Research Article

On 2-Absorbing Primary Fuzzy Ideals of Commutative Rings

Deniz Sönmez,¹ Gürsel Yeşilot,¹ Serkan Onar,¹ Bayram Ali Ersoy,¹ and Bijan Davvaz²

¹Department of Mathematics, Yildiz Technical University, Davutpaşa, Istanbul, Turkey

²Department of Mathematics, Yazd University, Yazd, Iran

Correspondence should be addressed to Deniz Sönmez; dnmzguel@hotmail.com

Received 6 April 2017; Accepted 20 July 2017; Published 23 August 2017

Academic Editor: Leonid Shaikh

Copyright © 2017 Deniz Sönmez et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this work, we give a characterization of generalizations of prime and primary fuzzy ideals by introducing 2-absorbing fuzzy ideals and 2-absorbing primary fuzzy ideals and establish relations between 2-absorbing (primary) fuzzy ideals and 2-absorbing (primary) ideals. Furthermore, we give some fundamental results concerning these notions.

1. Introduction

The fundamental concept of fuzzy set was introduced by Zadeh [1] in 1965. In 1982, Liu introduced the notion of fuzzy ideal of a ring [2]. Mukherjee and Sen have continued the study of fuzzy ideals by introducing the notion of prime fuzzy ideals [3]. To the present day, fuzzy algebraic structures have been developed and many interesting results were obtained.

Prime ideals and primary ideals play a significant role in commutative ring theory. Because of this importance, the concept of 2-absorbing ideals, which is a generalization of prime ideals [4], and the concept of 2-absorbing primary ideals, which is a generalization of primary ideals [5], were introduced. While the prime fuzzy ideals and primary fuzzy ideals have been investigated [3, 6], the concepts of 2-absorbing fuzzy ideals and 2-absorbing primary fuzzy ideals have not been studied yet. In this paper, we introduce the 2-absorbing fuzzy ideals and 2-absorbing primary fuzzy ideals and some generalizations of 2-absorbing primary fuzzy ideals and describe some properties of 2-absorbing primary fuzzy ideals.

Let R be a commutative ring with identity. Recall that a proper ideal I of R is called a 2-absorbing ideal if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$ [4] and a proper ideal I of R is called a 2-absorbing primary ideal if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$ [5]. Based on these definitions, a nonconstant fuzzy ideal μ of R is called a 2-absorbing fuzzy

ideal of R if for any fuzzy points x_r, y_s, z_t of R , $x_r y_s z_t \in \mu$ implies that either $x_r y_s \in \mu$ or $x_r z_t \in \mu$ or $y_s z_t \in \mu$ and a nonconstant fuzzy ideal μ of R is said to be a 2-absorbing primary fuzzy ideal of R if for any fuzzy points x_r, y_s, z_t of R , $x_r y_s z_t \in \mu$ implies that either $x_r y_s \in \mu$ or $x_r z_t \in \sqrt{\mu}$ or $y_s z_t \in \sqrt{\mu}$. It is shown that if μ is a 2-absorbing primary fuzzy ideal of R , then $\sqrt{\mu}$ is 2-absorbing fuzzy ideal. We introduce the notions of *weakly completely 2-absorbing fuzzy ideal* and *weakly completely 2-absorbing primary fuzzy ideal*, which is a weakened status of the 2-absorbing fuzzy ideals and 2-absorbing primary fuzzy ideal, respectively. Then relationship between the 2-absorbing primary fuzzy ideals and weakly completely 2-absorbing primary fuzzy ideals is analyzed. Based on the definition of the level set, the transition of 2-absorbing primary ideals of R and 2-absorbing primary fuzzy ideals of R is examined. By a 2-absorbing primary ideal of R , a 2-absorbing primary fuzzy ideal is established (Proposition 20). For a ring homomorphism $f : R \rightarrow S$, it is shown that $f^{-1}(\xi)$ is a 2-absorbing primary fuzzy ideal of R , where ξ is 2-absorbing primary fuzzy ideal of S . If μ is 2-absorbing primary fuzzy ideal of R , which is constant on $\text{Ker } f$, then it is proved that $f(\mu)$ is a 2-absorbing primary fuzzy ideal of S . It is shown under what condition the intersection of the collection of 2-absorbing primary fuzzy ideals is 2-absorbing primary fuzzy ideal. It is shown that the intersection of two 2-absorbing primary fuzzy ideals need not be a 2-absorbing primary fuzzy ideal if this condition is not satisfied (Example 27). Also, it is proved that union of a

directed collection of 2-absorbing primary fuzzy ideals of R is 2-absorbing primary fuzzy ideal.

2. Preliminaries

We assume throughout that all rings are commutative with $1 \neq 0$. Unless stated otherwise $L = [0, 1]$ stands for a complete lattice. Z denotes the ring of integers, $L(R)$ denotes the set of fuzzy sets of R , and $LI(R)$ denotes the set of fuzzy ideals of R . For $\mu, \xi \in L(R)$, we say $\mu \subseteq \xi$ if and only if $\mu(x) \leq \xi(x)$ for all $x \in R$. When $r \in L$, $x, y \in R$ we define $x_r \in L(R)$ as follows:

$$x_r(y) = \begin{cases} r & x = y, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

and x_r is referred to as fuzzy point of R .

