Research Article

Integral Sliding Modes with Nonlinear $\mathcal{H}_\infty$ -Control for Time-Varying Minimum-Phase Underactuated Systems with Unmatched Disturbances

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This paper presents a methodology for controlling nonlinear time-varying minimum-phase underactuated systems affected by matched and unmatched perturbations. The proposed control structure consists of an integral sliding mode control coupled together with a global nonlinear $\mathcal{H}_\infty$ -control for rejecting vanishing and nonvanishing matched perturbations and for attenuating the unmatched ones, respectively. It is theoretically proven that, using the proposed controller, the origin of the free-disturbance nonlinear system is asymptotically stabilized, while the matched disturbances are rejected whereas the $\mathcal{L}_2$ -gain of the corresponding nonlinear system with unmatched perturbation is less than a given disturbance attenuation level γ with respect to a given performance output. The capability of the designed controller is verified through a flexible joint robot manipulator typically affected by both classes of external perturbations. In order to assess the performance of the proposed controller, an existing sliding modes controller based on a nonlinear integral-type sliding surface is also implemented. Both controllers are then compared for trajectory tracking tasks. Numerical simulations show that the proposed approach exhibits better performance.

1. Introduction

Much research in recent years has focused on the stabilization and control of mechanical systems operating under uncertain conditions such as external disturbances, uncertain parameters of the plant, and parasitic dynamics. These problems are present in real-world applications revealing, for example, instability, limit cycles, steady-state error, poor repeatability, or imprecisions.

In spite of the rich and diverse literature on the matter (see, e.g., [1–3]), the unmatched disturbances are still a challenging problem, faced by control engineers, that adversely affect the performance of any system to be controlled. This kind of disturbances cannot be trivially neglected since they can be aroused by unavoidable noise in the measurements or perturbing the output as well. Moreover, disturbances acting in the nonactuated part of an underactuated mechanical

system (e.g., pendulums, car-like robots, biped robots, and unmanned aerial vehicles) are a typical example where the unmatched disturbances must be counteracted. Indeed, the problem becomes more complicated for the motion control of this kind of systems since unmodeled dynamics can emerge (see, e.g., [4]).

Sliding modes are long recognized as a powerful control method to *reject* vanishing and nonvanishing uniformly bounded matched disturbances and plant uncertainties. However, unmatched disturbances are not counteracted. On the other hand, nonlinear $\mathcal{H}_\infty$ control has the capability of *attenuating* both matched and unmatched disturbances [5, 6]. There have been many results dealing with unmatched disturbances; however, integral sliding modes have begun to receive a growing interest. For example, Kumar et al. [7] solve the regulation problem for a Stewart robot using a smooth integral sliding mode (ISM) controller, which drives

the position error to the origin in finite time, while the closed-loop system is demonstrated to be robust against matched disturbances only. Mahieddine et al. [8] propose a sliding mode controller that allows attenuating matched and unmatched uncertainties in nonlinear systems and also reducing the chattering in the control signal. Han et al. [9] developed output feedback-based sliding mode control schemes for linear time-delayed systems considering both matched and unmatched uncertainties. A detailed revision of ISM addressing the unmatched disturbances is presented in [10].

In relevant works, Osuna et al. [11] make a $\mathcal{L}_2$ -gain analysis for hybrid mechanical systems operating under unilateral constraints and admitting sliding modes and collision phenomena. Rubagotti et al. [12] prove that the definition of a suitable sliding manifold and the generation of sliding modes upon it guarantee that matched disturbances are completely rejected while the unmatched ones are not amplified. Besides, a linear control with nonlinear compensation is proposed against matched disturbances. Cao and Xu [13] present a nonlinear ISM controller where the unmatched disturbances are not amplified, but a linear controller is also used. Castaños and Fridman [14] show the robustness properties of an integral sliding mode controller ensuring rejection of the matched disturbances, while unmatched perturbations are not amplified using also a linear $\mathcal{H}_\infty$ control. Galvan-Guerra and Fridman [15] proposed an output ISM for linear time-variant systems, where the main goal is to eliminate the matched perturbation. In [16], an adaptive ISM control for a class of nonlinear uncertain and invariant systems is proposed to eliminate the quantization sensitivity parameters and matched perturbations, which is accomplished by using an integral sliding function from the local dynamics of the plant. Chen et al. [17] propose a nonlinear ISM fault tolerant control where an optimal control is used against matched disturbances.

The aforementioned literature includes a linear $\mathcal{H}_\infty$ control to attenuate unmatched disturbances to easily find a suboptimal solution through an algebraic Riccati equation (ARE). On the other hand, a physical phenomenon is better described by its nonlinear dynamic equations. However, the local solution of these dynamic equations is required for the ARE to be solved. Besides, hard computational work is also entailed for verifying the Hamilton-Jacobi-Isaacs inequality and obtaining a global solution.

In this paper, an ISM control combined together with a nonlinear $\mathcal{H}_\infty$ control is presented. The proposed controller allows rejecting matched bounded disturbances and attenuating the effect of the unmatched ones. The synthesized controller, that admits a time-varying input matrix, was applied for solving the tracking control problem for 1 degree of freedom (DOF) flexible joint robot (FJR) manipulator, which consists of a single link interconnected by an elastic revolute joint. The formulation of the nonlinear $\mathcal{H}_\infty$ -control problem is confined to nonautonomous affine systems, and it requires a controller design that guarantees both the internal asymptotic stability of the closed-loop system and its dissipativity with respect to admissible external disturbances. In contrast to previous $\mathcal{H}_\infty$ works, a strict Lyapunov function

was proposed in this paper to ensure a global solution of the $\mathcal{H}_\infty$ control problem, by means of the verification of the Hamilton-Jacobi-Isaacs inequality, thus avoiding a hard numerical computation of a partial differential equation or a solution of the corresponding differential Riccati equation [18, 19] where the linearization around the equilibrium point of the plant is required. In order to assess the performance of the proposed controller, we implemented a controller, introduced by Cao and Xu in [13], based on a nonlinear integral-type sliding surface. The results of both, the proposed controller and the Cao-Xu controller, were compared in a trajectory tracking task.

