Research Article

ε -Properly Efficiency of Multiobjective Semidefinite Programming with Set-Valued Functions

Chun Hong Yuan¹ and Wei Dong Rong²

¹College of Mathematics Science, Inner Mongolia Normal University, Huhhot 010022, China

²College of Mathematics Science, Inner Mongolia University, Huhhot 010021, China

Correspondence should be addressed to Chun Hong Yuan; yuanchunhong1979@126.com

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In this paper, we study the ε -properly efficiency of multiobjective semidefinite programming with set-valued functions. Firstly, we obtain the scalarization theorems under the condition of the generalized cone-subconvexlikeness. Then, we establish the alternative theorem which contains matrixes and vectors, the ε -Lagrange multiplier theorems, and the ε -proper saddle point theorems of the primal programming under some suitable conditions.

1. Introduction

Vector optimization with set-valued functions has been used widely in many fields, such as economics and engineering. In recent years, set-valued optimization problem has aroused extensive concerns among the researchers [1–6]. Many authors focus their attention on how to get the approximate solutions of optimization problems. Chicco et al. [7] proposed a new type of solution based on the upper comprehensive sets and discussed the existence of optimal points in multicriteria situations. Zhao and Yang [8] obtained some useful properties of E -Benson proper efficiency. Gutiérrez et al. [9] presented some properties of strict efficient solutions in vector optimization related to the ε -efficiency notion. In locally convex Hausdorff topological vector spaces, ε -strongly efficient solutions of vector optimization with set-valued maps were discussed by Wang [10]. Rong and Wu [11] proposed the concept of ε -weak efficient solutions and derived scalarization results, saddle point theorems, and Lagrangian multipliers theorems.

Semidefinite programming involves optimization problems with a linear objective function over semidefinite constraints. It shares many interesting properties with linear programming. Semidefinite programming unifies several standard problems (linear and quadratic programming) and finds many applications in engineering [12].

By combining approximate solutions of vector optimization problems with multiobjective semidefinite programming, ε -properly efficiency of multiobjective semidefinite programming with set-valued functions is discussed in this paper. The rest of the paper is organized as follows: In Section 2, we introduce some notations and definitions used throughout the text. In Section 3, we derive scalarization theorems expressing the necessary and sufficient optimality conditions for ε -properly efficient solutions of the primal programming. Under the condition of the generalized cone-subconvexlikeness, the alternative theorem which contains matrix and vector and the ε -Lagrange multiplier theorems are established in Section 4. In Section 5, the ε -proper saddle point results are presented.

2. Preliminaries

Let $v = \{v_1, \dots, v_m\}^T \in R^m$ and

$$R_+^m = \{v \in R^m : v_i \geq 0, i = 1, \dots, m\}. \quad (1)$$

Obviously, $R_+^m \subset R^m$ is a pointed closed convex cone with nonempty interior.

Let A be a nonempty subset of R^m ; the generated cone of A is defined as

$$\text{cone}(A) = \{\alpha a : \alpha \geq 0, a \in A\}. \quad (2)$$

$\text{cl}(A)$ and $\text{int}(A)$ represent the closure and the interior of A , respectively. The positive dual cone A^+ of A is defined as

$$A^+ = \{y^* \in R^m : y \cdot y^* \geq 0, \forall y \in A\}, \quad (3)$$

and the strict positive dual cone A^{+i} of A is defined as

$$A^{+i} = \{y^* \in R^m : y \cdot y^* > 0, \forall y \in A \setminus \{0\}\}, \quad (4)$$

where $y \cdot y^* = y^T y^*$. Obviously, we see that $A^{+i} \subset A^+$. Let $A \subset R_+^m$ be a nonempty subset and let $\varepsilon \in R_+^m$; we define the following two important sets:

$$\begin{aligned} \varepsilon - \text{PMin}(A, R_+^m) \\ = \{\bar{y} \in A : \text{clcone}(A + R_+^m + \varepsilon - \bar{y}) \cap (-R_+^m) = \{0\}\}, \end{aligned} \quad (5)$$

$$\begin{aligned} \varepsilon - \text{PMax}(A, R_+^m) \\ = \{\bar{y} \in A : \text{clcone}(A - R_+^m - \varepsilon - \bar{y}) \cap R_+^m = \{0\}\}. \end{aligned} \quad (6)$$

Let S^p be a set of real symmetric matrixes of p -order and S_+^p be a set of real symmetric positive semidefinite matrixes of p -order. For matrixes $K_1, K_2 \in S^p$,

$$\begin{aligned} K_1 \geq K_2 &\iff \\ K_1 - K_2 &\in S_+^p. \end{aligned} \quad (7)$$

The dot product of K_1 and K_2 is defined as $K_1 \cdot K_2 = \text{tr}(K_1 K_2)$, where $\text{tr}(K_1 K_2)$ is the trace of the matrix $K_1 K_2$. For two vectors $a, b \in R^m$, their dot product denotes $a \cdot b = a^T b$.

