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Review Article

Applications of Group Theoretical Methods to Non-Newtonian Fluid Flow Models: Survey of Results

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The present review is intended to encompass the applications of symmetry based approaches for solving non-Newtonian fluid flow problems in various physical situations. Works which deal with the fundamental science of non-Newtonian fluids that are analyzed using the Lie group method and conditional symmetries are reviewed. We provide the mathematical modelling, the symmetries deduced, and the solutions obtained for all the models considered. This survey includes, as far as possible, all the articles published until 2015. Only papers published by a process of peer review in archival journals are reviewed and are grouped together according to the specific non-Newtonian models under investigation.

1. Introduction

The scientific and applications appeal of non-Newtonian fluid mechanics has necessitated a deeper study of its theory. There has been considerable focus in the study of the physical behavior and properties of non-Newtonian fluids over the past several decades. One particular reason for this interest is the wide range of applications of such models, both natural and industrial. These applications range from the extraction of crude oil from petroleum products to the polymer industry. Spin coating is a classic example where the coating fluids are typically non-Newtonian. A non-Newtonian fluid is one whose flow curve (shear stress versus shear rate) is nonlinear or does not pass through the origin, that is, where the apparent viscosity, shear stress divided by shear rate, is not constant at a given temperature and pressure but is dependent on flow conditions such as flow geometry and shear rate and sometimes even on the kinematic history of the fluid element under consideration. Such fluids may be conveniently grouped into three general classes as follows:

(1) Fluids for which the rate of shear at any point is determined only by the value of the shear stress at that point at that instant: these fluids are variously known as *time independent*, *purely viscous*, *inelastic*, or *generalized Newtonian fluids* (GNF).

(2) More complex fluids on which the relation between shear stress and shear rate depends: in addition, based upon the duration of shearing and their kinematic history, they are known as *time-dependent fluids*.

(3) Those substances exhibiting characteristics of both ideal fluids and elastic solids and showing partial elastic recovery, after deformation, are categorized as *viscoelastic fluids*.

Due to the complex physical structure of non-Newtonian fluids, there is not a single constitutive expression which describes the physical and mathematical properties of all nonlinear fluids. For this reason, many non-Newtonian fluid models for constitutive equations are available with most of them being empirical and semiempirical.

There are three diverse motivations for analyzing the flow behavior of non-Newtonian fluids: firstly, to extend the results of the flow models of Newtonian fluids to various classes of non-Newtonian fluids; secondly, to study the flow structure of non-Newtonian fluids as they occur in industry

under conditions which arise there; thirdly, to construct solutions of complicated nonlinear equations as exact solutions: these, when reported, facilitate the verification of complicated numerical codes and are also helpful in stability analysis. Consequently, the exact (closed-form) solutions of the flow models of non-Newtonian fluids are physically very significant. The most challenging task that we need to address when dealing with flow problems of non-Newtonian fluids is that the governing equations of these models are of a high order, nonlinear, and complicated in nature. Such fluids are modelled by constitutive equations which vary greatly in complexity. Thus, the resulting nonlinear equations are not easy to solve exactly. Several methods have been developed in recent years to obtain the solutions of these fluid models. Some of the techniques are the variational iteration method, Adomian decomposition method, homotopy analysis method, homotopy perturbation method, simplest equation method, semi-inverse variational method, and the exponential function method, amongst others. There are also the Lie symmetry and conditional symmetry group methods which are the main focus of this review.

Lie symmetry methods for differential equation were originated in the 1870s and were introduced by the Norwegian mathematician Marius Sophus Lie. Lie's theory is useful for solving differential equations that admit sufficient number of symmetries in a systematic way. Lie group methods are capable of handling a large number of equations. The application of this method neither depends on the type of the equation nor on the number of variables involved in the equations. Lie's theory is a general procedure which can be applied to any class of differential equations. However, if one peruses the literature on Lie's methods, we observe that this method and its extensions have rarely been applied in comparison with the wealth of differential equations in practical and theoretical problems.

The Lie symmetries of differential equations naturally form a group. Such groups are called Lie groups and are invertible point transformations of both the dependent and independent variables of the differential equations. Lie pointed out in his work that these groups are of great importance in understanding and constructing solutions of differential equations. Lie demonstrated that many techniques for finding solutions can be unified and extended by considering symmetry groups. Today, the Lie symmetry approach to differential equations is widely applied in various fields of mathematics, mechanics, physics, and the applied sciences and many results published in these areas demonstrate that Lie's theory is an efficient tool for solving nonlinear problems formulated in terms of differential equations. The primary objective of the Lie symmetry analysis advocated by Lie is to find one or several parameters of local continuous transformations leaving the equations invariant and then exploit them to obtain reductions and the so-called invariant or similarity solutions, and the usefulness of this approach has been widely illustrated by several researchers in different contexts. An extension of this approach is the conditional symmetry approach which is also very useful.

Motivated by the above-mentioned facts, the purpose of the present survey is to provide a detailed review of those studies which deal with the flow models of non-Newtonian fluids and solved using the group theoretic approaches. We have presented the mathematical modelling of each of the problem under review together with the symmetries deduced and the solutions obtained for that particular problem.

2. Symmetry Methods for Differential Equations

In this section, we briefly discuss the main aspects of the Lie symmetry method for differential equations with some words on conditional or nonclassical symmetries.

2.1. Symmetry Transformations of Differential Equations. A transformation under which a differential equation remains invariant (unchanged) is called a symmetry transformation of the differential equation.

Consider a kth order ($k \geq 1$) system of differential equations

$$F^{\sigma}\left(x, u, u_{(1)}, \ldots, u_{(k)}\right) = 0, \quad \sigma = 1, \ldots, m, \tag{1}$$

where $u = (u^1, \ldots, u^m)$, called the dependent variable, is a function of the independent variable $x = (x^1, \ldots, x^n)$ and $u_{(1)}$, $u_{(2)}$ up to $u_{(k)}$ are the collection of all first-order and second-order up to kth order derivatives of u.

A transformation of the variables x and u, namely.

$$\begin{aligned} \overline{x}^i &= f^i(x, u), \\ \overline{u}^\alpha &= g^\alpha(x, u), \\ & \qquad i = 1, \ldots, n; \ \alpha = 1, \ldots, m, \end{aligned} \tag{2}$$

is called a symmetry transformation of system (1) if (1) is form-invariant in the new variables $\overline{x}$ and $\overline{u}$; that is,

$$F^{\sigma}\left(\overline{x}, \overline{u}, \overline{u}_{(1)}, \ldots, \overline{u}_{(k)}\right) = 0, \quad \sigma = 1, \ldots, m, \tag{3}$$

whenever

$$G^{\sigma}\left(x, u, u_{(1)}, \ldots, u_{(k)}\right) = 0, \quad \sigma = 1, \ldots, m. \tag{4}$$

For example, the first-order Abel equation of the second kind

$$f\frac{df}{dy} = y^3 + yf \tag{5}$$

has symmetry transformations

$$\begin{aligned} \overline{y} &= ay, \\ \overline{f} &= a^2 f, \\ & \qquad a \in \mathbb{R}^+. \end{aligned} \tag{6}$$

2.2. Lie Symmetry Method for Partial Differential Equations. Here we discuss the classical Lie symmetry method to obtain all possible symmetries of a system of partial differential equations.

Let us consider a pth order system of partial differential equations in n independent variables $x = (x_1, \ldots, x_n)$ and m dependent variable $u = (u_1, \ldots, u_m)$, namely.

$$E\left(x, u, u_{(1)}, \ldots, u_{(p)}\right) = 0, \tag{7}$$

where $u^{\alpha}_{(k)}$, $1 \leq k \leq p$, denotes the set of all kth order derivative of u, with respect to the independent variables defined by

$$u^{\alpha}_{(k)} = \left\{ \frac{\partial^k u}{\partial x_{i_1}, \ldots, x_{i_k}} \right\}, \tag{8}$$

with

$$1 \leq i_1, i_2, \ldots, i_k \leq n. \tag{9}$$

For finding the symmetries of (7), we first construct the group of invertible transformations depending on the real parameter a, which leaves (7) invariant; namely,

$$\begin{aligned} \overline{x}_1 &= f^1(x, u, a), \\ &\vdots \\ \overline{x}_n &= f^n(x, u, a), \\ \overline{u}^{\alpha} &= g^{\alpha}(x, u, a). \end{aligned} \tag{10}$$

The above transformations have the closure property, are associative, admit inverses and identity transformation, and are said to form a one-parameter group.

Since a is a small parameter, transformations (10) can be expanded in terms of a series expansion as

$$\begin{aligned} \overline{x}_1 &= x_1 + a\xi_1(x, u) + O\left(a^2\right), \\ &\vdots \\ \overline{x}_n &= x_n + a\xi_n(x, u) + O\left(a^2\right), \\ \overline{u}_1 &= u_1 + a\eta_1(x, u) + O\left(a^2\right), \\ &\vdots \\ \overline{u}_m &= u_m + a\eta_n(x, u) + O\left(a^2\right). \end{aligned} \tag{11}$$

Transformations (11) are the infinitesimal transformations and the finite transformations are found by solving the Lie equations

$$\begin{aligned} \xi_1(\overline{x}, \overline{u}) &= \frac{d\overline{x}_1}{da}, \\ &\vdots \\ \xi_n(\overline{x}, \overline{u}) &= \frac{d\overline{x}_n}{da}, \\ \eta(\overline{x}, \overline{u}) &= \frac{d\overline{u}}{da}, \end{aligned} \tag{12}$$

with the initial conditions

$$\begin{aligned} \overline{x}_1(\overline{x}, \overline{u}, a)\big|_{a=0} &= x_1, \\ &\vdots \\ \overline{x}_n(\overline{x}, \overline{u}, a)\big|_{a=0} &= x_n, \\ \overline{u}_1(\overline{x}, \overline{u}, a)\big|_{a=0} &= u_1, \\ &\vdots \\ \overline{u}_m(\overline{x}, \overline{u}, a)\big|_{a=0} &= u_m, \end{aligned} \tag{13}$$

where $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)$ and $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_m)$.

Transformations (10) can be denoted by the Lie symmetry generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x, u)\frac{\partial}{\partial u^{\alpha}}, \tag{14}$$

where the functions ξ^i $(i = 1, \ldots, n)$ and η^{α} $(\alpha = 1, \ldots, m)$ are the coefficient functions of the operator X.

