Research Article

Existence, Multiplicity, and Stability of Positive Solutions of a Predator-Prey Model with Dinosaur Functional Response

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We investigate the property of positive solutions of a predator-prey model with Dinosaur functional response under Dirichlet boundary conditions. Firstly, using the comparison principle and fixed point index theory, the sufficient conditions and necessary conditions on coexistence of positive solutions of a predator-prey model with Dinosaur functional response are established. Secondly, by virtue of bifurcation theory, perturbation theory of eigenvalues, and the fixed point index theory, we establish the bifurcation of positive solutions of the model and obtain the stability and multiplicity of the positive solution under certain conditions. Furthermore, the local uniqueness result is studied when b and d are small enough. Finally, we investigate the multiplicity, uniqueness, and stability of positive solutions when $k > 0$ is sufficiently large.

1. Introduction

The dynamic relationship between predator and their prey is one of dominant themes in ecology and mathematical ecology. Some models have been studied from various views and obtained many good results (see [1–16] and the references therein). In this paper, we are concerned with the predator-prey model with Dinosaur functional response under the homogeneous Dirichlet boundary conditions as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= u(a - u - bve^{-ku}), \quad x \in \Omega, \\ \frac{\partial v}{\partial t} - \Delta v &= v(c - v + due^{-ku}), \quad x \in \Omega, \\ u = v = 0, \quad x &\in \partial\Omega \\ u(x, 0) &= u_0(x), \\ v(x, 0) &= v_0(x), \\ x &\in \partial\Omega, \end{aligned} \quad (1)$$

where Ω is a bounded domain in R^n ($n \geq 1$) with smooth boundary $\partial\Omega$ and u, v stand for the densities of prey and predators, respectively. a, b, c, d, k are positive constants. a and c denote prey intrinsic growth rate and predator intrinsic growth rate and then decrease to zero when $u \rightarrow \infty$, as follows:

$$\begin{aligned} \frac{d}{du} ue^{-ku} &> 0, \quad 0 \leq u < \frac{1}{k}, \\ \frac{d}{du} ue^{-ku} &< 0, \quad u > \frac{1}{k}. \end{aligned} \quad (2)$$

The Dinosaur reaction term is an improvement or simplification of the Ivlev-type reaction term, and the change on the species density of prey better than Ivlev-type functional response $h(1 - e^{-ku})$ can be explained. It is easy to see that the Dinosaur reaction term describes prey focus on the fight against predators when the species density of prey is large enough, so as to achieve better defense or disguise itself. To see more biological significance of systems with Ivlev-type functional responses, one can resort to [17–28] and their

contents. The research on existence and uniqueness of the limit cycle of a predator-prey model with Ivlev response can be found in [22, 23]. The permanence and existence and stability of positive periodic solutions of the model were studied in [24–26]. The spatial pattern formation of the model was investigated by using Hopf bifurcation in [27, 28]. Some dynamical behavior analysis of the Ivlev response predator-prey systems was established in [17–21]. However, the researches on system (1) are very few. Hence, this paper mainly aims at establishing the existence, bifurcation, and multiplicity of positive solutions on the corresponding elliptic equations to system (1).

The research on the steady-states in reaction-diffusion model is the hot point question [1–4, 9, 10, 13, 16]. In the present paper, we study the steady-state problem corresponding to (1), with the specific form as follows:

$$\begin{aligned} -\Delta u &= u(a - u - bve^{-ku}), \quad x \in \Omega, \\ -\Delta v &= v(c - v + due^{-ku}), \quad x \in \Omega, \\ u &= v = 0, \quad x \in \partial\Omega. \end{aligned} \quad (3)$$

Motivated by the papers [1–4], in the present paper, we mainly consider the positive solution of (3). In Section 2, we give some basic results and calculate the fixed point index by the fixed point index theory. In Section 3, we apply the results obtained in Section 2 to study the existence of positive solutions. In Section 4, by virtue of bifurcation theory, perturbation theory of eigenvalues, and the fixed point index theory, we establish the bifurcation of positive solutions of (3) and obtain the stability and multiplicity of the positive solution under certain conditions. Furthermore, the local uniqueness result is studied when b and d are small enough. Finally, we investigate the multiplicity, uniqueness, and stability of positive solutions when $k > 0$ is sufficiently large.

2. Preliminaries

In this section, we mainly calculate the fixed point index by the fixed point index theory, in the following, and set up some definitions which are used in last paper.

Let E be a Banach space. $W \subset E$ is called a wedge if W is a closed convex set $\alpha W \subset W$ for all $\alpha \geq 0$. For any $y \in W$, define

$$\begin{aligned} W_y &= \{x \in E : \exists r = r(x) > 0, \text{ s.t. } y + rs \in W\}, \\ S_y &= \{x \in \overline{W}_y : -x \in \overline{W}_y\}, \end{aligned} \quad (4)$$

and we always assume that $E = \overline{W} - \overline{W}$. Let $T : W_y \rightarrow W_y$ be a compact linear operator on E . We say that T has property α on $\overline{W}_y$ if there exist $t \in (0, 1)$ and $\omega \in \overline{W}_y \setminus S_y$ such that $\omega - tT\omega \in S_y$. Suppose that $F : W \rightarrow W$ is a compact operator for the fixed point $y_0 \in W$, and F is Fréchet differentiable at y_0 . Set $L = F'(y_0)$ as the Fréchet differentiable of F at y ; hence $L : \overline{W} \rightarrow \overline{W}$.

Lemma 1 (see [29, 30]). *Assuming that $I - L$ is invertible on E , then one has*

- (i) L has property α on $\overline{W}$, and then $\text{index}_w(F, y_0) = 0$.
- (ii) L does not have property α on $\overline{W}$, and then $\text{index}_w(F, y_0) = (-1)^\sigma$, where σ is the sum of algebra multiplicities of the eigenvalues of L which are greater than one.

Let $q(x) \in L^\infty$, $k > 0$ denote the principal eigenvalue of the following problem:

$$-\Delta(\phi) + q(x)\phi = \lambda\phi, \quad x \in \Omega, \quad \phi = 0, \quad x \in \partial\Omega, \quad (5)$$

and, by $\lambda_1(q(x))$, it follows from [3, 4] that $\lambda_1(q(x))$ is strictly increasing in the case that $q_1(x) \leq q_2(x)$ and $q_1(x) \neq q_2(x)$ implies $\lambda_1(q_1(x)) < \lambda_1(q_2(x))$. For convenience we denote $\lambda_1(0)$ by λ_1 . Let $r(T)$ be the spectral radius of linear operator T on Banach space.