Also, for $\mu \in L(R)$ and $t \in L$, define μ_t as follows:

$$\mu_t = \{x \in R : \mu(x) \geq t\}. \quad (2)$$

Definition 1 (see [2]). A fuzzy subset μ of a ring R is called a fuzzy ideal of R if for all $x, y \in R$ the following conditions are satisfied:

- (i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$, $\forall x, y \in R$.
- (ii) $\mu(xy) \geq \mu(x) \vee \mu(y)$, $\forall x, y \in R$.

Let μ be any fuzzy ideal of R ; $x, y \in R$, and let 0 be the additive identity of R . Then it is easy to verify the following:

- (i) $\mu(0) \geq \mu(x)$, $\mu(x) = \mu(-x)$ and $\mu_t \subset \mu_s$, where $s, t \in \text{Im}(\mu)$ and $t > s$.
- (ii) If $\mu(0) = \mu(x - y)$, then $\mu(x) = \mu(y)$, $\mu(x) = s$ iff $x \in \mu_s$, and $x \notin \mu_t$, $\forall t > s$.

Definition 2 (see [7]). Let μ be any fuzzy ideal of R . The ideals μ_t , $(\mu(0) \geq t)$ are called level ideals of μ .

Definition 3 (see [3]). A fuzzy ideal μ of R is called prime fuzzy ideal if for any two fuzzy points x_r, y_s of R , $x_r y_s \in \mu$ implies either $x_r \in \mu$ or $y_s \in \mu$.

Definition 4 (see [6]). Let μ be a fuzzy ideal of R . Then $\sqrt{\mu}$, called the radical of μ , is defined by $\sqrt{\mu}(x) = \bigvee_{n \geq 1} \mu(x^n)$.

Definition 5 (see [6]). A fuzzy ideal μ of R is called primary fuzzy ideal if for $x, y \in R$, $\mu(xy) > \mu(x)$ implies $\mu(xy) \leq \mu(y^n)$ for some positive integer n .

Theorem 6 (see [6]). Let μ be fuzzy ideal of a ring R . Then $\sqrt{\mu}$ is a fuzzy ideal of R .

Definition 7 (see [3]). Let R be a ring. Then a nonconstant fuzzy ideal μ is said to be weakly completely prime fuzzy ideal iff for $x, y \in R$, $\mu(xy) = \max\{\mu(x), \mu(y)\}$.

Theorem 8 (see [8]). If μ and ξ are two fuzzy ideals of R , then $\sqrt{\mu \cap \xi} = \sqrt{\mu} \cap \sqrt{\xi}$.

Theorem 9 (see [8]). Let $f : R \rightarrow S$ be a ring homomorphism and let μ be a fuzzy ideal of R such that μ is constant on $\text{Ker } f$ and let ξ be a fuzzy ideal of S . Then,

- (i) $\sqrt{f(\mu)} = f(\sqrt{\mu})$,
- (ii) $\sqrt{f^{-1}(\xi)} = f^{-1}(\sqrt{\xi})$.

Definition 10 (see [4]). A nonzero proper ideal I of a commutative ring R with $1 \neq 0$ is called a 2-absorbing ideal if whenever $a, b, c \in R$ with $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$.

Definition 11 (see [5]). A proper ideal I of R is called a 2-absorbing primary ideal of R if whenever $a, b, c \in R$ with $abc \in I$, then either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Theorem 12 (see [5]). If I is a 2-absorbing primary ideal of R , then $\sqrt{I}$ is a 2-absorbing ideal of R .

Definition 13 (see [9]). An element $1 > \alpha \in L$ is called a 2-absorbing element if for any $x, y, z \in L$, $x \wedge y \wedge z < \alpha$ implies either $x \wedge y < \alpha$ or $x \wedge z < \alpha$ or $y \wedge z < \alpha$.

3. 2-Absorbing Primary Fuzzy Ideals

Definition 14. Let μ be a nonconstant fuzzy ideal of R . Then μ is called a 2-absorbing fuzzy ideal of R if for any fuzzy points x_r, y_s, z_t of R , $x_r y_s z_t \in \mu$ implies that either $x_r y_s \in \mu$ or $x_r z_t \in \mu$ or $y_s z_t \in \mu$.

Theorem 15. Every prime fuzzy ideal of R is a 2-absorbing fuzzy ideal.

Proof. The proof is straightforward. $\square$

Lemma 16. Let μ be a fuzzy ideal and $t \in [0, \mu(0)]$. If μ is a 2-absorbing fuzzy ideal, then μ_t is a 2-absorbing ideal of R .

Proof. Let μ be a 2-absorbing fuzzy ideal and $t \in [0, \mu(0)]$. If $xyz \in \mu_t$ for any $x, y, z \in R$, then $\mu(xyz) \geq t$. Thus, $(xyz)_t(xyz) = t \leq \mu(xyz)$ and $(xyz)_t = x_t y_t z_t \in \mu$. Since μ is a 2-absorbing fuzzy ideal, we have $x_t y_t = (xy)_t \in \mu$ or $x_t z_t = (xz)_t \in \mu$ or $y_t z_t = (yz)_t \in \mu$. Hence, $xy \in \mu_t$ or $xz \in \mu_t$ or $yz \in \mu_t$. Therefore, μ_t is a 2-absorbing ideal of R . $\square$

The following example shows that the converse of the lemma need not be true.