Operator (14) is a symmetry generator of (7) if

$$X^{[p]}E\big|_{E=0} = 0, \tag{15}$$

where $X^{[p]}$ represents the pth prolongation of the operator X and is given by

$$\begin{aligned} X^{[1]} &= X + \sum_{i=1}^{n} \zeta^{\alpha}_{x_i} \frac{\partial}{\partial u_{x_i}}, \\ X^{[2]} &= X^{[1]} + \sum_{i=1}^{n}\sum_{j=1}^{n} \zeta^{\alpha}_{x_i x_j} \frac{\partial^2}{\partial u_{x_i x_j}}, \\ &\vdots \\ X^{[p]} &= X^{[1]} + \cdots + X^{[p-1]} \\ &\quad + \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \zeta^{\alpha}_{x_{i_1} \cdots x_{i_p}} \frac{\partial^p}{\partial u_{x_{i_1} \cdots x_{i_p}}}, \end{aligned} \tag{16}$$

with

$$\begin{aligned} u_{x_i} &= \frac{\partial u}{\partial x_i}, \\ u_{x_{i_1} \cdots x_{i_k}} &= \frac{\partial^k u}{\partial x_{i_1} \cdots x_{i_k}}. \end{aligned} \tag{17}$$

In the above equations, the additional coefficient functions satisfy the following relations:

$$\begin{aligned}
\zeta^{\alpha}_{x_i} &= D_{x_i}(\eta) - \sum_{j=1}^{n} u_{x_j} D_{x_i}(\xi^j),\\
\zeta^{\alpha}_{x_i x_j} &= D_{x_j}(\eta^{x_i}) - \sum_{k=1}^{n} u_{x_i x_k} D_{x_j}(\xi^k),\\
&\vdots\\
\zeta^{\alpha}_{x_{i_1}\cdots x_{i_p}} &= D_{x_{i_p}}(\eta^{x_{i_1}\cdots x_{i_{p-1}}}) - \sum_{j=1}^{n} u_{x_{i_1}\cdots x_{i_{p-1}} x_j} D_{x_{i_p}}(\xi^j),
\end{aligned} \tag{18}$$

where D_{x_i} denotes the total derivative operator and is given by

$$D_{x_i} = \frac{\partial}{\partial x_i} + u_{x_i}\frac{\partial}{\partial u} + \sum_{j=1}^{n} u_{x_i x_j}\frac{\partial}{\partial x_j} + \cdots. \tag{19}$$

The determining equation (15) results in a polynomial in terms of the derivatives of the dependent variable u. After separation of (15) with respect to the partial derivatives of u and their powers, one obtains an overdetermined system of linear homogeneous partial differential equations for the coefficient functions ξ^i's and η^α's. By solving the overdetermined system, one has the following cases:

(i) There is no symmetry, which means that the Lie point symmetry generators given by ξ^i and η^α are all zero.

(ii) The point symmetry has $r \neq 0$ arbitrary constants; in this case, we obtain r generators of symmetry which forms an r-dimensional Lie algebra of point symmetries.

(iii) The point symmetry admits some finite number of arbitrary constants and arbitrary functions, in which case we obtain an infinite-dimensional Lie algebra.

2.3. Example on the Lie Symmetry Method. Here we illustrate the use of the Lie symmetry method on the well-known Korteweg-de Vries equation given by

$$f_t + f_{xxx} + ff_x = 0. \tag{20}$$

We seek for an operator of the form

$$X = \xi^1(t,x,f)\frac{\partial}{\partial t} + \xi^2(t,x,f)\frac{\partial}{\partial x} + \eta(t,x,f)\frac{\partial}{\partial f}. \tag{21}$$

Equation (21) is a symmetry generator of (20) if

$$X^{[3]}(f_t + f_{xxx} + ff_x)\big|_{f_t=-f_{xxx}-ff_x} = 0. \tag{22}$$

The third prolongation in this case is

$$X^{[3]} = X + \zeta^t\frac{\partial}{\partial f_t} + \zeta^x\frac{\partial}{\partial f_x} + \zeta^{xxx}\frac{\partial}{\partial f_{xxx}}. \tag{23}$$

Therefore, the determining equation (22) becomes

$$\left(\zeta^t + \zeta^{xxx} + f_x\zeta + f\zeta^x\right)\big|_{f_t=-f_{xxx}-ff_x} = 0. \tag{24}$$

Using the definitions of ζ^t, ζ^x, and ζ^{xxx} into (24) lead to an overdetermined system of linear homogenous system of partial differential equations given by

$$\begin{aligned}
f_{txx}f_x &: \xi^1_f = 0,\\
f_{txx} &: \xi^1_x = 0,\\
f^2_{xx} &: \xi^2_f = 0,\\
f_{xx}f_x &: \eta_{ff} = 0,\\
f_{xx} &: 3\eta_{xf} - 3\xi^2_{xx} = 0,\\
f_t &: -3\xi^2_x + \xi^1_t = 0,\\
f_x &: \eta - f\xi^2_x + 3\eta_{xxf} - \xi^2_{xxf} + f\xi^1_t - \xi^2_t = 0,\\
I &: f\eta_x + \eta_{xxx} + \eta_t = 0.
\end{aligned} \tag{25}$$

By solving system (25), we find four Lie point symmetries which are generated by the following generators:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\\
X_2 &= \frac{\partial}{\partial x},\\
X_3 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial f},\\
X_4 &= -3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + 2f\frac{\partial}{\partial f}.
\end{aligned} \tag{26}$$

2.4. Nonclassical Symmetry Method for Partial Differential Equations. Here we present a brief version of the nonclassical symmetry method for partial differential equations. In last few years, the interest in nonclassical group method has increased. There are mathematical problems appearing in applications that do not admit Lie point symmetries but have nonclassical symmetries. Therefore, this approach is helpful in obtaining exact solutions.

We begin by considering a kth order partial differential equation

$$G\left(x, u, u^{(1)}, \ldots, u^{(k)}\right) = 0, \tag{27}$$

in n independent variables $x = (x_1, \ldots, x_n)$ and one dependent variable u, with $u^{(k)}$ denoting the derivatives of the u with respect to x up to order k defined by

$$u^{(k)} = \left[\frac{\partial^k u}{\partial x_{i_1}, \ldots, x_{i_k}}\right], \tag{28}$$

with

$$1 \le i_1, \ldots, i_k \le n. \tag{29}$$

Suppose that $\mathbf{X}$ is a field of vectors which consists of dependent and independent variables:

$$\mathbf{X} = \xi^1(x,u)\frac{\partial}{\partial x_1} + \cdots + \xi^n(x,u)\frac{\partial}{\partial x_n} + \eta(x,u)\frac{\partial}{\partial u}, \quad (30)$$

where ξ^i and η are the coefficient functions of the vector field $\mathbf{X}$.

Suppose that the vector field $\mathbf{X}$ is the nonclassical symmetry generator of (27). Then the solution

$$u = f(x_1, x_2, \ldots, x_n) \quad (31)$$

of (27) is an invariant solution of (27) under a one-parameter subgroup generated by $\mathbf{X}$ if the condition

$$\Phi(x,u) = \eta(x,u) - \sum_{i=1}^{n}\xi^i(x,u)\frac{\partial u^\alpha}{\partial x_i} = 0 \quad (32)$$

holds together with (27). The condition given in (32) is known as an *invariant surface condition.* Thus, the invariant solution of (27) is obtained by solving the invariant surface condition (32) together with (27).

For (27) and (32) to be compatible, the kth prolongation $\mathbf{X}^{[k]}$ of the generator $\mathbf{X}$ must be tangent to the intersection of G and the surface Φ; that is,

$$\mathbf{X}^{[k]}(G)\big|_{G\cap\Phi} = 0. \quad (33)$$

If (32) is satisfied, then the operator $\mathbf{X}$ is called a nonclassical infinitesimal symmetry of the kth order partial differential equation (27).

For the case of two independent variables, t and y, two cases arise, namely. when $\xi^1 \neq 0$ and $\xi^1 = 0$.

When $\xi^1 \neq 0$, the operator is

$$\mathbf{X} = \frac{\partial}{\partial t} + \xi^2(t,y,u)\frac{\partial}{\partial y} + \eta(t,y,u)\frac{\partial}{\partial u}, \quad (34)$$

and thus

$$\Phi = u_t - \eta + \xi^2 u_y = 0 \quad (35)$$

is the invariant surface condition.

When $\xi^1 = 0$, the operator is

$$\mathbf{X} = \frac{\partial}{\partial y} + \eta(t,y,u)\frac{\partial}{\partial u}, \quad (36)$$

and hence

$$\Phi = u_y - \eta = 0 \quad (37)$$

is the invariant surface condition.

2.5. Example on the Nonclassical Symmetry Method. We illustrate the use of the nonclassical symmetry method on the well-known heat equation

$$u_t = u_{xx}. \quad (38)$$

Consider the infinitesimal operator

$$X = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \eta(t,x,u)\frac{\partial}{\partial u}. \quad (39)$$

The invariant surface condition is

$$\begin{aligned}\Phi(t,x,u) &= \eta(t,x,u) - \xi(t,x,u)\frac{\partial u}{\partial x} - \tau(t,x,u)\frac{\partial u}{\partial t}\\ &= 0.\end{aligned} \quad (40)$$

One can assume without loss of generality that $\tau = 1$, so that (40) takes the form

$$\Phi(t,x,u) = \eta(t,x,u) - \xi(t,x,u)\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0. \quad (41)$$

The nonclassical symmetries determining equations are

$$X^{[2]}u_t - u_{xx}\big|_{\text{Eq.}(38)=0,\Phi=0} = 0, \quad (42)$$

where $X^{[2]}$ is the usual third prolongation of operator X.

Applying the method to the heat PDE (38) with $\tau = 1$ yields

$$\begin{aligned}\xi &= \xi(x,t),\\ \eta &= A(x,t)u + B(x,t),\end{aligned} \quad (43)$$

where

$$\begin{aligned}A_t + 2A\xi_x - A_{xx} &= 0,\\ B_t + 2B\xi_x - B_{xx} &= 0,\\ \xi_t + 2\xi\xi_x - \xi_{xx} + 2A_x &= 0.\end{aligned} \quad (44)$$

The solution of system of (44) gives the following nonclassical infinitesimals:

$$\begin{aligned}\xi &= -\frac{w_t v - w v_t}{w_x v - w v_x},\\ \tau &= 1,\\ \eta &= \frac{v_t w_x - v_x w_t}{w_x v - w v_x}(u - u_0) + \xi f_x + f_t,\end{aligned} \quad (45)$$

where w, v, and f satisfy the heat equation.

3. Power-Law Fluid Flow Problems

In this section, all those problems dealing with the flow of a power-law fluid and solved by using the Lie symmetry approach are discussed.

The Cauchy stress tensor for a power-law fluid is written as

$$\mathbf{T} = -p\mathbf{I} + \mu\left(\left|\sqrt{\frac{1}{2}\operatorname{tr}\mathbf{A}_1^2}\right|^{n-1}\right)\mathbf{A}_1, \quad (46)$$

where p is the fluid pressure, $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity of the fluid, tr is the trace, and the first Rivlin-Ericksen tensor $\mathbf{A}_1$ is given by

$$\mathbf{A}_1 = (\operatorname{grad} \mathbf{V}) + (\operatorname{grad} \mathbf{V})^T, \tag{47}$$

in which $\mathbf{V}$ is the fluid velocity. It should be noted that n is the power-law index. If $n = 1$, (46) represents a viscous fluid. Furthermore, (46) represents shear thinning behavior when $n < 1$ and shear thickening for $n > 1$.

3.1. Solution of the Rayleigh Problem for a Power-Law Non-Newtonian Conducting Fluid via Group Method [1]. Abd-el-Malek et al. [1] studied the magnetic Rayleigh problem where a semi-infinite plate is given an impulsive motion and thereafter moves with constant velocity in a non-Newtonian power-law fluid of infinite extent. The governing nonlinear model was solved by means of the Lie group approach.

The governing problem describing the flow model [1] is given by

$$\frac{\partial u}{\partial t} - \gamma \frac{\partial}{\partial y} \left\{ \left[\left(\frac{\partial u}{\partial y} \right)^2 \right]^{(n-1)/2} \frac{\partial u}{\partial y} \right\} + MH^2 u = 0, \tag{48}$$

with the boundary and initial conditions

$$\begin{aligned} u(0,t) &= V, \\ u(\infty,t) &= 0, \\ & \qquad t > 0, \\ u(y,0) &= 0, \quad y > 0. \end{aligned} \tag{49}$$

The method of solution employed in [1] depends on the application of a one-parameter group of transformations to the partial differential equation (48). The one-parameter group, which transforms the PDE (48) and the boundary conditions (49), is of the form [1]

$$\begin{aligned} \overline{y} &= \left(h^t\right)^{1/(n+1)} y, \\ \overline{t} &= h^t t, \\ \overline{u} &= u, \\ \overline{H} &= \left(\frac{1}{\sqrt{h^t}} \right) H. \end{aligned} \tag{50}$$

Under transformations (50), the two independent variables reduce by one and the partial differential equation (48) is transformed into an ordinary differential equation. The reduced ordinary differential equation was then solved numerically.