This paper contributes to the following:

- (i) Presenting the design of a new controller by combining ISM control and a nonlinear $\mathcal{H}_\infty$ control for time-varying minimum-phase underactuated systems affected by both matched and unmatched disturbances
- (ii) Developing a rigorous stability analysis, with a global solution to $\mathcal{H}_\infty$ control and verifying the Hamilton-Jacobi-Isaacs inequality
- (iii) Detailing a procedure to implement the Cao-Xu controller in a 1-DOF FJR manipulator
- (iv) Presenting a comparative analysis of both controllers by means of numerical simulations with a trajectory tracking task

This paper is organized as follows. Section 2 presents the problem statement and synthesis of ISM control and $\mathcal{H}_\infty$ control for a class of time-varying systems. In Section 3, the combined ISM and nonlinear $\mathcal{H}_\infty$ tracking control is developed for a single pendulum with elastic joint affected by matched and unmatched perturbation. Here, the details regarding the implementation of the Cao-Xu controller are also presented. In Section 4, the performance of both controllers is evaluated in a simulation study. Finally, conclusions are provided in Section 5.

2. Controller Design

This work aims to design a controller for nonlinear time-varying minimum-phase underactuated systems being affected by matched and unmatched perturbations. Let us denote $x_d(t)$ as the desired reference signal and $x(t)$ as the output of the nonlinear system to be controlled. The control problem can now be defined as follows.

Control Problem. Given a smooth reference signal $(C^\infty) x_d(t)$, find a control law $u(t)$ such that

$$\lim_{t \rightarrow \infty} \|x_d(t) - x(t)\| = 0 \quad (1)$$

holds for the free-disturbance case (i.e., $w_m(t) = w_u(t) = 0$), and the $\mathcal{L}_2$ -gain of the perturbed nonlinear system is less than a positive constant level γ with respect to a given performance output $z(t) \in \mathbb{R}^l$ to be controlled.

The block diagram of the closed-loop system with the proposed controller is depicted in Figure 1. The performance

FIGURE 1: Block diagram of the closed-loop system with the proposed controller, which consists of a combination of a $\mathcal{H}_\infty$ -control and an ISM control.

output $z(t)$ is nothing other than a vector of variables, including the nonmeasurable ones, where the disturbances must be attenuated. The effect of the matched perturbation $w_m(t)$ is canceled out by means of an ISM controller $u_1(t)$. However, the resulting closed-loop system is still affected by the unmatched perturbation $w_u(t)$. Then, using an $\mathcal{H}_\infty$ -control, the $\mathcal{L}_2$ -gain of the closed-loop system is made less than a given attenuation level γ .

The next three definitions are provided in order to clarify the $\mathcal{L}_2$ -gain concept.

Definition 1 (space $\mathcal{L}_2$, [20]). The space $\mathcal{L}_2$ consists in the set of all piecewise-continuous function u such that

$$\int_0^\infty u(t)^T u(t) dt = \int_0^\infty \|u(t)\|^2 dt < \infty. \quad (2)$$

Definition 2 ($\mathcal{L}_2$ norm, [20]). $\mathcal{L}_2$ norm of all piecewise-continuous inputs $u(t)$ is given by

$$\|u(t)\|_{\mathcal{L}_2} = \left(\int_0^\infty \|u(t)\|^2 dt \right)^{1/2}. \quad (3)$$

Definition 3 ($\mathcal{L}_2$ -gain, [19]). Let $z \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ be piecewise-continuous functions, denoting the output and the input of a system, respectively. Viewing $z(t)$ and $w(t)$ as finite-energy signals, $\mathcal{L}_2$ -gain is then defined as the ratio between $\mathcal{L}_2$ norm of the output and $\mathcal{L}_2$ norm of the input bounded by γ ; that is,

$$\frac{\|z(t)\|_{\mathcal{L}_2}}{\|w(t)\|_{\mathcal{L}_2}} \leq \gamma, \quad (4)$$

and equivalently

$$\frac{\|z(t)\|_{\mathcal{L}_2}^2}{\|w(t)\|_{\mathcal{L}_2}^2} \leq \gamma^2. \quad (5)$$

Functions in $\mathcal{L}_2[0, \infty)$ represent signals having finite energy over the infinite time interval $[0, \infty)$ and therefore the number γ in inequalities (4) and (5) can be interpreted as (an upper bound of) the ratio between the energies of output and input.

The steps required to design the controller $u(t)$ are detailed in the following.

Consider a nonautonomous nonlinear system of the form

$$\dot{x} = f(x, t) + B(x, t)u + B(x, t)w_m + B^\perp(x, t)w_u, \quad (6)$$

where $t \geq 0$ is the time variable, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $B(x, t) \in \mathbb{R}^{n \times m}$, and $B^\perp(x, t) \in \mathbb{R}^{n \times (n-m)}$; the elements $w_m(t) \in \mathbb{R}^m$ and $w_u \in \mathbb{R}^{n-m}$ represent the matched and unmatched perturbations, respectively, which are uniformly bounded in t ; that is,

$$\begin{aligned} \|w_m(t)\|_\infty &\leq W_m^+, \\ \|w_u(t)\|_\infty &\leq W_u^+, \end{aligned} \quad (7)$$

where W_m^+ , W_u^+ are positive constants, known *a priori*.

The functions $f(x, t)$, $B(x, t)$, and $B^\perp(x, t)$ are assumed to be piecewise continuous in t for all x and continuously differentiable in x for all t . The matrix $B^\perp(x, t)$ is assumed to be full rank, with image orthogonal to the image of $B(x, t)$, that is, $B^T(x, t)B^\perp(x, t) = 0$ for all x and for all t [21]. From now on, it is assumed that $B(x, t)$ fulfills the following assumption.