Example 17. Let $R = Z$, the ring of integers. Define the fuzzy ideal μ of Z by

$$\mu(x) = \begin{cases} 1, & x = 0, \\ \frac{1}{4}, & x \in 4Z - 0, \\ 0, & x \in Z - 4Z. \end{cases} \quad (3)$$

Then μ_t is (0) , $4Z$, Z in case $t \geq 1$, $t \geq 1/4$, $t \geq 0$, respectively. Thus, it is seen that μ_t is 2-absorbing ideal for

all $t \in \text{Im } \mu$. Since $2_{1/2}2_{1/2}1_{1/4}(4) = 1/4 \leq \mu(4) = 1/4$ so $2_{1/2}2_{1/2}1_{1/4} \in \mu$ but $2_{1/2}2_{1/2}(4) = 1/2 > \mu(4) = 1/4$ and $2_{1/2}1_{1/4}(2) = 1/4 > \mu(2) = 0$. Hence, μ is not 2-absorbing fuzzy ideal.

Definition 18. Let μ be a nonconstant fuzzy ideal of R . Then μ is said to be a 2-absorbing primary fuzzy ideal of R if $x_r y_s z_t \in \mu$ implies that either $x_r y_s \in \mu$ or $x_r z_t \in \sqrt{\mu}$ or $y_s z_t \in \sqrt{\mu}$ for any fuzzy points x_r, y_s, z_t .

Theorem 19. Every primary fuzzy ideal of R is a 2-absorbing primary fuzzy ideal of R .

Proof. It is clear from the definition of primary fuzzy ideal. $\square$

Proposition 20. Let I be a 2-absorbing primary ideal of R and $1 \neq \alpha \in L$ a 2-absorbing element. If μ is the fuzzy ideal of R defined by

$$\mu(x) = \begin{cases} 1, & x \in I, \\ \alpha, & x \notin I, \end{cases} \quad (4)$$

for all $x \in R$, then μ is a 2-absorbing primary fuzzy ideal of R .

Proof. Assume that $x_r y_s z_t \in \mu$ but $x_r y_s \notin \mu$ and $x_r z_t \notin \sqrt{\mu}$ and $y_s z_t \notin \sqrt{\mu}$ for any $x, y, z \in R$. Then $\mu(xy) < r \wedge s$ and $\mu((yz)^n) \leq \sqrt{\mu}(yz) < s \wedge t$ and $\mu((xz)^n) \leq \sqrt{\mu}(xz) < r \wedge t$ for all $n \geq 1$. In this case, $\mu(xy) = \alpha$ and $xy \notin I$, $\mu((yz)^n) = \alpha$ and $(yz)^n \notin I$ so $yz \notin \sqrt{I}$, $\mu((xz)^n) = \alpha$, and $(xz)^n \notin I$ so $xz \notin \sqrt{I}$. Since I is a 2-absorbing primary ideal of R , then we get $xyz \notin I$ and so $\mu(xyz) = \alpha$. By our assumption we get $(xyz)_{r \wedge s \wedge t} = x_r y_s z_t \in \mu$ and $r \wedge s \wedge t \leq \mu(xyz) = \alpha$. Thus, $r \wedge s \leq \alpha$ or $s \wedge t \leq \alpha$ or $r \wedge t \leq \alpha$, since α is 2-absorbing element, which is a contradiction. Hence, μ is a 2-absorbing primary fuzzy ideal of R . $\square$

Theorem 21. Every 2-absorbing fuzzy ideal of R is a 2-absorbing primary fuzzy ideal.

Proof. The proof is straightforward by the definition of the 2-absorbing fuzzy ideal. $\square$

The following example shows that the converse of Theorem 21 is not true.

Example 22. Let $\mu \in LI(R)$ be defined as

$$\mu(x) = \begin{cases} 1, & x \in 8Z, \\ 0, & x \notin 8Z. \end{cases} \quad (5)$$

By Theorem 19 and Proposition 20 μ is a 2-absorbing primary fuzzy ideal. For $2 \in Z$, $2_1 2_1 2_1 \in \mu$ but $2_1 2_1(4) = 1 > \mu(4) = 0$ so $2_1 2_1 \notin \mu$. Thus, μ is not a 2-absorbing fuzzy ideal.

Lemma 23. Let μ be a fuzzy ideal and $t \in [0, \mu(0)]$. If μ is a 2-absorbing primary fuzzy ideal, then μ_t is a 2-absorbing primary ideal of R .

Proof. If $xyz \in \mu_t$ for any $x, y, z \in R$, then $\mu(xyz) \geq t$. Thus, $(xyz)_t(xyz) = t \leq \mu(xyz)$ and $(xyz)_t = x_t y_t z_t \in \mu$. Since μ is a 2-absorbing primary fuzzy ideal, we have $x_t y_t = (xy)_t \in \mu$ or $x_t z_t = (xz)_t \in \sqrt{\mu}$ or $y_t z_t = (yz)_t \in \sqrt{\mu}$. Hence, $xy \in \mu_t$ or $xz \in \sqrt{\mu_t} = \sqrt{\mu}_t$ or $yz \in \sqrt{\mu_t} = \sqrt{\mu}_t$. Therefore, μ_t is a 2-absorbing primary ideal. $\square$

Note that if μ_t is a 2-absorbing primary ideal of R , then μ need not be 2-absorbing primary fuzzy ideal of R . In Example 17, μ is not 2-absorbing fuzzy ideal and also it is not 2-absorbing primary fuzzy ideal by Theorem 21, although μ_t is a 2-absorbing primary ideal of R .

Proposition 24. If μ is a 2-absorbing primary fuzzy ideal of R , then $\sqrt{\mu}$ is a 2-absorbing fuzzy ideal of R .