3.2. Invariant Solutions of the Unidirectional Flow of an Electrically Charged Power-Law Non-Newtonian Fluid over a Flat Plate in Presence of a Transverse Magnetic Field [2]. Wafo Soh [2] investigated a boundary value problem for a nonlinear diffusion equation arising in the study of a charged power-law non-Newtonian fluid through a time-dependent transverse magnetic field. Two families of exact invariant solutions were obtained by use of the Lie symmetry method.

The governing equation describing the flow model is given by [2]

$$\frac{\partial u}{\partial t} - n\gamma \left(\frac{\partial u}{\partial y} \right)^{n-1} \frac{\partial^2 u}{\partial y^2} + MH^2(t)\, u = 0. \tag{51}$$

The relevant boundary and initial conditions are

$$\begin{aligned} u(0,t) &= V(t), \quad t > 0, \\ u(\infty,t) &= 0, \quad t > 0, \\ u(y,0) &= 0, \quad y > 0. \end{aligned} \tag{52}$$

The symmetry Lie algebra of PDE (51) is five-dimensional and is spanned by the operators [2]

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y}, \\ X_2 &= y \frac{\partial}{\partial y} + \frac{n+1}{n-1} u \frac{\partial}{\partial u}, \quad n \neq 1, \\ X_3 &= L_2'(t) \frac{\partial}{\partial u}, \\ X_4 &= \frac{1}{L_n'(t)} \frac{\partial}{\partial t} - \frac{MH^2}{L_n'(t)} u \frac{\partial}{\partial u}, \\ X_5 &= \frac{L_n(t)}{L_n'(t)} \frac{\partial}{\partial t} - \left[MH^2 \frac{L_n(t)}{L_n'(t)} + \frac{1}{n-1} \right] u \frac{\partial}{\partial u}, \\ & \qquad n \neq 1, \end{aligned} \tag{53}$$

where

$$L_n(t) = \int_0^t d\tau e^{(1-n)M \int_0^\tau H^2(\lambda) d\lambda}. \tag{54}$$

With the use of the above symmetries, the group invariant solution for the PDE (51) found in [2] is

$$u(y,t) = L_n^{1/(1-n)}(t)\, L_2'(t)\, \psi(y), \tag{55}$$

with $\psi(y)$ given by

$$\psi(y) = \left[\left(\frac{n-1}{n+1} \right) \left(\frac{1+n}{2\gamma n(1-n)} \right)^{1/(1+n)} y + 1 \right]^{(n+1)/(n-1)}. \tag{56}$$

3.3. Unsteady Boundary Layer Flow of Power-Law Fluid on Stretching Sheet Surface [3]. Yürüsoy [3] treated the unsteady boundary layer equations of a power-law fluid over a stretching sheet. By the use of similarity transformations, the governing system of partial differential equations reduced to a nonlinear ordinary differential equation system. Finally, the resulting system of reduced ordinary differential equations was solved using a combination of the Runge-Kutta algorithm and shooting technique.

The governing equations describing the flow model [3] are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=(2m+1)\,2^{m}\left|\frac{\partial u}{\partial y}\right|^{2m}\frac{\partial^{2}u}{\partial y^{2}}+\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}, \tag{57}$$

where u and v are the velocity components inside the boundary layer and $U(x,t)$ is the velocity outside the boundary layer.

The boundary conditions for flow over a stretching sheet are

$$u(x,0,t)=A(x,t),\quad v(x,0,t)=0,\quad u(x,\infty,t)=0. \tag{58}$$

By use of the Lie group method, the similarity transformations for the reduction of the above system of PDEs are given by [3]

$$\xi=yt^{(2m-1)/(2m+2)}x^{-m/(m+1)},\quad v=x^{m/(m+1)}t^{-(4m+1)/(2m+2)}g(\xi),\quad u=\frac{x}{t}f(\xi),\quad A(x,t)=\lambda\frac{x}{t}. \tag{59}$$

Transformation (59) transforms the two-dimensional unsteady boundary layer equation problem into ordinary differential equations. The reduced ordinary differential equations have been solved numerically using a variable step size Runge-Kutta subroutine combined with a shooting technique.

3.4. Axisymmetric Spreading of a Thin Power-Law Fluid under Gravity on a Horizontal Plane [4]. Nguetchue and Momoniat [4] studied a nonlinear PDE modelling the axisymmetric spreading under gravity of a thin power-law fluid on a horizontal surface. The model equation was reduced to a nonlinear second-order ordinary differential equation for the spatial variable. Then Lie symmetry analysis applied to the nonlinear ordinary differential equation enabled its linearization and solution.

The equation modelling the height of a thin power-law fluid film on a horizontal plane in presence of gravity is given by [4]

$$\frac{\partial h}{\partial t}=\frac{1}{(\beta+1)x}\frac{\partial}{\partial x}\left[xh^{\beta+1}\left(\frac{\partial h}{\partial x}\right)^{\beta-1}\right]. \tag{60}$$

Here $h(x,t)$ is the film height and β is the power-law fluid parameter. The Lie point symmetry generator for the PDE (60) is [4]

$$X=\left[(1-\beta)At+B\right]\frac{\partial}{\partial t}+Ah\frac{\partial}{\partial h}. \tag{61}$$

The invariant solution of PDE (60) corresponding to the symmetry generator (61) found in [4] is

$$h(x,t)=\frac{3A^{1/3}x^{2/3}}{2(3At-B)^{1/3}}. \tag{62}$$

3.5. Symmetry Reductions of a Flow with Power-Law Fluid and Contaminant-Modified Viscosity [5]. Moitsheki et al. [5] have analyzed a system dealing with nonreactive pollutant transport along a single channel. Constitutive equations obeying a power-law fluid are utilized in the description of the mathematical problem. Invariant solutions which satisfy physical boundary conditions have been constructed using the Lie group approach.

The dimensionless governing equations that describe the flow model are [5]

$$\frac{\partial u}{\partial t}=K+M\frac{\partial}{\partial y}\left[c^{\lambda}\left(\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial u}{\partial y}\right],\quad \frac{\partial c}{\partial t}=\frac{1}{R}\frac{\partial}{\partial y}\left(c^{\lambda}\frac{\partial c}{\partial y}\right)+S(y,t). \tag{63}$$

Here R is the Schmidt number and K is the imposed constant pressure axial gradient. The Lie point symmetries of the above system corresponding to different forms of the source term $S(y,t)$ are given in Table 1 of [5]. The invariant solutions of system (63) found in [5] are of the form

$$u(y,t)=Kt+4\gamma_1R\gamma_3\left[\frac{1}{2}\cos\left(\frac{y+c_2}{2c_1}\right)\sin\left(\frac{y+c_2}{2c_1}\right)+\left(\frac{y+c_2}{4c_1}\right)\right]+c_3,\quad c(y,t)=\frac{t}{2\gamma_2R}\sec^{2}\left(\frac{y+c_2}{2c_1}\right). \tag{64}$$

3.6. Scaling Group Transformation under the Effect of Thermal Radiation Heat Transfer of a Non-Newtonian Power-Law Fluid over a Vertical Stretching Sheet with Momentum Slip Boundary Condition [6]. An analysis has been conducted to study the problem of heat transfer of a power-law fluid over a vertical stretching sheet with slip boundary condition by Mutlag et al. [6]. The partial differential equations governing the physical model have been converted into a set of nonlinear coupled ordinary differential equations using scaling group of transformations. These reduced equations are then solved numerically using the Runge-Kutta-Fehlberg fourth-fifth order numerical method.

The dimensionless forms of the governing equations of the flow model [6] are

$$\begin{aligned} &\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\ &u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial}{\partial y}-\left|\frac{\partial u}{\partial y}\right|^{n}\frac{\partial^{2}u}{\partial y^{2}}\pm\lambda x\theta,\\ &\frac{\theta}{x}u+u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}=\frac{R_{e}^{2/(1+n)}}{LU_{r}}\left[\alpha+\frac{16\sigma_{1}T_{\infty}^{3}}{3\rho c_{p}k_{1}}\right]\frac{\partial^{2}\theta}{\partial y^{2}}. \end{aligned} \tag{65}$$

The boundary conditions specified to solve the above system of PDEs are

$$\begin{aligned} &u=x+\frac{aU_{r}^{n-2}R_{e}^{1/(1+n)}}{L^{n}}\left|-\frac{\partial u}{\partial y}\right|^{n}\frac{\partial u}{\partial y},\\ &v=0,\\ &\theta=1 \qquad \text{at } y=0,\\ &u=0,\\ &\theta=0 \qquad \text{as } y\longrightarrow\infty. \end{aligned} \tag{66}$$

The scaling symmetry operator for the system of PDEs (65) is calculated as [6]

$$X=\left(\frac{n+1}{2n}\right)\frac{\partial}{\partial x}+\left(\frac{n-1}{2n}\right)\frac{\partial}{\partial y}+u\frac{\partial}{\partial u}. \tag{67}$$

The corresponding similarity transformations are

$$\begin{aligned} &\eta=yx^{(1-n)/(1+n)},\\ &\psi=x^{2n/(1+n)}f(\eta),\\ &\theta=\theta(\eta). \end{aligned} \tag{68}$$

Transformation (67) transforms the system of PDEs (65) into a nonlinear system of ODEs. The reduced ordinary differential equations are solved numerically.

3.7. Lie Group Analysis of a Non-Newtonian Fluid Flow over a Porous Surface [7]. Akgül and Pakdemirli [7] investigated the two-dimensional unsteady squeezed flow over a porous surface for a power-law non-Newtonian fluid. Lie Group theory was applied on the model equations. Then, a partial differential system with three independent variables was converted into an ordinary differential system, via application of two successive symmetry generators. The ordinary differential equations were then solved numerically.

The problem describing the flow model [7] is given by

$$\begin{aligned} &\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\ &\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=F(x,t)+n\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial^{2}u}{\partial y^{2}},\\ &\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{P_{r}}\frac{\partial^{2}\theta}{\partial y^{2}}, \end{aligned} \tag{69}$$

with

$$\begin{aligned} &u(x,0,t)=S(x,t),\\ &v(x,0,t)=V,\\ &T(x,0,t)=1,\\ &u(x,\infty,t)=U(x,t),\\ &T(x,\infty,t)=0. \end{aligned} \tag{70}$$

The symmetries for the system of PDEs (69) found in [7] are

$$\begin{aligned} &\xi_{1}=\frac{3}{2}ax+b(t),\\ &\xi_{2}=\frac{a}{2}y,\\ &\xi_{3}=at+d,\\ &\eta_{1}=\frac{a}{2}u+b',\\ &\eta_{2}=-\frac{a}{2}v,\\ &\eta_{3}=0. \end{aligned} \tag{71}$$

Symmetries (71) are used to reduce the nonlinear system of PDEs (69) to a nonlinear system of ODEs which was then solved using a numerical approach.

3.8. Flow of Power-Law Fluid over a Stretching Surface: A Lie Group Analysis [8]. The investigation of the boundary layer flow of power-law fluid over a permeable stretching surface was made by Jalil and Asghar [8]. The use of Lie group analysis reveals all possible similarity transformations of the problem. The similarity transformations have been utilized to reduce the governing system of nonlinear PDEs to a nonlinear boundary value problem.