Assumption 1. $\text{rank}(B) = m$ for all $(x, t) \in \mathbb{R}^{n+1}$.

Let us assume the next structure for the controller $u(t)$ in (6) is as follows:

$$u = u_0 + u_1, \quad (8)$$

where $u_1(t)$ will be designed as an ISM controller, with the aim of rejecting the matched perturbation $w_m(t)$. The nominal control $u_0(t)$ will be designed to drive the system trajectories to the origin, while attenuating the effect due to the unmatched perturbation $w_u(t)$.

2.1. Design of the Integral Sliding Modes Control. Let us define the sliding surface $s(x, t)$ as

$$\begin{aligned} s(x, t) &= D \left[x(t) - x(t_0) - \int_{t_0}^t [f(x, \tau) + B(x, \tau)u_0] d\tau \right], \end{aligned} \quad (9)$$

with $D \in \mathbb{R}^{m \times n}$ being a constant matrix.

Equation (9) can be seen as a penalizing factor of the difference between the actual trajectories and the trajectories of the system (6) in the absence of matched perturbations and in the presence of the control u_0 , projected along D . The sliding mode begins from the initial time t_0 ; that is, $s(x, t) = 0$ for all $t \geq t_0$. Now we assume the following.

Assumption 2. The product DB is nonsingular for all $(x, t) \in \mathbb{R}^{n+1}$.

In order to drive the trajectories of (6) to the sliding surface (9), the following feedback controller is proposed:

$$u_1 = -\Gamma \text{sign}(s), \quad (10)$$

with the matrix Γ satisfying the condition

$$\Gamma > W_m^+ + \frac{\|B^\perp(x, t)\|}{\|B(x, t)\|} W_u^+. \quad (11)$$

Once the trajectories reach the sliding surface, the dynamics of $\dot{s}$ within the set $\Delta = \{x \in \mathbb{R}^n : s = \dot{s} = 0\}$ are

$$\dot{s} = DB[u_1 + w_m] + DB^\perp w_u = 0, \quad (12)$$

and, then, the equivalent control $u_{1_{\text{eq}}}$ is obtained by solving (12) for u_1 [22]

$$u_{1_{\text{eq}}} = -w_m - [DB]^{-1} DB^\perp w_u. \quad (13)$$

By substituting (13) into (6), the equation of motion (6) is reduced to

$$\dot{x}_0 = f(x_0, t) + B(x_0, t)u_0 + g_1(x_0, t)w_u, \quad (14)$$

with

$$g_1(x_0, t) = [I - B[DB]^{-1}D]B^\perp. \quad (15)$$

Notice that, in (14), the unmatched disturbances w_u are still present. Then, a nonlinear $\mathcal{H}_\infty$ -controller is considered to attenuate the effects due to w_u .

2.2. Nonlinear $\mathcal{H}_\infty$ -Control Design. Let us consider the non-autonomous nonlinear systems of the form

$$\begin{aligned} \dot{x}_0 &= f(x_0, t) + B(x_0, t)u_0 + g_1(x_0, t)w_u, \\ z &= h_1(x_0, t) + k_{12}(x_0, t)u_0, \\ y &= h_2(x_0, t), \end{aligned} \quad (16)$$

where $z(t) \in \mathbb{R}^l$ is the performance output to be controlled and $y(t) \in \mathbb{R}^p$ is the output of the system (16).

In the forthcoming analysis, the following assumptions are considered [19].

Assumption 3. The functions $f(x_0, t)$, $B(x_0, t)$, $g_1(x_0, t)$, $h_1(x_0, t)$, $h_2(x_0, t)$, and $k_{12}(x_0, t)$ are assumed to be piecewise continuous in t for all x and continuously differentiable in x for all t .

Assumption 4. $f(0, t) = 0$, $h_1(0, t) = 0$, and $h_2(0, t) = 0$ for all $t > t_0$.

Assumption 5. $h_1^T(x_0, t)k_{12}(x_0, t) = 0$ and $k_{12}^T(x_0, t)k_{12}(x_0, t) = I$.

Assumption 3 guarantees the well-posedness of the system (16), while being enforced by integrable exogenous inputs, and allows including nonsmooth nonlinearities. Assumption 4 ensures that the origin is an equilibrium point of the nondriven ($u_0 = 0$) disturbance-free ($w_u = 0$) dynamic system (16). Assumption 5 is a simplified assumption inherited from the standard $\mathcal{H}_\infty$ -control problem [19].

By considering the full-information case, the static state-feedback controller

$$u_0(t) = \mathcal{K}(x_0, t) \quad (17)$$

is said to be an admissible controller if equilibrium point of the closed-loop system (16)-(17) is asymptotically stable when $w_u = 0$. Besides, given the disturbance attenuation level $\gamma > 0$, the system (16)-(17) is said to have $\mathcal{L}_2$ -gain less than γ if the response z , resulting from w_u for initial state $x_0(t_0) = 0$, satisfies

$$\int_{t_0}^{t_1} \|z(t)\|^2 dt < \gamma^2 \int_{t_0}^{t_1} \|w_u(t)\|^2 dt, \quad (18)$$

for all $t_1 > t_0$ and all piecewise-continuous functions $w_u(t)$. Thus, an admissible controller of the form (17) constitutes a solution of the $\mathcal{H}_\infty$ -control problem if there exists a neighborhood U of the origin such that inequality (18) is satisfied for all $t_1 > t_0$ and all piecewise-continuous functions $w_u(t)$ for which the state trajectory of the corresponding closed-loop system, starting from the initial point $x_0(t_0) = 0$, remains in U for all $t \in [t_0, t_1]$.

Let us consider the following hypothesis [19].