It is easy to prove the following results.

Proposition 1. (i) Let $K_1 \in S_+^p, K_2 \in S_+^p$; then $K_1 \cdot K_2 \geq 0$.
(ii) Let $K_1 \in S^p$; if $K_1 \cdot K_2 \geq 0$ for any $K_2 \in S_+^p$, then $K_1 \in S_+^p$.

We now state the following multiobjective semidefinite programming with set-valued functions (MSDP) to be studied in the present paper:

$$\begin{aligned} \min \quad & F(x), \\ \text{s.t.} \quad & G(x) \cap (-S_+^p) \neq \emptyset, \\ & 0 \in H(x), \\ & x \in X_0, \end{aligned} \quad (\text{MSDP})$$

where $F : R^n \rightarrow 2^{R^m}, G : R^n \rightarrow 2^{S^p}$, and $H : R^n \rightarrow 2^{R^q}$ are set-valued functions with nonempty value, and $X_0 \subset R^n, X_0 \neq \emptyset$. We use

$$X = \{x \in X_0 : G(x) \cap (-S_+^p) \neq \emptyset, 0 \in H(x)\} \quad (8)$$

to represent the feasible set of (MSDP) and assume that X is nonempty. Let

$$F(X) = \bigcup_{x \in X} F(x). \quad (9)$$

Definition 2. A point $\bar{x} \in X$ is said to be an ε -properly efficient solution of (MSDP), if

$$F(\bar{x}) \cap \varepsilon - \text{PMin}(F(X), R_+^m) \neq \emptyset. \quad (10)$$

An ordered pair $(\bar{x}, \bar{y})$ is said to be an ε -properly efficient pair of (MSDP), if

$$\bar{y} \in F(\bar{x}) \cap \varepsilon - \text{PMin}(F(X), R_+^m). \quad (11)$$

The following assumptions, (H1) and (H2), for (MSDP) will be imposed in the rest of this paper:

(H1) ((H1.1)–(H1.3)):

(H1.1) $\text{clcone}(F(X_0) + R_+^m)$ is a convex set

(H1.2) $\forall \lambda \in (0, 1), x_1, x_2 \in X_0$:

$$\lambda G(x_1) + (1 - \lambda) G(x_2) \subset G(X_0) + S_+^p \quad (12)$$

(H1.3) $\forall \lambda \in (0, 1), x_1, x_2 \in X_0$:

$$\lambda H(x_1) + (1 - \lambda) H(x_2) \subset H(X_0) \quad (13)$$

(H2) (Slater constraint qualification): for any $\Omega \in S_+^p$ and $\xi \in R^q$, with $(\Omega, \xi) \neq (0, 0)$, there exists $x_0 \in X_0$, such that

$$(\Omega \cdot G(x_0) + \xi \cdot H(x_0)) \cap R_- \neq \emptyset, \quad (14)$$

where $R_- = \{r \in R : r < 0\}$

Remark 3. F is said to be generalized R_+^m -subconvexlike on X_0 , if assumption (H1.1) is true [1]. G is said to be S_+^p -convexlike on X_0 , if assumption (H1.2) is true [2].

3. Scalarization Theorems

The following scalar minimization problem of (MSDP) is discussed in this section.

$$\begin{aligned} \min \quad & u \cdot F(x) = \{u \cdot y : y \in F(x)\}, \quad u \in R^m \setminus \{0\}, \\ \text{s.t.} \quad & x \in X. \end{aligned} \quad (P_u)$$

Definition 4. An ordered pair $(\bar{x}, \bar{y})$ is said to be an $u \cdot \varepsilon$ -optimal pair of (P_u) , if there exist $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$, such that

$$u \cdot \bar{y} \leq u \cdot y + u \cdot \varepsilon, \quad \forall y \in F(X). \quad (15)$$

Lemma 5 (see [13]). Let U be a closed convex pointed cone in R^n and V be a closed convex cone in R^n with $V \neq \emptyset$; then $(-U) \cap V = \{0\}$ if and only if $U^{+i} \cap V^+ \neq \emptyset$.

Theorem 6. Let $\bar{u} \in \text{int } R_+^m$; if $(\bar{x}, \bar{y})$ is an $u \cdot \varepsilon$ -optimal pair of (P_u) , then $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (MSDP).

