} t)^h}{r_{i,i} h!} + 1 \right] e^{-\mu_i t} dA(t) \right\} \\
& = \sum_{i=1}^N \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left(\frac{1}{1-r_{0,0}} - \frac{1}{1-r_{i,i}} \right) \\
& \quad - \sum_{i=1}^N \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left\{ \frac{1}{(1-r_{0,0}) r_{0,0}} [\widetilde{A}(\mu_i(1-r_{0,0})) - \widetilde{A}(\mu_i)] + \frac{1}{1-r_{0,0}} \widetilde{A}(\mu_i) \right\} \\
& \quad + \sum_{i=1}^N \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left\{ \frac{1}{(1-r_{i,i}) r_{i,i}} [\widetilde{A}(\mu_i(1-r_{i,i})) - \widetilde{A}(\mu_i)] + \frac{1}{1-r_{i,i}} \widetilde{A}(\mu_i) \right\} \\
& = \sum_{i=1}^N \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left\{ \frac{1}{1-r_{0,0}} + \left(\frac{1}{r_{0,0}} - \frac{1}{r_{i,i}} \right) \widetilde{A}(\mu_i) - \frac{1}{(1-r_{0,0}) r_{0,0}} \widetilde{A}(\mu_i(1-r_{0,0})) \right\}.
\end{aligned} \tag{35}$$

Therefore, from (14),

$$\begin{aligned}
 \delta_0 &= \frac{1}{1-r_{0,0}} - \frac{\bar{A}(\theta)}{1-r_{0,0}} \\
 &- \sum_{i=1}^N \frac{q_i \theta}{(1-r_{0,0}) r_{0,0} [\theta - \mu_i (1-r_{0,0})]} [\bar{A}(\mu_i (1-r_{0,0})) \\
 &- \bar{A}(\theta)] + \sum_{i=1}^N \frac{q_i \theta}{r_{0,0} (\theta - \mu_i)} [\bar{A}(\mu_i) - \bar{A}(\theta)] \\
 &+ \sum_{i=1}^N \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left\{ \frac{1}{1-r_{0,0}} + \left(\frac{1}{r_{0,0}} - \frac{1}{r_{i,i}} \right) \bar{A}(\mu_i) \right. \\
 &- \left. \frac{1}{(1-r_{0,0}) r_{0,0}} \bar{A}(\mu_i (1-r_{0,0})) \right\} = 1 \\
 &- \sum_{i=1}^N \frac{r_{0,i}}{r_{0,0} - r_{i,i}} \left\{ \frac{1}{r_{0,0}} [\bar{A}(\mu_i (1-r_{0,0})) - \bar{A}(\mu_i)] \right. \\
 &+ \left. \frac{1}{r_{i,i}} \bar{A}(\mu_i) - 1 \right\} + \sum_{i=1}^N \frac{q_i \theta}{r_{0,0} (\theta - \mu_i)} [\bar{A}(\mu_i) - \bar{A}(\theta)] \\
 &- \sum_{i=1}^N \frac{q_i \theta}{r_{0,0} [\theta - \mu_i (1-r_{0,0})]} [\bar{A}(\mu_i (1-r_{0,0})) - \bar{A}(\theta)] \\
 &= 1 - \sum_{i=1}^N \delta_i.
 \end{aligned} \tag{36}$$

Finally, the expressions for γ_i , $i = 1, 2, \dots, N$, can be easily obtained by

$$\begin{aligned}
 \gamma_i &= \sum_{k=1}^{\infty} r_{i,i}^{k-1} \left(1 - \sum_{h=0}^k b_{h,i} \right) \\
 &= \frac{1}{1-r_{i,i}} - \frac{1}{(1-r_{i,i}) r_{i,i}} \int_0^{\infty} \sum_{h=1}^{\infty} \frac{(\mu_i r_{i,i} t)^h}{h!} dA(t) \\
 &- \frac{1}{1-r_{i,i}} \bar{A}(\mu_i) \\
 &= \frac{1}{1-r_{i,i}} - \frac{1}{(1-r_{i,i}) r_{i,i}} [\bar{A}(\mu_i (1-r_{i,i})) - \bar{A}(\mu_i)] \\
 &- \frac{1}{1-r_{i,i}} \bar{A}(\mu_i) = \frac{\bar{A}(\mu_i)}{r_{i,i}}.
 \end{aligned} \tag{37}$$