Proof. Let x_r, y_s, z_t be any fuzzy points of R such that $x_r y_s z_t \in \sqrt{\mu}$ and $x_r y_s \notin \sqrt{\mu}$. Since $x_r y_s z_t \in \sqrt{\mu}$, we get $r \wedge s \wedge t = x_r y_s z_t(xyz) \leq \sqrt{\mu}(xyz)$. By the definition of $\sqrt{\mu}$, $\sqrt{\mu}(xyz) = \bigvee_{n \geq 1} \mu(x^n y^n z^n) \geq r \wedge s \wedge t$. Then there is a $k \in \mathbb{Z}^+$ such that $r \wedge s \wedge t \leq \mu(x^k y^k z^k) = \mu((xyz)^k)$. It implies that $(x_r y_s z_t)^k \in \mu$. If $x_r y_s \notin \sqrt{\mu}$, then, for all $k \in \mathbb{Z}^+$, $(x_r y_s)^k = x_r^k y_s^k \notin \mu$. Since μ is a 2-absorbing primary fuzzy ideal, we conclude $x_r z_t \in \sqrt{\mu}$ or $y_s z_t \in \sqrt{\mu}$. Thus, $\sqrt{\mu}$ is a 2-absorbing fuzzy ideal. $\square$

Definition 25. Let μ be a 2-absorbing primary fuzzy ideal of R . Then $\gamma = \sqrt{\mu}$ is a 2-absorbing fuzzy ideal by Proposition 24. We say that μ is a γ -2-absorbing primary fuzzy ideal of R .

Theorem 26. Let $\mu_1, \mu_2, \dots, \mu_n$ be γ -2-absorbing primary fuzzy ideals of R for some 2-absorbing fuzzy ideal γ of R . Then $\mu = \bigcap_{i=1}^n \mu_i$ is a γ -2-absorbing primary fuzzy ideal of R .

Proof. Suppose that $x_r y_s z_t \in \mu$ and $x_r y_s \notin \mu$. Then $x_r y_s \notin \mu_j$ for some $n \geq j \geq 1$ and $x_r y_s z_t \in \mu_j$ for all $n \geq j \geq 1$. Since μ_j is a γ -2-absorbing primary fuzzy ideal, we have $y_s z_t \in \sqrt{\mu_j} = \gamma = \bigcap_{i=1}^n \sqrt{\mu_i} = \sqrt{\bigcap_{i=1}^n \mu_i} = \sqrt{\mu}$ or $x_r z_t \in \sqrt{\mu_j} = \gamma = \bigcap_{i=1}^n \sqrt{\mu_i} = \sqrt{\bigcap_{i=1}^n \mu_i} = \sqrt{\mu}$. Thus, μ is a γ -2-absorbing primary ideal of R . $\square$

In the following example, we show that if μ_1, μ_2 are 2-absorbing primary fuzzy ideals of a ring R , then $\mu_1 \cap \mu_2$ need not to be a 2-absorbing primary fuzzy ideal of R .

Example 27. Let $R = \mathbb{Z}$, the ring of integers. Define the fuzzy ideals μ_1 and μ_2 of $\mathbb{Z}$ by

$$\mu_1(x) = \begin{cases} 1, & x \in 50Z, \\ 0, & x \notin 50Z, \end{cases} \quad (6)$$

and by

$$\mu_2(x) = \begin{cases} 1, & x \in 75Z, \\ 0, & x \notin 75Z. \end{cases} \quad (7)$$

Here μ_1 and μ_2 are 2-absorbing primary fuzzy ideals of Z by Proposition 20. But it is not difficult to show that $\mu_1 \cap \mu_2$ is not a 2-absorbing primary fuzzy ideal of Z . Since

$$(\mu_1 \cap \mu_2)(x) = \begin{cases} 1, & x \in 150Z, \\ 0, & x \notin 150Z, \end{cases} \quad (8)$$

then $25_1.3_1.2_1 \in \mu_1 \cap \mu_2$ but $25_1.3_1 \notin \mu_1 \cap \mu_2$, $25_1.2_1 \notin \mu_1 \cap \mu_2$, and $2_1.3_1 \notin \mu_1 \cap \mu_2$. Moreover, by the definition of

$$\sqrt{\mu_1 \cap \mu_2} = \begin{cases} 1, & x \in 30Z, \\ 0, & x \notin 30Z, \end{cases} \quad (9)$$

we conclude that $25_1.3_1.2_1 \in \mu_1 \cap \mu_2$ but $25_1.3_1 \notin \sqrt{\mu_1 \cap \mu_2}$, $25_1.2_1 \notin \sqrt{\mu_1 \cap \mu_2}$, and $2_1.3_1 \notin \sqrt{\mu_1 \cap \mu_2}$. Hence, $\mu_1 \cap \mu_2$ is not a 2-absorbing primary fuzzy ideal of Z .

Theorem 28. Let μ be a fuzzy ideal of R . If $\sqrt{\mu}$ is a prime fuzzy ideal of R , then μ is a 2-absorbing primary fuzzy ideal of R .

Proof. Assume that $x_r y_s z_t \in \mu$ and $x_r y_s \notin \mu$ for any $x, y, z \in R$ and $r, s, t \in [0, 1]$. Since $x_r y_s z_t \in \mu$ and R is commutative ring, we have $x_r y_s z_t z_t = (x_r z_t)(y_s z_t) \in \mu \subseteq \sqrt{\mu}$. Thus, $x_r z_t \in \sqrt{\mu}$ or $y_s z_t \in \sqrt{\mu}$ since $\sqrt{\mu}$ is a prime fuzzy ideal of R . Hence, we conclude that μ is a 2-absorbing primary fuzzy ideal of R . $\square$

Corollary 29. If μ is a prime fuzzy ideal of R , then μ^n is 2-absorbing primary fuzzy ideal of R for any $n \in \mathbb{Z}^+$.