Hypothesis 1. For some positive γ and some positive definite function $F(x_0)$, there exists a locally Lipschitz continuous positive definite decrescent, radially unbounded solution $V(x_0, t)$ of the Hamilton-Jacobi-Isaacs (HJI) inequality

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_0} f(x_0, t) + \gamma^2 \alpha_1^T(x_0, t) \alpha_1(x_0, t) \\ - \alpha_2^T(x_0, t) \alpha_2(x_0, t) + h_1^T(x_0, t) h_1(x_0, t) \\ + F(x_0) \leq 0, \end{aligned} \quad (19)$$

specified with

$$\begin{aligned} \alpha_1(x_0, t) &= \frac{1}{2\gamma^2} g_1^T(x_0, t) \left[\frac{\partial V}{\partial x_0} \right]^T, \\ \alpha_2(x_0, t) &= -\frac{1}{2} B^T(x_0, t) \left[\frac{\partial V}{\partial x_0} \right]^T. \end{aligned} \quad (20)$$

Provided that Hypothesis 1 holds, the next theorem shows the state-feedback solution of the $\mathcal{H}_\infty$ -control problem derived in terms of the solution of the HJI inequality (19) [19].

Theorem 6. Consider the system (16), with Assumptions 1–3. Let Hypothesis 1 be satisfied. Then, the static state-feedback controller

$$u_0(t) = \alpha_2(x_0, t) \quad (21)$$

is globally admissible and, with respect to a given output $z(t)$, the $\mathcal{L}_2$ -gain of the closed-loop system (16), (21) is less than γ .

Summarizing, the following result is obtained.

Theorem 7. Consider system (6) with Assumptions 1–5 and Hypothesis 1. Let us assume that (7) holds. Then, the controller $u(t)$

$$u(x, t) = u_0(x_0, t) + u_1(x, t), \quad (22)$$

specified by the ISM controller

$$\begin{aligned} u_1 &= -\Gamma \operatorname{sign}(s), \\ s(x, t) &= D \left[x(t) - x(t_0) - \int_{t_0}^{t_1} [f(x, r) + B(x, r) u_0] dr \right], \end{aligned} \quad (23)$$

together with the static state-feedback $\mathcal{H}_\infty$ -control

$$u_0(t) = \alpha_2(x_0, t), \quad (24)$$

makes the equilibrium point of the unmatched disturbance-free closed-loop system (6), (10), and (21) asymptotically stable. Besides, the $\mathcal{L}_2$ -gain of the perturbed nominal system (16) is less than γ with respect to output $z(t)$.

Proof. The conditions for the trajectories of (6) to converge to the manifold $s(x, t) = \dot{s}(x, t) = 0$ in (9) and the sliding mode to exist on this manifold may be derived based on the Lyapunov function

$$V(s) = \frac{1}{2} s^T s, \quad (25)$$

whose time derivative along the solution of the closed-loop system (6) and (10), is given by

$$\dot{V} = s^T \dot{s} = s^T [DB(x, t) [u_1 + w_m] + DB^\perp(x, t) w_u]. \quad (26)$$

Substituting (10) into the latter equation yields

$$\begin{aligned} \dot{V} &= s^T [DB(x, t) [-\Gamma \operatorname{sign}(s) + w_m] + DB^\perp(x, t) w_u] \\ &\leq -\|s^T D\| [\|B(x, t)\| [\Gamma - W_m^+] - \|B^\perp(x, t)\| W_u^+]; \end{aligned} \quad (27)$$

and, then, $\dot{V}$ proves to be negative definite for any Γ satisfying (11). The proof is completed by following the same line of reasoning of Theorem 22 from [19], which allows concluding asymptotic stability using $\mathcal{H}_\infty$ -control for the full-information case employing system (16). $\square$

3. Application to a Flexible Joint Robot Manipulator

In order to support the applicability and performance of the proposed controller (22) given in Theorem 7, let us consider a 1-DOF FJR manipulator described by the following differential equations [23]:

$$m\ddot{q} + g \sin(q) + k[q - \theta] = w_u, \quad (28)$$

$$J\ddot{\theta} - k[q - \theta] = \tau + w_m, \quad (29)$$

where $q(t), \dot{q}(t) \in \mathbb{R}$ are the link position and velocity, respectively; $\theta(t), \dot{\theta}(t)$ are the joint position and velocity, respectively; $m > 0$ is the link mass; $g > 0$ is the gravity constant; $k > 0$ is the spring stiffness constant; $J > 0$ is the rotor inertia; $\tau \in \mathbb{R}$ is the torque applied at the joint; and w_m, w_u are the matched and unmatched perturbations, respectively.

The tracking control problem for this system can be established as follows.

Control Problem. Given the smooth desired reference signal $q_d(t) \in \mathbb{R}$, find a control law $\tau(t)$ such that, for any initial condition $q(t_0)$, the limit

$$\lim_{t \rightarrow \infty} \|q_d(t) - q(t)\| = 0 \quad (30)$$

holds for the disturbance-free system (28)-(29), and the $\mathcal{L}_2$ -gain of the perturbed nominal system (28)-(29) is less than γ with respect to a given performance output $z(t)$.

In order to apply the proposed controller stated in Theorem 7 to a FJR manipulator, let us consider the block diagram depicted in Figure 2. First, the dynamic equation (29) is decoupled from (28) using the controller $\tau(t)$. This decoupling controller combines the ISM control (10) and the $\mathcal{H}_\infty$ -control (24) together with a properly defined reference signal $\theta_d(t)$ and its first and second time derivatives. The reference signal $\theta_d(t)$ is generated using the FJR manipulator output measurements and the outputs generated by $\mathcal{H}_\infty$ and ISM controllers. In the following, the steps required to obtain the controller $\tau(t)$ are described.

3.1. Stability Analysis. The first step to follow consists of designing a control law $\tau(t)$ that decouples the dynamic equation (29) from (28). To this end, let us consider the control law

$$\tau = k[\theta - q] + k_{df}\dot{\tilde{\theta}} + k_{pf}\tilde{\theta} + J\ddot{\theta}_d + u, \quad (31)$$

where $k_{pf}, k_{df} \in \mathbb{R}^+$ are positive constants, $\tilde{\theta} = \theta_d - \theta$ is the joint position error, $\dot{\tilde{\theta}}$ is the joint velocity error, $\ddot{\theta}_d$ is the second time derivative of the joint reference signal $\theta_d(t)$, and u is an auxiliary control input. Both u and θ_d are to be defined later.