Lemma 2 (see [24, 30]). *Suppose that $q(x) \in C^1$, and P is large enough such that $\forall x \in \Omega$, $-q(x) + P > 0$. Then the following results hold:*

- (i) $\lambda_1(q_1(x)) < 0 \Rightarrow r[(-\Delta + P)^{-1}(-q(x) + P)] > 1$.
- (ii) $\lambda_1(q_1(x)) > 0 \Rightarrow r[(-\Delta + P)^{-1}(-q(x) + P)] < 1$.
- (iii) $\lambda_1(q_1(x)) = 0 \Rightarrow r[(-\Delta + P)^{-1}(-q(x) + P)] = 1$.

Next, we present some a priori estimate by maximum principle which is similar to [3, 4]; its proof will be omitted.

Theorem 3. *Any positive solution (u, v) of (3) has an a priori boundary; that is,*

$$\begin{aligned} u(x) &\leq a, \\ v(x) &\leq c + \frac{d}{ke}. \end{aligned} \quad (6)$$

Consider the following equation:

$$-\Delta\theta = \theta(\rho - \theta), \quad x \in \Omega, \quad \theta = 0, \quad x \in \partial\Omega; \quad (7)$$

it is easy to see that (7) has unique positive solution denoted by θ_ρ when $\rho > \lambda_{1,k}$. Hence, if $a > \lambda_1$, then (3) has a semitrivial solution $(\theta_a, 0)$; if $c > \lambda_1$, (3) has a semitrivial solution $(0, \theta_c)$.

Now, some notations are introduced as follows:

- (1) $E : c(\overline{\Omega}) \oplus c(\overline{\Omega})$, here $c(\overline{\Omega}) := \{\phi \in C(\overline{\Omega}) : \phi = 0, \quad x \in \partial\Omega, \quad v = 0\}$.
- (2) $W : K \oplus K$, here $K := \{\phi \in C(\overline{\Omega}) : 0 \leq \phi(x), \quad x \in \Omega\}$.
- (3) $D : \{(u, v) \in E : u \leq a + 1, \quad v \leq v + d/ke + 1\}$.
- (4) $D' := \text{int } D \cap W$.

Define a compact operator $F : D' \rightarrow W$ by the following form:

$$F(u, v) = (-\Delta + P)^{-1} \begin{pmatrix} u(a - u - bve^{-ku}) + Pu \\ v(c - v + due^{-kv}) + Pv \end{pmatrix}, \quad (8)$$

where P is sufficiently large constant such that $P > \max\{a + b(c + d/ke), c + d/ke\}$. It follows that (3) has a positive solution in W if and only if F has a positive fixed point in D' . Hence, $(0, 0)$, $(\theta_a, 0)$, $(0, \theta_c)$ are the only isolated fixed points of F if $a, c > \lambda_1$. Therefore, $\text{index}_w(F, (0, 0))$, $\text{index}_w(F, (\theta_a, 0))$, $\text{index}_w(F, (0, \theta_c))$ are well-defined, and the indices can be calculated, similar to Lemmas 2.3-2.4 in [3]; we can directly give Lemmas 4 and 5 as follows.

Lemma 4. Suppose $a > \lambda_1$.

- (i) $\text{index}_w(F, D') = 1$.
- (ii) $\text{index}_w(F, (0, 0)) = 0$.
- (iii) If $c > \lambda_1(-d\theta_a e^{-k\theta_a})$, then $\text{index}_w(F, (\theta_a, 0)) = 0$.
- (iv) If $c < \lambda_1(-d\theta_a e^{-k\theta_a})$, then $\text{index}_w(F, (\theta_a, 0)) = 1$.

Lemma 5. Suppose $c > \lambda_1$. (i) If $a > \lambda_1(b\theta_c)$, then $\text{index}_w(F, (0, \theta_c)) = 0$; (ii) if $a < \lambda_1(b\theta_c)$, then $\text{index}_w(F, (0, \theta_c)) = 1$.

3. Existence of Positive Solutions

In the section, we apply the result of Lemmas 4 and 5 obtained in Section 2 to study the existence and nonexistence of positive solutions of (3).

Theorem 6. (i) If $c > \lambda_1$, $a > \lambda_1(b\theta_c)$, then (3) has at least one positive solution.

(ii) If $c < \lambda_1$, then (3) has a positive solution if and only if $a > \lambda_1$ and $c > \lambda_1(-d\theta_a e^{-k\theta_a})$.

Proof. (i) According to Lemmas 4 and 5 and the additivity property of the index, we have

$$\begin{aligned} \text{index}_w(F, D') - \text{index}_w(F, (0, 0)) \\ - \text{index}_w(F, (\theta_a, 0)) - \text{index}_w(F, (0, \theta_c)) = 1. \end{aligned} \quad (9)$$

Hence, (3) has at least one positive solution.

(ii) If $c < \lambda_1$, then (3) has no solution taking the form $(0, v)$ ($v > 0$). Supposing $a > \lambda_1$ and $c > \lambda_1(-d\theta_a e^{-k\theta_a})$, note that $c < \lambda_1$, and by Lemma 5, we can get the following:

$$\begin{aligned} \text{index}_w(F, D') - \text{index}_w(F, (0, 0)) \\ - \text{index}_w(F, (\theta_a, 0)) = 1. \end{aligned} \quad (10)$$

Hence, (3) has at least one positive solution. So the sufficiency is proved. Conversely, suppose that (u, v) is a positive solution of (3); then $a < \lambda_1$, $u < \theta_a$. This is because (u, v) satisfies

$$-\Delta v = v(c - v + due^{-kv}), \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega. \quad (11)$$

It follows that $0 = \lambda_1(-c + v - due^{-kv}) > \lambda_1(-c - d\theta_a e^{-k\theta_a})$. This completes the proof of Theorem 6. $\square$

Theorem 7. (i) If $a \leq \lambda_1$, then (3) has no positive solutions; if $a \leq \lambda_1$ and $c \leq \lambda_1$, then (3) has no nonnegative nonzero solution.

(ii) If $c \leq \lambda_1$, then the conditions $a > \lambda_1$, $c + d/ke > \lambda_1$ are necessary for the existence of positive solution of (3). In addition, as $a < 1/k$, the condition $c > \lambda_1(-d\theta_a e^{-k\theta_a})$ can take place of the necessary condition $c + d/ke > \lambda_1$.

(iii) If $c > \lambda_1$, then the conditions $a > \max\{\lambda_1(b\theta_c), \lambda_1(b\theta_c)e^{-k\theta_a}\}$ are necessary for the existence of positive solutions of (3).