The governing equations of the flow model [8] are

$$\begin{aligned} &\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\ &u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial}{\partial y}\left|\left(-\frac{\partial u}{\partial y}\right)\right|^{n}. \end{aligned} \tag{72}$$

The boundary conditions are

$$\begin{aligned} u &= u_x(x), \\ v &= \frac{V_0 v_w(x)}{U_0}\left(\frac{\rho U_0^{2-n} L^n}{K}\right)^{1/(n+1)} \\ &\qquad\qquad \text{at } y = 0, \\ u &= 0 \quad \text{as } y \longrightarrow \infty. \end{aligned} \tag{73}$$

The form of the infinitesimals is found to be [8]

$$\begin{aligned} \xi_1 &= a + b(x), \\ \xi_2 &= \frac{b + (n-2)c}{(n+1)} y + \gamma(x), \\ \varphi_1 &= cu, \\ \varphi_2 &= \frac{(2n-1)c - nb}{(n+1)} v + u\gamma'(x). \end{aligned} \tag{74}$$

Symmetries (74) are used to compute the appropriate similarity transformations which were then used to reduce the nonlinear system of the above PDEs to a nonlinear boundary value problem. The reduced boundary value problem was solved numerically.

3.9. Group Invariant Solution for a Preexisting Fracture Driven by a Power-Law Fluid in Impermeable Rock [9]. The effect of power-law rheology on hydraulic fracturing has been studied by Fareo and Mason [9]. With the aid of lubrication theory and the PKN approximation, a partial differential equation for the fracture half-width was derived. By using a linear combination of the Lie symmetry generators of the governing equation, the group invariant solution was obtained and the problem was reduced to a boundary value problem for an ordinary differential equation.

The mathematical problem describing the preexisting fracture driven by a power-law fluid in impermeable rock [9] is given by

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left[h^{(2n+1)/n}\left(-\frac{\partial h}{\partial x}\right)^{1/n}\right] = 0, \tag{75}$$

with

$$\begin{aligned} \frac{dV}{dt} &= 2\left(-\frac{\partial h(0,t)}{\partial x}\right)^{1/n} h^{(2n+1)/n}(0,t), \\ h(L(t),t) &= 0. \end{aligned} \tag{76}$$

The symmetry Lie algebra of (75) is spanned by the operators [9]

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= t\frac{\partial}{\partial t} - \left(\frac{n}{n+2}\right) h\frac{\partial}{\partial h}, \\ X_4 &= x\frac{\partial}{\partial x} + \left(\frac{n+1}{n+2}\right) h\frac{\partial}{\partial h}. \end{aligned} \tag{77}$$

The group invariant solutions of the PDE (75) found in [9] are of the form

$$\begin{aligned} h(x,t) &= \frac{1}{L(t)}\left[1 - u^{n+1}\right]^{1/(n+2)}, \\ h(x,t) &= L(t)^{1/(n+2)}\left[1 - u\right]^{1/(n+2)}, \end{aligned} \tag{78}$$

where the particular values of $L(t)$ are given in [9]. The Lie symmetries given in (77) were also utilized to perform various reductions of PDE (75) which was then solved numerically.

4. Sisko Fluid Flow Problems

In this section, we investigate all those models which deal with the flow of a Sisko fluid and solved with the aid of the Lie group approach.

The Cauchy stress tensor $\mathbf{T}$ for a Sisko fluid model is given by

$$\mathbf{T} = -p\mathbf{I} + \left[a + b\left|\sqrt{\frac{1}{2}\operatorname{tr}\mathbf{A}_1^2}\right|^{n-1}\right]\mathbf{A}_1, \tag{79}$$

where $\mathbf{V}$ is the velocity vector, $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor, and a and b are the material constants. The model is a combination of viscous and power-law models. For $a = 0$, the model exhibits power-law behavior whereas for $b = 0$, the flow is Newtonian and $n > 0$ is a characteristic of the non-Newtonian behavior of the fluid.

4.1. Rayleigh Problem for a MHD Sisko Fluid [10]. Molati et al. [10] studied the problem of unsteady unidirectional flow of an incompressible Sisko fluid bounded by a suddenly moved plate. The fluid is magnetohydrodynamic (MHD) in the presence of a time-dependent magnetic field applied in the transverse direction of the flow. The nonlinear governing flow model was solved analytically using the Lie symmetry approach.

The problem describing the flow model [10] is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + L\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial u}{\partial y}\right] - M^2 H^2(t)\, u, \tag{80}$$

with

$$u(t,0) = g(t), \quad t > 0,$$
$$u(t,y) \longrightarrow 0 \quad \text{as } y \longrightarrow \infty, \ t > 0, \tag{81}$$
$$u(0,y) = 0, \quad y > 0.$$

The symmetry Lie algebra of the PDE (80) is three-dimensional and spanned by the operators [10]

$$X_1 = \frac{\partial}{\partial y},$$
$$X_2 = (2t+\beta)\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}, \tag{82}$$
$$X_3 = \left(\frac{2t+\beta}{2+\beta}\right)^{-M^2H_0^2/2}\frac{\partial}{\partial u},$$

where

$$H_0 = \sqrt{2t+\beta}H(t). \tag{83}$$

The similarity solution from the invariants of X_2 assumes the form [10]

$$u(t,y) = (2t+\beta)^{1/2}F(\gamma) \quad \text{with } \gamma = y(2t+\beta)^{-1/2}. \tag{84}$$

Invariant (84) is used to reduce the PDE (80) into a nonlinear ODE. The reduced ODE together with suitable boundary conditions was solved by employing a numerical approach.

4.2. Reduction and Solutions for MHD Flow of a Sisko Fluid in a Porous Medium [11]. Mamboundou et al. [11] obtained the analytical solutions for magnetohydrodynamic (MHD) flow of a Sisko fluid in a semi-infinite porous medium. The governing nonlinear differential equation was solved by employing the symmetry method.

The governing equation of the flow model [11] is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\left[\left(1 + b\left(\frac{\partial u}{\partial y}\right)^{n-1}\right)\frac{\partial u}{\partial y}\right] - \frac{1}{K}\left[1 + b\left(\frac{\partial u}{\partial y}\right)^{n-1}\right] - M^2u. \tag{85}$$

The relevant boundary and initial conditions are

$$u(0,t) = V(t), \quad t > 0,$$
$$u(\infty,t) = 0, \quad t > 0, \tag{86}$$
$$u(y,0) = h(y), \quad y > 0.$$

The above PDE admits the Lie point symmetry generators [11]

$$X_1 = \frac{\partial}{\partial t},$$
$$X_1 = \frac{\partial}{\partial y}. \tag{87}$$

The travelling wave solutions of the PDE (85) were constructed corresponding to the symmetry generators (87) and is given by [11]

$$u(y,t) = \exp\left[\frac{1}{nK}\left(1 - n - nKM^2\right)t - \frac{1}{\sqrt{nK}}y\right]. \tag{88}$$

4.3. Stokes' First Problem for Sisko Fluid over a Porous Wall [12]. The study of time-dependent flow of an incompressible Sisko fluid over a wall with suction or blowing was performed by Hayat et al. [12]. The magnetohydrodynamic nature of the fluid was taken into account by applying a variable magnetic field. The resulting nonlinear problem was solved by invoking the symmetry approach.

The problem governing the flow model [12] in a nondimensional form is given by

$$\frac{\partial u}{\partial t} - S\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\left(1 + L\left|\frac{\partial u}{\partial y}\right|^{n-1}\right)\frac{\partial u}{\partial y}\right] - M^2H^2(t)u, \tag{89}$$

$$u(t,0) = 1, \quad t > 0, \tag{90}$$

$$\lim_{y\to\infty} u(t,y) = 0, \quad t > 0, \tag{91}$$

$$u(y,0) = 0, \quad y > 0. \tag{92}$$

The symmetry analysis of (89) revealed that extra symmetries are admitted for the cases

$$H = 0,$$
$$H = \text{Constant},$$
$$H = \frac{h_0}{\sqrt{t}},$$
$$H = \frac{h_0}{\sqrt{\alpha t + t_0}}, \tag{93}$$
$$H = \frac{h_0}{\sqrt{2t + t_0}}, \quad \text{where } h_0, t_0 \ni \mathfrak{R}.$$

The reductions of PDE (89) for these cases lead to nonlinear ordinary differential equations. However, the imposed boundary conditions are not invariant under the admitted Lie point symmetries. Hence, the governing model was then solved by making use of numerical techniques.

4.4. Boundary Layer Equations and Lie Group Analysis of a Sisko Fluid [13]. Sari et al. [13] recently derived the boundary layer equations for a Sisko fluid. Using Lie group theory, a symmetry analysis of the equations was performed. A partial differential system is transferred to an ordinary differential system using symmetries and the resulting reduced equations were numerically solved.

The dimensionless form of the boundary layer equations for a Sisko fluid is [13]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\varepsilon_1\frac{\partial^2 u}{\partial y^2}+\varepsilon_2\left(\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial^2 u}{\partial y^2}. \tag{94}$$

The classical boundary conditions for the problem are [13]

$$u(x,0)=0,$$
$$v(x,0)=0, \tag{95}$$
$$u(x,\infty)=U(x).$$

The infinitesimals of the above system of PDEs are [13]

$$\xi_1=3ax+b,$$
$$\xi_2=ay,$$
$$\eta_1=au, \tag{96}$$
$$\eta_2=-av.$$

The corresponding similarity transformations are

$$\xi=\frac{y}{x^{1/3}},$$
$$u=x^{1/3}f(\xi), \tag{97}$$
$$v=\frac{g(\xi)}{x^{1/3}}.$$

Transformations (97) are used to reduce the above PDE system to an ordinary differential system. The reduced ordinary differential system was solved by using a numerical method.

4.5. Analytic Approximate Solutions for Time-Dependent Flow and Heat Transfer of a Sisko Fluid [14]. The purpose of this study was to find analytic approximate solutions for unsteady flow and heat transfer of a Sisko fluid. Translational symmetries were utilized in [14] to find a family of travelling wave solutions of the governing nonlinear problem.

In dimensionless form, the governing problem takes the form [14]

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial y}\left[\left(1+b\left(-\frac{\partial u}{\partial y}\right)^{n-1}\right)\frac{\partial u}{\partial y}\right], \tag{98}$$

$$\frac{\partial \theta}{\partial t}=\frac{1}{P_r}\frac{\partial^2\theta}{\partial t^2}+E_c\left[1+b\left(-\frac{\partial u}{\partial y}\right)^{n-1}\right]\left(\frac{\partial u}{\partial y}\right)^2, \tag{99}$$

with the boundary conditions

$$\begin{aligned}
&u(0,t)=V_1(t),\\
&\theta(0,t)=V_2(t),\\
&\qquad t>0,\\
&u(\infty,t)=0,\\
&\theta(\infty,t)=0,\\
&\qquad t>0,\\
&u(y,0)=h_1(y),\\
&\theta(y,0)=h_2(y),\\
&\qquad y>0.
\end{aligned} \tag{100}$$

Equation (98) admits the Lie point symmetry generators $X=\partial/\partial t$ and $Y=\partial/\partial y$. The generator $X-cY$ which represents a family of travelling wave with constant wave speed c has been used in [14] to perform reduction of the above system of PDEs into nonlinear system of ODEs. The reduced system of ODEs was solved by homotopy analysis method.

