Global Dynamics of an SIRS Epidemic Model with Distributed Delay on Heterogeneous Network

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A novel epidemic SIRS model with distributed delay on complex network is discussed in this paper. The formula of the basis reproductive number $R_0$ for the model is given, and it is proved that the disease dies out when $R_0 < 1$ and the disease is uniformly persistent when $R_0 > 1$. In addition, a unique endemic equilibrium for the SIRS model exists when $R_0 > 1$, and a set of sufficient conditions on the global attractiveness of the endemic equilibrium for the system is given.

1. Introduction

Following the seminal work on small-world network by Watts and Strogatz [1], and the scale-free network, in which the probability of $p(k)$ for any node with $k$ links to other nodes is distributed according to the power law $p(k) = Ck^{-\gamma}$ ($2 < \gamma \leq 3$), suggested by Barabási and Albert [2], the spreading of epidemic disease on heterogeneous network, that is, scale-free network, has been studied by many researchers [3–23].

Compared with the ordinary differential equation (ODE) models (see [3–18] and references therein), more realistic models should be retarded functional differential equation (RFDE) models which can include some of the past states of these systems. Time delay plays an important role in the process of the epidemic spreading; for instance, the incubation period of the infectious diseases, the infection period of infective members, and the immunity period of the recovered individuals can be represented by time delays [24]. However, less attention has been paid to the epidemic models with time delays on heterogeneous network [19–22].

Zou et al. constructed a delayed SIR model without birth rate and death rate on scale-free network [19]. In the model, the discrete delay in model represents the incubation period during which the infectious agents develop in the vector [21]. However, the assumption that the incubation period of an infective vector is determinate is somewhat idealized. And it is interesting to discuss the spreading of disease by using functional differential equation model with distributed delay [25]. Motivated by the work of Zou et al. [19] and Wang et al. [22], considering the fact the immune individual may become the susceptible individual [15], we will present a novel functional differential equation SIRS model with distributed delay on heterogeneous network in this paper to investigate the epidemic spreading, where the distributed delay represents the incubation period of an infective vector.

We consider the whole population and their contacts on network in which every individual is considered as a node in the network. Suppose the size of the network is a constant $N$ during the period of epidemic spreading; we also suppose that the degree of each node is time invariant; let $S_k(t)$, $I_k(t)$, and $R_k(t)$ be the relative density of susceptible nodes, infected nodes, and recovered nodes of connectivity $k$ at time $t$, respectively, where $k = m, m+1, \ldots, n$ in which $m$ and $n$ are the minimum and maximum number of contact each node, respectively.

In the process of the epidemic propagation via vector (such as mosquito), when a susceptible vector is infected by an infected nodes, there is a delay $\mu$ during which the
infectious agents develop in the vector, and infected vector becomes itself infectious after the delay. At the same time, the vector’s usual activities are in a limited range; that is, if a vector is infected by an infected node, its usual activities are in the vicinity of the infected node. Furthermore, if the vector population size is large enough, we can suppose that the number of the infectious vector population in the vicinity of the infected nodes with degree \( k (k = 1, 2, \ldots, n) \) at any time \( t \) is simply proportional to the number of the infected nodes with degree \( k \) at time \( t - \mu \) [25, 26]. Let the kernel function \( f(u) \) denote the probability that an susceptible vector who is infected at time \( t - \mu \) and becomes infective at time \( t \). Meanwhile, let \( \lambda(k) \) be the correlated \((k\text{-dependent})\) infection rate such as \( \lambda k \) and \( \lambda c(k) \) [11]. The susceptible nodes may acquire temporary immunity and the removal rate from the susceptible nodes to the recovered nodes is given by \( \delta \). And \( \mu \) is removal rate from the recovered nodes to the susceptible nodes because the recovered nodes lose the temporary immunity. In addition, the infected nodes are cured with rate \( r \). The dynamical equations for the density \( S_k(t) \), \( I_k(t) \), and \( R_k(t) \), at the mean-field level, satisfy the following set of functional differential equations when \( t > 0 \):

\[
\begin{align*}
\dot{S}_k(t) &= -\lambda(k) S_k(t) \int_0^{\infty} \Theta(t-u) f(u) du - \delta S_k(t), \\
\dot{I}_k(t) &= \lambda(k) S_k(t) \int_0^{\infty} \Theta(t-u) f(u) du - r I_k(t), \\
\dot{R}_k(t) &= r I_k(t) - \mu R_k(t) + \delta S_k(t)
\end{align*}
\]

with

\[
S_k(t) + I_k(t) + R_k(t) = 1, \quad k = m, m + 1, \ldots, n
\]