Proof. (i) Suppose that (u, v) is a positive solution of (3); then (u, v) satisfies

$$\begin{aligned} -\Delta u = u(a - u - bve^{-ku}) \leq a(a - u), \\ x \in \Omega, \quad u = 0, \quad x \in \partial\Omega, \end{aligned} \quad (12)$$

so we have $a = \lambda_1(u + bve^{-ku})$. According to the comparison principle of eigenvalues, we have $a > \lambda_1$, a contradiction. Next, we suppose that (u, v) is a nonnegative nonzero solution of (3). If $u \not\equiv 0$, $v \equiv 0$, then $a > \lambda_1$, a contradiction to the previous proof. Similarly, it is easy to see that $c > \lambda_1$, when $u \equiv 0$, $v \not\equiv 0$, which derives a contradiction.

(ii) Suppose that (u, v) is a positive solution of (3); it follows that $a > \lambda_1$, and then the positive semitrivial solution θ_a exists. Thanks to

$$\begin{aligned} -\Delta u = u(a - u - bve^{-ku}) \leq a(a - u), \\ x \in \Omega, \quad u = 0, \quad x \in \partial\Omega, \end{aligned} \quad (13)$$

u is a lower solution of (3), and due to the uniqueness of θ_a , $u < \theta_a$. Thus, since v satisfies the problem

$$-\Delta v = v(c - v + due^{-kv}), \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega \quad (14)$$

it follows from Lemma 2(iii) that

$$0 = \lambda_1(-c + v - due^{-kv}) > \lambda_1(-c - d\theta_a e^{-k\theta_a}). \quad (15)$$

Moreover, as $a < 1/k$, due to $u < \theta_a \leq a < 1/k$, by the second equation of (3), we get $c > \lambda_1(-d\theta_a e^{-k\theta_a})$.

(iii) Supposing that (u, v) is a positive solution of (3), by the proof of (ii), θ_a exists and $u \leq \theta_a$. Similarly, the given condition $c > \lambda_1$ can deduce the existence of the positive θ_c of (3), $\theta_c \leq v$. Similar to the proof of (i), we also obtain the fact that $a = \lambda_1(u + bve^{-ku})$ has the minimum at $u = 0$, $v = \theta_c$ or $u = \theta_a$, $v = \theta_c$, which derives the desired result.

This completes the proof of Theorem 7. $\square$

Theorem 8. If one of the following conditions holds, then (3) has no positive solutions:

- (i) $b \geq e^{ka}$ and $a \leq c$.
- (ii) $b < e^{ka}$ and $a + d/ke \leq be^{ka}(c + d/ke)$.

Proof. (i) Assume that there is a positive solution (u, v) of (3). Recalling the results of Theorem 3, as $b \geq e^{ka}$ and $a \leq c$, we get

$$\begin{aligned} 0 &= \lambda_1 (-a + u + bve^{-ku}) \\ &\geq \lambda_1 (-c + v - due^{-ku} - v + bve^{-ku} + u + due^{-ku}) \\ &\geq \lambda_1 [-c + v - due^{-ku} - (1 - be^{-ku})v] \\ &\geq \lambda_1 [-c + v - due^{-ku}], \end{aligned} \quad (16)$$

which is a contradiction to Lemma 2(iii).

(ii) Assume that there is a positive solution (u, v) of (3). By Theorem 3, if $b < e^{ka}$ and $a + d/ke \leq be^{ka}(c + d/ke)$, we have

$$\begin{aligned} 0 &= \lambda_1 (-a + u - bve^{-ku}) \geq \lambda_1 (-c + v - due^{-ku} - a \\ &\quad + c - v + u + bve^{-ku} + due^{-ku}) \geq \lambda_1 \left[-c + v \right. \\ &\quad \left. - due^{-ku} - a + c - \left(c + \frac{d}{ke} \right) (1 - be^{-ka}) \right] \\ &\geq \lambda_1 [-c + v - due^{-ku}], \end{aligned} \quad (17)$$

which is a contradiction to Lemma 2(iii).

This completes the proof of Theorem 8. $\square$

4. Bifurcation and Multiplicity

In this section, taking a as a main bifurcation parameter, we shall prove that (3) has at least two positive solutions when the parameters involved in (3) satisfy some ranges. In particular, the uniqueness of positive solutions is established when b and d are small enough. By the local bifurcation theory [6, 31] or Section 13 in [30], the branch of positive solutions of (3) bifurcates from $(0, \theta_c, a)$ (or $(\theta_a, 0, c)$) when $c > \lambda_1$ (or $a > \lambda_1$) is established. Now we give the following Lemma where the proof is similar to Theorem 3.8 in [3].

Lemma 9. (i) If $c > \lambda_1$ and $a \geq \lambda_1(b\theta_c)$, then (3) has a positive solution which bifurcates from $(0, \theta_c, a)$ if and only if $a = \tilde{a}$, and here $\tilde{a} := \lambda_1(b\theta_c)$.

(ii) If $a > \lambda_1$ and $\lambda_1(-d\theta_a e^{-k\theta_a}) \leq c \leq \lambda_1$, then (3) has a positive solution which bifurcates from $(\theta_a, 0, c)$ if and only if $c = \tilde{c}$, and here $\tilde{c} := \lambda_1(-d\theta_a e^{-k\theta_a})$.

More precisely, all positive solutions near $\tilde{a} := \lambda_1(b\theta_c)$ are defined by

$$\begin{aligned} \{(u(s), v(s), a(s)) = s\Phi + O(s^2), \theta_c + s\Psi \\ + O(s^2), \tilde{a} + a_1 s + O(s^2) : 0 < s < \sigma\}, \end{aligned} \quad (18)$$

where Φ is the principal eigenfunction of $\tilde{a}$, and it follows from the following problem:

$$\begin{aligned} -\Delta \xi + (\tilde{a} - b\theta_c)\xi &= 0, \quad x \in \Omega, \\ -\Delta \eta + (c - 2\theta_c)\eta &= -d\theta_c \xi, \quad x \in \Omega, \\ \xi = \eta &= 0, \quad x \in \partial\Omega, \end{aligned} \quad (19)$$

where $\Psi = (-\Delta + c - 2\theta_c)^{-1}(d\theta_c\Phi)$.

Remark 10. In Section 3, by Theorems 6 and 7, we can know the sufficient condition and necessary condition on the existence of positive solutions and find that there exists a gap between $a > \lambda_1(b\theta_c)$ and $a > \lambda_1(b\theta_c)e^{-k\theta_a}$.

Next, using the constant a as a main bifurcation parameter, we get the following theorem which establishes the multiplicity and stability results of positive solution for (3) in the gap.

Theorem 11. Suppose $c > \lambda_1$ and $\int_{\Omega} \Phi^3(1 - bk\theta_c) < 0$. There exists small enough $\varepsilon > 0$ such that the local bifurcation of the positive solution $(u(s), v(s))$ bifurcates from $(0, \theta_c, \tilde{a})$ which is nondegenerate and unstable for $a \in (\tilde{a} - \varepsilon, \tilde{a})$ and $0 < d \ll 1$. In addition, (3) has at least two positive solutions.