Let us consider the following change of coordinates:

$$\begin{aligned} x_1 &= \tilde{q} = q_d - q, \\ x_2 &= \dot{\tilde{q}} = \dot{q}_d - \dot{q}, \\ x_3 &= \tilde{\theta} = \theta_d - \theta, \\ x_4 &= \dot{\tilde{\theta}} = \dot{\theta}_d - \dot{\theta} \end{aligned} \quad (32)$$

with $x = [x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4$.

By using (28), (29), and (32), the next error dynamics state-space representation is obtained

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \ddot{q}_d - m^{-1} [w_u - g \sin(q_d - x_1) \\ &\quad - k[q_d - x_1 - \theta_d + x_3]] = m^{-1} [m\ddot{q}_d \\ &\quad + g \sin(q_d - x_1) + k[x_3 - x_1 + q_d - \theta_d]] \end{aligned}$$

FIGURE 2: Block diagram of the methodology used for applying the proposed controller (22) to a FJR manipulator.

$$\begin{aligned}
 & -m^{-1}w_u, \\
 \dot{x}_3 &= x_4, \\
 \dot{x}_4 &= -J^{-1} [k_{pf}x_3 + k_{df}x_4 + u + w_m].
 \end{aligned} \tag{33}$$

$$B^\perp = \begin{bmatrix} 0 \\ -m^{-1} \\ 0 \\ 0 \end{bmatrix}. \tag{36}$$

Now, let us define the reference signal $\theta_d(t)$ as follows:

$$\begin{aligned}
 \theta_d &= q_d - x_1 \\
 &+ k^{-1} [g \sin(q_d - x_1) + m\ddot{q}_d + k_{pc}x_1 + k_{dc}x_2].
 \end{aligned} \tag{34}$$

Using (34), the error dynamics (33) can be rewritten in the next state-space form:

$$\begin{aligned}
 \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} x_2 \\ m^{-1} [-k_{pc}x_1 - k_{dc}x_2 + kx_3] + [-m^{-1}] w_u \\ x_4 \\ -J^{-1} [k_{pf}x_3 + k_{df}x_4 + u + w_m] \end{bmatrix}.
 \end{aligned} \tag{35}$$

The expression (35) can now be written in the form (6), with

$$\begin{aligned}
 f(x, t) &= \begin{bmatrix} x_2 \\ m^{-1} [-k_{pc}x_1 - k_{dc}x_2 + kx_3] \\ x_4 \\ -J^{-1} [k_{pf}x_3 + k_{df}x_4] \end{bmatrix}, \\
 B(x, t) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ -J^{-1} \end{bmatrix},
 \end{aligned}$$

Let us select the matrix $D \in \mathbb{R}^{1 \times 4}$ in (9) as

$$D = [0 \ 0 \ 0 \ 1]. \tag{37}$$

From (36)-(37), it is clear that Assumptions 1 and 2 are satisfied; that is, $\text{rank}(B) = 1$ and $DB = -J^{-1}$ is nonsingular. Then, by assuming that w_u, w_m satisfy inequalities (7), controller (10) drives the trajectories of system (35) towards the sliding surface s given in (9). Then, the closed-loop system (35) and (10) can be expressed in the form (16), with $g_1(x_0, t)$ given by

$$g_1(x, t) = [0 \ -m^{-1} \ 0 \ 0]^T. \tag{38}$$

Since the full-information case is considered, the performance output $z(t)$ to be controlled and the output $y(t)$ are selected as

$$\begin{aligned}
 z &= \rho \begin{bmatrix} 0 \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ 0_{4 \times 1} \end{bmatrix} u_0, \\
 y &= x,
 \end{aligned} \tag{39}$$

with $\rho \in \mathbb{R}^+$ a positive constant. Then, the elements h_1, k_{12}, h_2 are given by

$$\begin{aligned}
 h_1 &= \rho \begin{bmatrix} 0 \\ x \end{bmatrix}, \\
 k_{12} &= \begin{bmatrix} 1 \\ 0_{4 \times 1} \end{bmatrix}, \\
 h_2 &= x.
 \end{aligned} \tag{40}$$

Note that the functions g_1, h_1, h_2, k_{12} given in (38) and (40) satisfy Assumptions 3 and 5. Besides, the functions f, h_1, h_2 given in (36) and (40) satisfy Assumption 4.

Now, let us define the smooth functions $F(x)$ and $V(x)$ as

$$F(x) = \varepsilon [x_1^2 + x_2^2 + x_3^2 + x_4^2], \quad (41)$$

$$V(x) = \frac{1}{2}\beta_1 x_1^2 + \frac{1}{2}\beta_2 x_2^2 + \frac{1}{2}\beta_3 x_3^2 + \frac{1}{2}\beta_4 x_4^2 + \beta_5 x_1 x_2 + \beta_6 x_3 x_4, \quad (42)$$

with $\varepsilon, \beta_i, i = 1, \dots, 6$, being positive constants. Besides, the constants β_1, β_2 are defined as

$$\begin{aligned} \beta_1 &= \frac{\beta_2 k_{pc}}{m} + \frac{\beta_5 k_{dc}}{m} - \frac{\beta_2 \beta_5}{2\gamma^2 m^2}, \\ \beta_2 &= \frac{k_{pf} \beta_4 + k_{df} \beta_6}{J} + \frac{\beta_4 \beta_6}{2J^2}. \end{aligned} \quad (43)$$

Functions $F(x)$ and $V(x)$ are positive definite if the inequalities

$$\begin{aligned} \beta_1 \beta_2 - \beta_5^2 &> 0, \\ \beta_3 \beta_4 - \beta_6^2 &> 0, \end{aligned} \quad (44)$$

hold.