4.6. Self-Similar Unsteady Flow of a Sisko Fluid in a Cylindrical Tube Undergoing Translation [15]. The governing equation for unsteady flow of a Sisko fluid in a cylindrical tube due to translation of the tube wall is modelled in [15]. The reduction of the nonlinear problem was carried out by using Lie group approach. The partial differential equation is transformed into an ordinary differential equation, which was integrated numerically.

The unsteady flow of a Sisko fluid in a cylindrical tube due to impulsive motion of tube is governed by [15]

$$\begin{aligned}
\frac{\partial w}{\partial t}&=\frac{\partial^2 w}{\partial r^2}+bn\frac{\partial^2 w}{\partial r^2}\left(\frac{\partial w}{\partial r}\right)^{n-1}\\
&\quad+\frac{1}{r}\left[\frac{\partial w}{\partial r}+b\left(\frac{\partial w}{\partial r}\right)^n\right],
\end{aligned} \tag{101}$$

subject to the boundary conditions

$$\begin{aligned}
&w(1,t)=V(t),\quad t>0,\\
&\frac{\partial w}{\partial r}(0,t)=0,\quad t>0,\\
&W(r,0)=W(r),\quad y>0.
\end{aligned} \tag{102}$$

The Lie point symmetries for the PDE (101) are spanned by the operators [15]

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= w\frac{\partial}{\partial w}, \\ X_3 &= 2t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + w\frac{\partial}{\partial w}, \\ X_4 &= (w + br)\frac{\partial}{\partial w}, \\ X_5 &= d(t,r)\frac{\partial}{\partial w}, \end{aligned} \tag{103}$$

where $d(t,r)$ satisfies the linear partial differential equation (101). The operator X_3 has been used in [15] to deduce the similarity transformations

$$w(r,t) = rf(\varsigma), \quad \varsigma = \frac{r}{\sqrt{t}}. \tag{104}$$

The similarity transformations (104) are employed to reduce the partial differential equation (101) into a nonlinear ordinary differential equation. The reduced ordinary differential together with suitable boundary and initial conditions was solved by shooting method.

5. Jeffrey Fluid Flow Problems

Here we discuss the problems dealing with the flow of a Jeffrey fluid that are solved using the Lie group approach.

The constitutive equations for an incompressible Jeffrey fluid model are

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{105}$$

with

$$\mathbf{S} = \frac{\mu}{1+\lambda_1}\left[\dot{\gamma} + \lambda_2\ddot{\gamma}\right], \tag{106}$$

where $\mathbf{T}$ and $\mathbf{S}$ are the Cauchy stress tensor and the extra stress tensor, respectively, p is the pressure, $\mathbf{I}$ is the identity tensor, λ_1 is the ratio of relaxation to retardation times, λ_2 is the retardation time, $\dot{\gamma}$ is the shear rate, and the dots over the quantities indicate differentiation with respect to time.

5.1. Lie Point Symmetries and Similarity Solutions for an Electrically Conducting Jeffrey Fluid [16]. The only model available in the literature dealing with the flow of a Jeffrey fluid and solved by employing the Lie symmetry approach was studied by Mekheimer et al. [16]. In their work, the equations for the two-dimensional incompressible fluid flow of an electrically conducting Jeffrey fluid are studied. A Lie symmetry analysis was performed and the group invariant solutions were derived.

The governing equations of the model [16] are

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \epsilon_1\left(\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y}\right) - \epsilon_2 u, \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \epsilon_1\left(\frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y}\right), \end{aligned} \tag{107}$$

with

$$\begin{aligned} S_{xx} &= \frac{2}{1+\lambda_1}\left[1 + \lambda_2\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\right]\frac{\partial u}{\partial x}, \\ S_{xy} &= \frac{1}{1+\lambda_1}\left[1 + \lambda_2\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\right] \\ &\quad \cdot\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \\ S_{yy} &= \frac{2}{1+\lambda_1}\left[1 + \lambda_2\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\right]\frac{\partial v}{\partial y}. \end{aligned} \tag{108}$$

The relevant boundary conditions are of the form [16]

$$\begin{aligned} u(x,0,0) &= U_0, \\ u(x,\infty,t) &= 0, \\ \frac{\partial u(x,0,0)}{\partial y} &= 0, \\ v(x,0,0) &= -V_0, \\ p(x,\infty,0) &= P_0, \end{aligned} \tag{109}$$

where U_0 is the velocity of the plate, V_0 is the magnetic fluid penetrating into the plate, and P_0 is the pressure deep in the magnetic fluid. The symmetries of the system of PDEs (107) found in [16] are

$$\begin{aligned} \xi_1 &= \gamma(t), \\ \xi_2 &= a_1, \\ \xi_3 &= a_2, \\ \eta_1 &= \gamma'(t), \\ \eta_2 &= 0, \\ \eta_3 &= \delta(t) - x\left(\gamma''(t)' + \epsilon_2\gamma'(t)\right), \end{aligned} \tag{110}$$

where a_1 and a_2 are the arbitrary constants and $\gamma(t)$ and $\delta(t)$ are the arbitrary functions of the variable t only. With the use

of symmetries given in (110), the group invariant solutions for the system of PDEs (107) are [16]

$$\begin{aligned} u(x,y,t) &= \frac{U_0}{(\alpha_2-\alpha_1)}\left[\alpha_2 \exp\left[\alpha_1 (y-Ct)\right]\right. \\ &\quad \left.-\alpha_1 \exp\left[\alpha_2 (y-Ct)\right]\right], \\ v(x,y,t) &= \frac{U_0}{(\alpha_2-\alpha_1)}\left[\alpha_2 \exp\left[\alpha_1 (y-Ct)\right]\right. \\ &\quad \left.-\alpha_1 \exp\left[\alpha_2 (y-Ct)\right]\right]-(U_0+P_0), \\ p(x,y,t) &= \frac{U_0}{(\alpha_2-\alpha_1)(1+\lambda_1)}\alpha_1\epsilon_1+\left(1+\lambda_1\right. \\ &\quad \left.-2\epsilon_1\alpha_1^2\lambda_2\right)(C+U_0+V_0)\times\alpha_2\exp\left[\alpha_1(y-Ct)\right] \\ &\quad -\left[\alpha_2\epsilon_1+\left(1+\lambda_1-2\epsilon_1\alpha_1^2\lambda_2\right)(C+U_0+V_0)\right. \\ &\quad \left.\times\alpha_1\exp\left[\alpha_2(y-Ct)\right]\right]+P_0. \end{aligned} \tag{111}$$

6. Williamson Fluid Flow Problems

In this section, we investigate the problems which deal with the flow of a Williamson fluid which are solved using the Lie symmetry approach.

The Cauchy stress tensor $\mathbf{T}$ for a Williamson fluid model is given by

$$\mathbf{T} = \left[\mu_\infty + \frac{\mu_0-\mu_\infty}{1+\lambda|\dot{\gamma}|}\right]\dot{\gamma}, \tag{112}$$

where

$$\begin{aligned} \dot{\gamma} &= \begin{bmatrix} 2\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} & 2\dfrac{\partial v}{\partial y} \end{bmatrix}, \\ |\dot{\gamma}| &= \left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]^{1/2}. \end{aligned} \tag{113}$$

Here μ_0 and μ_∞ are the limiting viscosities at zero and at infinite shear rate, respectively, and λ is a rheological parameter.

6.1. Boundary Layer Theory and Symmetry Analysis of a Williamson Fluid [17]. The first study available in the literature dealing with the flow of a Williamson fluid and solved by employing the Lie group approach was performed by Aksoy et al. [17]. In [17], the boundary layer equations for a Williamson fluid are derived for the first time. Using Lie group theory, a symmetry analysis of the equations was performed. The partial differential system was converted to an ordinary differential system via symmetries and the resulting equations were numerically solved.

The governing problem of the flow model [17] is

$$\begin{aligned} &\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \\ &u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} \\ &\quad = U\frac{dU}{dx}+k_2\frac{\partial^2 u}{\partial y^2} \\ &\quad +(k_1-k_2)\left[1+k_3\left|\frac{\partial u}{\partial y}\right|\right]^{-1}\frac{\partial^2 u}{\partial y^2} \\ &\quad -(k_1-k_2)k_3\left[1+k_3\left|\frac{\partial u}{\partial y}\right|\right]^{-2}\frac{\partial^2 u}{\partial y^2}\frac{\partial u}{\partial y}, \end{aligned} \tag{114}$$

where $\varepsilon_1 = k_1\delta^2$, $\varepsilon_2 = k_2\delta^2$, and $\varepsilon_3 = k_1\delta$. The classical boundary conditions for the problem are

$$\begin{aligned} u(x,0) &= 0, \\ v(x,0) &= 0, \\ u(x,\infty) &= U(x). \end{aligned} \tag{115}$$

The infinitesimals of the above system of PDEs are [17]

$$\begin{aligned} \xi_1 &= 3ax+b, \\ \xi_2 &= ay, \\ \eta_1 &= au, \\ \eta_2 &= -av. \end{aligned} \tag{116}$$

The corresponding similarity transformations are

$$\begin{aligned} \xi &= \frac{y}{x^{1/3}}, \\ u &= x^{1/3}f(\xi), \\ v &= \frac{g(\xi)}{x^{1/3}}. \end{aligned} \tag{117}$$

The similarity transformations (117) are used to reduce the above PDE system into a system of nonlinear ordinary differential equations. The reduced ordinary differential system was solved by using numerical techniques.

6.2. Boundary Layer Flow of Williamson Fluid with Chemically Reactive Species Using Scaling Transformation and Homotopy Analysis Method [18]. The study of Williamson fluid flow with a chemically reactive species was made recently by Khan et al. [18]. The governing equations of Williamson model in two-dimensional flows were constructed by using scaling group transformation. The series solution of the system of reduced nonlinear ordinary differential equations (ODEs) was obtained by using homotopy analysis method.

The equations governing the model [18] are

$$\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\
&u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{dp}{dx}+\frac{1}{R_e}\frac{\partial^2 u}{\partial y^2}+2\frac{R_e}{W_e}\left(\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial y^2},\\
&u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=\gamma\frac{\partial^2 C}{\partial y^2},
\end{aligned} \tag{118}$$

where W_e is the Weissenberg number and R_e is a Reynolds number. The boundary conditions for the problem are

$$\begin{aligned}
u(x,0)&=0,\\
v(x,0)&=0,\\
u(x,\infty)&=U(x),\\
C(x,0)&=1,\\
C(x,\infty)&=0.
\end{aligned} \tag{119}$$

The Lie point symmetries of the system of PDEs (118) are [18]

$$\begin{aligned}
\xi_1&=3ax+b,\\
\xi_2&=ay,\\
\eta_1&=au,\\
\eta_2&=-av,\\
\eta_3&=aU,\\
\eta_4&=aC.
\end{aligned} \tag{120}$$

The corresponding similarity transformations are

$$\begin{aligned}
\eta&=\frac{y}{x^{1/3}},\\
u&=x^{1/3}f(\eta),\\
v&=\frac{g(\eta)}{x^{1/3}},\\
U&=x^{1/3},\\
C&=\phi(\eta).
\end{aligned} \tag{121}$$

The similarity transformations (121) are utilized in [18] to reduce the above PDE system into a system of nonlinear ordinary differential equations. The reduced ordinary differential system was solved analytically by homotopy analysis method.

7. Second-Grade Fluid Flow Problems

In this section, we present the studies related to flow of a second-grade fluid model that are solved by the Lie symmetry reduction method.

The constitutive equation for an incompressible homogeneous Rivlin-Ericksen fluid of second grade is given by the following relation:

$$\mathbf{T}=-p\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2, \tag{122}$$

where p is the pressure of the fluid, $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity, and α_i $(i=1,2)$ are the material moduli and denote the first and second normal stress coefficients which are not always constants.

