A No-Equilibrium Hyperchaotic System and Its Fractional-Order Form

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1. Introduction

It is now well established from a variety of studies that a hyperchaotic system is specified by having at least two positive Lyapunov exponents [1]. Hyperchaotic systems have been applied in various areas due to their higher level of complexity with respect to chaotic systems [2–6]. There has been an increasing amount of literature on hyperchaos [7, 8]. Authors have introduced and studied different hyperchaotic systems such as switched hyperchaotic system [9], four-wing hyperchaotic attractor [10], hyperchaotic Chua’s circuits [11], and hyperchaotic Lü attractor [12]. It is noted that there are countable numbers of equilibrium points in such reported hyperchaotic systems.

It is interesting that Wang et al. found a hyperchaotic system without equilibrium [13], which is different from normal hyperchaotic systems. Following up the first discovery of Wang et al., other hyperchaotic systems without equilibrium were presented [14–17]. Recent studies have attempted to explore special features of no-equilibrium hyperchaotic systems. Wang et al. investigated multiwing nonequilibrium attractors in simplified hyperchaotic systems [18]. Bao et al. constructed a memristive hyperchaotic system which does not display any equilibrium [19]. This memristive system generated coexisting hidden attractors. Moreover, fractional-order systems without equilibria were discovered in [20, 21]. It is worth noting that attractors in such no-equilibrium hyperchaotic system are “hidden” from the viewpoint of computations [22, 23]. More recent attention has focused on hidden attractors due to their importance in both theoretical issues and engineering applications [24–27].

The aim of this study is to introduce a new hyperchaotic system without equilibrium. The organization of the paper is as follows. The new no-equilibrium system is described in the next section. Section 3 presents dynamics of the new system without equilibrium. Fractional derivation effect on the proposed system is investigated in Section 4. Our conclusions are drawn in the last section.
2. The Model of the New System without Equilibrium

Wei and Wang have introduced a special system which is different from the original Lorenz and Lorenz-like systems [28]. Wei-Wang system is given by

\[
\begin{align*}
\dot{x} &= a(y - x), \\
\dot{y} &= -by + nxz + cw, \\
\dot{z} &= d - e^{xy}, \\
\dot{w} &= -my,
\end{align*}
\]

in which \(x, y, z,\) and \(w\) are state variables while \(a, b, d,\) and \(n\) are parameters. There are two equilibrium points in system (1) for \(d > 0\) and \(n \neq 0:\) \(E_1 (\sqrt{n}d, \sqrt{n}d, b/n)\) and \(E_2 (-\sqrt{n}d, -\sqrt{n}d, b/n).\) Moreover, Wei-Wang system (1) is chaotic when both equilibrium points \(E_{1,2}\) are stable [28].

Motivated by the noticeable features of model (1), we construct a novel four-dimensional system by introducing an additional fourth state \(w\) as follows:

\[
\begin{align*}
\dot{x} &= a(y - x), \\
\dot{y} &= -by + nz + cw, \\
\dot{z} &= d - e^{xy}, \\
\dot{w} &= -my.
\end{align*}
\]

in which \(x, y, z,\) and \(w\) are four state variables. In system (2), positive parameters are \(a, b, c, d, n, m,\) and \(d \neq 1.\) System (2) is invariant under the transformation \((x; y; z; w) \rightarrow (-x; -y; z; -w)\). It means that the system has rotational symmetry around the \(z\)-axis.

Obviously, it is simple to find equilibrium points of system (2) by solving \(\dot{x} = 0, \dot{y} = 0, \dot{z} = 0,\) and \(\dot{w} = 0.\) Thus we get

\[
\begin{align*}
a(y - x) &= 0, \\
-by + nz + cw &= 0, \\
d - e^{xy} &= 0, \\
-my &= 0.
\end{align*}
\]

Therefore, there is not any equilibrium in system (2).