Using (38) and (42), the functions α_1, α_2 in (20) are given by

$$\begin{aligned} \alpha_1 &= -\frac{1}{2\gamma^2 m} [\beta_5 x_1 + \beta_2 x_2], \\ \alpha_2 &= \frac{1}{2J} [\beta_6 x_3 + \beta_4 x_4], \end{aligned} \quad (45)$$

and the left hand side of the HJI inequality (19), defined as $\mathcal{H}$, is given by

$$\mathcal{H} = -\sum_{i=1}^4 \delta_i x_i^2 + \frac{\beta_2 k}{m} x_2 x_3 + \frac{\beta_5 k}{m} x_1 x_3, \quad (46)$$

with

$$\begin{aligned} \delta_1 &= \frac{\beta_5 k_{pc}}{m} - \frac{\beta_5^2}{4\gamma^2 m^2} - \rho^2 - \varepsilon, \\ \delta_2 &= \frac{\beta_2 k_{dc}}{m} - \frac{\beta_2^2}{4\gamma^2 m^2} - \beta_5 - \rho^2 - \varepsilon, \\ \delta_3 &= \frac{\beta_6 k_{pf}}{J} + \frac{\beta_6^2}{4J^2} - \rho^2 - \varepsilon, \\ \delta_4 &= \frac{\beta_4 k_{df}}{J} + \frac{\beta_4^2}{4J^2} - \beta_6 - \rho^2 - \varepsilon. \end{aligned} \quad (47)$$

After completing squares, the expression (46) can be upper bounded as follows:

$$\begin{aligned} \mathcal{H} &\leq -\left[\delta_1 - \frac{\beta_5 k_{pc}}{2m}\right] x_1^2 - \left[\delta_2 - \frac{\beta_2 k_{dc}}{2m}\right] x_2^2 \\ &\quad - \left[\delta_3 - \frac{\beta_5 k^2}{2mk_{pc}} - \frac{\beta_2 k^2}{2mk_{dc}}\right] x_3^2 - \delta_4 x_4^2; \end{aligned} \quad (48)$$

and, then, $\mathcal{H}$ will be negative semidefinite if $k_{pc}, k_{dc}, k_{pf}, k_{df}$ fulfill the following inequalities:

$$\begin{aligned} k_{pc} &\geq \frac{2m}{\beta_5} \left[\frac{\beta_5^2}{4\gamma^2 m^2} + \rho^2 + \varepsilon \right], \\ k_{dc} &\geq \frac{2m}{\beta_2} \left[\frac{\beta_2^2}{4\gamma^2 m^2} + \beta_5 + \rho^2 + \varepsilon \right], \\ k_{pf} &\geq \frac{J}{\beta_6} \left[\frac{\beta_5 k^2}{2mk_{pc}} + \frac{\beta_2 k^2}{2mk_{dc}} - \frac{\beta_6^2}{4J^2} + \rho^2 + \varepsilon \right], \\ k_{df} &\geq \frac{J}{\beta_4} \left[\beta_6 + \rho^2 + \varepsilon - \frac{\beta_4^2}{4J^2} \right]. \end{aligned} \quad (49)$$

Then, the following theorem can be stated.

Theorem 8. Let us consider the system (28)-(29), with the reference signal $\theta_d(t)$ defined by (34). Assume that there exist positive constants $\beta_i, i = 1, \dots, 6$, with β_1, β_2 being defined by (43). If the inequalities (44) and (49) hold, controller (31), with $u = u_0 + u_1$, where u_1 is given by (10) and u_0 is given by

$$u_0(t) = \alpha_2(x, t) = \frac{1}{2J} [\beta_6 x_3 + \beta_4 x_4], \quad (50)$$

accomplishes the control problem (30) for the disturbance-free system (28)-(29) and makes the $\mathcal{L}_2$ -gain of the perturbed nominal system (28)-(29) less than γ with respect to the output $z(t)$ given in (40).

3.2. Comparison with a Nonlinear Controller. The performance of the proposed controller was compared with that corresponding to the nonlinear controller developed by Cao and Xu [13], modified to deal also with matched and unmatched uncertainties. The next development details the procedure to implement the Cao-Xu controller in a 1-DOF FJR manipulator.

Let us consider the unperturbed version of the 1-DOF FJR manipulator described in (28)-(29) (i.e., $w_u = w_m = 0$ for all $t \geq 0$), that is,

$$m\ddot{q} + g \sin(q) + k[q - \theta] = 0, \quad (51)$$

$$J\ddot{\theta} - k[q - \theta] = \tau. \quad (52)$$

Under the same line of reasoning presented in Section 3.1, the dynamics of (52) are decoupled from (51) using the following control law:

$$\tau = k[\theta - q] + J\ddot{\theta}_d + k_{d1}\dot{\theta} + k_{p1}\tilde{\theta}. \quad (53)$$

Substituting (53) into (52) yields

$$J\ddot{\tilde{\theta}} + k_{d1}\dot{\tilde{\theta}} + k_{p1}\tilde{\theta} = 0, \quad (54)$$

which can be written in state-space form as

$$\frac{d}{dt} \begin{bmatrix} \tilde{\theta} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} \dot{\tilde{\theta}} \\ -\frac{1}{J} [k_{p1}\tilde{\theta} + k_{d1}\dot{\tilde{\theta}}] \end{bmatrix}. \quad (55)$$

By a proper selection of the gains k_{p1} , k_{d1} , the origin $[\tilde{\theta}, \dot{\tilde{\theta}}]^T = 0_{2 \times 1}$ of the system (55) is exponentially stable, that is, $\theta(t)$ converges exponentially to $\theta_d(t)$.