Then the proof is completed. $\square$

Remark 5. If $N = 1$, let $\mu_1 = \mu$; we have

$$R = \begin{pmatrix} r_{0,0} & r_{0,1} \\ 0 & r_{1,1} \end{pmatrix}, \tag{38}$$

where $r_{0,0} = \bar{A}(\theta)$, $r_{1,1}$ is the unique root in the range $0 < z < 1$ of equation $z = \bar{A}(\mu(1-z))$, and $r_{0,1} = (\theta/(\theta - \mu(1-r_{0,0}))) (r_{1,1} - r_{0,0})$, which are derived by Tian et al. [6].

Remark 6. If $N = 1$, let $\mu_1 = \mu$; then

$$B[R] = \begin{bmatrix} \Delta_0 & \beta_0 & \beta_1 \\ \gamma_0 & 0 & \delta_1 \\ \gamma_1 & 0 & \Delta_1 \end{bmatrix}, \tag{39}$$

where

$$\begin{aligned}
 \Delta_0 &= 1 - r_{0,0} - \frac{\theta}{\theta - \mu} (\bar{A}(\mu) - r_{0,0}), \\
 \beta_0 &= r_{0,0}, \\
 \beta_1 &= \frac{\theta}{\theta - \mu} (\bar{A}(\mu) - r_{0,0}), \\
 \gamma_0 &= 1 - \delta_1, \\
 \delta_1 &= \frac{\theta}{\theta - \mu (1 - r_{0,0})} \left(\frac{\bar{A}(\mu)}{r_{0,0}} - \frac{\bar{A}(\mu)}{r_{1,1}} \right) \\
 &- \frac{\theta}{r_{0,0} (\theta - \mu)} (\bar{A}(\mu) - r_{0,0}), \\
 \gamma_1 &= \frac{\bar{A}(\mu)}{r_{1,1}}, \\
 \Delta_1 &= 1 - \frac{\bar{A}(\mu)}{r_{1,1}}.
 \end{aligned} \tag{40}$$

From (21) and (22), we have

$$\begin{aligned}
 \pi_{k,0} &= (1 - r_{1,1}) \sigma (r_{0,0})^k, \quad k \geq 0, \\
 \pi_{k,1} &= (1 - r_{1,1}) \sigma \beta (r_{1,1}^k - r_{0,0}^k), \quad k \geq 1,
 \end{aligned} \tag{41}$$

where

$$\begin{aligned}
 \sigma &= \frac{\theta - \mu (1 - r_{0,0})}{\theta - \mu (1 - r_{1,1})}, \\
 \beta &= \frac{\theta}{\theta - \mu (1 - r_{0,0})}.
 \end{aligned} \tag{42}$$

All the above results coincide with the results derived in [6].

5. The Stationary Distribution at Arbitrary Epochs

In this section, we will derive the limiting distribution of $L(t)$ by using the method of the semi-Markov process (SMP). To this end, we consider another stochastic process $\{(Z(t), K(t)), t \geq 0\}$, where $Z(t) = L_\ell, \tau_\ell \leq t < \tau_{\ell+1}$, and $K(t) = J_\ell, \tau_\ell \leq t < \tau_{\ell+1}$. Clearly, $(Z(t), K(t))$ is a SMP, and (L_ℓ, J_ℓ) is its embedded Markov chain.