It is easy to see that no-equilibrium system (2) is dissipative with an exponential contraction rate

\[
\frac{dV}{dt} = e^{-(a + b)y},
\]

because of

\[
\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{w}}{\partial w} = -a - b < 0.
\]

However, it is noted that the dissipativity in the sense of Levinson should be studied further [29, 30].

It is interesting that the system without equilibrium exhibits hyperchaotic behavior for \(a = n = 1, b = m = 0.5, c = 0.2,\) and \(d = 2.5\) and initial conditions \((x(0), y(0), z(0), w(0)) = (0.1, 0.1, 0.1, 0.1)\) as illustrated in Figure 1. The Lyapunov exponents of the no-equilibrium system are \(L_1 = 0.1515, L_2 = 0.0112, L_3 = 0,\) and \(L_4 = -1.6623.\) We have applied Wolf’s method [31] to calculate the Lyapunov exponents. The time of computation is 10,000 and the initial conditions are \((x(0), y(0), z(0), w(0)) = (0.1, 0.1, 0.1, 0.1).\) It is noted that the values of Lyapunov exponents are not the same for any initial point on invariant set [32]. Computations of Lyapunov exponents should be considered seriously [33–35].

3. Dynamical Behavior of the System without Equilibrium

Dynamics of the system has been investigated by considering the effect of parameters on system’s behavior. Our simulations show that no-equilibrium system (2) displays rich dynamics for the parameter \(a.\) We have studied the dynamics of no-equilibrium system (2) by varying the value of parameter \(a.\) Bifurcation diagram and Lyapunov exponents of no-equilibrium system (2) are presented in Figures 2 and 3, respectively. For \(a > 2.276,\) the system without equilibrium only generates periodical behavior. However, no-equilibrium system (2) displays different attractors such as periodic, quasiperiodic, chaotic, and hyperchaotic attractors for \(a < 2.276.\)

We have also investigated the multistability of the new no-equilibrium system by using the continuation diagram [36]. As can be seen in Figure 4, a window of multistability dynamics is identified in the range of \(a\) from 1.905 to 2.215. For example, Figure 5 illustrates the coexisting attractor in no-equilibrium system (2) for \(a = 2.1.\)

From the viewpoint of applications, the amplitude control of a chaotic signal is an important topic [37–40]. Interestingly, no-equilibrium system (2) is an amplitude-controllable system. Here we illustrate this special feature of no-equilibrium system (2).

We introduce a single control parameter \(k_w\) into system (2):

\[
\begin{align*}
\dot{x} &= a(y - x), \\
\dot{y} &= -by + nz + cw - k_w, \\
\dot{z} &= d - e^{xy}, \\
\dot{w} &= -my.
\end{align*}
\]

The control parameter \(k_w\) is used to boost the amplitude of the variable \(w.\) Therefore, we can change the signal \(w\) easily, for example, from a bipolar signal to unipolar one as illustrated in Figure 6.

In order to control the amplitudes of variables \(x\) and \(y,\) the coefficient \(k_{xy}\) is included into system (2) as follows:

\[
\begin{align*}
\dot{x} &= a(y - x), \\
\dot{y} &= -by + nz + cw, \\
\dot{z} &= d - e^{k_{xy}}, \\
\dot{w} &= -my.
\end{align*}
\]
Figure 1: Projections of hyperchaotic attractors in (a) $x$-$y$ plane, (b) $x$-$z$ plane, and (c) $x$-$w$ plane for $a = n = 1, b = m = 0.5, c = 0.2, d = 2.5$, and initial conditions $(x(0), y(0), z(0), w(0)) = (0.1, 0.1, 0.1, 0.1)$.

Figure 2: Bifurcation diagram of no-equilibrium system (2) for $b = m = 0.5, c = 0.2, d = 2.5, n = 1$, and $a \in [0.8, 2.8]$.