Proof. Firstly, we prove that any positive solutions bifurcated from $(0, \theta_c)$ are nondegenerate and unstable. To complete this, we need only show that there exists a sufficiently small $\varepsilon > 0$ such that, for $a \in (\tilde{a} - \varepsilon, \tilde{a})$, any positive solution $(u(s), v(s))$ of (3) is nondegenerate and the linearized eigenvalue problem

$$\begin{aligned} -\Delta \xi - \xi(a(s) - 2u(s) + bv(s)e^{-ku(s)} \\ + bu(s)v(s)ke^{-ku(s)}) + bu(s)e^{-ku(s)}\eta = \mu\xi, \end{aligned} \quad x \in \Omega, \quad (20)$$

$$\begin{aligned} -\Delta \eta - \xi(dv(s)e^{-ku(s)} - du(s)v(s)ke^{-ku(s)}) - (c \\ - 2v(s) + du(s)e^{-ku(s)})\eta = \mu\eta, \quad x \in \Omega \end{aligned}$$

has a unique eigenvalue $\hat{\mu}$ such that $\text{Re}(\hat{\mu}) < 0$ with multiplicity one. Suppose that $\{\varepsilon_n > 0\}$, $\{d_n\}$, $\{s_n > 0\}$ are the sequences which converge to 0 as $n \rightarrow \infty$. Thanks to $a = \tilde{a} + a_1 s + O(s^2)$, there exists the sequence $\{a_n\}$ such that $a_n \in (\tilde{a} - \varepsilon_n, \tilde{a})$ as $n \rightarrow \infty$. Set (u_n, v_n) as a solution of (3). So linearized problem (20) can be denoted by

$$\begin{aligned} -\Delta \xi_n - \xi_n(a_n - u_n - bv_n e^{-ku_n} + bu_n v_n ke^{-ku_n}) \\ + bu_n e^{-ku_n} \eta_n = \mu_n \xi_n, \quad x \in \Omega, \\ -\Delta \eta_n - \xi_n(d_n v_n e^{-ku_n} - d_n u_n v_n ke^{-ku_n}) \\ - (c - 2v_n + d_n u_n ke^{-ku_n}) \eta_n = \mu_n \eta_n, \quad x \in \Omega, \\ \xi_n = \eta_n = 0, \quad x \in \partial\Omega. \end{aligned} \quad (21)$$

Notice that, as $n \rightarrow \infty$, problem (21) converges to the following problem:

$$\begin{aligned} \Delta \xi - (\tilde{a} - b\theta_c) \xi &= \mu \xi, \quad x \in \Omega, \\ -\Delta \eta - (c - 2\theta_c) \eta &= \mu \eta, \quad x \in \Omega, \\ \xi &= \eta = 0, \quad x \in \partial\Omega, \end{aligned} \quad (22)$$

and since $c = \lambda_1(\theta_c) < \lambda_1(2\theta_c)$, (22) has 0 as a simple eigenvalue with corresponding eigenfunction $(\Phi, 0)$. Moreover, all the other eigenvalues are positive and stand apart from 0. Thus, it follows from perturbation theory [32] that problem (21) has a unique eigenvalue μ_n which is near to zero for large n . In particular, all the other eigenvalues of problem (21) have positive real parts and stand apart from 0. Note that μ_n is simple real eigenvalue which converges to zero and we can denote the corresponding eigenfunction (ξ_n, η_n) such that

$$\lim_{n \rightarrow \infty} (\xi_n, \eta_n) \longrightarrow (\Phi, 0). \quad (23)$$

Now we prove that $\mu_n < 0$ for large n in following. Multiplying the first equation of (21) with Φ and integrating on Ω , we get

$$\begin{aligned} & - \int_{\Omega} \Delta \xi_n \Phi \\ & - \int_{\Omega} \Phi \xi_n (a_n - 2u_n - bv_n e^{-ku_n} + bu_n v_n m e^{-ku_n}) \\ & + \int_{\Omega} bu_n e^{-ku_n} \eta_n \Phi = \int_{\Omega} \mu_n \xi_n \Phi. \end{aligned} \quad (24)$$

Meanwhile, multiplying the first equation of (3) with $(u, v) = (u_n, v_n)$ by ξ_n and integrating, we obtain

$$- \int_{\Omega} \Delta u_n \xi_n = \int_{\Omega} u_n \xi_n (a_n - u_n - bv_n e^{-ku_n}), \quad (25)$$

Thanks to $u_n = s_n \Phi + O(s^2)$, adopting variational principle, by the above equation, we have

$$- \int_{\Omega} \Delta \xi_n \Phi = \int_{\Omega} \Phi \xi_n (a_n - u_n - bve^{-ku_n}) + O(s_n^2). \quad (26)$$

According to (24) and (26), we have

$$\begin{aligned} & - \int_{\Omega} \xi_n \Phi u_n (1 - bv_n m e^{-ku_n}) - \int_{\Omega} bu_n e^{-ku_n} \eta_n \Phi \\ & = \int_{\Omega} \Phi \mu \xi_n + O(s_n^2). \end{aligned} \quad (27)$$

Taking $(u(s), v(s)) = (s\Phi + O(s^2), \theta_c + s\Psi + O(s^2))$ into (27) and so dividing by s_n and taking the limit, we obtain

$$\lim_{n \rightarrow \infty} \frac{\mu_n}{s_n} = \frac{\int_{\Omega} \Phi^3 (1 - bk\theta_c)}{\int_{\Omega} \Phi^2}; \quad (28)$$

recalling that $\int_{\Omega} \Phi^3 (1 - bm\theta_c) < 0$, it is easy to see that $\mu < 0$ for large n . This proves our claim.