Let $\omega_{k,i}$ denote the time that SMP $(Z(t), K(t))$ resides in state (k, i) , where $(k, i) \in \Omega$. By the definition of SMP, we have $P\{\omega_{k,i} \leq t\} = A(t)$, and $E(\omega_{k,i}) = \lambda^{-1}$, for all $(k, i) \in \Omega$. Let

$$f_{k,i} = \lim_{t \rightarrow \infty} P\{Z(t) = k, K(t) = i\}. \tag{43}$$

Then

$$f_{k,i} = \frac{\pi_{k,i} E(\omega_{k,i})}{\sum_{(h,i) \in \Omega} \pi_{h,i} E(\omega_{h,i})} \quad (44)$$

(see Gross et al. [25], P. 299).

$$p_k = \sum_{(h,i) \in \Omega} f_{h,i} \sum_{j=0}^N \int_0^\infty P \{ \text{required changes in } t \text{ to bring state from } (h,i) \text{ to } (k,j) \} \lambda (1 - A(t)) dt. \quad (45)$$

For $k \geq 2$, from (22), we have

$$\begin{aligned} p_k &= \sum_{h=k-1}^\infty \pi_{h,0} \sum_{j=1}^N q_j \int_0^\infty \left\{ \int_0^t \frac{[\mu_j(t-x)]^{h+1-k}}{(h+1-k)!} e^{-\mu_j(t-x)} \theta e^{-\theta x} dx \right\} \\ &\quad \cdot \lambda (1 - A(t)) dt + \pi_{k-1,0} \int_0^\infty e^{-\theta t} \lambda (1 - A(t)) dt \\ &\quad + \sum_{i=1}^N \sum_{h=k-1}^\infty \pi_{h,i} \int_0^\infty \frac{(\mu_i t)^{h+1-k}}{(h+1-k)!} e^{-\mu_i t} \lambda (1 - A(t)) dt \\ &= \pi_{1,0} r_{0,0}^{k-2} \sum_{j=1}^N q_j \int_0^\infty \left\{ \int_0^t e^{-\mu_j(1-r_{0,0})(t-x)} \theta e^{-\theta x} dx \right\} \lambda (1 \\ &\quad - A(t)) dt + \pi_{1,0} r_{0,0}^{k-2} \frac{\lambda}{\theta} (1 - \bar{A}(\theta)) + \sum_{i=1}^N \sum_{h=k-1}^\infty \left(\pi_{1,0} r_{0,i} \right. \\ &\quad \cdot \left. \frac{r_{0,0}^{h-1} - r_{i,i}^{h-1}}{r_{0,0} - r_{i,i}} + \pi_{1,i} r_{i,i}^{h-1} \right) \int_0^\infty \frac{(\mu_i t)^{h+1-k}}{(h+1-k)!} e^{-\mu_i t} \lambda (1 - A(t)) dt \\ &= \pi_{1,0} r_{0,0}^{k-2} \theta \lambda \sum_{j=1}^N \frac{q_j}{\theta - \mu_j (1 - r_{0,0})} \left[\frac{1 - \bar{A}(\mu_j (1 - r_{0,0}))}{\mu_j (1 - r_{0,0})} \right. \\ &\quad \left. - \frac{1 - \bar{A}(\theta)}{\theta} \right] + \pi_{1,0} r_{0,0}^{k-2} \lambda \frac{1 - \bar{A}(\theta)}{\theta} \\ &\quad + \pi_{1,0} r_{0,0}^{k-2} \lambda \sum_{i=1}^N \frac{r_{0,i}}{(r_{0,0} - r_{i,i}) \mu_i (1 - r_{0,0})} [1 - \bar{A}(\mu_i (1 - r_{0,0}))] \\ &\quad - \pi_{1,0} \lambda \sum_{i=1}^N \frac{r_{0,i} r_{i,i}^{k-2}}{(r_{0,0} - r_{i,i}) \mu_i} + \lambda \sum_{i=1}^N \frac{\pi_{1,i} r_{i,i}^{k-2}}{\mu_i}. \end{aligned} \quad (46)$$