Proof. Taking arbitrarily $y \in \text{clcone}(F(X) + R_+^m + \varepsilon - \bar{y}) \cap (-R_+^m)$, we have that $y \in (-R_+^m)$ and there exist sequences $y_n \subset F(X)$, $r_n \subset R_+^m$, and $\lambda_n \subset R_+$, such that

$$y = \lim_{n \rightarrow \infty} \lambda_n (y_n + r_n + \varepsilon - \bar{y}). \quad (16)$$

Therefore, we obtain

$$\bar{u} \cdot y = \lim_{n \rightarrow \infty} \lambda_n (\bar{u} \cdot y_n + \bar{u} \cdot r_n + \bar{u} \cdot \varepsilon - \bar{u} \cdot \bar{y}). \quad (17)$$

Since $(\bar{x}, \bar{y})$ is an $u \cdot \varepsilon$ -optimal pair of (P_u) , by Definition 4, we have

$$\bar{u} \cdot \bar{y} \leq \bar{u} \cdot y_n + \bar{u} \cdot \varepsilon. \quad (18)$$

From $r_n \in R_+^m$ and $\bar{u} \in \text{int } R_+^m$, we obtain $\bar{u} \cdot r_n \geq 0$. Therefore, the right side of (17) is nonnegative; sequentially $\bar{u} \cdot y \geq 0$. On the other hand, from $y \in (-R_+^m)$, we have also $\bar{u} \cdot y \leq 0$. Thus $\bar{u} \cdot y = 0$. Notice that $\bar{u} \in \text{int } R_+^m$, and we obtain $y = 0$. Then

$$\text{clcone}(F(X) + R_+^m + \varepsilon - \bar{y}) \cap (-R_+^m) = \{0\}. \quad (19)$$

The proof is complete by Definition 2 and (5). $\square$

Theorem 7. *If $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (MSDP), and $F + \varepsilon - \bar{y}$ is generalized R_+^m -subconvexlike on X , then there exists $\bar{u} \in \text{int } R_+^m$, such that $(\bar{x}, \bar{y})$ is an $\bar{u} \cdot \varepsilon$ -optimal pair of (P_u) .*

Proof. If $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (MSDP), by Definition 2 and (5), we have

$$\text{clcone}(F(X) + R_+^m + \varepsilon - \bar{y}) \cap (-R_+^m) = \{0\}. \quad (20)$$

Since $F + \varepsilon - \bar{y}$ is generalized R_+^m -subconvexlike on X , from Remark 3, we have that $\text{clcone}(F(X) + R_+^m + \varepsilon - \bar{y})$ is convex. By Lemma 5, we obtain

$$\text{int } R_+^m \cap (\text{clcone}(F(X) + R_+^m + \varepsilon - \bar{y}))^+ \neq \emptyset; \quad (21)$$

hence there exists $\bar{u} \in \text{int } R_+^m \cap (\text{clcone}(F(X) + R_+^m + \varepsilon - \bar{y}))^+$. Since $F(X) + \varepsilon - \bar{y} \subset \text{clcone}(F(X) + \varepsilon - \bar{y} + R_+^m)$, we obtain

$$\bar{u} \cdot (y + \varepsilon - \bar{y}) \geq 0, \quad \forall y \in F(X). \quad (22)$$

By Definition 4, $(\bar{x}, \bar{y})$ is an $\bar{u} \cdot \varepsilon$ -optimal pair of (P_u) . $\square$

4. ε -Lagrange Multiplier Theorems

The map $T : S^p \rightarrow R^m$ is defined as $TK = (T_1 \cdot K, \dots, T_m \cdot K)^T$, where $K \in S^p$, $T_i \in S^p$ ($i = 1, \dots, m$). It is easy to prove that T , denoted by $T = (T_1, \dots, T_m)^T$, is a linear operator from S^p to R^m . A linear operator T is said to be nonnegative, denoted by $T \geq 0$, if $T_i \in S_+^p$ ($i = 1, \dots, m$). Let L denote the set of all linear operators T , and let L_+ denote the set of all nonnegative linear operators; obviously, $L_+ \subset L$.

Remark 8. Let a linear operator $T \in L_+$, and let a matrix $A \leq 0$; by Proposition 1(i), we know that the vector $TA \leq 0$.

The scalar valued Lagrangian function of (MSDP) is defined as

$$l(x, \phi, \Omega, \xi) = \phi \cdot F(x) + \Omega \cdot G(x) + \xi \cdot H(x), \quad (23)$$

where $x \in X_0$, $\phi \in R_+^m$, $\Omega \in S_+^p$, and $\xi \in R^q$. The vector valued Lagrangian function of (MSDP) is defined as

$$L(x, T, M) = F(x) + TG(x) + MH(x), \quad (24)$$

where $x \in X_0$, $T \in L^+$, and $M \in R^{m \times q}$.