(2) due to the fact that the number of total nodes with degree \( k \) is a constant \( p(k)N \) during the period of epidemic spreading. The dynamics of \( n \) groups of SIRS subsystems are coupled through the function \( \Theta(t) \), which represents the probability that any given link points to an infected site. Assuming that the network has no degree correlations [3, 11], we have

\[
\Theta(t) = \frac{1}{\langle k \rangle} \sum_{k=m}^{n} \phi(k) p(k) I_k(t),
\]

(3) where \( \langle k \rangle = \sum_k p(k)k \) stands for the average node degree and \( \phi(k) = ak^{\alpha}/(1 + bk^{\alpha}) \) [7] \((0 \leq \alpha < 1, \quad a > 0, \quad b \geq 0)\) denotes an infected node with degree \( k \) occupied edges which can transmit the disease. If \( b \neq 0 \), \( \phi(k) \) gradually become saturated with the increase of degree \( k \), that is, \( \lim_{k \to \infty} \phi(k) = b/a \).

The kernel function \( f(u) \) is nonnegative and continuous on \([0, \infty)\) and satisfies

\[
\begin{align*}
\int_0^{\infty} f(u) du &= 1, \\
\int_0^{\infty} f(u) e^{\alpha u} du &= < +\infty,
\end{align*}
\]

(4) where \( \alpha \) is a positive number. And there are many types of kernel functions such as

1. the gamma distribution \( f(u) = \frac{u^{\gamma-1}/(\sqrt{\pi}b^\beta)}{e^{-u/b}} \) where \( b > 0 \) is a real number and \( n > 1 \) is an integer, especially when \( n = 1, b = 1, \) and then \( f(u) = e^{-u} \),

2. the uniform distribution

\[
f(u) = \begin{cases} 
1/h, & 0 \leq u \leq h \\
0, & u > h,
\end{cases}
\]

(5) where \( h > 0 \) is real number,

3. the Delta-distribution \( f(u) = \delta(u - \tau), \) where \( \tau > 0 \) is real constant.

Define the following Banach space of fading memory type (see [27] and references therein):

\[
C = \left\{ \phi \in C((-\infty, 0], \mathbb{R}) : \phi(s) \cdot e^{\alpha s} \text{ is uniformly continuous for } s \right\}
\]

(6) with norm \( \|\phi\| = \sup_{s \leq 0} |\phi(s)| e^{\alpha s} \), and let \( \phi_i \in C \) be such that \( \phi_i(s) = \phi(t + s), \quad s \in (-\infty, 0] \).

Consider system (1) in phase space \( X = (C \times C \times C^{n-m+1}) \). Standard theory of functional differential equation implies system (1) has a unique solution satisfying the initial conditions

\[
\begin{align*}
S_k(0) &= \phi_{1k}(s), \\
I_k(0) &= \phi_{2k}(s), \\
R_k(0) &= \phi_{3k}(s),
\end{align*}
\]

(7) where \( \phi_{1k}(s), \phi_{2k}(s), \ldots, \phi_{3n}(s) \) \( E^{-1} \in X \).

It can be verified that solutions of system (1) in \( X \) with initial conditions above remain positive for all \( t \geq 0 \).

The rest of this paper is organized as follows. The dynamical behaviors of the SIRS model with distributed delay are discussed in Section 2. Numerical simulations and discussions are offered to demonstrate the main results in Section 3.
2. Dynamical Behaviors of the Model

Since $S_k(t) + I_k(t) + R_k(t) = 1$, system (1) is equivalent to the following system (8):

$$
\dot{S}_k(t) = -\lambda(k) S_k(t) \int_0^{+\infty} \Theta(t - u) f(u) \, du - \delta S_k(t) + \mu \left( 1 - S_k(t) - I_k(t) \right),
$$

$$
\dot{I}_k(t) = \lambda(k) S_k(t) \int_0^{+\infty} \Theta(t - u) f(u) \, du - r I_k(t).
$$

Thus we only discuss system (8) if we want to discuss the dynamical behaviors of system (1).

Denote

$$
R_0 = \frac{\mu}{(\mu + \delta) r} \langle \lambda(k) \varphi(k) \rangle,
$$

where $\langle f(k) \rangle = \sum_k f(k)p(k)$ in which $f(k)$ is a function.

Note that we can obtain from the first equation of system (8) that

$$
\dot{S}_k(t) \leq \mu - (\mu + \delta) S_k(t).
$$

By the standard comparison theorem in the theory of differential equations, we have

$$
\lim_{t \to +\infty} \sup S_k(t) \leq \frac{\mu}{\mu + \delta}.
$$

Hence we know

$$
D_0 = \left\{ (S_m, I_m, \ldots, S_n, I_n) \in R^{2(n-m+1)}_+ \mid 0 < S_k, I_k, S_k + I_k \leq 1, \ 0 \leq S_k \leq \frac{\mu}{\mu + \delta} \right\}
$$

is positively invariant with respect to system (8), and every forward orbit in $R^{2(n-m+1)}_+$ eventually enters $D_0$.