Next, in order to prove the existence of at least two positive solutions, we may use reduction to absurdity and

FIGURE 1: Bifurcation sketch map.

suppose that (3) has a unique coexistence state $(\hat{u}, \hat{v})$; it follows from local bifurcation theory that solutions must be positive solutions bifurcated from $(0, \theta_c)$ near $\tilde{a}$; moreover, $(\hat{u}, \hat{v})$ is nondegenerate and the corresponding linearized eigenvalue problem has a unique eigenvalue $\hat{\mu}$ such that $\text{Re}(\hat{\mu}) < 0$ with multiplicity one. Thanking to the above facts, it is easy to see that $I - F'(\hat{u}, \hat{v})$ is invertible and does not have property α on $\overline{W}$, and it follows from Lemma 1(ii) that $\text{index}_w(F, (\hat{u}, \hat{v})) = (-1)^1 = -1$. Thus, applying Lemmas 4-5 and the additivity property of the index, we obtain

$$\begin{aligned} 1 &= \text{index}_w(F, D') \\ &= \text{index}_w(0, 0) + \text{index}_w(F, (\theta_a, 0)) \\ &\quad + \text{index}_w(F, (\hat{u}, \hat{v})) = 0 + 0 + 1 - 1 = 0, \end{aligned} \quad (29)$$

which derives a contradiction. The proof is completed. $\square$

Remark 12. Theorem 6 gives the multiplicity of positive solutions when $k \geq 0$. Note that since $\int_{\Omega} \Phi^3 (1 - bm\theta_c) < 0$ with $a_1 < 0$ for $d \ll 1$, there exists $\tilde{a} = \tilde{a}(s) \in (\lambda_1, \tilde{a})$. Remark 10 shows that there is no positive solution of (1) if $a > \lambda_1(b\theta_c e^{-k\theta_c})$. Hence, by establishing a bifurcation sketch map as in Figure 1, we can easily get the fact that there must be at least two coexistence states for $a \in (\tilde{a}, \tilde{a})$ and some $\tilde{a} \in (\lambda_1(b\theta_c e^{-k\theta_c}), \lambda_1(b\theta_c))$.

Theorem 13. Suppose that $c > \lambda_1$, $a > \lambda_1 + b(c + d/(ke))$, and $b(c + d/(ke))k < 1$. If

$$\Gamma b^2 + \Gamma(ak + 1)^2 d^2 + 2akbd + 4bk \left(c + \frac{d}{(ke)} \right) \leq 4, \quad (30)$$

where

$$\Gamma := \max \left\{ \sup_{\Omega} \frac{\theta_a}{\theta_c}, \sup_{\Omega} \frac{\theta_{c+d/(ke)}}{\theta_{a-b(c+d/(ke))}} \right\}, \quad (31)$$

then (3) has a unique positive solution.

Proof. We suppose that (u_1, v_1) and (u_2, v_2) are two positive solutions of (3). Due to $c > \lambda_1$ and $a > \lambda_1 + b(c + d/(ke))$,

it follows from the comparison argument for elliptic problem that we have the following results:

$$\begin{aligned}\theta_{a-b(c+d/(ke))} &\leq u_i \leq \theta_a; \\ \theta_c &\leq v_i \leq \theta_{c+d/(ke)}, \\ i &= 1, 2.\end{aligned}\quad (32)$$

Setting $A := u_1 - u_2$; $B := v_1 - v_2$, according to system (3) and the differential mean value theorem, there exists $\xi_1, \xi_2 \in [u_1, u_2]$ such that A and B satisfy

$$\begin{aligned}-\Delta A - A(a - u_1 - bv_1 e^{-ku_1}) + u_2 A \\ - u_2 b(v_2 k e^{-k\xi_1} A - e^{-ku_1} B) &= 0, \quad x \in \Omega, \\ -\Delta B - B(c - v_1 + du_1 e^{-ku_1}) + v_2 B \\ - (e^{-ku_2} - u_1 k e^{k\xi_2}) v_2 d A &= 0, \quad x \in \Omega, \\ A = B = 0, \quad x \in \partial\Omega.\end{aligned}\quad (33)$$

In view of the fact that (u_1, v_1) is a solution of (3), 0 is the principal eigenvalue for the following two eigenvalue problems:

$$\begin{aligned}-\Delta\chi - \chi(a - u_1 - bv_1 e^{-ku_1}) &= 0, \\ x \in \Omega, \quad \chi &= 0, \quad x \in \partial\Omega, \\ -\Delta\omega - \omega(c - v_1 + du_1 e^{-ku_1}) &= 0, \\ x \in \Omega, \quad \omega &= 0, \quad x \in \partial\Omega.\end{aligned}\quad (34)$$

Hence, applying Rayleigh's formula for the principal eigenvalue, it follows that

$$\begin{aligned}\int_{\Omega} [|\nabla A|^2 - A^2(a - u_1 - bv_1 e^{-ku_1})] &\geq 0, \\ \int_{\Omega} [|\nabla B|^2 - B^2(c - v_1 + du_1 e^{-ku_1})] &\geq 0.\end{aligned}\quad (35)$$

Multiplying two equations of (33) by A and B , respectively, and integrating on Ω , we get the fact that

$$\begin{aligned}\int_{\Omega} [(1 - v_2 b k e^{k\xi_1}) u_2 A^2 \\ + (u_2 b e^{-ku_1} + u_1 v_2 k d e^{-k\xi_2} - v_2 d e^{-ku_2}) AB \\ + v_2 B^2] &\leq 0.\end{aligned}\quad (36)$$

Thanks to $1 - v_2 b k e^{-k\xi_1} \geq 1 - b(c + d/(ke))k > 0$, thus

$$\begin{aligned}I := (u_2 b e^{-ku_1} + u_1 v_2 k d e^{-k\xi_2} - v_2 d e^{-ku_2})^2 - 4 \left(1 \right. \\ \left. - v_2 b k e^{-k\xi_1} \right) u_2 v_2 = (u_2 b e^{-ku_1})^2 + (u_1 k e^{-k\xi_2} \\ - e^{-ku_2})^2 v_2^2 d^2 + 2 (u_2 b e^{-ku_1}) (u_1 k e^{-k\xi_2} - e^{-ku_2}) v_2 d \\ - 4 (1 - v_2 b k e^{-k\xi_1}) u_2 v_2 \leq (u_2 b e^{-ku_1})^2 + (u_1 k e^{-k\xi_2}\end{aligned}$$

$$\begin{aligned}- e^{-ku_2})^2 v_2^2 d^2 + 2 u_1 u_2 k b e^{-(ku_1 + k\xi_2)} v_2 d \\ + 4 u_2 v_2^2 b k e^{-k\xi_1} - 4 u_2 v_2 = u_2 v_2 \left[\frac{u_2}{v_2} (b e^{-ku_1})^2 \right. \\ \left. + (u_1 k e^{-k\xi_2} + e^{-ku_2}) \frac{v_2}{u_2} d^2 + 2 u_1 k b d e^{-(ku_1 + k\xi_2)} \right. \\ \left. + 4 v_2 k b e^{-k\xi_1} - 4 \right] \leq 0.\end{aligned}\quad (37)$$

Hence, the integral in (36) has a nonnegative value, and then $A, B \equiv 0$ which show the desired results. $\square$

5. Stability and Multiplicity of Positive Solutions

In this section, the multiplicity, stability, and some uniqueness of positive solutions of (3) are considered by using k as a parameter. Next, some lemmas to obtain the main results of this section are given. These lemmas will show that there exist some upper and lower solutions which do not depend on k and indicate the nondegeneracy at any solution of (3) with certain hypothesis. In the following, an asymptotic result is given firstly.