Now we consider the following two unconstrained optimization problems:

$$\min_{x \in X_0} l(x, \phi, \Omega, \xi), \quad (\text{SP})(\phi, \Omega, \xi)$$

$$\min_{x \in X_0} L(x, T, M). \quad (\text{SMP})(T, M)$$

Lemma 9 (see [1]). *Let $A \subset X_0$; then $\text{clcone}(F(A) + \text{int } R_+^m) = \text{clcone}(F(A) + R_+^m)$.*

Lemma 10. *Let $A \subset X_0$; then $F(A) \cap (-\text{int } R_+^m) = \emptyset$ if and only if $\text{clcone}(F(A) + R_+^m) \cap (-\text{int } R_+^m) = \emptyset$.*

Proof. Let $F(A) \cap (-\text{int } R_+^m) = \emptyset$. If $\text{clcone}(F(A) + R_+^m) \cap (-\text{int } R_+^m) \neq \emptyset$, from Lemma 9, we have

$$\text{clcone}(F(A) + \text{int } R_+^m) \cap (-\text{int } R_+^m) \neq \emptyset. \quad (25)$$

It follows that there exist $\bar{y} \in \text{clcone}(F(A) + \text{int } R_+^m) \cap (-\text{int } R_+^m)$ and sequences $y_n \in \text{cone}(F(A) + \text{int } R_+^m)$, such that $\lim_{n \rightarrow \infty} y_n = \bar{y}$. Thus there exist $\alpha_n \in R_+$, $y'_n \in F(A)$, and $r_n \in \text{int } R_+^m$, such that

$$y_n = \alpha_n (y'_n + r_n). \quad (26)$$

In view of $\bar{y} \in -\text{int } R_+^m$ and $\lim_{n \rightarrow \infty} y_n = \bar{y}$, there exists a natural number N , such that $n > N$, $y_n \in -\text{int } R_+^m$. It is easy to check that if $n > N$, then $\alpha_n \neq 0$. From (26) we have

$$y'_n = \frac{y_n}{\alpha_n} - r_n \in -\text{int } R_+^m - \text{int } R_+^m \subset -\text{int } R_+^m, \quad (27)$$

$n > N$.

This implies

$$y'_n \in F(A) \cap (-\text{int } R_+^m), \quad n > N. \quad (28)$$

This is a contradiction to the assumption.

Conversely, the sufficient condition can be proved by considering $F(A) \subset \text{clcone}(F(A) + R_+^m)$. $\square$

Lemma 11 (alternative theorem). *Suppose that the set-valued functions F , G , and H satisfy assumptions (H1) and (H2) on X_0 ; then exactly one of the following statements is true:*

- (i) *There exists $x \in X_0$, such that $F(x) \cap (-\text{int } R_+^m) \neq \emptyset$, $G(x) \cap (-S_+^p) \neq \emptyset$, and $0 \in H(x)$.*
- (ii) *There exists $(\phi, \Omega, \xi) \in (R_+^m \times S_+^p \times R^q)$, $\phi \neq 0$, such that*

$$\phi \cdot F(x) + \Omega \cdot G(x) + \xi \cdot H(x) \geq 0, \quad \forall x \in X_0. \quad (29)$$

Proof. Obviously, (i) and (ii) cannot hold simultaneously. Otherwise, there exist $\bar{x} \in X_0$, $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x})$, and $\bar{w} \in H(\bar{x})$, such that

$$\begin{aligned} \bar{y} &\in -\text{int } R_+^m, \\ \bar{z} &\in -S_+^p, \\ \bar{w} &= 0. \end{aligned} \quad (30)$$

By Proposition 1, there exists $(\phi, \Omega, \xi) \in R_+^m \times S_+^p \times R^q$, $\phi \neq 0$, such that

$$\phi \cdot \bar{y} + \Omega \cdot \bar{Z} + \xi \cdot \bar{w} < 0, \quad (31)$$

and this contradicts (ii). Now we show that if (i) is not true, then (ii) is true. Let

$$\begin{aligned} S = \{ & (u, V, w) \in R^m \times S^p \times R^q \mid \exists x \in X_0, \text{ s.t. } u \\ & \in \text{clcone}(F(x) + R_+^m) + \text{int } R_+^m, V \in G(x) \\ & + S_+^p, w \in H(x) \}. \end{aligned} \quad (32)$$

It is easy to show that S is a convex set and $\text{int } S \neq \emptyset$. If (i) is not true, then $(0, 0, 0) \notin S$. Otherwise, assume that $(0, 0, 0) \in S$; from (32) there exists $x \in X_0$, such that

$$\begin{aligned} \text{clcone}(F(x) + R_+^m) \cap (-\text{int } R_+^m) & \neq \emptyset, \\ G(x) \cap (-S_+^p) & \neq \emptyset, \\ 0 & \in H(x). \end{aligned} \quad (33)$$