Now, we will prove that if $\theta(t) = \theta_d(t)$, this implies the exponential convergence of $q(t)$ to the desired link reference signal $q_d(t)$. With this aim, let us define the desired rotor reference signal $\theta_d(t)$ as follows:

$$\theta_d(t) = \frac{1}{k} [k_{p2}\tilde{q} + k_{d2}\dot{\tilde{q}} + m\ddot{q}_d + g \sin(q)] + q. \quad (56)$$

Now, by substituting (56) into (51), assuming that $\theta = \theta_d$, the following expression is obtained:

$$m\ddot{\tilde{q}} + k_{d2}\dot{\tilde{q}} + k_{p2}\tilde{q} = 0, \quad (57)$$

whose state-space representation is

$$\frac{d}{dt} \begin{bmatrix} \tilde{q} \\ \dot{\tilde{q}} \end{bmatrix} = \begin{bmatrix} \dot{\tilde{q}} \\ -\frac{1}{m} [k_{p2}\tilde{q} + k_{d2}\dot{\tilde{q}}] \end{bmatrix}. \quad (58)$$

A proper selection of the gains k_{p2} , k_{d2} allows concluding the exponential convergence of the origin $[\tilde{q}, \dot{\tilde{q}}]^T = 0_{2 \times 1}$. Thus, the controller (53), with $\theta_d(t)$ defined as in (56), exponentially stabilizes the unperturbed nonlinear system (51)-(52), as was specified for the controller presented in [13].

Consider now the perturbed system (28)-(29), represented in its state-space form as

$$\dot{x}(t) = \bar{f}(x, t) + \bar{B}(x, t) \{ [1 + \Delta B_m] u + \Delta f_m \} + \Delta f_u, \quad (59)$$

where $x(t) \in \mathbb{R}^4$ is defined in (32) and

$$\bar{f}(x, t) = \begin{bmatrix} x_2 \\ -\frac{k_{p2}}{m}x_1 - \frac{k_{v2}}{m}x_2 + \frac{k}{m}x_3 \\ x_4 \\ \frac{k_{p2}}{J}x_1 + \frac{k_{v2}}{J}x_2 - \frac{k}{J}x_3 + \frac{m}{J}\ddot{q}_d + \frac{g}{J}\sin(q_d - x_1) + \ddot{\theta}_d \end{bmatrix}, \quad (60)$$

$$\bar{B}(x, t) = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{J} \end{bmatrix}^T,$$

$$\Delta f_u = \begin{bmatrix} 0 & -\frac{w_u}{m} & 0 & 0 \end{bmatrix}^T,$$

$$\Delta B_m = 0,$$

$$\Delta f_m = w_m.$$

Thus, the controller presented in [13] to be applied to the latter 1-DOF FJR manipulator is

$$\tau(t) = \begin{cases} \bar{k}(x, t) - \bar{p}(x, t) \frac{[\overline{DB}(x, t)]^T \sigma(x, t)}{\|[\overline{DB}(x, t)]^T \sigma(x, t)\|}, & \text{if } \sigma \neq 0, \\ \bar{k}(x, t), & \text{if } \sigma = 0, \end{cases} \quad (61)$$

where $\sigma(x, t)$ is the nonlinear integral-type sliding surface defined as

$$\sigma(x, t) = \overline{D}x(t) - \overline{D}x(t_0) - \int_{t_0}^t [\overline{D}\bar{f}(x, \zeta) + \overline{D}\bar{B}\bar{k}(x, \zeta)] d\zeta = 0, \quad (62)$$

$$\bar{k}(x, t) = k[\theta - q] + J\ddot{\theta}_d + k_{d1}\dot{\tilde{\theta}} + k_{p1}\tilde{\theta},$$

and $\overline{D} \in \mathbb{R}^{1 \times 4}$ is a constant row vector.

The switching surface must also fulfill the following inequality:

$$\begin{aligned} \bar{\rho}(x, t) &> \frac{1}{\epsilon_b} \left\{ \rho_m(x, t) + [1 - \epsilon_b] \|k(x, t)\| \right. \\ &\quad \left. + \left\| [\overline{DB}(x, t)]^{-1} \overline{D} \right\| \rho_u \right\}, \end{aligned} \quad (63)$$

provided $\|\Delta f_m\| \leq \rho_m$, $\|\Delta B_m\| \leq 1 - \epsilon_b$, $\epsilon_b > 0$, and $\|\Delta f_m\| \leq \rho_u$.

4. Simulation Results

This section presents the numerical simulation results obtained using the proposed controller and the Cao-Xu controller. In the following the proposed method and the Cao-Xu controller will be referred to as the iSMH controller and the Cao-Xu controller, respectively. Simulations were performed using the software MATLAB-SIMULINK®, using the solver ode45 Dormand-Prince. The parameters of the model (28)-(29) were set to $m = 1.0001$ [kg], $k = 100$ [Nm/rad], $J = 0.02$ [kgm²], and $g = 9.81$ [m/s²], and all the initial conditions were set to zero.

The numerical simulations considered two trajectories to follow:

- (1) A sinusoidal desired link reference signal described by

$$q_d(t) = \sin(3t) \text{ [rad]} \quad (64)$$

- (2) A combination of sinusoids for the desired link reference signal; that is,

$$q_d(t) = \pi \sin(2t) + 1.5 \sin(3t) + 2 \sin(5t) \text{ [rad]} \quad (65)$$

In order to provide a set of realistic gains for tuning the controllers, the parameters corresponding to an existing platform of Quanser [24] were considered. Specifically, the Maxon 273759 DC motor was taken as reference model, which has a torque constant $k_m = 0.119$ [Nm/A]. Besides, it was assumed that the motor drives an harmonic gearbox with gear ratio of $g_r = 100$. The previous motor's specifications allow obtaining the required voltage for each controller, in other words, the voltage v demanded by each controller is obtained by using the following equation:

$$v = \frac{\tau}{k_m g_r}. \quad (66)$$

FIGURE 3: First simulation. Link position error $\tilde{q}$ and joint position error $\tilde{\theta}$ corresponding to the iSMH controller and the Cao-Xu controller, respectively, using the link reference signal (64).