**Theorem 1.** System (8) has always a disease-free equilibrium $E_0(\mu/(\mu + \delta), \ldots, \mu/(\mu + \delta), 0, \ldots, 0)$. System (8) has a unique endemic equilibrium $E_+ (S^*_m, S^*_m, \ldots, S^*_n, I^*_m, I^*_m, \ldots, I^*_n)$ when $R_0 > 1$.

**Proof.** Obviously, the disease-free equilibrium $E_0$ of system (8) always exists. Now we discuss the existence of the endemic equilibrium of system (8). Combined with $\int_0^{+\infty} f(u) \, du = 1$, it is easy to know that the equilibrium $E_+$ satisfies

$$
-\lambda(k) S^*_k \Theta^* - \delta S^*_k + \mu \left( 1 - S^*_k - I^*_k \right) = 0.
$$

$$
\lambda(k) S^*_k \Theta^* - r I^*_k = 0.
$$

From (13), we obtain that

$$
I^*_k = \frac{\lambda(k) \mu \Theta^*}{(\mu + \delta) r + \lambda(k) (r + \mu) \Theta^*}.
$$

Substituting it into (14), we obtain the self-consistency equality

$$
\Theta^* = \frac{1}{(k)} \sum_k \varphi(k) p(k) \lambda(k) \mu \Theta^* \left( \frac{1}{(k)} \sum_k \varphi(k) p(k) \right).
$$

and it can be verified that (15) has a unique positive solution when $R_0 > 1$ using the same proof as for Theorem 1 in [21]; consequently, system (8) has a unique endemic equilibrium $E_+$ since (13) and (15) hold.

**Theorem 2.** If $R_0 < 1$, the disease-free equilibrium $E_0$ of system (8) is globally attractive.

**Proof.** Obviously, we need only discuss global attractiveness of system (8) in $D_0$.

Consider the following Lyapunov function

$$
V(t) = \frac{1}{2} V_1^2(t) + \frac{R_0 r}{2} V_2(t),
$$

where

$$
V_1(t) = \int_0^{+\infty} \Theta^* (t - u) f(u) \, du,
$$

$$
V_2(t) = \int_0^{+\infty} f(u) \int_{t-2u}^{t-u} \Theta^2(s) \, ds \, du.
$$

Calculating the derivative of $V(t)$ along solution of (8), for $t > T_1$, we get

$$
\dot{V}(t)|_{(0)} = \int_0^{+\infty} \Theta(t - u) \left( \frac{1}{(k)} \sum_k \varphi(k) p(k) \lambda(k) \right)
$$

$$
\cdot \left( \int_0^{+\infty} \Theta(t - 2u) f(u) \, du - r I_k(t - u) \right) \, du
$$

$$
+ \frac{R_0 r}{2} \left( V_1(t) - \int_0^{+\infty} f(u) \Theta^2(t - 2u) \, du \right)
$$

$$
\leq \int_0^{+\infty} \Theta(t - u) \left( \frac{1}{(k)} \sum_k \varphi(k) p(k) \right)
$$

$$
\cdot \left( \lambda(k) \frac{\mu}{\mu + \delta} \int_0^{+\infty} \Theta(t - 2u) f(u) \, du
$$

$$
- r I_k(t - u) \right) \, du + \frac{R_0 r}{2} \left( V_1(t)
$$

$$
- \int_0^{+\infty} f(u) \Theta^2(t - 2u) \, du \right) = R_0 r \int_0^{+\infty} \Theta(t - u)$$
\[ \dot{X}(t)=\frac{K}{2} \int_{-\infty}^{\infty} f'(t) \Theta^2(t-u) du - \int_{0}^{\infty} f'(t) \Theta^2(t) \]
\[ -2u) du \leq \frac{K}{2} \int_{0}^{\infty} (\Theta^2(t-u) + \Theta^2(t) - 2u) f(u) du - \frac{K}{2} \int_{0}^{\infty} \Theta^2(t-2u) du \]
\[ = r(R_0 - 1) V_1(t). \]

Thus \( \dot{V}(t)|_{\theta}=0 \) when \( R_0 < 1 \), and \( \dot{V}(t)|_{\theta}=0 \) if and only if \( V_1(t)=0 \). Note that the fact \( \dot{V}(t) \) means \( I_k=0 \); moreover, \( \lim_{t \to \infty} S(t) = \mu/(\mu+\delta) \); the largest invariant set of \( \dot{V}(t)|_{\theta}=0 \) is a singleton \( E_0 \). Hence the disease-free equilibrium \( E_0 \) is globally attractive when \( R_0 < 1 \) according to the LaSalle Invariance Principle [28, Chapter 2, Theorem 5.3].