By Lemma 10, there exists $x \in X_0$, such that

$$\begin{aligned} F(x) \cap (-\text{int } R_+^m) & \neq \emptyset, \\ G(x) \cap (-S_+^p) & \neq \emptyset, \\ 0 & \in H(x), \end{aligned} \quad (34)$$

which is inconsistent with the assumed condition. Hence by the separation theorem of convex sets, there exists $(\phi, \Omega, \xi) \in (R^m \times S^p \times R^q) \setminus \{0, 0, 0\}$, such that

$$\phi \cdot u + \Omega \cdot V + \xi \cdot w \geq 0, \quad \forall (u, v, w) \in \text{cl } S. \quad (35)$$

Thus for any $x \in X_0$, $y \in F(x)$, $Z \in G(x)$, $w \in H(x)$, $r \in \text{int } R_+^m$, $s \in S_+^p$, $\varepsilon > 0$, and $\eta > 0$, we have

$$\phi \cdot (y + \varepsilon r) + \Omega \cdot (Z + \eta s) + \xi \cdot w \geq 0. \quad (36)$$

Letting $\varepsilon \rightarrow +\infty$ in (36), we obtain

$$\phi \cdot r \geq 0, \quad \forall r \in \text{int } R_+^m; \quad (37)$$

hence $\phi \in R_+^m$. Letting $\eta \rightarrow +\infty$ in (36), we have also $\Omega \in S_+^p$. Now letting $\varepsilon \rightarrow 0$ and $\eta \rightarrow 0$ in (36), we have

$$\begin{aligned} \phi \cdot y + \Omega \cdot Z + \xi \cdot w & \geq 0, \\ \forall x \in X_0, y \in F(x), Z \in G(x), w \in H(x), \end{aligned} \quad (38)$$

which implies that there exists $(\phi, \Omega, \xi) \in (R_+^m \times S_+^p \times R^q) \setminus \{0, 0, 0\}$, such that

$$\phi \cdot F(x) + \Omega \cdot G(x) + \xi \cdot H(x) \geq 0, \quad \forall x \in X_0. \quad (39)$$

At last we find $\phi \neq 0$. Contrarily, taking $\phi = 0$ in (39), we obtain

$$\Omega \cdot G(x) + \xi \cdot H(x) \geq 0, \quad \forall x \in X_0, \quad (40)$$

and this contradicts (H2). Therefore, (ii) holds. $\square$

Theorem 12. Let $\bar{x} \in X$, $\bar{y} \in F(\bar{x})$; suppose that $F + \varepsilon - \bar{y}$, G , and H satisfy (H1) and (H2) on X_0 and $F + \varepsilon - \bar{y}$ is generalized R_+^m -subconvexlike on X . If $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (MSDP), then there exists $(\phi, \Omega, \xi) \in \text{int } R_+^m \times S_+^p \times R^q$, such that

$$\phi \cdot \bar{y} \leq l(x, \phi, \Omega, \xi) + \phi \cdot \varepsilon, \quad \forall x \in X_0. \quad (41)$$

Proof. If $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (MSDP) and $F + \varepsilon - \bar{y}$ is generalized R_+^m -subconvexlike on X , by Theorem 7 and Definition 4, there exists $\bar{u} \in \text{int } R_+^m$, such that

$$\bar{u} \cdot \bar{y} \leq \bar{u} \cdot F(x) + \bar{u} \cdot \varepsilon, \quad \forall x \in X. \quad (42)$$

Therefore the following conditions are not true:

$$\begin{aligned} \bar{u} \cdot (F(x) + \varepsilon - \bar{y}) \cap (-\text{int } R_+) & \neq \emptyset, \\ G(x) \cap (-S_+^p) & \neq \emptyset, \\ 0 & \in H(x), \\ \forall x \in X_0. \end{aligned} \quad (43)$$

Since $(F + \varepsilon - \bar{y})$ is generalized R_+^m -subconvexlike on X_0 , we can easily prove that $\bar{u} \cdot (F + \varepsilon - \bar{y})$ is generalized R_+ -subconvexlike on X_0 . Hence $\bar{u} \cdot (F + \varepsilon - \bar{y})$, G , and H satisfy (H1) and (H2) on X_0 . From Lemma 11, there exists $(\bar{\alpha}, \Omega, \xi) \in (R_+ \times S_+^p \times R^q)$, $\bar{\alpha} \neq 0$, such that

$$\begin{aligned} \bar{\alpha} \bar{u} \cdot (F(x) + \varepsilon - \bar{y}) + \Omega \cdot G(x) + \xi \cdot H(x) & \geq 0, \\ \forall x \in X_0. \end{aligned} \quad (44)$$

Let $\phi = \bar{\alpha} \bar{u}$; we have $\phi \in \text{int } R_+^m$, and (44) can be written as

$$\phi \cdot \bar{y} \leq l(x, \phi, \Omega, \xi) + \phi \cdot \varepsilon, \quad \forall x \in X_0. \quad (45)$$

This completes the proof. $\square$

Theorem 13. Let $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$. If there exists $(\phi, \Omega, \xi) \in \text{int } R_+^m \times S_+^p \times R^q$, such that

$$\phi \cdot \bar{y} \leq l(x, \phi, \Omega, \xi) + \phi \cdot \varepsilon, \quad \forall x \in X_0, \quad (46)$$

then $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (MSDP).