The controller gains for both, the iSMH and Cao-Xu controllers, was tuned to obtain a similar behavior for the link and rotor position errors. These gains fulfill the constraints inherent of each controller. Furthermore, it is assumed that the magnitude of the maximum allowable voltage was 10 [volts]. In all simulations, the matched perturbation $w_m(t)$ was selected as a square signal with amplitude of 1 [rad] and frequency of 1 [rad/s]. The unmatched perturbation was chosen as a sinusoidal signal with unitary amplitude and frequency of 1 [rad/s].

For the first simulation, using the reference signal (64), the gains corresponding to the iSMH controller (31) and Cao-Xu controller (61) were set to

$$k_{pc} = 50,$$

$$k_{dc} = 5,$$

$$k_{pf} = 10,$$

$$k_{df} = 1,$$

$$\beta_4 = 5,$$

$$\beta_6 = 100,$$

$$\Gamma = 0.05,$$

$$k_{p1} = 150,$$

$$k_{d1} = 5,$$

$$k_{p2} = 50,$$

$$k_{d2} = 5,$$

$$\bar{p} = 6.$$

(67)

The link position error $\tilde{q}$ and joint position error $\tilde{\theta}$ for both controllers are depicted in Figure 3. Notice that the position errors are similar as was prespecified. However, it can be observed that a more oscillatory behavior is exhibited by the Cao-Xu controller.

FIGURE 4: First simulation. Behavior of the sliding surfaces s , σ and the voltage v demanded by the iSMH and the Cao-Xu controllers using the reference signal (64).

In Figure 4, the behavior of the sliding surfaces s , σ and the voltage demanded by each controller are shown. It is clear that the system trajectories are closer to the ideal sliding mode using the iSMH controller. Besides, the voltage demanded by the Cao-Xu controller is bigger than that required by the proposed controller.

For the second simulation, using the reference signal (65), the gains corresponding to the iSMH controller (31) and Cao-Xu controller (61) were set to

$$k_{pc} = 60,$$

$$k_{dc} = 5,$$

$$k_{pf} = 10,$$

$$k_{df} = 0.5,$$

$$\beta_4 = 5,$$

$$\beta_6 = 20,$$

$$\Gamma = 0.1,$$

$$k_{p1} = 700,$$

$$k_{d1} = 30,$$

$$k_{p2} = 45,$$

$$k_{d2} = 20,$$

$$\bar{\rho} = 7.$$

(68)

The link position error and joint position error for the iSMH and Cao-Xu controllers are depicted in Figure 5. Again, the behavior of the position errors is similar for both controllers. However, it is observed again that a more oscillatory behavior is exhibited by the Cao-Xu controller.

In Figure 6, the behavior of the sliding surfaces s , σ and the voltage demanded by each controller are shown. It is

FIGURE 5: Second simulation. Link position error $\tilde{q}$ and joint position error $\tilde{\theta}$ corresponding to the iSMH controller and the Cao-Xu controller, respectively, using the link reference signal (65).

clear that the system is closer to the ideal sliding mode using the iSMH controller, and the voltage demanded by the Cao-Xu controller is slightly greater than that required by the proposed controller.

The previous simulation results show that, although similar responses may be obtained using any of the controllers, the iSMH requires less energy to achieve the control objective, while the oscillatory behavior is significantly less than that of the Cao-Xu controller. The reason for such behavior is that the proposed controller attenuates the effect of the unmatched uncertainty w_u by means of the $\mathcal{H}_\infty$ -control. To illustrate this, Figure 7 shows the effect of the $\mathcal{H}_\infty$ -control in the performance of the closed-loop system. The first 2.5 seconds show the behavior of the joint position error, link position error, and the sliding surface when the $\mathcal{H}_\infty$ -control is absent. After this time interval, the behavior of these signals is shown but now turning on the $\mathcal{H}_\infty$ -control. It is observed that the vibration at the joint is considerably reduced by introducing the $\mathcal{H}_\infty$ -control. The unmatched

perturbation affects the performance of the ISM control. However, when the $\mathcal{H}_\infty$ -control is applied, the effect of $w_u(t)$ on the performance of the closed-loop system is reduced, which shows the advantage of using the proposed controller.

The previous numerical simulations validate the proposed methodology for controlling underactuated nonlinear time-varying systems affected by matched and unmatched perturbations. Besides, it was observed that it was convenient to use the combination of the ISM control with a $\mathcal{H}_\infty$ -control because the latter reduces vibrations and allows the ISM controller to have a better performance, even in presence of unmatched disturbances.

It is important to remark that in practice, a saturation function can also be used instead of the signum function given in (10). From sliding mode control theory analysis, signum function allows satisfying the robustness property and the reachability condition, that is, finite time convergence. However, the chattering phenomenon emerges, caused by fast switching in real applications. Instead, saturation

FIGURE 6: Second simulation. Behavior of the sliding surfaces s, σ and the voltage v demanded by the iSMH and the Cao-Xu controllers using the reference signal (65).

FIGURE 7: Effect of using the $\mathcal{H}_\infty$ -controller. The first 2.5 s shows the behavior of the link position error (a), joint position error (b), and the sliding surface (c), when the $\mathcal{H}_\infty$ -control is absent. The following 2.5 s shows the behavior of the same signals when the $\mathcal{H}_\infty$ -control is applied.

functions are normally brought into play in order to avoid such undesirable phenomenon (see [25], e.g.).

5. Conclusion

In this paper, a controller for time-varying minimum-phase underactuated systems affected by matched and unmatched perturbations was presented. The proposed controller uses a combination of an ISM controller with $\mathcal{H}_\infty$ -control. A Lyapunov approach was used to prove theoretically asymptotic stability of the equilibrium of the disturbance-free closed-loop system when applying the proposed controller and the boundedness of the solutions of the closed-loop system for the perturbed case. It also included an example using a flexible joint robot manipulator to show how the proposed controller can be applied to an underactuated system. In order to assess the performance of the proposed controller, an existing nonlinear controller based on a nonlinear integral-type sliding surface was implemented. Numerical simulations have shown that the proposed controller has the best performance despite the presence of matched and unmatched perturbations.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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