Similarly, for $k = 1$, we have

$$\begin{aligned} p_1 &= \pi_{0,0} \theta \lambda \sum_{j=1}^N \frac{q_j}{\theta - \mu_j} \left[\frac{1 - \bar{A}(\mu_j)}{\mu_j} - \frac{1 - \bar{A}(\theta)}{\theta} \right] \\ &\quad + \pi_{1,0} r_{0,0}^{-1} \theta \lambda \sum_{j=1}^N \frac{q_j}{\theta - \mu_j (1 - r_{0,0})} \left[\frac{1 - \bar{A}(\mu_j (1 - r_{0,0}))}{\mu_j (1 - r_{0,0})} \right. \\ &\quad \left. - \frac{1 - \bar{A}(\theta)}{\theta} \right] - \pi_{1,0} r_{0,0}^{-1} \theta \lambda \sum_{j=1}^N \frac{q_j}{\theta - \mu_j} \left[\frac{1 - \bar{A}(\mu_j)}{\mu_j} \right. \end{aligned}$$

Evidently, we have $f_{k,i} = \pi_{k,i}$; that is, the SMP and its embedded Markov chain have an identical limiting distribution.

Let $p_k = \lim_{t \rightarrow \infty} P\{L(t) = k\}$, $k = 0, 1, \dots$. From the theory of SMP (see Gross et al. [25], P. 299), we obtain

$$\begin{aligned} &\left[-\frac{1 - \bar{A}(\theta)}{\theta} \right] + \pi_{0,0} \frac{\lambda}{\theta} (1 - \bar{A}(\theta)) \\ &+ \pi_{1,0} \lambda \sum_{i=1}^N \frac{r_{0,i}}{(r_{0,0} - r_{i,i}) r_{0,0}} \left[\frac{1 - \bar{A}(\mu_i (1 - r_{0,0}))}{\mu_i (1 - r_{0,0})} \right. \\ &\quad \left. - \frac{1 - \bar{A}(\mu_i)}{\mu_i} \right] - \pi_{1,0} \lambda \sum_{i=1}^N \frac{r_{0,i} \bar{A}(\mu_i)}{(r_{0,0} - r_{i,i}) r_{i,i} \mu_i} \\ &+ \lambda \sum_{i=1}^N \frac{\pi_{1,i} \bar{A}(\mu_i)}{r_{i,i} \mu_i}. \end{aligned} \quad (47)$$

Finally, for $k = 0$, we can obtain $p_0 = 1 - \sum_{k=1}^\infty p_k$.

Then, the expected stationary system size at an arbitrary epoch, denoted by $E(L)$, can be given by

$$E(L) = \sum_{k=0}^\infty k p_k. \quad (48)$$

Remark 7. If $N = 1$, let $\mu_1 = \mu$; substituting the results in Remarks 5 and 6 into (46) and (47), we obtain

$$\begin{aligned} p_1 &= (1 - r_{1,1}) \sigma \left\{ \frac{\lambda}{\mu} \beta + \frac{\lambda}{\theta} (1 - \beta) (1 - r_{0,0}) \right\}, \\ p_k &= (1 - r_{1,1}) \sigma \left\{ \frac{\lambda}{\mu} \beta r_{1,1}^{k-1} + \frac{\lambda}{\theta} r_{0,0}^{k-1} (1 - \beta) (1 - r_{0,0}) \right\}, \\ &\quad k \geq 2; \end{aligned} \quad (49)$$

then

$$p_0 = 1 - \sigma \left\{ \frac{\lambda}{\mu} \beta + \frac{\lambda}{\theta} (1 - \beta) (1 - r_{1,1}) \right\}, \quad (50)$$

which coincide with the result derived in [6].