**Lemma 3** (see [28, p273–280]). Let \( X \) be a complete metric space, \( X = X^0 \cup \partial X^0 \), where \( \partial X^0 \), assumed to be nonempty, is the boundary of \( X^0 \). Assume the \( C^0 \)-semigroup \( T(t) \) on \( X \) satisfies \( T(x) : X^0 \to X^0 \), \( T(x) : \partial X^0 \to \partial X^0 \) and

(i) there is a \( t_0 \) such that \( T(t) \) is compact for \( t > t_0 \);

(ii) \( T(t) \) is point dissipative in \( X \);

(iii) \( \overline{A}_3 \) is isolated and has an acyclic covering \( M \).

Then \( T(t) \) is uniformly persistent if and only if, for each \( M_i \in M \),

\[ W^s(M_i) \cap X^0 = \emptyset, \]

where \( \overline{A}_3 = \bigcup_{x \in A_3} \omega(x), \) and \( \omega(x) \) is the omega limit set of \( T(x) \) through \( x \), and \( A_3 \) is global attractor of \( T(0) \) in \( \partial X^0 \) in which \( T(0) = T(0)|_{\partial X^0} \).

**Theorem 4.** For system (8), if \( R_0 > 1 \), the disease-free equilibrium \( E_0 \) is unstable, and the disease is uniformly persistent; that is, there exists a positive constant \( e \) such that \( \lim_{t \to \infty} \inf I_k(t) > e, k = m, m+1, \ldots, n \).

**Proof.** Denote

\[ X = \{ (\bar{S}, \bar{V}) : \psi_k(\theta) \geq 0, \forall \theta \in (-\infty, 0], k = m, m+1, \ldots, n \}, \]
\[ X^0 = \{ (\bar{S}, \bar{V}) : \psi_k(\theta) > 0, \text{ for some } \theta \in (-\infty, 0], k = m, m+1, \ldots, n \}, \]

and consequently,

\[ \partial X^0 = \{ (\bar{S}, \bar{V}) : \psi_i(\theta) = 0, \forall \theta \in (-\infty, 0], i \in \{m, m+1, \ldots, n\} \}. \]

where \( (\bar{S}, \bar{V}) = (S_{m}, S_{m+1}, \ldots, S_{n}, \psi_{m}, \psi_{m+1}, \ldots, \psi_{n}) \).

Let \( (S_{m}(t), \ldots, S_{n}(t), I_{m}(t), \ldots, I_{n}(t)) = (S_{m}(t, \omega), \ldots, S_{n}(t, \omega), I_{m}(t, \omega), \ldots, I_{n}(t, \omega)) \) be the solution of (8) with initial function \( \omega = (\phi_{m}(s), \phi_{m+1}(s), \phi_{m+2}(s), \ldots, \phi_{n}(s)) \) and

\[ I_{m}(t + \theta, \omega), \ldots, I_{n}(t + \theta, \omega), \quad \theta \in (-\infty, 0]. \]

Obviously, \( X \) and \( X^0 \) are positively invariant set for \( T(t) \). \( T(t) \) is completely continuous for \( t > 0 \). Also, it follows from \( 0 < S_{m}, I_{k} \leq 1 \) for \( t > 0 \) that \( T(t) \) is point dissipative. \( E_{0} \) is the unique equilibrium of system (8) on \( \partial X^0 = X/X^0 \) and it is globally stable on \( \partial X^0 = X/X^0, \overline{A}_3 = \{ E_{0} \}, \) and \( E_{0} \) is isolated and acyclic. Finally, the proof will be done if we prove \( W^s(E_{0}) \cap X^0 = \emptyset \), where \( W^s(E_{0}) \) is the stable manifold of \( E_{0} \).

Suppose it is not true; then there exists a solution \( (\bar{S}, \bar{I}) \) in \( X^0 \) such that

\[ \lim_{t \to \infty} S_{k}(t) = \mu/(\mu+\delta), \]
\[ I_{k}(t), \quad k = m, m+1, \ldots, n. \]

Since \( R_0 > 1 \), we may choose \( 0 < \eta < 1 \) such that \( (\mu(1 - \eta))/((\lambda(k)\eta + \mu + \delta)) > 1 \). At the same time, there exists a \( T_{2} > 0 \) and \( 0 < \epsilon < \eta \) such that \( 0 \leq I_{k}(t) < \eta \) for \( t > T_{2} \) due to \( \lim_{t \to \infty} \inf I_{k}(t) = 0 \).