7.1. Lie Group Analysis of Creeping Flow of a Second-Grade Fluid [19]. Yürüsoy et al. [19] considered the steady plane creeping flow equations of a second-grade fluid in Cartesian coordinates. Lie group theory was applied to the equations of motion. The symmetries of the equations were found. Two different types of exact solutions were constructed for the model equation.

The equations governing the creeping flow of a second-grade fluid are [19]

$$\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\
&-\frac{\partial p}{\partial x}+\epsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\epsilon_1\left[5\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right.\\
&\quad+u\frac{\partial^3 u}{\partial x^3}+v\frac{\partial^3 u}{\partial y^3}+u\frac{\partial^3 u}{\partial y^2\partial x}+2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\\
&\quad\left.+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2}+v\frac{\partial^3 u}{\partial x^2\partial y}\right]=0,\\
&-\frac{\partial p}{\partial y}+\epsilon\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}\right)+\epsilon_1\left[5\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2}\right.\\
&\quad-v\frac{\partial^3 u}{\partial x\partial y^2}+u\frac{\partial^3 v}{\partial x^3}-v\frac{\partial^3 u}{\partial x^3}+2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}-\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2}\\
&\quad\left.+\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2}-u\frac{\partial^3 u}{\partial x^2\partial y}\right]=0,
\end{aligned} \tag{123}$$

where

$$\begin{aligned}
\epsilon&=\frac{1}{R_e}=\frac{\mu}{\rho UL},\\
\epsilon_1&=\frac{\alpha_1}{\rho L^2}.
\end{aligned} \tag{124}$$

The Lie point symmetries of the above system of PDEs (123) are [19]

$$\begin{aligned}
\xi_1&=ax+b,\\
\xi_2&=ay+c,\\
\eta_1&=au,\\
\eta_2&=av,\\
\eta_3&=d.
\end{aligned} \tag{125}$$

With the use of Lie point symmetries (125), the group invariant solutions for the system of PDEs (123) are [19]

$$\begin{aligned} u(x,y) &= c_1\left(\frac{\alpha\epsilon_1}{\epsilon}\right)^2 \exp\left[-\left(\frac{\epsilon}{\alpha\epsilon_1}\right)(y-mx)\right] \\ &\quad + c_2(y-mx)+c_3, \\ v(x,y) &= m\left[c_1\left(\frac{\alpha\epsilon_1}{\epsilon}\right)^2 \exp\left[-\left(\frac{\epsilon}{\alpha\epsilon_1}\right)(y-mx)\right]\right. \\ &\quad \left. + c_2(y-mx)+c_3\right]+\alpha, \\ p(x,y) &= 2\left(1+m^2\right)^2\epsilon_1 c_1\left(\frac{\alpha\epsilon_1}{\epsilon}\right) \\ &\quad \cdot\left(\begin{array}{c}\frac{1}{2}\frac{\alpha\epsilon_1}{\epsilon}c_1\exp\left[-\left(\frac{\epsilon}{\alpha\epsilon_1}\right)(y-mx)\right] \\ -c_2\exp\left[-\left(\frac{\epsilon}{\alpha\epsilon_1}\right)(y-mx)\right]\end{array}\right)+c_4. \end{aligned} \tag{126}$$

7.2. Similarity Solutions for Creeping Flow and Heat Transfer in Second-Grade Fluids [20]. The steady plane creeping flow and heat transfer equations of a second-grade fluid in Cartesian coordinates are modelled by Yürüsoy [20]. Lie group theory was employed for the equations of motion. The symmetries of the equations were deduced. The equations admit a scaling symmetry, translation symmetries, and an infinite parameter dependent symmetry. New exact analytical solutions are found for the model equations.

The equations of the flow model [20] are

$$\begin{aligned} 0 &= \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}, \\ 0 &= -\frac{\partial p}{\partial x}+\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\delta\left[5\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right. \\ &\quad + u\frac{\partial^3 u}{\partial x^3}+v\frac{\partial^3 u}{\partial y^3}+u\frac{\partial^3 u}{\partial y^2\partial x}+2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} \\ &\quad \left.+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2}+v\frac{\partial^3 u}{\partial x^2\partial y}\right], \\ 0 &= -\frac{\partial p}{\partial y}+\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}\right)+\delta\left[5\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}\right. \\ &\quad -\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2}-v\frac{\partial^3 u}{\partial x\partial y^2}+u\frac{\partial^3 v}{\partial x^3}-v\frac{\partial^3 u}{\partial x^3}+2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} \\ &\quad \left.-\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2}-u\frac{\partial^3 u}{\partial x^2\partial y}\right], \\ 0 &= \frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\delta_1^*\left[4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right] \\ &\quad +\delta_2^*\left[\frac{\partial^2 u}{\partial x^2}\left(4u\frac{\partial u}{\partial x}-v\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial y}\right)\right. \\ &\quad +\frac{\partial^2 u}{\partial y^2}\left(v\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial y}\right)+\frac{\partial^2 v}{\partial x^2}\left(u\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial x}\right) \\ &\quad \left.+\frac{\partial^2 u}{\partial x\partial y}\left(4v\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial x}+u\frac{\partial u}{\partial y}\right)\right]. \end{aligned} \tag{127}$$

The infinitesimals for the system of PDEs (127) are [20]

$$\begin{aligned} \xi_1 &= ax+b, \\ \xi_2 &= ay+c, \\ \eta_1 &= au, \\ \eta_2 &= av, \\ \eta_3 &= d, \\ \eta_4 &= 2a\theta+\gamma(x,y), \quad \text{with } \frac{\partial^2\gamma}{\partial x^2}+\frac{\partial^2\gamma}{\partial y^2}=0. \end{aligned} \tag{128}$$

The invariant solutions for the system of PDEs (127) deduced in [20] are

$$\begin{aligned} u(x,y) &= (a\delta)^2\exp\left[-\frac{(y-mx)}{a\delta}\right]+c_2(y-mx) \\ &\quad +c_3, \\ v(x,y) &= m\left[c_1(a\delta)^2\exp\left[-\frac{(y-mx)}{a\delta}\right]+c_2(y\right. \\ &\quad \left.-mx)+c_3\right]+\alpha, \\ p(x,y) &= 2\left(1+m^2\right)^2\delta c_1(a\delta)\left[\frac{1}{2}(a\delta)c_1\right. \\ &\quad \left.\cdot\exp\left[-\frac{2(y-mx)}{a\delta}\right]-c_2\exp\left[-\frac{(y-mx)}{a\delta}\right]\right] \\ &\quad +c_4, \\ \theta(x,y) &= -\frac{m^4+2m^2+1}{1+m^2}\left[\delta_1^*\left\{c_1\frac{(a\delta)^4}{4}\right.\right. \\ &\quad \cdot\exp\left[-\frac{2(y-mx)}{a\delta}\right]-2c_1c_2(a\delta)^3 \\ &\quad \left.\cdot\exp\left[-\frac{(y-mx)}{a\delta}\right]+c_2^2\frac{(y-mx)^2}{2}\right\} \\ &\quad +\alpha\delta_2^*\left\{c_1c_2(a\delta)^2\exp\left[-\frac{(y-mx)}{a\delta}\right]-c_1^2\frac{(a\delta)^3}{4}\right. \\ &\quad \left.\left.\cdot\exp\left[-\frac{2(y-mx)}{a\delta}\right]\right\}\right]+c_5(y-mx)+c_6. \end{aligned} \tag{129}$$

7.3. Lie Symmetry Analysis and Some New Exact Solutions for Rotating Flow of a Second-Order Fluid on a Porous Plate [21]. The Lie symmetry analysis and the basic similarity reductions are performed for the rotating flow of a second-order fluid on a porous plate by Fakhar et al. [21]. Two new exact solutions to these equations were generated from the similarity transformations.

The equations governing the rotating flow of a second-order fluid on a porous plate are [21]

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t\partial z} - W_0\frac{\partial^2 u}{\partial z^2} - 2\Omega\frac{\partial v}{\partial z} \\
&\quad = \nu\frac{\partial^3 u}{\partial z^3} + \beta\left(\frac{\partial^4 u}{\partial t\partial z^3} - W_0\frac{\partial^4 u}{\partial z^4}\right), \\
&\frac{\partial^2 v}{\partial t\partial z} - W_0\frac{\partial^2 v}{\partial z^2} + 2\Omega\frac{\partial u}{\partial z} \\
&\quad = \nu\frac{\partial^3 v}{\partial z^3} + \beta\left(\frac{\partial^4 v}{\partial t\partial z^3} - W_0\frac{\partial^4 v}{\partial z^4}\right),
\end{aligned} \tag{130}$$

where u and v are the velocity components.

The Lie point symmetries for the above system of PDEs are spanned by the operators [21]

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= \frac{\partial}{\partial z}, \\
X_3 &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \\
X_4 &= v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \\
X_5 &= f_1(t,z)\frac{\partial}{\partial u}, \\
X_5 &= f_2(t,z)\frac{\partial}{\partial v}.
\end{aligned} \tag{131}$$

With the use of symmetries (131), two different types of exact solutions were deduced and these are given by

$$\begin{aligned}
&u(t,z) = 2ic_1 e^{\lambda_1(t-z)}\sin\left[\mu_1(t-z)\right], \\
&v(t,z) = 2ic_1' e^{\lambda_1(t-z)}\sin\left[\mu_1(t-z)\right], \\
&u(t,z) \\
&\quad = e^{t-z}\left[A_1\sin\left(z-2t-\theta_1\right) + A_0\sin\left(z-\theta_0\right)\right], \\
&v(t,z) \\
&\quad = e^{t-z}\left[-\left\{A_1\cos\left(z-2t-\theta_1\right) + A_0\cos\left(z-\theta_0\right)\right\}\right].
\end{aligned} \tag{132}$$

7.4. Some New Exact Solutions for MHD Aligned Creeping Flow and Heat Transfer in Second-Grade Fluids by Using Lie Group Analysis [22]. Afify [22] carried out the Lie group analysis and the basic similarity reductions for the MHD aligned flow and heat transfer in a second-grade fluid. Two new exact solutions were constructed for the model equations.