Proof. Let μ be a prime fuzzy ideal and $x_r y_s z_t \in \mu^n$ but $x_r y_s \notin \mu^n$ for any $n \in \mathbb{Z}^+$. Since $x_r y_s z_t \in \mu^n$ and R is commutative ring, we conclude that $x_r y_s z_t z_t = (x_r z_t)(y_s z_t) \in \mu^n \subseteq \mu$. Hence, $x_r z_t \in \mu = \sqrt{\mu^n}$ or $y_s z_t \in \mu = \sqrt{\mu^n}$ since μ is prime fuzzy ideal of R . $\square$

Theorem 30. Let $\{\mu_i : i \in I\}$ be a directed collection of 2-absorbing primary fuzzy ideals of R . Then the fuzzy ideal $\mu = \bigcup_{i \in I} \mu_i$ is a 2-absorbing primary fuzzy ideal of R .

Proof. Suppose $x_r y_s z_t \in \mu$ and $x_r y_s \notin \mu$ for some x_r, y_s, z_t fuzzy points of R . Then there are some $j \in I$ such that $x_r y_s z_t \in \mu_j$ and $x_r y_s \notin \mu_j$ for all $j \in I$. Since μ_j is 2-absorbing primary fuzzy ideal, we have $y_s z_t \in \sqrt{\mu_j}$ or $x_r z_t \in \sqrt{\mu_j}$. Thus, $y_s z_t \in \sqrt{\mu_j} \subseteq \bigcup_{i \in I} \sqrt{\mu_i} = \sqrt{\bigcup_{i \in I} \mu_i} = \mu$ or $x_r z_t \in \sqrt{\mu_j} \subseteq \bigcup_{i \in I} \sqrt{\mu_i} = \sqrt{\bigcup_{i \in I} \mu_i} = \mu$. $\square$

Theorem 31. Let $f : R \rightarrow S$ be a ring homomorphism. If ξ is a 2-absorbing primary fuzzy ideal of S , then $f^{-1}(\xi)$ is a 2-absorbing primary fuzzy ideal of R .

Proof. Assume that $x_r y_s z_t \in f^{-1}(\xi)$, where x_r, y_s, z_t are any fuzzy points of R . Then $r \wedge s \wedge t \leq f^{-1}(\xi)(xyz) = \xi(f(xyz)) = \xi(f(x)f(y)f(z))$. Let $f(x) = a$, $f(y) = b$, and $f(z) = c \in S$. Thus, we get that $r \wedge s \wedge t \leq \xi(abc)$ and $a_r b_s c_t \in \xi$. Since ξ is a 2-absorbing primary fuzzy ideal, we conclude that $a_r b_s \in \xi$ or $a_r c_t \in \sqrt{\xi}$ or $b_s c_t \in \sqrt{\xi}$. If $a_r b_s \in \xi$, then $r \wedge s \leq \xi(ab) =$

$\xi(f(x)f(y)) = \xi(f(xy)) = f^{-1}(\xi)(xy)$; hence, we conclude that $x_r y_s \in f^{-1}(\xi)$.

If $a_r c_t \in \sqrt{\xi}$, then, for some $n \in \mathbb{Z}^+$, $r \wedge t \leq \xi(a^n c^n) = \xi(f(x)^n f(z)^n) \leq \bigvee_{n \geq 1} \xi(f(x)^n f(z)^n) = \sqrt{\xi}(f(x)f(z)) = \sqrt{\xi}(f(xz)) = f^{-1}(\sqrt{\xi})(xz) = \sqrt{f^{-1}(\xi)}$. Thus, we get that $x_r z_t \in \sqrt{f^{-1}(\xi)}$. By a similar way, it can be seen that $y_s z_t \in \sqrt{f^{-1}(\xi)}$. $\square$

Theorem 32. Let $f : R \rightarrow S$ be a surjective ring homomorphism. If μ is a 2-absorbing primary fuzzy ideal of R which is constant on $\text{Ker } f$, then $f(\mu)$ is a 2-absorbing primary fuzzy ideal of S .

Proof. Suppose that $a_r b_s c_t \in f(\mu)$, where a_r, b_s, c_t are any fuzzy points of S . Since f is a surjective ring homomorphism, there exist $x, y, z \in R$ such that $f(x) = a$, $f(y) = b$, $f(z) = c$. Thus, $a_r b_s c_t(abc) = r \wedge s \wedge t \leq f(\mu)(abc) = f(\mu)(f(x)f(y)f(z)) = f(\mu)(f(xyz)) = \mu(xyz)$ because μ is constant on $\text{Ker } f$. Then we get $x_r y_s z_t \in \mu$. Since μ is a 2-absorbing primary fuzzy ideal, we conclude $x_r y_s \in \mu$ or $x_r z_t \in \sqrt{\mu}$ or $y_s z_t \in \sqrt{\mu}$.