When \( t > T_{2} \), we obtain from the first equation of system (8) that

\[ \dot{S}_{k}(t) = -\lambda(k) S_{k}(t) \eta + \mu(1 - S_{k}(t) - \eta) - \delta S_{k}(t) \]
\[ = \mu(1 - \eta) - (\lambda(k) \eta + \mu + \delta) S_{k}(t). \]

Hence there exist a \( T_{3} > T_{2} \) such that the following equality holds when \( t > T_{3} \):

\[ S_{k}(t) \geq \frac{\mu(1 - \eta)}{\lambda(k) \eta + \mu + \delta}. \]

For \( t > T_{2} \), we have from (3) and (26) that

\[ \dot{\psi}(t) = \frac{1}{K} \sum_{k} \phi(k) p(k) I_{k}(t) = \frac{1}{K} \sum_{k} \phi(k) p(k) \cdot \left[ \lambda(k) S_{k}(t) \int_{0}^{\infty} \psi(t - \tau) f(u) du - r I_{k}(t) \right] \]
\[ \geq \frac{\mu(1 - \eta)}{\lambda(k) \eta + \mu + \delta} \cdot \left[ \lambda(k) S_{k}(t) \int_{0}^{\infty} \psi(t - \tau) f(u) du - r I_{k}(t) \right]. \]
By \( (\mu(1 - \eta))((\lambda(k)\eta + \mu + \delta)r)((\lambda(k)\varphi(k))/\langle k \rangle) > 1 \) and the comparison principle furthermore, it is easy to see that \( \lim_{t \to \infty, \Theta(t)} = +\infty \), contradicting \( \lim_{t \to \infty, \Theta(t)} = 0 \) as \( \lim_{t \to \infty, \Theta(t)} = 0 \). Hence \( \lim_{t \to \infty, \Theta(t)} = 0 \); moreover, there exists \( k_0 \in [m, m + 1, \ldots, n] \) such that \( \lim_{t \to \infty, \Theta(t)} = 0 \), contradicting \( \lim_{t \to \infty, \Theta(t)} = 0 \), \( k = m, m + 1, \ldots, n \).

Hence, the infection is uniformly persistent according to Lemma 3; that is, there exists a \( e \) and a positive constant such that \( \lim_{t \to \infty} \inf I_k(t) > e \), and the disease-free equilibrium \( E_0 \) is unstable accordingly. This completes the proof. \( \square \)

At last, let us discuss the global stability of the endemic equilibrium of system (8) by constructing suitable Lyapunov function.

**Theorem 5.** If \( R_0 > 1, \delta < r \) and \( I_k^* < \mu/((\mu + \delta)(\delta/r)), k = m, m + 1, \ldots, n \), the endemic equilibrium \( E_* \) of system (8) is globally asymptotically attractive.

**Proof.** For convenience, we still discuss system (1). According to (13) and \( S_k(t) + I_k(t) + R_k(t) = 1, k = m, m + 1, \ldots, n \), we know

\[
\mathcal{D}_0 = \left\{ (S_m, I_m, R_m, \ldots, S_n, I_n, R_n) \in \mathbb{R}^{2(n-m+1)} \mid 0 < S_k, I_k, R_k, S_k + I_k + R_k = 1, 0 \leq S_k \leq \frac{\mu}{\mu + \delta} \right\}
\]

(28)

is positively invariant with respect to system (8), and every forward orbit in \( \mathcal{D}_0 \) eventually enters \( \mathcal{D}_0 \).

Thus we just need to discuss the global attractiveness of system (1) in \( \mathcal{D}_0 \).

Denote \( R_k^* = 1 - S_k^* - I_k^* \), and then \( E^* (S_m^*, I_m^*, R_m^*, S_{m+1}^*, I_{m+1}^*, R_{m+1}^*, \ldots, S_n^*, I_n^*, R_n^*) \) is the endemic equilibrium of system (1). System (1) may be rewritten as follows:

\[
\dot{S}_k(t) = -\sum_{l=m}^{n} \beta_{kl}S_k S_l(t) \int_{0}^{\infty} I_l(t-u)(u)du - \delta S_k(t) + \mu R_k(t),
\]

(29)

\[
\dot{I}_k(t) = \sum_{l=m}^{n} \beta_{kl}S_k S_l(t) \int_{0}^{\infty} I_l(t-u)(u)du - r I_k(t),
\]

\[
\dot{R}_k(t) = r - (\mu + r) R_k(t) - (\delta - r) S_k(t),
\]

where \( \beta_{kl} = \lambda(l)\varphi(l)p(l)/\langle k \rangle \), \( l = m, m + 1, \ldots, n \).

