Research Article

Bäcklund Transformations between the KdV Equation and a New Nonlinear Evolution Equation

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We first give a Bäcklund transformation from the KdV equation to a new nonlinear evolution equation. We then derive two Bäcklund transformations with two pseudopotentials, one of which is from the KdV equation to the new equation and the other from the new equation to itself. As applications, by applying our Bäcklund transformations to known solutions, we construct some novel solutions to the new equation.

1. Introduction

In 1895 two Dutch mathematicians Korteweg and de Vries derived a nonlinear wave equation which is now called KdV equation and adopts the canonical form [1]

$$u_t = 6uu_x + u_{xxx}. \quad (1)$$

Since the KdV equation (1) possesses the solitary wave solution

$$u(x, t) = 2\kappa^2 \operatorname{sech}^2(\kappa(x + 4\kappa^2 t - x_0)), \quad (2)$$

where κ and x_0 are constants, it provides a theoretical confirmation of the existence of the solitary wave observed in 1834 by the Scottish engineer John Scott Russell on the Union canal. In 1965 Zabusky and Kruskal [2] discovered that the interaction of two solitary wave solutions is elastic, and therefore they called this kind of solutions *solitons*. In 1967 Gardner et al. [3] related the solution of the Cauchy initial value problem for the KdV equation (1) to the inverse scattering problem for a one-dimensional linear Schrödinger equation and derived the analytical expressions of n -soliton solutions. The n -soliton solutions to the KdV equation (1) can also be derived from the Darboux [4] and Bäcklund transformations [5], which have been extended and applied to a large variety of nonlinear evolution equations (see [6–13]).

It is well known that the KdV equation (1) is connected to the potential KdV equation

$$v_t = 3v_x^2 + v_{xxx} \quad (3)$$

via $v = \int u \, dx$. By use of this connection, Wahlquist and Estabrook [14] obtained the following Bäcklund transformation for the KdV equation (1), as well as for the potential KdV equation (3).

Theorem 1 (see [14]). *If v is a solution of the potential KdV equation (3), then the system*

$$\begin{aligned} v'_x &= \lambda - v_x - \frac{1}{2}(v' - v)^2, \\ v'_t &= 2\lambda^2 - 2\lambda v_x - v_x^2 - v_{xxx} + 2v_{xx}(v' - v) \\ &\quad - (\lambda + v_x)(v' - v)^2 \end{aligned} \quad (4)$$

on v' is integrable, where λ is an arbitrary constant. Moreover, v' also satisfies (3). So the integrable system (4) defines a Bäcklund transformation $v \mapsto v'$ for the potential KdV equation (3), and it also gives a Bäcklund transformation $u \mapsto u'$ for the KdV equation (1) which is defined by

$$u \mapsto v = \int u \, dx \mapsto v' \mapsto u' = v'_x. \quad (5)$$

Note that (4) is a Lax pair of the potential KdV equation (3). In this paper, we first give another Lax pair of (3) and show that it defines a Bäcklund transformation from the KdV equation (1) to a new nonlinear evolution equation (7) (see Theorem 2). Then by combining the Bäcklund transformations given in Theorems 1 and 2, we derive two Bäcklund transformations defined by two pseudopotentials (see Theorem 8): one is from (1) to (7) (see (22)) and the other from (7) to itself (see (23)). As applications, by applying our Bäcklund transformations to known solutions of (1), (3), or (7), we construct some novel solutions to (7) (see Examples 6, 7, 11, and 12).

2. Bäcklund Transformation with One Pseudopotential

In this section, we prove the following result.