Proof. If $x \in X$, there exist $Z_x \in G(x)$ and $w_x \in H(x)$, such that $Z_x \in -S_+^p$ and $w_x = 0$; then we have $\Omega \cdot Z_x \leq 0$, $\xi \cdot w_x = 0$, and

$$\begin{aligned} \phi \cdot F(x) + \Omega \cdot Z_x + \xi \cdot w_x & \leq \phi \cdot F(x) + \phi \cdot \varepsilon, \\ \forall x \in X. \end{aligned} \quad (47)$$

By (46) and (47), we obtain

$$\phi \cdot \bar{y} \leq \phi \cdot F(x) + \phi \cdot \varepsilon, \quad \forall x \in X. \quad (48)$$

From $\phi \in \text{int } R_+^m$ and Definition 4, we know that $(\bar{x}, \bar{y})$ is an $\phi \cdot \varepsilon$ -optimal pair of (P_ϕ) . Therefore, $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (MSDP) by Theorem 6. $\square$

Theorem 14. Let $\bar{x} \in X$, $\bar{y} \in F(\bar{x})$, and $0 \in G(\bar{x})$, and let $F + \varepsilon - \bar{y}$, G , and H satisfy (H1) and (H2) on X_0 ; $F + \varepsilon - \bar{y}$ is also generalized R_+^m -subconvexlike on X . If $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (MSDP), then there exist $T \in L^+$ and $M \in R^{m \times q}$, such that $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (SMP)(T, M).

Proof. If $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (MSDP), by Theorem 12, there exists $(\phi, \Omega, \xi) \in \text{int } R_+^m \times S_+^p \times R^q$, such that

$$\phi \cdot \bar{y} \leq \phi \cdot F(x) + \Omega \cdot G(x) + \xi \cdot H(x) + \phi \cdot \varepsilon, \quad (49)$$

$$\forall x \in X_0.$$

Taking $c \in \text{int } R_+^m$ satisfying $\phi \cdot c = 1$, we can define a map $T : S^p \rightarrow R^m$ as

$$TA = (\Omega \cdot A)c = (c_1 \Omega \cdot A, \dots, c_m \Omega \cdot A)^T. \quad (50)$$

Then $T \geq 0$, $TG(x) = (\Omega \cdot G(x))c$. Let $M = c\xi^T \in R^{m \times q}$, $MH(x) = (\xi \cdot H(x))c$. From (49) and $\phi \cdot c = 1$, we have

$$\phi \cdot \bar{y} \leq \phi \cdot (F(x) + TG(x) + MH(x) + \varepsilon), \quad (51)$$

$$\forall x \in X_0.$$

Since $\bar{x} \in X$, $0 \in G(\bar{x})$, and $0 \in H(\bar{x})$, $\bar{y} \in F(\bar{x}) + TG(\bar{x}) + MH(\bar{x})$. By Definition 4, $(\bar{x}, \bar{y})$ is a $\phi \cdot \varepsilon$ -optimal pair of the scalar problem:

$$\min_{x \in X_0} \phi \cdot L(x, T, M). \quad (52)$$

From $\phi \in \text{int } R_+^m$, we obtain that $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (SMP)(T, M) by Theorem 6. $\square$

Theorem 15. Let $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$; if there exist $T \in L^+$ and $M \in R^{m \times q}$, such that $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (SMP)(T, M), then $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (MSDP).

Proof. If there exist $T \in L^+$ and $M \in R^{m \times q}$, such that $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (SMP)(T, M), by Definition 2 and (5), we have

$$\text{clcone}(L(X_0, T, M) + \varepsilon - \bar{y} + R_+^m) \cap (-R_+^m) = \{0\}. \quad (53)$$

Let $\bar{Z} \in G(\bar{x}) \cap (-S_+^p)$; we have $-T\bar{Z} \in R_+^m$ and $R_+^m - T\bar{Z} \subset R_+^m + R_+^m \subset R_+^m$. Hence, we obtain

$$R_+^m \subset T\bar{Z} + R_+^m \subset TG(X) + R_+^m; \quad (54)$$

then

$$F(X) + \varepsilon - \bar{y} + R_+^m \subset F(X) + TG(X) + M0 + R_+^m$$

$$+ \varepsilon - \bar{y} \quad (55)$$

$$\subset L(X, T, M) + \varepsilon - \bar{y} + R_+^m.$$

This implies

$$0 \in \text{clcone}(F(X) + \varepsilon - \bar{y} + R_+^m)$$

$$\subset \text{clcone}(L(X_0, T, M) + \varepsilon - \bar{y} + R_+^m). \quad (56)$$