Lemma 14. Suppose that $a > \lambda_1$, $c > \lambda_1$. For any given small $\varepsilon > 0$, there exists $K(\varepsilon)$ such that for $k > K(\varepsilon)$, (3) has at least one positive solution (u, v) which satisfies

$$\begin{aligned}\theta_{a-\varepsilon} &\leq u \leq \theta_a, \\ \theta_c &\leq v \leq \theta_{c+\varepsilon}.\end{aligned}\quad (38)$$

Proof. Set $\underline{U} = (\underline{u}, \underline{v}) = (\theta_{a-\varepsilon}, \theta_c)$, $\overline{U} = (\overline{u}, \overline{v}) = (\theta_a, \theta_{c+\varepsilon})$. It is easy to see that functions $u(a - u - bve^{-ku})$ and $v(c - v + due^{-kv})$ satisfy the Lipschitz continuous in $\{\underline{U}, \overline{U}\}$. If we can show that $\underline{U}$ and $\overline{U}$ are the upper and lower solutions of (3), respectively, then (3) has at least one positive solution $(u, v) \in \{\underline{U}, \overline{U}\}$ with $k > K$. Thanks to Chapter 10 Comparison Theorems and Monotonicity Methods in [30] for nonquasimonotone functions, it suffices to require that the following hold:

$$\begin{aligned}\Delta \overline{u} + \overline{u}(a - \overline{u} - b\overline{v}e^{-k\overline{u}}) &\leq 0, \quad x \in \Omega, \\ \Delta \overline{v} + \overline{v}(c - \overline{u} - d\overline{e}e^{-k\overline{u}}) &\leq 0, \quad x \in \Omega, \\ \Delta \underline{u} + \underline{u}(a - \underline{u} - b\underline{v}e^{-k\underline{u}}) &\geq 0, \quad x \in \Omega, \\ \Delta \underline{v} + \underline{v}(c - \underline{v} - d\underline{u}e^{-k\underline{u}}) &\geq 0, \quad x \in \Omega, \\ u(x, 0) &= u_0(x), \\ v(x, 0) &= v_0(x), \\ x &\in \Omega.\end{aligned}\quad (39)$$

The first and fourth inequalities in (39) are obvious. Next we check the second and third of (21), by the direct calculation as follows:

$$\begin{aligned}
 & \Delta \bar{v} + \bar{v} (c - \bar{u} - d u e^{-k u}) \\
 &= \Delta \bar{v} + \bar{v} (c + \varepsilon + \bar{v}) + \bar{v} (-\varepsilon + b u e^{-k u}) \\
 &= \bar{v} (-\varepsilon + b u e^{-k u}), \\
 & \Delta u + u (a - u - b v e^{-k u}) \\
 &= \Delta u + u (a - \varepsilon - u) + u (\varepsilon - b v e^{-k u}) \\
 &= u (\varepsilon - b v e^{-k u}).
 \end{aligned} \tag{40}$$

Therefore, for $k \gg 1$, the inequalities hold when

$$\begin{aligned}
 & \Delta \bar{v} + \bar{v} (c - \bar{v} + d u e^{-k u}) \leq 0, \\
 & \Delta u + u (a - u - b v e^{-k u}) \geq 0.
 \end{aligned} \tag{41}$$

□

Lemma 15. Suppose that $a > \lambda_1$, $c > \lambda_1$.

(i) The positive solution of (3) meets (38) and

$$(u, v) \longrightarrow (\theta_a, \theta_c) \quad \text{as } k \longrightarrow \infty. \tag{42}$$

(ii) There exists sufficiently large $K(\varepsilon)$ such that, for $k > K(\varepsilon)$, the positive solution of (3) which meets (38) is nondegenerate and linearly stable.

Proof. (i) As $k \rightarrow \infty$, it is clear to see that the operator $F(u, v)$ defined in Section 2 converges to the following operator:

$$\widehat{A} = (-\Delta + p)^{-1} \begin{pmatrix} u(a - u) + P u \\ v(c - v) + P v \end{pmatrix}; \tag{43}$$

therefore, $(u, v) \rightarrow (\theta_a, \theta_c)$.

(ii) Suppose that the result does not hold. Then we can obtain $k \rightarrow \infty$; there exists μ_n with $\text{Re}(\mu_n) < 0$ and $(\xi_n, \eta_n) \neq 0$ with $\|\xi_n\|_{L^2}^2 + \|\eta_n\|_{L^2}^2 = 1$ such that

$$\begin{aligned}
 & -\Delta \xi_n - \xi_n (a - u_n - b v_n e^{-k_n u_n} + b u_n v_n k_n e^{-k_n u_n}) \\
 &+ b u_n e^{-k_n u_n} \eta_n = \mu_n \xi_n, \quad x \in \Omega, \\
 & -\Delta \eta_n - \xi_n (d v_n e^{-k_n u_n} - d u_n v_n k_n e^{-k_n u_n}) \\
 &- (c - 2 v_n + d u_n e^{-k_n u_n}) \eta_n = \mu_n \eta_n, \quad x \in \Omega, \\
 & \xi_n = \eta_n = 0, \quad x \in \partial \Omega,
 \end{aligned} \tag{44}$$

where (u_n, v_n) is a positive solution of (1) as $k = k_n$ which meets (38). By multiplying $\bar{\xi}$ and $\bar{\eta}$ to two equations of (3), respectively, and integrating on Ω , we get

$$\begin{aligned}
 \mu_n &= \int_{\Omega} |\nabla \xi_n|^2 \\
 &- \int_{\Omega} |\xi_n|^2 (a - u_n - b v_n e^{-k_n u_n} + b u_n v_n k_n e^{-k_n u_n}) \\
 &+ \int_{\Omega} b u_n e^{-k_n u_n} \eta_n \bar{\xi}_n + \int_{\Omega} |\nabla \eta_n|^2 \\
 &- \xi_n \bar{\eta}_n (d v_n e^{-k_n u_n} - d u_n v_n k_n e^{-k_n u_n}) \\
 &- \int_{\Omega} (c - 2 v_n + d u_n e^{-k_n u_n}) |\eta_n|^2,
 \end{aligned} \tag{45}$$

where $\bar{\xi}_n$ and $\bar{\eta}_n$ are the complex conjugates of ξ_n and η_n , respectively. By the above equation, it is easy to see that $\text{Im}(\mu_n)$ and $\text{Re}(\mu_n)$ are bounded. Then we can suppose that $\mu_n \rightarrow \mu$ and $\text{Re}(\mu_n) \leq 0$. Since ξ_n and η_n are also bounded, so let $\eta_n \rightarrow \eta$, $\xi_n \rightarrow \xi$. Taking the limit on (44), we get

$$\begin{aligned}
 & -\Delta \xi - \xi (a - 2\theta_a) = \mu \xi, \quad x \in \Omega, \\
 & -\Delta \eta - \eta (c - 2\theta_c) = \mu \eta, \quad x \in \Omega, \\
 & \xi = \eta = 0, \quad x \in \partial \Omega.
 \end{aligned} \tag{46}$$