Theorem 2. *If v is a solution of (3), then the system on ϕ*

$$\begin{aligned}\phi_x &= (\phi + v)^2, \\ \phi_t &= -v_x^2 + 2v_{xx}(\phi + v) + 2v_x(\phi + v)^2\end{aligned}\quad (6)$$

is integrable. Moreover, $w = (\phi + v)^2$ satisfies the nonlinear evolution equation

$$w^2 w_t + 6w^3 w_x - \frac{3}{4}w_x^3 + \frac{3}{2}w w_x w_{xx} - w^2 w_{xxx} = 0. \quad (7)$$

Proof. From (3) and (6) we have

$$\begin{aligned}\phi_{xt} &= 2(\phi + v) \\ &\quad \cdot \left(-v_x^2 + 2v_{xx}(\phi + v) + 2v_x(\phi + v)^2 + 3v_x^2 + v_{xxx} \right), \\ \phi_{tx} &= -2v_x v_{xx} + 2v_{xxx}(\phi + v) + 2v_{xx}(\phi + v)^2 \\ &\quad + (2v_{xx} + 4v_x(\phi + v))((\phi + v)^2 + v_x).\end{aligned}\quad (8)$$

Therefore $\phi_{xt} = \phi_{tx}$; that is, (6) is integrable.

From the first equation of (6) we have

$$\begin{aligned}v &= \phi_x^{1/2} - \phi, \\ v_x &= \frac{1}{2}\phi_x^{-1/2}\phi_{xx} - \phi_x, \\ v_{xx} &= -\frac{1}{4}\phi_x^{-3/2}\phi_{xx}^2 + \frac{1}{2}\phi_x^{-1/2}\phi_{xxx} - \phi_{xx}.\end{aligned}\quad (9)$$

Substituting the above equations into the second equation of (6) yields that ϕ satisfies

$$\phi_t = -3\phi_x^2 - \frac{3}{4}\phi_x^{-1}\phi_{xx}^2 + \phi_{xxx}, \quad (10)$$

and therefore

$$\phi_{xt} = -6\phi_x\phi_{xx} + \frac{3}{4}\phi_x^{-2}\phi_{xx}^3 - \frac{3}{2}\phi_x^{-1}\phi_{xx}\phi_{xxx} + \phi_{xxxx}. \quad (11)$$

So $w = (\phi + v)^2 = \phi_x$ satisfies (7). $\square$

To our knowledge, (7) has not been reported in literature previously.

Remark 3. If w is a solution of (7), then

$$v = w^{1/2} - \int w \, dx \quad (12)$$

is a solution of (3).

Remark 4. The integrable system (6) gives a Bäcklund transformation from the potential KdV equation (3) to (7)

$$v \mapsto w = (\phi + v)^2, \quad (13)$$

which is defined by the pseudopotential ϕ [15].

Remark 5. The integrable system (6) also defines a Bäcklund transformation with the pseudopotential ϕ :

$$u \mapsto v = \int u \, dx \mapsto w = (\phi + v)^2, \quad (14)$$

which takes a solution u of the KdV equation (1) to a solution w of (7).

Substituting a known solution v of (3) into the integrable system (6), one can get a solution $w = (\phi + v)^2$ of (7). In the next two examples, from stationary and kink solutions of (3), we generate solutions of (7).

Example 6. Substituting the stationary solution of the potential KdV equation (3)

$$v(x, t) = \frac{2}{x} \quad (15)$$

into the integrable system (6) yields the following rational solution of (6),

$$\phi(x, t) = -\frac{4x^3 - 12t}{x^4 - 12xt}, \quad (16)$$

and therefore of (7)

$$w(x, t) = \frac{4(x^3 + 6t)^2}{x^2(x^3 - 12t)^2}. \quad (17)$$

The profiles of the solution (17) at $t = -1, 0, 1$ are shown in Figure 1. Note that, at $t \neq 0$, the solution curve has three pieces; as t approaches 0, the piece located in the middle becomes smaller and smaller and finally disappears.

Example 7. Substituting the kink solution (see Figure 2) of the potential KdV equation (3)

$$v(x, t) = 2k \tanh(kx + 4k^3 t), \quad (18)$$

FIGURE 1: Profiles of solution (17).