Figure 3: Three largest Lyapunov exponents of no-equilibrium system (2) for $b = m = 0.5, c = 0.2, d = 2.5, n = 1$, and $a \in [0.8, 2.8]$. 
Figure 4: Continuations of new no-equilibrium system (2) for $b = m = 0.5$, $c = 0.2$, $d = 2.5$, and $n = 1$. Forward continuation (black): increasing the bifurcation parameter $a$ from 1.5 to 2.5 and starting with initial conditions $(x(0), y(0), z(0), w(0)) = (0.1, 0.1, 0.1, 0.1)$. Backward continuation (red): decreasing the bifurcation parameter $a$ from 2.5 to 1.5 and starting with initial conditions $(x(0), y(0), z(0), w(0)) = (-0.1, -0.1, 0.1, -0.1)$.

Figure 5: Coexistence of attractors in no-equilibrium system (2) for $a = 2.1$, $b = m = 0.5$, $c = 0.2$, $d = 2.5$, and $n = 1$ in $x$-$y$ plane: (a) periodic state and (b) quasiperiodic state.

Figure 6: Varying attractors of no-equilibrium system (6) for $a = n = 1$, $b = m = 0.5$, $c = 0.2$, and $d = 2.5$ for different boosting constants: $k_w = 0$ (black), $k_w = -1.6$ (blue), and $k_w = 1.6$ (red).
Figure 7: Controllable attractors of no-equilibrium system (7) for $a = n = 1$, $b = m = 0.5$, $c = 0.2$, and $d = 2.5$ when changing the coefficient $k_{xy}$: $k_{xy} = 1$ (black), $k_{xy} = 0.25$ (blue), and $k_{xy} = 4$ (red).

Figure 8: The phase portrait in the planes $(x, y)$, $(x, z)$, and $(x, w)$ of fractional-order system (8) at $a = n = 1$, $b = m = 0.5$, $c = 0.2$, and $d = 2.5$ and for specific values of commensurate fractional-order $q$: (a) $q = 0.945$ and (b) $q = 0.948$. The initial conditions used are $(x(0), y(0), z(0), w(0)) = (0.1, 0.1, 0.1, 0.1)$. 
Here we can consider the coefficient $k_{xy}$ as an amplitude controller. Attractor of no-equilibrium system (2) is controlled as shown in Figure 7.

4. Fractional Derivation Effect on the System without Equilibrium

In this section, we focus on the effect of commensurate fractional derivation on the hyperchaotic system (2) when $a = n = 1, b = m = 0.5, c = 0.2, and d = 2.5$ (see Figure 1). The fractional-order form of system (2) is obtained by replacing the integer-order derivatives by fractional-order ones. The mathematical description of the fractional-order version of system (2) is expressed as

\[
\frac{d^q x}{dt^q} = a(y - x),
\]

\[
\frac{d^q y}{dt^q} = -by + nxz + cw,
\]

\[
\frac{d^q z}{dt^q} = d - e^{xy},
\]

\[
\frac{d^q w}{dt^q} = -my,
\]

where $q$ is the derivative order satisfying $0 < q < 1$. The fractional-order form of system (2) has no-equilibrium points; therefore the effect of fractional derivation on the hyperchaotic system (8) can only be numerically investigated. Here, the Adams-Bashforth-Moulton predictor-corrector scheme is used \[41, 42\]. This method is based on the Caputo definition of the fractional-order derivative, given by \[43\]

\[
\frac{d^q X_i}{dt^q} = \frac{1}{\Gamma(q-n)} \int_0^t \frac{\dot{u}(t')}{(t-t')^n} dt',
\]

where $u = x, y, z, w$ and $\Gamma(\cdot)$ is the Gamma function. We perform the numerical simulations of fractional-order system (8) for different fractional-order $q (0 < q < 1)$.
We present in Figure 8 the phase portraits in the planes \((x, y), (x, z),\) and \((x, w)\) obtained for two specific values of commensurate fractional-order \(q\).

For \(q = 0.945\), the fractional-order system (8) displays a point attractor as shown in Figure 8(a). When the fractional derivative order increases, the fractional-order system (8) presents a transient chaos at \(q = 0.948\) (see Figure 8(b)). The transient chaos is confirmed in Figure 9 which depicts the time series of the corresponding phase portraits of fractional-order system (8) shown in Figure 8(b).