Note that the endemic equilibrium of system (1) satisfies

\[
-\sum_{l=m}^{n} \beta_{kl}S_k S_l^* - \delta S_k^* + \mu R_k^* = 0,
\]

\[
\sum_{l=m}^{n} \beta_{kl}S_k^* I_l^* = r I_k^* + (\mu + r) R_k^* + (\delta - r) S_k^* = r.
\]

(30)

We have from (29) and (30) that

\[
S_k(t) = -\sum_{l=m}^{n} \beta_{kl}S_k(t) \int_{0}^{\infty} I_l(t-u)du
-\sum_{l=m}^{n} \beta_{kl}S_k(t) \int_{0}^{\infty} I_l(t-u)du
-\delta (S_k(t) - S_k^*) + \mu (R_k(t) - R_k^*),
\]

(31)

\[
I_k(t) = \sum_{l=m}^{n} \beta_{kl}S_k(t) \int_{0}^{\infty} I_l(t-u)du
- \frac{1}{I_k^*} \left( \sum_{l=m}^{n} \beta_{kl}S_k(t) \int_{0}^{\infty} I_l(t-u)du \right) I_k(t),
\]

\[
R_k(t) = -(\mu + r)(R_k(t) - R_k^*) + (\delta - r)(S_k(t) - S_k^*).
\]

Let us consider

\[
V_k(t) = S_k(t) - S_k^* \ln \frac{S_k(t)}{S_k^*} + I_k(t) - I_k^* - I_k^*
\]

\[
-\ln \frac{I_k(t)}{I_k^*} + \sum_{l=m}^{n} \beta_{kl}S_k(t) \int_{0}^{\infty} f(u)du
\]

\[
\cdot \left( I_k(t) - I_k^* \int_{0}^{\infty} f(u)du \right) + \frac{\mu}{S_k^* (r - \delta)} (R_k(t) - R_k^*)^2.
\]

(32)

Calculating the derivative of \( V_k(t) \) along solution of (31), we get

\[
V_k(t) \bigg|_{(32)} = -\delta (1 - \frac{S_k^*}{S_k}) (S_k - S_k^*) + \mu \left( 1 - \frac{S_k^*}{S_k} \right)
\]

\[
(R_k - R_k^*) + \sum_{l=m}^{n} \beta_{kl}S_k(t) \int_{0}^{\infty} \left( 2 - \frac{S_k^*}{S_k} - \frac{I_k(t)}{I_k^*} \right)
\]

\[
- \frac{S_k I_k(t - \tau) I_k^*}{S_k^* I_k(t) I_k^*} - \ln \frac{I_k(t)}{I_k^*} + \ln \frac{I_k(t - \tau)}{I_k^*} + \frac{I_k(t)}{I_k^*}.
\]
\[ \cdot \int f(u) \, du + \mu (R_k - R_k^*)(1 - \frac{S_k}{S_k^*}) \]
\[ - \frac{\mu (r + \mu)}{S_k^* (r - \delta)} (R_k - R_k^*)^2. \]

Since
\[ \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* \int_{0}^{\infty} \left( 2 - \frac{S_k}{S_k^*} \frac{I_l I_l^*}{I_l^*} - \frac{S_k I_l (t-u) I_l^*}{S_k^* I_l I_l^*} \right) \left( \ln \frac{l}{l_l} + \ln \left( \frac{l_l (t-u) + l}{l_{l_l}} \right) \right) f(u) \, du \]
\[ = \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* \left( H \left( \frac{l}{l_{l_l}} \right) - H \left( \frac{l}{l_{l_{l_l}}} \right) \right) \]
\[ - \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* + \sum_{l=m}^{n} \sum_{l_1=m}^{n} \beta_{kl} S_k^* I_l^* \]
\[ \int_{0}^{\infty} G \left( \frac{S_k I_l (t-u) I_l^*}{S_k^* I_l I_l^*} \right) f(u) \, du \]
\[ + \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* G \left( \frac{S_k}{S_k^*} \right) \]
\[ - \delta \left( 1 - \frac{S_k^*}{S_k} \right) (S_k - S_k^*) = -\delta S_k^* \left( G \left( \frac{S_k}{S_k^*} \right) \right) \]
\[ + G \left( \frac{S_k}{S_k^*} \right) \mu \left( 1 - \frac{S_k^*}{S_k} \right) (R_k - R_k^*) + \mu (R_k - R_k^*) \]

\[ \cdot \left( 1 - \frac{S_k}{S_k^*} \right) = -\mu (R_k - R_k^*) \left( G \left( \frac{S_k}{S_k^*} \right) \right) \]
\[ + G \left( \frac{S_k}{S_k^*} \right) = -\mu R_k G \left( \frac{S_k}{S_k^*} \right) - \mu R_k G \left( \frac{S_k^*}{S_k^*} \right) \]
\[ + \mu R_k^* \left( G \left( \frac{S_k}{S_k^*} \right) + G \left( \frac{S_k^*}{S_k^*} \right) \right) \]