The equations governing the model [22] are

$$\begin{aligned}
0 &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \\
0 &= -\frac{\partial p}{\partial x} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \delta\left[5\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right. \\
&\quad + u\frac{\partial^3 u}{\partial x^3} + v\frac{\partial^3 u}{\partial y^3} + u\frac{\partial^3 u}{\partial y^2\partial x} + 2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x} \\
&\quad \left. + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2} + v\frac{\partial^3 u}{\partial x^2\partial y}\right] - \delta_3\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right), \\
0 &= -\frac{\partial p}{\partial y} + \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x\partial y}\right) + \delta\left[5\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}\right. \\
&\quad - \frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2} - v\frac{\partial^3 u}{\partial x\partial y^2} + u\frac{\partial^3 v}{\partial x^3} - v\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} \\
&\quad \left. - \frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2} - u\frac{\partial^3 u}{\partial x^2\partial y}\right] - \delta_3 H_1\left(\frac{\partial H_2}{\partial x}\right. \\
&\quad \left. - \frac{\partial H_1}{\partial y}\right), \\
0 &= \frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y}, \\
0 &= \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \delta_1^*\left[4\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right] \\
&\quad + \delta_2^*\left[\frac{\partial^2 u}{\partial x^2}\left(4u\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial y}\right)\right. \\
&\quad + \frac{\partial^2 u}{\partial y^2}\left(v\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial^2 v}{\partial x^2}\left(u\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial x}\right) \\
&\quad \left. + \frac{\partial^2 u}{\partial x\partial y}\left(4v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} + u\frac{\partial u}{\partial y}\right)\right], \\
&\frac{\partial^3 H_1}{\partial x^2\partial y} + \frac{\partial^3 H_1}{\partial y^3} - \frac{\partial^3 H_1}{\partial x^3} - \frac{\partial^3 H_2}{\partial x\partial y^2} = \delta_4\left[v\frac{\partial H_1}{\partial x^2}\right. \\
&\quad + \frac{\partial^2 v}{\partial x^2}H_1 + v\frac{\partial^2 H_1}{\partial y^2} + \frac{\partial^2 v}{\partial y^2}H_1 - u\frac{\partial H_2}{\partial x^2} - \frac{\partial^2 u}{\partial x^2}H_2 \\
&\quad \left. - u\frac{\partial^2 H_2}{\partial y^2} - \frac{\partial^2 u}{\partial y^2}H_2\right].
\end{aligned} \tag{133}$$

The symmetries for the system of PDEs (133) are [22]

$$\begin{aligned}
\xi_1 &= ax + b, \\
\xi_2 &= ay + c, \\
\eta_1 &= au, \\
\eta_2 &= av, \\
\eta_3 &= d,
\end{aligned}$$

$$\eta_4 = 2a\theta + \gamma(x, y), \quad \text{with } \frac{\partial^2\gamma}{\partial x^2} + \frac{\partial^2\gamma}{\partial y^2} = 0,$$
$$\eta_5 = aH_1 + n,$$
$$\eta_6 = aH_2 + s. \tag{134}$$

With the use of the symmetries given in (134), the invariant solutions for the system of PDEs (133) are [22]

$$\begin{aligned}
u(x, y) &= a_1(y - mx) + a_2,\\
v(x, y) &= m[a_1(y - mx) + a_2],\\
H_1(x, y) &= a_3(y - mx)^2 + a_4(y - mx) + a_5,\\
H_2(x, y) &= m\left[a_3(y - mx)^2 + a_4(y - mx) + a_5\right],\\
p(x, y) &= \frac{-\delta_3}{2}\left(1 + m^2\right)\\
&\quad \cdot \left(a_3(y - mx)^2 + a_4(y - mx) + a_5\right)^2 + a_6,\\
\theta(x, y) &= -\left[\frac{\delta_1^* a_1^2\left(1 + m^2\right)(y - mx)^2}{2}\right]\\
&\quad + a_7(y - mx) + a_8.
\end{aligned} \tag{135}$$

The same model was investigated again very recently by Khan et al. [23]. The travelling wave symmetry reduction was performed in [23] to reduce the governing system of PDEs (133) and thereafter the same family of exact solutions was found as in [22].

7.5. Symmetry Analysis for Steady Boundary Layer Stagnation-Point Flow of Rivlin-Ericksen Fluid of Second-Grade Subject to Suction [24]. Abd-el-Malek and Hassan [24] studied the steady two-dimensional boundary layer stagnation-point flow of Rivlin-Ericksen fluid of second grade with a uniform suction that is carried out via symmetry approach. By using the Lie group method for the given system of nonlinear partial differential equations, the symmetries of the equations were obtained. Using these symmetries, the solution of the given equations was constructed.

The dimensionless form of the governing equations is [24]

$$\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\\
&u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\\
&\quad = U\frac{dU}{dx} + \frac{\partial^2 u}{\partial y^2}\\
&\qquad + k\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3}\right],\\
&u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{P_r}\frac{\partial^2 T}{\partial y^2}.
\end{aligned} \tag{136}$$

The relevant boundary conditions are

$$\begin{aligned}
&u = 0,\\
&v = \frac{-v_0}{\sqrt{a\nu}},\\
&T = 1\\
&\qquad\qquad \text{at } y = 0,\\
&u \longrightarrow x,\\
&\frac{\partial u}{\partial y} = 0,\\
&T \longrightarrow 0\\
&\qquad\qquad \text{as } y \longrightarrow \infty.
\end{aligned} \tag{137}$$

The system of nonlinear partial differential equations (136) has the three-parameter Lie group of point symmetries generated by [24]

$$\begin{aligned}
X_1 &= x\frac{\partial}{\partial x} + \Psi\frac{\partial}{\partial \Psi} + U\frac{\partial}{\partial U},\\
X_2 &= \frac{\partial}{\partial \Psi},\\
X_3 &= \frac{\partial}{\partial y},
\end{aligned} \tag{138}$$

where Ψ denotes the stream function. The symmetries given in (138) are used to reduce the nonlinear system of PDEs (136) to a nonlinear system of ODEs. The resulting system of nonlinear differential equations was solved numerically using a shooting method coupled with a Runge-Kutta scheme.

7.6. Application of the Lie Groups of Transformations for an Approximate Solution of MHD Flow of a Second-Grade Fluid [25]. Islam et al. [25] investigated the problem of steady boundary layer flow of a viscous incompressible electrically conducting second-grade fluid over a stretching sheet. The Lie symmetry method was utilized to reduce the governing partial differential equation into an ordinary differential equation and then numerical solutions were obtained.

The governing equations of the model [25] are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{139}$$

$$\begin{aligned}
&u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\\
&\quad = \frac{\partial^2 u}{\partial y^2} - k\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3}\right]\\
&\qquad - Mu,
\end{aligned} \tag{140}$$

where M is known as a Hartman number. The imposed boundary conditions are

$$\begin{aligned} u &= 0, \\ v &= 0, \\ &\quad \text{at } y = 0, \\ u &\longrightarrow 0, \quad \text{as } y \longrightarrow \infty. \end{aligned} \tag{141}$$

The generator of the one-parameter infinitesimal Lie group of point transformations found in [25] is

$$X_1 = x\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial \psi}, \tag{142}$$

where ψ is the stream function. The Lie point symmetry has been utilized to reduce the above nonlinear system of PDEs to a nonlinear boundary value problem. The resulting nonlinear boundary value problem was solved numerically.

8. Modified Second-Grade Fluid Flow Problems

In this section, we analyze the problems dealing with the flow of a modified second-grade fluid which are solved using the Lie symmetry approach.

The Cauchy stress tensor for a modified second-grade fluid model is given by

$$\mathbf{T} = -p\mathbf{I} + \mu\Pi^{m/2}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{143}$$

where p is the pressure, $\mathbf{I}$ is the identity matrix, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first and second Rivlin-Ericksen tensors, respectively, and μ, m, α_1, and α_2 are material moduli that may be constants or temperature dependent. For both the models, when $m = 0$, $\alpha_1 = \alpha_2 = 0$, the fluid is Newtonian and hence μ represents the usual viscosity. The situation when $m = 0$ corresponds to the second-grade fluid and that of $\alpha_1 = \alpha_2 = 0$ to the power-law fluid. The tensor Π is defined as

$$\Pi = \frac{1}{2}\operatorname{tr}\mathbf{A}_1^2. \tag{144}$$

8.1. Boundary Layer Equations and Stretching Sheet Solutions for the Modified Second-Grade Fluid [26]. The boundary layer equations for a modified second-grade fluid model were derived by Aksoy et al. [26]. The symmetries of the boundary layer equations were calculated using the Lie group approach. With the use of one of the symmetries, the partial differential system was transferred to an ordinary differential system. The ordinary differential equations were then numerically integrated for the stretching sheet boundary conditions.

The boundary layer equations for a modified second-grade fluid model are [26]

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\ &\quad = U\frac{dU}{dx} + (m+1)\left(\frac{\partial u}{\partial y}\right)^m \frac{\partial^2 u}{\partial y^2} \\ &\qquad + k_1\left[v\frac{\partial^3 u}{\partial y^3} + u\frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}\right], \end{aligned} \tag{145}$$

For $m = 0$, the equations represent the boundary layers of a standard second-grade fluid and for $k_1 = 0$, the equations represent the boundary layers of a power-law fluid.

The Lie point symmetries for the above system of PDEs are spanned by the operators [26]

$$\begin{aligned} \xi_1 &= (1-m)\,ax + b, \\ \xi_2 &= c(x), \\ \eta_1 &= au, \\ \eta_2 &= c'(x)\,u + amv. \end{aligned} \tag{146}$$

The symmetry operators (146) are used to reduce the above PDE system into a system of nonlinear ODEs. The reduced ordinary differential systems were solved by using a numerical approach for the specific boundary conditions.

9. Power-Law Second-Grade Fluid Flow Problems

Here we discuss the problems dealing with the flow of a power-law second-grade fluid model solved using the Lie symmetry method.

The Cauchy stress tensor for a modified second-grade fluid model is given by

$$\mathbf{T} = -p\mathbf{I} + \Pi^{m/2}\left(\mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2\right), \tag{147}$$

where p is the pressure, $\mathbf{I}$ is the identity matrix, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first and second Rivlin-Ericksen tensors, respectively, and μ, m, α_1, and α_2 are the material constants.

9.1. Symmetries of Boundary Layer Equations of Power-Law Fluids of Second Grade [27]. The only model available dealing with the flow of a power-law second-grade fluid and solved by employing the Lie group method was investigated by Pakdemirli et al. [27]. They derived the boundary layer equations for an incompressible power-law second-grade fluid. Symmetries of the boundary layer equations are found by using Lie theory. By using one of the symmetries, namely, the scaling symmetry, the partial differential system was transformed into an ordinary differential system, which was then numerically integrated under the classical boundary layer conditions.

The equations governing the flow model [27] are

$$\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\
&u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+(m+1)\left(\frac{\partial u}{\partial y}\right)^{m}\frac{\partial^2 u}{\partial y^2}\\
&\quad+k_1\left(\frac{\partial u}{\partial y}\right)^{m-1}\left\{m\left[\frac{\partial^2 u}{\partial y^2}\left(u\frac{\partial u}{\partial x\partial y}+v\frac{\partial^2 u}{\partial y^2}\right.\right.\right.\\
&\quad\left.\left.+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}-2\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial x\partial y}\right)\right]+\frac{\partial u}{\partial y}\left(v\frac{\partial^3 u}{\partial y^3}\right.\\
&\quad\left.\left.+u\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right)\right\}.
\end{aligned}\tag{148}$$

For $m=0$, the equations represent the boundary layers of a second-grade fluid and for $k_1=0$, the equations represent the boundary layers of a power-law fluid.

The symmetries for the system of PDEs (148) are [27]

$$\begin{aligned}
\xi_1&=\frac{m+2}{m}a_1x+a_3,\\
\xi_2&=a_1y+a_2(x),\\
\eta_1&=\frac{m+2}{m}a_1u,\\
\eta_2&=a_2'(x)u+a_1v.
\end{aligned}\tag{149}$$

The infinitesimals given in (149) are used to reduce the above system of PDEs into a system of nonlinear ODEs. The reduced ordinary differential system was solved by using a numerical approach for the classical boundary conditions.

10. Maxwell Fluid Flow Problems

In this section, we present those problems which are related to flow of a Maxwell fluid and solved by using the Lie group method.

The Cauchy stress tensor $\mathbf{T}$ in an incompressible Maxwell fluid is given by

$$\begin{aligned}
\mathbf{T}&=-p\mathbf{I}+\mathbf{S},\\
\mathbf{S}+\lambda\left(\dot{\mathbf{S}}-\mathbf{LS}-\mathbf{SL}^T\right)&=\mu\mathbf{A},
\end{aligned}\tag{150}$$

where $-p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra stress tensor, $\mathbf{A}$ is the first Rivlin-Ericksen tensor, $\mathbf{L}$ is the velocity gradient, μ is the dynamic viscosity, λ is the relaxation time, and the dot denotes the material time differentiation.