6. The Stationary Waiting Time Distribution

Let W and $\bar{W}(s)$ denote the stationary waiting time of an arbitrary customer and its LST, respectively. Due to the

memoryless property of exponential distribution, the waiting time distribution of an arbitrary customer is given by

$$\begin{aligned} W(t) &= P\{W \leq t\} \\ &= \pi_{0,0}V(t) + \sum_{k=1}^{\infty} \pi_{k,0} \sum_{i=1}^N q_i [V * B_i^{k*}(t)] \\ &\quad + \sum_{i=1}^N \sum_{k=1}^{\infty} \pi_{k,i} B_i^{k*}(t), \end{aligned} \quad (51)$$

where $V(t)$ denotes the distribution of the vacation time, $B_i(t)$ denotes the distribution of the service time in service phase i , star $*$ denotes convolution, and $B_i^{k*}(t)$ denotes the k -fold convolution of $B_i(t)$.

Then, from (22) and (51), we obtain the LST of the waiting time distribution as follows:

$$\begin{aligned} \widetilde{W}(s) &= \pi_{0,0} \widetilde{V}(s) + \sum_{k=1}^{\infty} \pi_{k,0} \sum_{i=1}^N q_i \widetilde{V}(s) [\widetilde{B}_i(s)]^k \\ &\quad + \sum_{i=1}^N \sum_{k=1}^{\infty} \pi_{k,i} [\widetilde{B}_i(s)]^k = \pi_{0,0} \frac{\theta}{\theta + s} \\ &\quad + \pi_{1,0} \sum_{k=1}^{\infty} r_{0,0}^{k-1} \sum_{i=1}^N \frac{\theta}{\theta + s} \left(\frac{\mu_i}{\mu_i + s} \right)^k \\ &\quad + \sum_{i=1}^N \sum_{k=1}^{\infty} \left(\pi_{1,0} r_{0,i} \frac{r_{0,0}^{k-1} - r_{i,i}^{k-1}}{r_{0,0} - r_{i,i}} + \pi_{1,i} r_{i,i}^{k-1} \right) \left(\frac{\mu_i}{\mu_i + s} \right)^k \\ &= \frac{\theta}{\theta + s} \left(\pi_{0,0} + \sum_{i=1}^N \frac{\pi_{1,0} \mu_i}{\mu_i (1 - r_{0,0}) + s} \right) \\ &\quad + \sum_{i=1}^N \frac{\pi_{1,i} \mu_i}{\mu_i (1 - r_{i,i}) + s} \\ &\quad + \sum_{i=1}^N \frac{\pi_{1,0} r_{0,i} \mu_i}{r_{0,0} - r_{i,i}} \left(\frac{1}{\mu_i (1 - r_{0,0}) + s} \right. \\ &\quad \left. - \frac{1}{\mu_i (1 - r_{i,i}) + s} \right), \end{aligned} \quad (52)$$

where $\widetilde{V}(s)$ and $\widetilde{B}_i(s)$ denote the LSTs of $V(t)$ and $B_i(t)$, respectively.

By (52), the mean waiting time of an arbitrary customer can be given by

$$\begin{aligned} E(W) &= - \left. \frac{d\widetilde{W}(s)}{ds} \right|_{s=0} \\ &= \frac{\pi_{0,0}}{\theta} + \frac{N\pi_{1,0}}{\theta(1 - r_{0,0})} + \sum_{i=1}^N \frac{\pi_{1,0}}{\mu_i (1 - r_{0,0})^2} \\ &\quad + \sum_{i=1}^N \frac{\pi_{1,i}}{\mu_i (1 - r_{i,i})^2} \\ &\quad + \sum_{i=1}^N \frac{\pi_{1,0} r_{0,i}}{(r_{0,0} - r_{i,i}) \mu_i} \left(\frac{1}{(1 - r_{0,0})^2} - \frac{1}{(1 - r_{i,i})^2} \right). \end{aligned} \quad (53)$$