Thus, $r \wedge s \leq \mu(xy) = f(\mu)(f(xy)) = f(\mu)(f(x)f(y)) = f(\mu)(ab)$ so $a_r b_s \in f(\mu)$ or $r \wedge t \leq \sqrt{\mu}(xz) = \bigvee_{n \geq 1} \mu(x^n z^n) = \bigvee_{n \geq 1} f(\mu)(f(x)^n f(z)^n) = \bigvee_{n \geq 1} f(\mu)(a^n c^n) = \sqrt{f(\mu)}(ac)$ so $a_r c_t \in \sqrt{f(\mu)}$.

By a similar way, it is easy to see that $b_s c_t \in \sqrt{f(\mu)}$ if $y_s z_t \in \sqrt{\mu}$. $\square$

4. Weakly Completely 2-Absorbing Primary Fuzzy Ideals

Definition 33. (i) A nonconstant fuzzy ideal μ of R is called a weakly completely 2-absorbing fuzzy ideal of R if for all $a, b, c \in R$, $\mu(abc) \leq \mu(ab)$ or $\mu(abc) \leq \mu(ac)$ or $\mu(abc) \leq \mu(bc)$.

(ii) A nonconstant fuzzy ideal μ of R is called a weakly completely 2-absorbing primary fuzzy ideal of R if for all $a, b, c \in R$, $\mu(abc) \leq \mu(ab)$ or $\mu(abc) \leq \sqrt{\mu}(ac)$ or $\mu(abc) \leq \sqrt{\mu}(bc)$.

Theorem 34. Every weakly completely 2-absorbing fuzzy ideal of R is a weakly completely 2-absorbing primary fuzzy ideal.

Proof. The proof is straightforward. $\square$

The following example shows that the converse of Theorem 34 is not necessarily true.

Example 35. Let $R = \mathbb{Z}$, the ring of integers. Define the fuzzy ideal μ of $\mathbb{Z}$ by

$$\mu(x) = \begin{cases} 1, & x \in 8\mathbb{Z}, \\ 0, & x \notin 8\mathbb{Z}. \end{cases} \quad (10)$$

Assume that $\mu(xyz) > \mu(xy)$ for any $x, y, z \in R$. Thus, $\mu(xyz) = 1$ and $0 = \mu(xy)$ so we get $xyz \in 8\mathbb{Z}$ and $xy \notin$

8Z. Since 8Z is primary ideal of Z, we get $z \in 2Z$. By the definition of radical μ

$$\begin{aligned}\sqrt{\mu}(x) &= \begin{cases} 1, & x \in 2Z, \\ 0, & x \notin 2Z, \end{cases} \\ \sqrt{\mu}(xz) &= 1, \\ \sqrt{\mu}(yz) &= 1, \end{aligned} \quad (11)$$

hence, $\sqrt{\mu}(xz) \geq \mu(xyz)$ or $\sqrt{\mu}(yz) \geq \mu(xyz)$. Therefore, μ is a weakly completely 2-absorbing primary fuzzy ideal. But since $\mu(2.2.2) = 1 > 0 = \mu(2.2)$, we conclude that μ is not a weakly completely 2-absorbing fuzzy ideal.

Proposition 36. Every primary fuzzy ideal of R is a weakly completely 2-absorbing primary fuzzy ideal.

Proof. Let μ be a primary fuzzy ideal of R . Assume that $\mu(xyz) > \mu(xy)$ for any $x, y, z \in R$. Since μ is primary fuzzy ideal, we conclude $\sqrt{\mu}(z) \geq \mu(xyz)$. Since $\sqrt{\mu}$ is a fuzzy ideal, we have $\sqrt{\mu}(xz) \geq \sqrt{\mu}(z) \geq \mu(xyz)$ or $\sqrt{\mu}(yz) \geq \sqrt{\mu}(z) \geq \mu(xyz)$. So μ is a weakly completely 2-absorbing primary fuzzy ideal. $\square$

Lemma 37. Let μ be a fuzzy ideal of R . Then μ is a weakly completely 2-absorbing primary fuzzy ideal of R if and only if μ_t is a 2-absorbing primary ideal of R for all $t \in [0, \mu(0)]$.

Proof. Assume that $xyz \in \mu_t$ and $xy \notin \mu_t$ for any $x, y, z \in R$. We show that $yz \in \sqrt{\mu_t}$ or $xz \in \sqrt{\mu_t}$. Note that $\mu(xyz) \geq t > \mu(xy)$. Since μ is a weakly completely 2-absorbing primary fuzzy ideal of R , we have $\sqrt{\mu}(xz) \geq \mu(xyz) \geq t$ or $\sqrt{\mu}(yz) \geq \mu(xyz) \geq t$. Thus, $xz \in \sqrt{\mu_t}$ or $yz \in \sqrt{\mu_t}$. So we conclude that μ_t is a 2-absorbing primary ideal of R .

Conversely, assume that μ_t is a 2-absorbing primary ideal of R for all $t \in [0, \mu(0)]$. If $\mu(xyz) > \mu(xy)$ for any $x, y, z \in R$, then there is a $k \in [0, \mu(0)]$ such that $\mu(xyz) = k$ and $k = \mu(xyz) > \mu(xy)$. So $xyz \in \mu_k$ and $xy \notin \mu_k$. Since μ_k is a 2-absorbing primary ideal of R , we get that $xz \in \sqrt{\mu_k} = \sqrt{\mu_t}$ or $yz \in \sqrt{\mu_k} = \sqrt{\mu_t}$. Hence, $\sqrt{\mu}(xz) \geq k = \mu(xyz)$ or $\sqrt{\mu}(yz) \geq k = \mu(xyz)$. Therefore, μ is a weakly completely 2-absorbing primary fuzzy ideal of R . $\square$

Theorem 38. If μ is a weakly completely 2-absorbing primary fuzzy ideal of R , then $\sqrt{\mu}$ is a weakly completely 2-absorbing fuzzy ideal of R .