By the above expression and (53), we have

$$\text{clcone}(F(X) + \varepsilon - \bar{y} + R_+^m) \cap (-R_+^m) = \{0\}. \quad (57)$$

Therefore, $(\bar{x}, \bar{y})$ is an ε -properly efficient pair of (MSDP) by Definition 2 and (5). $\square$

5. ε -Proper Saddle Point Theorem

In this section, first we give the concept of ε -proper saddle point of the set-valued Lagrangian function; then we establish an ε -proper saddle point theorem.

Definition 16. Let $\bar{x} \in X_0$, $\bar{T} \in L_+$, and $\bar{M} \in R^{m \times q}$, and an ordered group $(\bar{x}, \bar{T}, \bar{M})$ is said to be an ε -proper saddle point of Lagrangian function $L(x, T, M)$ if

$$L(\bar{x}, \bar{T}, \bar{M}) \cap \varepsilon - \text{PMin}_{x \in X_0} L(x, \bar{T}, \bar{M}) \cap \varepsilon$$

$$- \text{PMax}_{T \geq 0, M} L(\bar{x}, T, M) \neq \emptyset. \quad (58)$$

Theorem 17. Let $\bar{x} \in X_0$, $\bar{T} \in L_+$, and $\bar{M} \in R^{m \times q}$. If $(\bar{x}, \bar{T}, \bar{M})$ is an ε -proper saddle point of Lagrangian function $L(x, T, M)$, then there exist $\bar{y} \in F(\bar{x})$ and $\bar{Z} \in G(\bar{x})$, such that the following conditions are satisfied:

- (i) $\bar{y} + \bar{T}\bar{Z} \in \varepsilon - \text{PMin}_{x \in X_0} L(x, \bar{T}, \bar{M})$.
- (ii) $G(\bar{x}) \subset -S_+^p$, $\bar{\omega} = 0$.
- (iii) $-\bar{T}\bar{Z} \in R_+^m \setminus (\varepsilon + \text{int } R_+^m)$.

Proof. By Definition 16, there exist $\bar{y} \in F(\bar{x})$, $\bar{Z} \in G(\bar{x})$, and $\bar{\omega} \in H(\bar{x})$, such that

$$\bar{y} + \bar{T}\bar{Z} + \bar{M}\bar{\omega} \in \varepsilon - \text{PMin}_{x \in X_0} L(x, \bar{T}, \bar{M}), \quad (59)$$

$$\bar{y} + \bar{T}\bar{Z} + \bar{M}\bar{\omega} \in \varepsilon - \text{PMax}_{T \geq 0, M} L(\bar{x}, T, M). \quad (60)$$

From (60) and (6), we have

$$\text{clcone}(F(\bar{x}) + TG(\bar{x}) + MH(\bar{x}) - \bar{y} - \bar{T}\bar{Z} - \bar{M}\bar{\omega}$$

$$- \varepsilon - R_+^m) \cap \text{int } R_+^m = \emptyset, \quad \forall T \geq 0, M \in R^{m \times q}. \quad (61)$$

That is,

$$(F(\bar{x}) + TG(\bar{x}) + MH(\bar{x}) - \bar{y} - \bar{T}\bar{Z} - \bar{M}\bar{\omega} - \varepsilon$$

$$- R_+^m) \cap \text{int } R_+^m = \emptyset, \quad \forall T \geq 0, M \in R^{m \times q}. \quad (62)$$

Taking $M = \bar{M}$ in (62), since $\bar{y} \in F(\bar{x})$, $\bar{Z} \in G(\bar{x})$, and $\bar{\omega} \in H(\bar{x})$, we have

$$(T\bar{Z} - \bar{T}\bar{Z} - \varepsilon - R_+^m) \cap \text{int } R_+^m = \emptyset, \quad \forall T \geq 0. \quad (63)$$

From the above expression and $0 \in R_+^m$, we obtain

$$T\bar{Z} - \bar{T}\bar{Z} - \varepsilon \notin \text{int } R_+^m, \quad \forall T \geq 0. \quad (64)$$

We assert $\bar{Z} \in -S_+^p$. Contrarily, by Proposition 1(ii), there exists $\Omega \in S_+^p \setminus \{0\}$, such that $\Omega \cdot \bar{Z} > 0$. Letting $c \in \text{int } R_+^m$, we define a map $T : S^p \rightarrow R^m$ as

$$TZ = \left(\frac{(\Omega \cdot Z)}{(\Omega \cdot \bar{Z})} \right) (c + \varepsilon) + \bar{T}Z. \quad (65)$$

It is easy to check that $T \in L_+$, and

$$T\bar{Z} - \bar{T}\bar{Z} - \varepsilon = c \in \text{int } R_+^m. \quad (66)$$

This contradicts (64), so $\bar{Z} \in -S_+^p$ and $-\bar{T}\bar{Z} \in R_+^m$. Taking $T = 0$ in (64), we obtain $-\bar{T}\bar{Z} \notin \varepsilon + \text{int } R_+^m$. Therefore, condition (iii) holds.