Remark 8. If $N = 1$, let $\mu_1 = \mu$; substituting the results in Remarks 5 and 6 into (52) and (53), we have

$$\begin{aligned} \widetilde{W}(s) &= (1 - r_{1,1}) \sigma \left\{ \frac{\theta}{\theta + s} \right. \\ &\quad \left. + \frac{\mu r_{0,0}}{\mu(1 - r_{0,0}) + s} \left(\frac{\theta}{\theta + s} - \beta \right) + \frac{\beta \mu r_{1,1}}{\mu(1 - r_{0,0}) + s} \right\}, \quad (54) \\ E(W) &= \frac{1}{\theta} + \frac{r_{1,1}}{\mu(1 - r_{1,1})}, \end{aligned}$$

which are derived in [6].

7. Mean Number of Customers in the System at the End of Phase 0

Let L_0 denote the mean number of customers in the system at the end of phase 0, and let A^+ represent the remaining interarrival time. The probability that no customer arrives during a vacation period is given by

$$\begin{aligned} P\{A^+ > V\} &= \int_0^{\infty} (1 - e^{-\theta t}) \lambda (1 - A(t)) dt \\ &= 1 - \frac{\lambda}{\theta} (1 - \widetilde{A}(\theta)). \end{aligned} \quad (55)$$

That is, the probability that at least one customer arrives during a vacation period is $\lambda(1 - \widetilde{A}(\theta))/\theta$.

In our model, if a vacation ends, the server finds no customer waiting, and a new vacation begins; otherwise, the system moves from phase 0 to some service phase i . Thus, the number of vacations in phase 0 is distributed according to geometric distribution with parameter $\lambda(1 - \widetilde{A}(\theta))/\theta$. Because the mean time of each vacation is $1/\theta$, the mean time that the system spends in phase 0, denoted by T_0 , can be obtained by

$$\begin{aligned} T_0 &= \sum_{k=1}^{\infty} \left[1 - \frac{\lambda(1 - \widetilde{A}(\theta))}{\theta} \right]^{k-1} \frac{\lambda(1 - \widetilde{A}(\theta))}{\theta} \frac{k}{\theta} \\ &= \frac{1}{\lambda(1 - \widetilde{A}(\theta))}, \end{aligned} \quad (56)$$

which is the product of the mean number of vacations that the system resides in phase 0 and the mean time of each vacation.

From (56), the mean number of customers in the system at the end of phase 0 is

$$L_0 = \frac{1}{\lambda(1 - \widetilde{A}(\theta))} \cdot \lambda = \frac{1}{1 - \widetilde{A}(\theta)}, \quad (57)$$

which is just the product of the mean time that the system spends in phase 0 and the arrival rate of customers in phase 0.

8. Cycle Analysis

We define a cycle as the time interval between two successive epochs at which phase i commences, $i = 0, 1, \dots, N$. Then

there are $(N + 1)$ types of cycles. The length of type- i cycle is denoted by C_i .

First, we derive the mean length of the type-0 cycle.

During the type-0 cycle, the system visits vacation phase 0 and certain service phase i one time, $i = 1, 2, \dots, N$, respectively. We know that the mean time that the system spends in phase 0 is T_0 (see (56)). And when the system jumps from phase 0 into certain service phase i with probability q_i , $i = 1, 2, \dots, N$, from (57), the mean time that the system spends in service phase i should be equal to the product of the mean number of customers in the system at the end of phase 0 and the mean busy period that one customer induces in service phase i . Let T_i represent the mean time that the system resides in service phase i . Then we have

$$T_i = \frac{1}{1 - \bar{A}(\theta)} \cdot \frac{1}{\mu_i - \lambda}, \quad i = 1, 2, \dots, N, \quad (58)$$

where $1/(\mu_i - \lambda)$ is the mean busy period that one customer induces in service phase i .