Proof. If μ is a weakly completely 2-absorbing primary fuzzy ideal, then by Lemma 37 μ_t is a 2-absorbing primary ideal of R for any $t \in [0, \mu(0)]$. By [5, Theorem 2.2], $\sqrt{\mu_t} = \sqrt{\mu_t}$ is 2-absorbing ideal of R . Then it is easy to see that from Definition 33 $\sqrt{\mu_t}$ is a 2-absorbing ideal of R if and only if $\sqrt{\mu}$ is a weakly completely 2-absorbing fuzzy ideal. $\square$

Definition 39. Let μ be fuzzy ideal of R . Then μ is called a K -2-absorbing primary fuzzy ideal of R if for all $x, y, z \in R$, $\mu(xyz) = \mu(0)$ implies that $\mu(xy) = \mu(0)$ or $\sqrt{\mu}(yz) = \mu(0)$ or $\sqrt{\mu}(xz) = \mu(0)$.

Proposition 40. Every weakly completely 2-absorbing primary fuzzy ideal is a K -2-absorbing primary fuzzy ideal.

Proof. Assume that μ is a weakly completely 2-absorbing primary fuzzy ideal. If $\mu(xyz) = \mu(0)$ for any $x, y, z \in R$, then $\mu(0) \geq \mu(xy) \geq \mu(xyz) = \mu(0)$ or $\mu(0) = \sqrt{\mu}(0) \geq \sqrt{\mu}(xz) \geq \mu(xyz) = \mu(0)$ or $\mu(0) = \sqrt{\mu}(0) \geq \sqrt{\mu}(yz) \geq \mu(xyz) = \mu(0)$ since μ is a weakly completely 2-absorbing primary fuzzy ideal. Hence, $\mu(xy) = \mu(0)$ or $\sqrt{\mu}(xz) = \mu(0)$ or $\sqrt{\mu}(yz) = \mu(0)$. We conclude that μ is a K -2-absorbing primary fuzzy ideal.

Note that the following example shows that a K -2-absorbing primary fuzzy ideal need not be a weakly completely 2-absorbing primary fuzzy ideal. $\square$

Example 41. Let $R = Z$, the ring of integers. Define the fuzzy ideal μ of Z by

$$\mu(x) = \begin{cases} 1, & x = 0, \\ \frac{1}{2}, & x \in 30Z - 0, \\ \frac{1}{3}, & x \in Z - 30Z. \end{cases} \quad (12)$$

Then μ is a K -2-absorbing primary fuzzy ideal. But since

$$\mu(2.3.5) = \frac{1}{2} > \bigvee \{\mu(2.3), \mu(2.5), \mu(3.5)\} = \frac{1}{3} \quad (13)$$

or

$$\begin{aligned}\mu(2.3.5) &= \frac{1}{2} > \bigvee \{\sqrt{\mu}(2.3), \sqrt{\mu}(2.5), \sqrt{\mu}(3.5)\} \\ &= \frac{1}{3}, \end{aligned} \quad (14)$$

then μ is not a weakly completely 2-absorbing primary fuzzy ideal.

Corollary 42. Every weakly completely prime fuzzy ideal is a weakly completely 2-absorbing primary fuzzy ideal.

Proof. Since every weakly completely prime fuzzy ideal is primary fuzzy ideal, by Proposition 36 every weakly completely prime fuzzy ideal is a weakly completely 2-absorbing primary fuzzy ideal. $\square$

Theorem 43. Every K -2-absorbing fuzzy ideal is a K -2-absorbing primary fuzzy ideal.

Proof. The proof is straightforward. $\square$

The following example shows that the converse of Theorem 43 is not true.

Example 44. Define the fuzzy ideal μ of Z by

$$\mu(x) = \begin{cases} 1, & x \in 8Z, \\ 0, & x \notin 8Z. \end{cases} \quad (15)$$

Then μ is a K -2-absorbing primary fuzzy ideal but since $\mu(2.2.2) = \sqrt{\mu}(2.2) = \bigvee_{n \geq 1} \mu(4^n) = 1 = \mu(0)$ and $\mu(2.2.2) = 1 = \mu(0) \neq \mu(4) = 0$, we have that μ is not a K -2-absorbing fuzzy ideal.

Theorem 45. Let $f : R \rightarrow S$ be a ring homomorphism. If ξ is a weakly completely 2-absorbing primary fuzzy ideal of S , then $f^{-1}(\xi)$ is a weakly completely 2-absorbing primary fuzzy ideal of R .