It follows from (i) that μ must be real number with $\mu \leq 0$. If $\xi \neq 0$ then μ is an eigenvalue of the following problem:

$$-\Delta \xi - \xi (a - 2\theta_a) = \mu \xi, \quad x \in \Omega, \quad \xi = 0, \quad x \in \partial \Omega. \tag{47}$$

Due to $\lambda_1(-a + 2\theta_a) > \lambda_1(-a + \theta_a) = 0$, $0 \geq \mu \geq \lambda_1(-a + 2\theta_a)$, which is a contradiction, so we get $\xi \equiv 0$. Similarly, we also can prove $\eta \equiv 0$, a contradiction to $(\xi_n, \eta_n) \neq 0$; the proof is complete. □

Next, by the condition $a > \lambda_1(b\theta_c)$, one can easily get the fact that $(a + \lambda_1(b\theta_c))/2 > \lambda_1(b\theta_c)$; then there exists a sufficiently small $\sigma > 0$ such that $(a + \lambda_1(b\theta_c))/2 > \lambda_1(b\theta_{c+\sigma})$, and then we can know that the problem

$$\begin{aligned}
 & -\Delta \omega = \omega \left(\frac{a + \lambda_1(b\theta_c)}{2} - b\theta_{c+\sigma} \right), \\
 & x \in \Omega, \quad \omega = 0, \quad x \in \partial \Omega
 \end{aligned} \tag{48}$$

has a unique positive solution $\tilde{\omega}$ by the result of (7).

Lemma 16. Suppose that $c > \lambda_1$ and $a > \lambda_1(b\theta_c)$. Then, for some σ , there exists a sufficiently large $K(\sigma)$ such that the following result holds when $K \geq K(\sigma)$.

(i) $u \geq \tilde{\omega}$; that is, $\tilde{\omega}$ is a lower solution for u .

(ii) A positive solution of (3) converges to (θ_a, θ_c) as $k \rightarrow \infty$. Meanwhile, the positive solution is nondegenerate and linearly stable.

Proof. (i) According to Theorem 3 and the comparison argument, we show $v \leq \theta_{c+d/(ke)}$; then there exists a sufficiently large $K(\sigma)$ such that $k \geq K(\sigma)$ with $d/(ke) < \sigma$; hence

$$\begin{aligned} -\Delta u &= u(a - u - bve^{-ku}) \\ &\geq u\left(\frac{a + \lambda_1(b\theta_c)}{2} - b\theta_{c+d/(ke)} - u\right) \\ &\geq u\left(\frac{a + \lambda_1(b\theta_c)}{2} - b\theta_{c+\sigma} - u\right). \end{aligned} \quad (49)$$

So the proof is complete.

(ii) As $k \rightarrow \infty$, $\bar{w}$ is a lower solution for u which does not depend on k ; the proof of (ii) is similar to Lemma 15, so we omit it. $\square$

Lemma 17. Suppose that $c \in (\lambda_1 - d/(ke), \lambda_1)$ and $a > \lambda_1 + \tau$; here $\tau := b(c + d/(ke))$. Then there exists a sufficiently large K such that $k > K$; the following claims hold:

- (i) $u \geq \theta_{a-\tau}$.
- (ii) The positive solution of (3) is nondegenerate and linearly stable.

Proof. (i) Applying the comparison argument for elliptic problem, we obtain $0 \leq v \leq \theta_{c+d/(ke)} \leq c + d/(ke)$, and so as k is large enough we get

$$\begin{aligned} -\Delta u &= u(a - u - bve^{-ku}) \\ &\geq u\left[a - b\left(c + \frac{d}{(ke)}\right) - u\right], \end{aligned} \quad (50)$$

which proves claim (i).

(ii) Let $c = c_n$, where $c_n \in (\lambda_1 - d/(ke), \lambda_1)$. Thanks to $u_n \geq \theta_{a-\tau_n}$, where $\tau_n := b(c_n + d/(k_n e))$, then $k_n \|u_n\|_\infty \rightarrow \infty$. Due to $c_n \in (\lambda_1 - d/(k_n e), \lambda_1)$, hence $(u_n, v_n) \rightarrow (\theta_a, 0)$. Since $\lambda_1 - d/(k_n e) \rightarrow \lambda_1$, so $c_n \rightarrow \lambda_1$.

Set $\hat{v}_n = v_n / \|v_n\|_\infty$; then we get

$$\begin{aligned} \Delta \hat{v}_n + \hat{v}_n(c_n - v_n + due^{-k_n u_n}) &= 0, \\ x \in \Omega, \quad \hat{v}_n &= 0, \quad x \in \partial\Omega. \end{aligned} \quad (51)$$

According to standard elliptic regularity theory and Sobolev embedding theorems, we may suppose that $\hat{v}_n \rightarrow \hat{v} > 0$. Taking the limit, the above equation satisfies

$$\begin{aligned} -\Delta \hat{v} &= \lambda_1 \hat{v}, \quad x \in \Omega; \\ \hat{v} &= 0, \quad x \in \partial\Omega, \\ \|\hat{v}\|_\infty &= 0. \end{aligned} \quad (52)$$

Hence, $\hat{v}$ must be the principal eigenfunction corresponding to λ_1 denoted by Φ_1 and $\|\Phi_1\|_\infty = 0$.

In the following, we prove that linearized eigenvalue problem (44) has no eigenvalue $\text{Re}(\mu_n) \leq 0$ as $n \gg 1$. Taking $n \rightarrow \infty$, problem (44) converges to the following:

$$\begin{aligned} -\Delta \xi - (a - 2\theta_a) \xi, \quad x &\in \Omega, \\ -\Delta \eta - \lambda_1 \eta &= \mu \eta, \quad x \in \Omega \\ \xi &= \eta = 0, \quad x \in \partial\Omega. \end{aligned} \quad (53)$$

Obviously, for (53), 0 is a simple eigenvalue with its corresponding eigenfunction $(\xi, \eta) = (0, \Phi_1)$. Applying the method of the proof in Theorem 11, it suffices to prove $\mu_n > 0$ for $n \gg 1$. Take an eigenfunction (ξ_n, η_n) such that $(\xi_n, \eta_n) \rightarrow (0, \Phi_1)$.