Therefore, the mean length of the type-0 cycle is

$$\begin{aligned} E(C_0) &= T_0 + \sum_{i=1}^N q_i T_i \\ &= \frac{1}{\lambda(1 - \bar{A}(\theta))} + \frac{1}{1 - \bar{A}(\theta)} \sum_{i=1}^N \frac{q_i}{\mu_i - \lambda}. \end{aligned} \quad (59)$$

Next, we consider the type- i cycle, $i = 1, 2, \dots, N$.

During the type- i cycle, the time that the system visits phase 0 follows a geometric distribution with parameter q_i . And it is easy to find that the mean length of type- i cycle during which the system visits phase 0 k times is

$$T_i + T_0 + (k - 1) \left(T_0 + \sum_{j=1, j \neq i}^N \frac{q_j}{1 - q_i} T_j \right). \quad (60)$$

Then we can obtain the mean length of type- i cycle as follows:

$$\begin{aligned} E(C_i) &= \sum_{k=1}^{\infty} (1 - q_i)^{k-1} \\ &\quad \cdot q_i \left[T_i + T_0 + (k - 1) \left(T_0 + \sum_{j=1, j \neq i}^N \frac{q_j}{1 - q_i} T_j \right) \right] \\ &= T_i + T_0 + \sum_{k=1}^{\infty} (1 - q_i)^{k-1} q_i (k - 1) \\ &\quad \cdot \left[T_0 + \frac{1}{1 - q_i} (\alpha - T_0 - q_i T_i) \right] = T_i + T_0 \\ &\quad + \frac{1 - q_i}{q_i} \left[T_0 + \frac{1}{1 - q_i} (\alpha - T_0 - q_i T_i) \right] = \frac{\alpha}{q_i}, \end{aligned} \quad (61)$$

where $\alpha = T_0 + \sum_{i=1}^N q_i T_i$.

FIGURE 1: The effect of λ on $E(W)$.

9. Numerical Examples

In this section, some numerical experiments are shown to explore the effect of the model parameters on the system performance measures. First, we assume that customers arrive according to a Poisson process with rate λ , and $N = 2$; that is, the system we consider translates into an $M/M/1$ queue with two service phases and a vacation phase. The model parameters are chosen as $\lambda = 1.2$, $\mu_1 = 2$, $\mu_2 = 2.5$, $q_1 = 0.4$, $q_2 = 0.6$, and $\theta = 1.4, 1.5, 1.6$, unless they are considered as variables or their values are given in the respective figures. Using Lemma 3 and Theorem 4, the stationary state probability $\{\pi_k, k \geq 0\}$ can be obtained. Then we can derive a variety of performance measures of the system.

Figure 1 demonstrates the effect of λ on $E(W)$ for different vacation rate θ . As intuitively expected, the mean waiting time of an arbitrary customer increases with the increase of λ for any θ . And for a fixed λ , $E(W)$ decreases as θ increases. This is obvious because as θ increases, the mean time of vacation decreases, and there are fewer customers in the system. The effect of μ_2 on $E(W)$ for different values of θ is illustrated in Figure 2. It can be observed that $E(W)$ decreases with the increase of μ_2 for different values of θ . And if μ_2 is fixed, as intuition tells us, $E(W)$ decreases as θ increases. In Figure 3, we plot the change trend of the mean waiting time of an arbitrary customer with varying values of q_1 . It is obvious that, for any θ , the larger q_1 is, the larger the mean waiting time $E(W)$ becomes. And for a fixed q_1 , $E(W)$ decreases with the increase of θ , as we expect.

Figure 4 demonstrates the effect of λ on $\pi_{0,0}$ for different vacation rate θ . It can be observed that, with the increase of λ , the probability that the system is empty declines for any θ , and if λ is fixed, $\pi_{0,0}$ increases with the increase of θ . Evidently, the change trends agree with the intuitive expectations. The effect of μ_2 on $\pi_{0,0}$ is illustrated in Figure 5. As intuitively

FIGURE 2: The effect of μ_2 on $E(W)$. FIGURE 4: The effect of λ on $\pi_{0,0}$. FIGURE 3: The effect of q_1 on $E(W)$. FIGURE 5: The effect of μ_2 on $\pi_{0,0}$.