Proof. Assume that $f^{-1}(\xi)(xyz) > f^{-1}(\xi)(xy)$ for any $x, y, z \in R$. Then $f^{-1}(\xi)(xyz) = \xi(f(xyz)) = \xi(f(x)f(y)f(z)) > f^{-1}(\xi)(xy) = \xi(f(xy)) = \xi(f(x)f(y))$. Since ξ is a weakly completely 2-absorbing primary fuzzy ideal of S , we conclude that $\xi(f(x)f(y)f(z)) = f^{-1}(\xi)(xyz) \leq \sqrt{\xi}(f(x)f(z)) = \sqrt{\xi}(f(xz)) = f^{-1}(\sqrt{\xi}(xz)) = \sqrt{f^{-1}(\xi)}(xz)$ or $\xi(f(x)f(y)f(z)) = f^{-1}(\xi)(xyz) \leq \sqrt{\xi}(f(y)f(z)) = \sqrt{\xi}(f(yz)) = f^{-1}(\sqrt{\xi}(yz)) = \sqrt{f^{-1}(\xi)}(yz)$. Thus, $f^{-1}(\xi)$ is a weakly completely 2-absorbing primary fuzzy ideal of R . $\square$

Theorem 46. Let $f : R \rightarrow S$ be a surjective ring homomorphism. If μ is a weakly completely 2-absorbing primary fuzzy ideal of R which is constant on $\text{Ker } f$, then $f(\mu)$ is a weakly completely 2-absorbing primary fuzzy ideal of S .

Proof. Assume that $f(\mu)(abc) > f(\mu)(ab)$ for any $a, b, c \in S$. Since f is surjective ring homomorphism, $f(x) = a$, $f(y) = b$, and $f(z) = c$ for some $x, y, z \in R$. Thus,

$f(\mu)(abc) = f(\mu)(f(x)f(y)f(z)) = f(\mu)(f(xyz)) > f(\mu)(ab) = f(\mu)(f(x)f(y)) = f(\mu)(f(xy))$. So, as μ is constant on $\text{Ker } f$, $f(\mu)(f(xyz)) = \mu(xyz)$ and $f(\mu)(f(xy)) = \mu(xy)$. This means that $f(\mu)(abc) = \mu(xyz) > \mu(xy) = f(\mu)(ab)$. Since μ is a weakly completely 2-absorbing primary fuzzy ideal of R , we have $\mu(xyz) = f(\mu)(f(x)f(y)f(z)) = f(\mu)(abc) \leq \sqrt{\mu}(xz) = f(\sqrt{\mu})(f(xz)) = f(\sqrt{\mu})(ac) = \sqrt{f(\mu)}(ac)$ or $\mu(xyz) = f(\mu)(f(x)f(y)f(z)) = f(\mu)(abc) \leq \sqrt{\mu}(yz) = f(\sqrt{\mu})(f(yz)) = f(\sqrt{\mu})(bc) = \sqrt{f(\mu)}(bc)$. Hence, $f(\mu)$ is a weakly completely 2-absorbing primary fuzzy ideal of R . $\square$

We state the following corollary without proof. Its proof is a result of Theorems 45 and 46.

Corollary 47. Let f be a homomorphism of a ring R onto a ring S . Then f induces a one-one inclusion preserving correspondence between the weakly completely 2-absorbing primary fuzzy ideals of R which is constant on $\text{Ker } f$ and the weakly completely 2-absorbing primary fuzzy ideals of S in such a way that if μ is a weakly completely 2-absorbing primary fuzzy ideal of R constant on $\text{Ker } f$, then $f(\mu)$ is the corresponding weakly completely 2-absorbing primary fuzzy ideal of S , and if ξ is a weakly completely 2-absorbing primary fuzzy ideal of S , then $f^{-1}(\xi)$ is the corresponding weakly completely 2-absorbing primary fuzzy ideal of R .

Remark 48. Note that the following diagram shows the transition between definitions of fuzzy ideals:

5. Conclusion

This article investigates the weakly completely 2-absorbing primary fuzzy ideal and 2-absorbing primary fuzzy ideal as a generalization of primary fuzzy ideal in commutative rings. Also some characterizations of 2-absorbing primary fuzzy ideal are obtained. Moreover, we see that a 2-absorbing primary fuzzy ideal by a 2-absorbing primary ideal of a commutative ring is established, so the transition between the two structures can be analyzed.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
- [2] W. J. Liu, "Fuzzy invariant subgroups and fuzzy ideals," *Fuzzy Sets and Systems*, vol. 8, no. 2, pp. 133–139, 1982.
- [3] T. K. Mukherjee and M. K. Sen, "Prime fuzzy ideals in rings," *Fuzzy Sets and Systems*, vol. 32, no. 3, pp. 337–341, 1989.
- [4] A. Badawi, "On 2-absorbing ideals of commutative rings," *Bulletin of the Australian Mathematical Society*, vol. 75, no. 3, pp. 417–429, 2007.
- [5] A. Badawi, U. Tekir, and E. Yetkin, "On 2-absorbing primary ideals in commutative rings," *Bulletin of the Korean Mathematical Society*, vol. 51, no. 4, pp. 1163–1173, 2014.
- [6] T. K. Mukherjee and M. K. Sen, "Primary fuzzy ideals and radical of fuzzy ideals," *Fuzzy Sets and Systems*, vol. 56, no. 1, pp. 97–101, 1993.
- [7] V. N. Dixit, R. Kumar, and N. Ajmal, "Fuzzy ideals and fuzzy prime ideals of a ring," *Fuzzy Sets and Systems*, vol. 44, no. 1, pp. 127–138, 1991.

- [8] F. I. Sidky and S. A. Khatab, "Nil radical of fuzzy ideal," *Fuzzy Sets and Systems*, vol. 47, no. 1, pp. 117–120, 1992.
- [9] F. Çallıalp, E. Yetkin, and U. Tekir, "On 2-absorbing primary and weakly 2-absorbing primary elements in multiplicative lattices," *Italian Journal of Pure and Applied Mathematics*, no. 34, pp. 263–276, 2015.

Hindawi
Submit your manuscripts at
<https://www.hindawi.com>