hold, where \( H(a) = -a + \ln a \) and \( G(a) = a - 1 - \ln a \geq 0 \), we can obtain from (30), (33), and (34) that

\[ \mathcal{V}_k(t) \big|_{(33)} \leq \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* \left( H \left( \frac{l}{l_l} \right) - H \left( \frac{l}{l_{l_l}} \right) \right) \]
\[ - \mu R_k G \left( \frac{S_k}{S_k^*} \right) + \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* G \left( \frac{S_k}{S_k^*} \right) \]
\[ \leq \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* \left( H \left( \frac{l}{l_l} \right) - H \left( \frac{l}{l_{l_l}} \right) \right) \]
\[ + (-\mu R_k + r I_k^*) G \left( \frac{S_k}{S_k^*} \right). \]

Furthermore, by \( r > \delta \) and \( S_k(t) \leq \mu/(\mu + \delta) \), we have from the last equation of system (31) that
\[ \dot{R}_k(t) \geq r - (\mu + r) R_k(t) - (r - \delta) \frac{\mu}{\mu + \delta} \]
\[ = (\mu + r) \left( \frac{\delta}{\mu + \delta} - R_k(t) \right). \]

Since \( I_k^* < \delta/(\mu + \delta)(\mu/r) \), we can take \( \varepsilon = \delta/(\mu + \delta) - I_k^* r/\mu > 0 \), and it follows from (42) that there exists a \( T_4 > T_1 \) such that \( R_k(t) \geq \delta/(\mu + \delta) - \varepsilon \) when \( t > T_4 \). Hence \(-\mu R_k + r I_k^* \leq -\mu(\delta/(\mu + \delta) - \varepsilon) + r I_k^* = 0 \) when \( t > T_4 \), that is,
\[ (-\mu R_k + r I_k^*) G \left( \frac{S_k}{S_k^*} \right) \leq 0. \]

In addition, the matrix
\[ (\beta_{kl} S_k^* I_l^*)_{(n-m+1) \times (n-m+1)} \]
\[ = \left( \begin{array}{cccc} \lambda(l) \varphi(l) \psi(l) \beta_{kl} S_k^* I_l^* \end{array} \right)_{(n-m+1) \times (n-m+1)} \]

is irreducible, so the following matrix is irreducible:

\[ B = \begin{pmatrix} \sum_{l=m}^{n} \beta_{m1} S_m^* I_l^* & -\beta_{m+1,m} S_m^* I_{m+1}^* & \cdots & -\beta_{nn+1,m} S_n^* I_{n+1}^* \\ -\beta_{m1,m} S_{m+1}^* I_m^* & \sum_{l=m}^{n} \beta_{m1} S_m^* I_{m+1}^* & \cdots & -\beta_{nm,m} S_n^* I_{m+1}^* \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_{mm} S_m^* I_m^* & -\beta_{m+1,m} S_{m+1}^* I_m^* & \cdots & \sum_{l=m}^{n} \beta_{ml} S_l^* I_l^* \end{pmatrix}. \]
Hence there exists a positive vector \( C = (c_1, c_2, \ldots, c_n) \) such that \( BC = 0 \) in which \( c_k \) is the cofactor of the \( k \)th diagonal of \( B, m \leq k \leq n \) [29, Lemma 2.1]. It follows from \( BC = 0 \) that

\[
\sum_{k=m}^{n} \sum_{l=1}^{n} c_k \beta_{kl} S_k^* I_l^* = c_k \sum_{l=1}^{n} \beta_{kl} S_k^* I_l^*, \quad k = m, m + 1, \ldots, n \tag{40}
\]

which leads to

\[
\sum_{k=m}^{n} \sum_{l=1}^{n} c_k \beta_{kl} S_k^* I_l^* H \left( \frac{I_k}{I_k^*} \right) = \sum_{k=m}^{n} \sum_{l=1}^{n} c_k \beta_{kl} S_k^* I_l^* H \left( \frac{I_k}{I_k^*} \right) = \sum_{k=m}^{n} c_k \sum_{l=m}^{n} \beta_{kl} S_k^* I_l^* H \left( \frac{I_k}{I_k^*} \right) \tag{41}
\]

\[
= \sum_{k=m}^{n} c_k \beta_{kk} S_k^* I_k^* H \left( \frac{I_k}{I_k^*} \right) = \sum_{k=m}^{n} c_k \beta_{kk} S_k^* I_k^* H \left( \frac{I_k}{I_k^*} \right),
\]

that is,

\[
\sum_{k=m}^{n} c_k \beta_{kk} S_k^* I_k^* \left( H \left( \frac{I_k}{I_k^*} \right) - H \left( \frac{I_k}{I_k^*} \right) \right) = 0. \tag{42}
\]