It is clearly seen in Figure 9 that the trajectories of fractional-order system (8) display chaotic behavior for \(t < 100\). For \(t > 100\), they converge to an equilibrium point. For \(q = 0.95\), the phase portraits in the planes \((x, y), (x, z),\) and \((x, w)\) are plotted in Figure 10.

From Figure 10, one can note that the fractional-order system (8) exhibits bistable double-scroll chaotic attractors at \(q = 0.95\). The chaotic behavior found in Figure 10 is confirmed in Figure 12 which presents the autocorrelation function of the outputs \(x(t), y(t), z(t),\) and \(w(t)\) of the corresponding phase portraits of fractional-order system (8) depicted in Figure 10. The coexistence between double-scroll chaotic and quasiperiodic attractors is shown in Figure 11 which presents the phase portraits in the planes \((x, y), (x, z),\) and \((x, w)\) for \(q = 0.99\).

For \(q = 0.99\) and using the initial conditions \((x(0), y(0), z(0), w(0)) = (0.1, 0.1, 0.1, 0.1)\), double-scroll chaotic attractor is obtained in the fractional-order system (8), while for \(q = 0.99\) and using the initial conditions \((x(0), y(0), z(0), w(0)) = (-0.1, -0.1, 0.1, -0.1)\), the fractional-order system (8) exhibits quasiperiodic attractor. The chaotic and quasiperiodic behaviors are confirmed in Figure 12 which presents the autocorrelation function of the outputs \(x(t), y(t), z(t),\) and \(w(t)\) of the corresponding phase portraits of fractional-order system (8) depicted in Figure 11. In order to know the dynamical behavior found in Figures 10 and 11, we...
Figure 11: The phase portrait in the planes $(x, y)$, $(x, z)$, and $(x, w)$ of fractional-order system (8) at $a = n = 1$, $b = m = 0.5$, $c = 0.2$, $d = 2.5$, and $q = 0.99$ for specific initial conditions $(x(0), y(0), z(0), w(0))$: (a) $(0.1, 0.1, 0.1, 0.1)$ and (b) $(-0.1, -0.1, 0.1, -0.1)$.

calculate autocorrelation function of the outputs $x(t)$, $y(t)$, $z(t)$, and $w(t)$. In Figure 12, we present the autocorrelation function

$$AC_u(t) = \frac{\langle [u(t) - \langle u(t) \rangle][u(t + t') - \langle u(t) \rangle] \rangle}{\langle u(t) - \langle u(t) \rangle \rangle^2},$$

where $u(t) = x(t), y(t), z(t), w(t)$, $\langle u(t) \rangle$ is the mean value of the amplitude along the trajectory, and $t'$ is the time shift. The coefficient $AC_u(t)$ is bounded in the range $(-1, 1)$ and it stays high for periodic, quasiperiodic, and chaotic cases and decays to zero in the case of the hyperchaotic attractor [44].

5. Conclusions

This paper introduces a new system, which has no equilibrium. However, different complex behaviors such as hyperchaos or coexistence of hidden attractors have been observed in such system. In addition, the new system without equilibrium is an amplitude-controllable system which is useful for practical applications. This study has found that commensurate fractional derivation affects the no-equilibrium system. Control and synchronization of such system should be studied in future works.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Figure 12: The autocorrelation functions of $x(t)$, $y(t)$, and $z(t)$ at $a = n = 1$, $b = m = 0.5$, $c = 0.2$, and $d = 2.5$ and for specific values of commensurate-order: (a) $q = 0.95$ and (b) $q = 0.99$. The curves in (a1) and (b1) are obtained using the initial conditions $(x(0), y(0), z(0), w(0)) = (0.1, 0.1, 0.1, 0.1)$ while the curves in (a2) and (b2) are obtained using the initial conditions $(x(0), y(0), z(0), w(0)) = (-0.1, -0.1, 0.1, -0.1)$.

References


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