Applying the integral equation on Ω which can be established by multiplying v_n to the second equation of (44) and integrating on Ω , we get

$$\begin{aligned} - \int_\Omega \Delta \eta_n v_n - \int_\Omega \xi_n v_n (dv_n e^{-k_n u_n} - du_n v_n k_m e^{-k_n u_n}) \\ - \int_\Omega (c - 2v_n + du_n e^{-k_n u_n}) \eta_n v_n = \int_\Omega \mu_n \eta_n v_n. \end{aligned} \quad (54)$$

Multiplying η_n to the second equation of (3) and using variational principle on Ω , we can deduce the following integral equation:

$$- \int_\Omega \Delta \eta_n v_n = \int_\Omega v_n \eta_n (c_n - v_n + du_n e^{-k_n u_n}). \quad (55)$$

By substituting $v_n = \hat{v}_n \|v_n\|_\infty$ into the above two equations and taking the limit, we can get

$$\lim_{n \rightarrow \infty} \frac{\mu_n}{\|v_n\|_\infty} = \frac{\int_\Omega \Phi_1^3}{\int_\Omega \Phi_1^2} > 0, \quad (56)$$

which get the desired result. $\square$

Theorem 18. (i) If $a \in (\lambda_1, \lambda_1(b\theta_c))$, $c > \lambda_1$, then there exists a sufficiently large $K = K(\varepsilon)$ such that (3) has at least two positive solutions for $k > K(\varepsilon)$.

(ii) If $a > \lambda_1(b\theta_c)$, $c > \lambda_1$, then there exists a sufficiently large $K = K(\sigma)$ such that (3) has a unique positive solution for $k > K(\varepsilon)$.

(iii) If $\lambda_1(-d\theta_a e^{-m\theta_a}) < c < \lambda_1$ and $a > \lambda_1 + \tau$, where $\tau := b(c + d/(ke))$, then there exists a sufficiently large $K = K(\sigma)$ such that (3) has a unique positive solution for $k > K(\varepsilon)$.

Proof. (i) Suppose the result is false; we can assume that (3) has a unique positive solution $(\hat{u}, \hat{v})$; then $(\hat{u}, \hat{v})$ must satisfy (38). As k is large enough, by Lemma 14, it is easy to see that $I - F'(\hat{u}, \hat{v})$ is invertible and has no property α on $\bar{w}(\hat{u}, \hat{v})$. Furthermore, F' has no real eigenvalue larger than or equal to one. It follows from Lemma 1(ii) that

$\text{index}_w(F, (\hat{u}, \hat{v})) = (-1)^0 = 1$. Applying Lemmas 4-5 and the additivity property of the index, we get

$$\begin{aligned} 1 &= \text{index}_w(F, D') \\ &= \text{index}_w(F, (0, 0)) + \text{index}_w(F, (\theta_a, 0)) \\ &\quad + \text{index}_w(F, (0, \theta_c)) + \text{index}_w(F, (\hat{u}, \hat{v})) \\ &= 0 + 0 + 1 + 1. \end{aligned} \quad (57)$$

Which derive a contradiction, so the result of (i) is established.

(ii) Suppose that (3) has many positive solutions with the hypothesis of (ii), thanks to the compactness of the operator F which has at most finitely many positive fixed points in the given region D' which are denoted by

$$U_i = (u_i, v_i), \quad i = 1, 2, \dots, k. \quad (58)$$

Similar to the proof of (i), it is easy to see that $\text{index}_w(F, U_i) = (-1)^0 = 1, i = 1, 2, \dots, k$. Finally, applying Lemmas 16(ii) and 1(ii), along with the additivity property of the index, we get

$$\begin{aligned} k &= \sum_{i=1}^k \text{index}_w(F, U_i) \\ &= \text{index}_w(F, D') + \text{index}_w(F, (0, 0)) \\ &\quad + \text{index}_w(F, (\theta_a, 0)) + \text{index}_w(F, (0, \theta_c)) \\ &= 1 - 0 - 0 - 0 = 1. \end{aligned} \quad (59)$$

Thus, the uniqueness is proved.

(iii) By $-d\theta_a e^{-k\theta_a} \leq d/(ke)$, we can get $c > \lambda_1 - d/(ke)$. The proof of (iii) is similar to (ii); just the semitrivial solution $(0, \theta_c)$ does not exist in this case. So we omit it. $\square$

Theorem 19. *If the assumption of Theorem 18(i) holds, then as $n \rightarrow \infty$, either $(u_n, v_n) \rightarrow (\theta_a, \theta_c)$ or $(k_n u_n, v_n) \rightarrow (U, \theta_c)$, where U is a positive solution of*

$$-\Delta U = U(a - b\theta_c e^{-U}), \quad x \in \Omega, \quad U = 0, \quad x \in \partial\Omega. \quad (60)$$

Proof. In the event that $k_n \|u_n\|_\infty \rightarrow \infty$, it is easy to see that $(u_n, v_n) \rightarrow (\theta_a, \theta_c)$ holds. Hence, we only investigate the case of $k_n \|u_n\|_\infty < \infty$. Let $k_n u_n = U_n$, it follows that

$$\begin{aligned} -\Delta U_n &= U_n \left(a - \frac{U_n}{k_n} - b v_n e^{-U_n} \right), \quad x \in \Omega, \\ -\Delta v_n &= v_n \left(c - v_n + d \frac{U_n}{k_n} e^{-U_n} \right), \quad x \in \Omega \\ U_n &= v_n = 0, \quad x \in \partial\Omega. \end{aligned} \quad (61)$$

Taking $n \rightarrow \infty$, the solution (U_n, v_n) of the above problem converges to a nonzero solution of the following problem:

$$\begin{aligned} -\Delta U &= U(a - b v e^{-U}), \quad x \in \Omega, \\ -\Delta v &= v(c - v), \quad x \in \Omega \\ U &= v = 0, \quad x \in \partial\Omega. \end{aligned} \quad (62)$$

Obviously, the result $(k_n u_n, v_n) \rightarrow (U, \theta_c)$ holds. $\square$

Remark 20. By Theorem 19, we can understand that the exact number of positive solutions is dependent on the multiplicity of positive solutions with the same hypothesis in Theorem 18(i).

6. Conclusion

The positive solution of a predator-prey model with Dinosaur functional response under Dirichlet boundary conditions is investigated. Firstly, using the comparison principle and fixed point index theory, the sufficient conditions and necessary conditions on coexistence of positive solutions on the predator-prey model with Dinosaur functional response are established. See Theorems 6–8. Secondly, by virtue of bifurcation theory, perturbation theory of eigenvalues, and the fixed point index theory, we establish the bifurcation of positive solutions to (3) and obtain the stability and multiplicity of the positive solution under certain conditions. See Theorem 11. Furthermore, the local uniqueness result is studied when b and d are small. See Theorem 13. Finally, we investigate the multiplicity, uniqueness, and stability of positive solutions when $k > 0$ is large. See Lemmas 15–17 and Theorems 18–19.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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