Define a Lyapunov function

\[
V(t) = \sum_{k=1}^{n} c_k V_k(t), \tag{43}
\]

where \( V_k(t) \) is defined by (32), and we have from (35), (37), and (42) that

\[
\dot{V}(t) \bigg|_{32} \leq 0. \tag{44}
\]

Moreover, \( \dot{V}(t) \bigg|_{32} = 0 \) if and only if \( S_k = S_k^* I_k = I_k^* \), \( R_k = R_k^* \). Therefore, LaSalle Invariance Principle [28, Chapter 2, Theorem 5.3] implies that the endemic equilibrium \( E^* \) of system (1) is globally attractive when \( R_0 > 1, \delta < r \) and \( I_k^* < \mu/\mu + \delta)\delta/r, \; k = m, m + 1, \ldots, n \). The proof is completed. \( \square \)

3. Numerical Simulation and Discussion

The basic reproductive number for system (8) (or (1)) is

\[
R_0 = \frac{\mu}{(\mu + \delta) r} \left\langle \frac{\lambda(k) \varphi(k)}{\langle k \rangle} \right\rangle. \tag{45}
\]

The equilibrium \( E_0 \) is globally attractive and the infection eventually disappears when \( R_0 < 1 \), and the infection will always exist when \( R_0 > 1 \). Note that \( R_0 \) is irrelevant to the distributed delay.

Extensive numerical simulations are carried out on scale-free model to demonstrate the mentioned theorems above. The simulations are based on system (8) and a scale-free networks in which the degree distribution is \( p(k) = C k^{-\gamma} \), and \( C \) satisfies \( \sum_{k=0}^{\infty} p(k) = 1 \). Supposing the network is finite one, the maximum connectivity \( n \) of any node is related to the network age, measured as the number of nodes \( N \) [3, 7]:

\[
n = mn^{\gamma/(\gamma - 1)}. \tag{46}
\]

\[\text{Figure 1: Dynamical behaviors of system (8) with } R_0 = 0.5547.\]

\[\text{Figure 2: Dynamical behaviors of system (8) with } r = 0.5 > \delta = 0.4, \text{ max } I_k^* = 0.3668 < \mu/\mu + \delta)r = 0.4800 \text{ and } R_0 = 1.294.\]

Let \( n = 100 \) and \( m = 1 \) is a suitable assumption. Meanwhile, let \( \varphi(k) = ak^\alpha/(1 + bk^\alpha) \) in which \( a = 0.5, \alpha = 0.75, b = 0.02 \), and \( \lambda(k) = \lambda k \). The initial functions are \( I_k(0) = 0.45, k = 2, 3, 4, 5 \) and \( I_k(0) = 0, k \neq 2, 3, 4, 5 \) for \( s \in (-\infty, 0] \).

Denote

\[
I(t) = \sum_k p(k) I_k(t). \tag{47}
\]

Obviously, \( I(t) \) is the relative average density of the infected nodes.

**Case 1.** Let \( \lambda = 0.3, \; r = 0.4, \; \delta = 0.7, \; \mu = 0.6, \; \gamma = 2.5 \), and \( f(u) = e^{-u} \); we can obtain from (45) that \( R_0 = 0.5064 < 1 \). Figure 1 shows the dynamical behaviors of system (8). The numerical simulation shows \( \lim_{t \to +\infty} I(t) = 0 \), it follows that \( \lim_{t \to +\infty} I_k(t) = 0 \), and the infection eventually disappears. The numerical result is consistent with Theorem 2.

**Case 2.** Let \( \lambda = 0.7, \; r = 0.5, \; \delta = 0.4, \; \mu = 0.6, \; \gamma = 2.5 \), and \( f(u) = e^{-u} \); we can obtain from (45) that \( R_0 = 1.4596 > 1 \) and max \( I_k^* = 0.3668 < \mu/\mu + \delta)r = 0.4800 \). Figure 2 shows the dynamic behaviors of system (8). The relative density \( I_k(t) \) and the relative average density \( I(t) \) converge to positive constant as \( t \to +\infty \), respectively, and
Moreover, let $\lambda = 0.1$, $r = 0.4$, $\delta = 0.7$, $\mu = 0.6$, $\gamma = 2.5$ and $f(u) = e^{-u}$, and $\delta > r$. Figure 3 shows the dynamic behaviors of system (8). The parameters of system (8) do not satisfy Theorem 5, but the relative density $I_k(t)$ and the relative average density $I(t)$ still converge to positive constant as $t \to +\infty$, respectively. Therefore, Theorem 5 has room for improvement.

### Competing Interests

The authors declare that they have no competing interests.

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### References