FIGURE 2: Profile of kink solution (18).

FIGURE 3: Profile of solution (20).

(where k is an arbitrary nonzero constant) into the integrable system (6) yields the following solution of (6),

$$\phi(x, t) = -k \left(\tanh(kx + 4k^3 t) + \coth(kx + 4k^3 t) \right), \quad (19)$$

and therefore of (7)

$$w(x, t) = k^2 \left(\tanh(kx + 4k^3 t) - \coth(kx + 4k^3 t) \right)^2. \quad (20)$$

Figure 3 illustrates the wave form of the solution (20) for $k = 1$.

3. Bäcklund Transformations with Two Pseudopotentials

By combining Theorems 1 and 2, we have the following result. The proof is obvious (see, for example, [16]).

Theorem 8. *If v is a solution of (3), then the system on θ and ϕ*

$$\theta_x = \lambda - v_x - \frac{1}{2}(\theta - v)^2,$$

$$\phi_x = (\theta + \phi)^2,$$

$$\begin{aligned} \theta_t &= 2\lambda^2 - 2\lambda v_x - v_x^2 - v_{xxx} + 2v_{xx}(\theta - v) \\ &\quad - (\lambda + v_x)(\theta - v)^2, \end{aligned}$$

$$\phi_t = -\theta_x^2 + 2\theta_{xx}(\theta + \phi) + 2\theta_x(\theta + \phi)^2 \quad (21)$$

is integrable, where λ is an arbitrary constant. Moreover, $w = (\theta + \phi)^2$ satisfies (7).

Remark 9. The integrable system (21) gives a Bäcklund transformation $u \mapsto w$ from the KdV equation (1) to (7) as follows:

$$u \mapsto v = \int u \, dx \mapsto w = (\theta + \phi)^2, \quad (22)$$

which is defined by two pseudopotentials θ and ϕ [15].

Remark 10. The integrable system (21) also defines a Bäcklund transformation $w \mapsto w'$ with two pseudopotentials θ and ϕ for (7) as follows:

$$w \mapsto v = w^{1/2} - \int w \, dx \mapsto w' = (\theta + \phi)^2. \quad (23)$$

FIGURE 4: Profile of one-soliton (24).

FIGURE 5: Profiles of w generated by applying Bäcklund transformation (22) to one-soliton (24).

Applying the Bäcklund transformation (22) or (23) to a known solution u of (1) or w of (7), it is usually not easy to get explicit formulas for θ and ϕ when solving the integrable system (21). But in this case, one can use an ordinary differential equation solver (see, for example, [13, 16]) to obtain numerical approximations of θ and ϕ . In the next two examples, we use this method.

Example 11. Applying the Bäcklund transformation (22) to the one-soliton solution (see Figure 4)

$$u(x, t) = 2k^2 \cosh^{-2}(kx + 4k^3 t) \quad (24)$$

of the KdV equation (1), where k is an arbitrary nonzero constant, we generate solution w of (7). Figure 5 consists of the wave forms of w for $\lambda = 0$ with initial values $\theta(150, 0) = 0$ and $\phi(150, 0) = 0$ at $t = 1, 2, 3$ (first row) and at $t = 5, 15, 25$ (second row), which shows the evolution with time of this solution.

Example 12. Applying the Bäcklund transformation (23) to the rational solution (17) of (7), we generate new solution w' of (7). Figure 6 shows the profiles of w' for $\lambda = 0$ with initial values $\theta(150, 0) = -1$ and $\phi(150, 0) = 1$ at $t = 1, 2, 3$ (first row), at $t = 5, 10, 20$ (second row), and at $t = 30, 40, 50$ (third row), which describes the evolution with time of this solution.

FIGURE 6: Profiles of w' generated by applying Bäcklund transformation (23) to rational solution (17).

Competing Interests

The author declares that they have no competing interests.

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