Now, we assert $G(\bar{x}) \subset -S_+^p$. Contrarily, there exists $\bar{Z} \in G(\bar{x})$, but $\bar{Z} \notin -S_+^p$. By Proposition 1(ii), there exists $\Omega \in S_+^p \setminus \{0\}$, such that $\Omega \cdot \bar{Z} > 0$. Letting $\bar{c} \in \text{int } R_+^m$, the map $\bar{T} : S^p \rightarrow R^m$ is defined as

$$\bar{T}Z = \left(\frac{(\Omega \cdot Z)}{(\Omega \cdot \bar{Z})} \right) (\bar{c} + \varepsilon). \quad (67)$$

Obviously, by $-\bar{T}\bar{Z} \in R_+^m$, we obtain

$$\bar{T}\bar{Z} - \bar{T}\bar{Z} - \varepsilon \in \bar{c} + R_+^m \subset \text{int } R_+^m + R_+^m \subset \text{int } R_+^m. \quad (68)$$

On the other hand, taking $T = \bar{T}$ in (64), we have

$$\bar{T}\bar{Z} - \bar{T}\bar{Z} - \varepsilon \notin \text{int } R_+^m, \quad (69)$$

and this contradicts (68), so $G(\bar{x}) \subset -S_+^p$.

Taking $T = \bar{T}$ in (62), since $\bar{y} \in F(\bar{x})$, $\bar{Z} \in G(\bar{x})$, and $0 \in R_+^m$, we have

$$(MH(\bar{x}) - \bar{M}\bar{w} - \varepsilon) \cap \text{int } R_+^m = \emptyset, \quad \forall M \in R^{m \times q}. \quad (70)$$

From $\bar{w} \in H(\bar{x})$, we obtain

$$\bar{M}\bar{w} - \bar{M}\bar{w} - \varepsilon \notin \text{int } R_+^m, \quad \forall M \in R^{m \times q}. \quad (71)$$

Suppose that $\bar{w} \neq 0$ and $\exists \xi \in R^q$, such that $\xi \cdot \bar{w} > 0$, letting $\bar{c} \in \text{int } R_+^m$. The map $\bar{M} \in R^{m \times q}$ is defined as

$$\bar{M}w = \left(\frac{(\xi \cdot w)}{(\xi \cdot \bar{w})} \right) (\bar{c} + \varepsilon) + \bar{M}w. \quad (72)$$

We obtain

$$\bar{M}\bar{w} - \bar{M}\bar{w} - \varepsilon = \bar{c} \in \text{int } R_+^m. \quad (73)$$

This contradicts (71); hence $\bar{w} = 0$. Therefore, condition (ii) holds. From the fact that $\bar{w} = 0$ and (59), condition (i) holds. $\square$

Theorem 18. *If an order group $(\bar{x}, \bar{T}, \bar{M})$ is an ε -proper saddle point of Lagrangian function $L(x, T, M)$ and $0 \in G(\bar{x})$, then there exist $\bar{y} \in F(\bar{x})$ and $\bar{Z} \in G(\bar{x})$, such that $(\bar{x}, \bar{y})$ is an ε' -properly efficient pair of (MSDP), where $\varepsilon' = \varepsilon - \bar{T}\bar{Z}$.*

Proof. By Theorem 17(i), there exist $\bar{y} \in F(\bar{x})$ and $\bar{Z} \in G(\bar{x})$, such that $\bar{y} + \bar{T}\bar{Z} \in \varepsilon - \text{PMin}_{x \in X_0} L(x, \bar{T}, \bar{M})$; from (5), we have

$$\begin{aligned} & \text{clcone} \left(L(X_0, T, M) + R_+^m - \bar{y} + (\varepsilon - \bar{T}\bar{Z}) \right) \cap (-R_+^m) \\ &= \{0\}, \quad \forall T \geq 0, M \in R^{m \times q}. \end{aligned} \quad (74)$$

By Theorem 17(iii), $-\bar{T}\bar{Z} \in R_+^m$; hence $\varepsilon' = \varepsilon - \bar{T}\bar{Z} \in R_+^m$. On the other hand, from the fact that $0 \in G(\bar{x})$, we obtain $\bar{y} \in L(X_0, T, M)$. Applying Theorem 15 yields the result. $\square$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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