expected, we can see that increased μ_2 leads to the larger $\pi_{0,0}$ for any given θ . Moreover, if μ_2 is fixed, $\pi_{0,0}$ increases with the increase of θ . In Figure 6, we plot the change trend of $\pi_{0,0}$ with q_1 varying. We can find that $\pi_{0,0}$ decreases as q_1 increases for any given θ . For a given q_1 , when q_1 is relatively small, $\pi_{0,0}$ increases with the increase of θ ; however, when q_1 is relatively large, $\pi_{0,0}$ decreases with the increase of θ . That is, when θ is larger, the influence of q_1 to $\pi_{0,0}$ is greater. This phenomenon can be explained from the fact that $\mu_1 < \mu_2$.

Then, in Figure 7, we present a comparison result of three queueing models: the $M/M/1$, $E_2/M/1$, and $D/M/1$ models, where E_2 stands for a 2-Erlangian arrival interval and D stands for a deterministic arrival interval. As demonstrated

in Figure 7, the mean waiting time $E(W)$ increases with the increase of λ for the three models, and for a fixed λ , the $M/M/1$ model has the largest value of $E(W)$, and the $D/M/1$ model has the smallest value.

Finally, Figures 8–10 demonstrate the effect of (μ_1, μ_2) and (q_1, q_2) on $E(W)$ for the above three interarrival distributions when $\bar{\mu}$ is fixed, where $\bar{\mu} = (\mu_1 + \mu_2)/2$. In Figure 8, we plot the change trend of $E(W)$ as μ_2 increases from 2 to 3 under the corresponding three models, given that $\bar{\mu} = 2.5$ and $q_1 = q_2 = 0.5$. We can observe that $E(W)$ is symmetrical about $\mu_2 = 2.5$ and the system efficiency is best when $\mu_2 = 2.5$ for each model. However, in Figures 9 and 10, we can see that if $\bar{\mu} = 2.5$, $q_1 = 0.2$, and $q_2 = 0.8$, $E(W)$ is not symmetrical and the

FIGURE 6: The effect of q_1 on $\pi_{0,0}$. FIGURE 7: The effect of λ on $E(W)$ for three interarrival time distributions ($\theta = 1.5$).

value of μ_2 that the system efficiency is best is greater than 2.5; if $\bar{\mu} = 2.5$, $q_1 = 0.8$, and $q_2 = 0.2$, $E(W)$ is not symmetrical as well and the value of μ_2 that the system efficiency is best is less than 2.5. Note that when $\mu_1 = \mu_2 = 2.5$, the model with two service phases actually translates into the model with only one service phase. From Figures 8–10, we can see that multiple service phases have important influence on the performance of the queueing system.

10. Conclusion

In this paper, we have carried out an analysis of $GI/M/1$ queue with vacations and multiple service phases. We have

FIGURE 8: The effect of μ_1 and μ_2 on $E(W)$ ($\bar{\mu} = 2.5$, $q_1 = q_2 = 0.5$, $\theta = 1.5$). FIGURE 9: The effect of μ_1 and μ_2 on $E(W)$ ($\bar{\mu} = 2.5$, $q_1 = 0.2$, $q_2 = 0.8$, $\theta = 1.5$).

obtained the distributions of the stationary system size at arrival and arbitrary epochs. Further, we have obtained the stationary waiting time distribution of an arbitrary customer. Some important performance measures of the system such as the mean waiting time of an arbitrary customer, the mean length of the type- i cycle, and the mean number of customers in the system at the end of phase 0 are obtained. Finally, the influence of the model parameters on several performance characteristics is demonstrated through some numerical experiments.

FIGURE 10: The effect of μ_1 and μ_2 on $E(W)$ ($\bar{\mu} = 2.5$, $q_1 = 0.8$, $q_2 = 0.2$, $\theta = 1.5$).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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