10.1. Lie Group Analysis and Similarity Solutions for Hydromagnetic Maxwell Fluid through a Porous Medium [28]. The only model available dealing with the flow of a Maxwell fluid and solved by employing the Lie group approach was studied by Mekheimer et al. [28]. The equations of a two-dimensional incompressible fluid flow for hydromagnetic Maxwell fluid through a porous medium were investigated in [28]. Lie group analysis was employed and group invariant solutions were obtained.

The equations governing the two-dimensional motion of an incompressible hydromagnetic Maxwell fluid through a porous medium are written as [28]

$$\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\
&\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]+\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x}\\
&\quad-\frac{1}{R}\nabla^2u+\left(1+\lambda\frac{\partial}{\partial t}\right)[M\theta(u-mv)]+\frac{1}{Rk}u=0,\\
&\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial x}\right]+\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial y}\\
&\quad-\frac{1}{R}\nabla^2v+\left(1+\lambda\frac{\partial}{\partial t}\right)[M\theta(v+mu)]+\frac{1}{Rk}v=0,
\end{aligned}\tag{151}$$

where $R=\rho LU/\mu$ is the Reynolds number, $M=\sigma B_0^2/\rho U$ is the Hartmann number, and L, U are the dimensionless length and velocity, respectively.

The infinitesimals for the system of PDEs (151) calculated in [28] are

$$\begin{aligned}
\xi_1&=a_2-a_1y,\\
\xi_2&=a_3+a_1x,\\
\xi_3&=a_4,\\
\eta_1&=-a_1v,\\
\eta_2&=a_2u,\\
\eta_2&=\delta(t),
\end{aligned}\tag{152}$$

where a_i $(i=1,\ldots,5)$ are arbitrary constants and $\delta(t)$ is an arbitrary function of the variable t only.

With the use of symmetries given in (152), the group invariant solutions for the above system of PDEs with suitable choice of boundary conditions found in [28] are

$$\begin{aligned}
u(x,y,t)&=\frac{U_0}{(\alpha_2-\alpha_1)}\left[\alpha_2\exp\left[\alpha_1(y+Wt)\right]\right.\\
&\quad\left.-\alpha_1\exp\left[\alpha_2(y+Wt)\right]\right],\\
v(x,y,t)&=\frac{V_0}{(\alpha_2-\alpha_1)}\left[\alpha_2\exp\left[\alpha_1(y+Wt)\right]\right.\\
&\quad\left.-\alpha_1\exp\left[\alpha_2(y+Wt)\right]\right],\\
p(x,y,t)&=\frac{mM\theta U_0}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}\left[-\alpha_1^2\exp\left[\alpha_2(y+Wt)\right]\right.\\
&\quad\left.+\alpha_2^2\exp\left[\alpha_1(y+Wt)\right]\right]-\left[p_0-p_a\right.\\
&\quad\left.+\frac{mM\theta U_0}{\alpha_1\alpha_2}(\alpha_1+\alpha_2)\right]\exp\left[-\frac{(y+Wt)}{C\lambda}\right]+P_0.
\end{aligned}\tag{153}$$

11. Micropolar Fluid Flow Problems

In this section, we recall the flow problems of micropolar fluid solved by the Lie group approach.

The field equations of the micropolar fluid dynamics are

$$\begin{aligned} &\operatorname{div}\mathbf{V} = 0, \\ &\rho\frac{d\mathbf{V}}{dt} = -\operatorname{grad} p - k\operatorname{curl}\mathbf{v} - (\mu + k)\operatorname{curl}\operatorname{curl}\mathbf{v}, \\ &\rho j\frac{\partial\mathbf{v}}{\partial t} = 2k\nu + k\operatorname{curl}\mathbf{V} - \gamma\operatorname{curl}\operatorname{curl}\mathbf{v} \\ &\qquad + (\alpha + \beta + \gamma)\operatorname{grad}(\operatorname{div}\mathbf{v}), \end{aligned} \tag{154}$$

where $\mathbf{V}$ is the velocity vector, $\mathbf{v}$ is the microrotation vector, p is the pressure of the fluid, ρ and j are the fluid density and the microgyration parameter, and (μ, k) and (α, β, γ) are the viscosity and gyroviscosity coefficients, respectively.

The stress tensor τ_{ij} and the couple stress tensor M_{ij} are given by

$$\begin{aligned} \tau_{ij} &= -p\delta_{ij} + (2\mu + k)e_{ij} + k\varepsilon_{ijm}(\varpi_m - \nu_m), \\ M_{ij} &= \alpha\nu_{k,k}\delta_{ij} + \beta\nu_{i,j} + \gamma\nu_{j,i}, \end{aligned} \tag{155}$$

where ϖ is the vorticity vector, δ_{ij} is the Kronecker delta, and ε_{ijm} is the alternating symbol.

11.1. Symmetries and Solution of a Micropolar Fluid Flow through a Cylinder [29]. Calmelet-Eluhu and Rosenhaus [29] considered the system of equations of motion for a micropolar fluid inside a cylinder. Classical Lie symmetries of the system of equations are studied and various classes of invariant solutions corresponding to different symmetry subgroups were obtained.

The equations governing the flow of a micropolar fluid inside a cylinder subject to longitudinal and rotational motion are [29]

$$\begin{aligned} &\frac{1}{r}U_\theta^2 = P_r, \\ &U_{\theta,t} = -KW_{z,r} + (1+K)\left(U_{\theta,rr} + \frac{1}{r}U_{\theta,r} - \frac{1}{r^2}U_\theta\right), \\ &U_{z,t} = K\left(\frac{1}{r}W_\theta + W_{\theta,r}\right) + (1+K)\left(U_{z,rr} + \frac{1}{r}U_{z,r}\right), \\ &JW_{r,t} - J\frac{U_\theta W_\theta}{r} \\ &\quad = -2KW_r + C\left(W_{r,rr} + \frac{1}{r}W_{r,r} - \frac{1}{r^2}W_r\right), \\ &JW_{\theta,t} + J\frac{U_\theta W_r}{r} \\ &\quad = -2KW_\theta - KU_{r,z} + \Gamma\left(W_{\theta,rr} + \frac{1}{r}W_{\theta,r} - \frac{1}{r^2}W_\theta\right), \\ &JW_{z,t} \\ &\quad = -2KW_z + K\left(U_{\theta,r} + \frac{1}{r}U_\theta\right) \\ &\qquad + \Gamma\left(W_{z,rr} + \frac{1}{r}W_{z,r}\right), \end{aligned} \tag{156}$$

where the dimensionless micropolar parameters are

$$\begin{aligned} G &= \frac{\alpha + \beta}{\mu R^2}, \\ J &= \frac{j}{R^2}, \\ K &= \frac{k}{\mu}, \\ \Gamma &= \frac{\gamma}{\mu R^2}, \\ C &= \Gamma + G. \end{aligned} \tag{157}$$

The system of PDEs (156)-(157) is invariant under the algebra of the following operators [29]:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= W_r\frac{\partial}{\partial W_r} + W_\theta\frac{\partial}{\partial W_\theta} + U_z\frac{\partial}{\partial U_z}. \end{aligned} \tag{158}$$

The symmetries given in (158) have been employed to construct the group invariant solutions given by [29]

$$\begin{aligned} W_r &= \sum_{n=1}^{\infty}\frac{a_n l_n}{b_n}J_1(p_n r)e^{-\lambda_n t}, \\ W_\theta &= \sum_{n=1}^{\infty}a_n l_n J_1(p_n r)e^{-\lambda_n t}, \\ U_z &= V + \sum_{n=1}^{\infty}l_n\frac{J_0(p_n) - J_0(p_n r)}{p_n}e^{-\lambda_n t}, \\ U_\theta &= wr, \\ W_z &= w\left(1 - e^{(-2K/r)t}\right). \end{aligned} \tag{159}$$

11.2. Analytic Solution for Flow of a Micropolar Fluid [30]. Shahzad et al. [30] discussed the time-independent equations for the two-dimensional incompressible micropolar fluid. Using group methods, the equations are reduced to ordinary differential equations and then solved analytically.

The governing problem of the flow model [30] is

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (\epsilon_1 + \epsilon_2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \epsilon_3\frac{\partial\sigma}{\partial y} \\ &\qquad - \epsilon_4\frac{\partial p}{\partial x}, \end{aligned}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=(\epsilon_1+\epsilon_2)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\epsilon_3\frac{\partial \sigma}{\partial x}-\epsilon_4\frac{\partial p}{\partial y},$$
$$u\frac{\partial \sigma}{\partial x}+v\frac{\partial \sigma}{\partial y}=\epsilon_5\left(\frac{\partial^2 \sigma}{\partial x^2}+\frac{\partial^2 \sigma}{\partial y^2}\right)-\epsilon_6\sigma+\epsilon_7\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right), \tag{160}$$

with

$$\begin{aligned} u(x,0)&=U_0,\\ u(x,\infty)&=0,\\ \frac{\partial u}{\partial x}(0,y)&=0,\\ v(x,0)&=-V_0,\\ \sigma(x,0)&=0,\\ \sigma(x,\infty)&=0,\\ p(x,\infty)&=p_0, \end{aligned} \tag{161}$$

where U_0 is the velocity of the plate, V_0 is the magnetic fluid penetrating into the plate, and p_0 is the pressure deep in the magmatic fluid. The infinitesimals for the system of PDEs (160) are [26]

$$\begin{aligned} \xi_1&=b,\\ \xi_2&=c,\\ \eta_1&=0,\\ \eta_2&=0,\\ \eta_3&=d,\\ \eta_2&=e. \end{aligned} \tag{162}$$

The infinitesimals given in (162) have been used to construct the group invariant solutions given as [30]

$$\begin{aligned} u(x,y)&=\frac{-U_0}{(\gamma_2-\gamma_1)}\left(\gamma_1 e^{-\alpha y}-\gamma_2 e^{-\beta y}\right),\\ v(x,y)&=m\left[\frac{-U_0}{(\gamma_2-\gamma_1)}\left(\gamma_1 e^{-\alpha y}-\gamma_2 e^{-\beta y}\right)\right]-mU_0-V_0,\\ \sigma(x,y)&=\frac{U_0}{(\gamma_2-\gamma_1)}\left(e^{-\alpha y}-e^{-\beta y}\right),\\ p(x,y)&=p_0, \end{aligned} \tag{163}$$

where

$$\begin{aligned} \gamma_1&=\frac{\epsilon_5\left(1+m^2\right)\alpha^2-\epsilon_6+C_1\alpha}{-\epsilon_7\left(1+m^2\right)\alpha},\\ \gamma_2&=\frac{\epsilon_5\left(1+m^2\right)\beta^2-\epsilon_6+C_1\beta}{-\epsilon_7\left(1+m^2\right)\beta}. \end{aligned} \tag{164}$$

11.3. Lie Group Analysis of Unsteady MHD Mixed Convection Boundary Layer Flow of a Micropolar Fluid along a Symmetric Wedge with Variable Surface Temperature Saturated Porous Medium [31]. The problem of unsteady mixed convection along a symmetric wedge in the presence of magnetic field was investigated by Mansour et al. [31]. Lie group theory was employed to reduce the governing system of nonlinear partial differential equations. The family of reduced ordinary differential equations was solved numerically using a fourth-order Runge-Kutta algorithm with a shooting technique.

The equations governing the unsteady MHD mixed convection boundary layer flow of a micropolar fluid along a symmetric wedge with variable surface temperature saturated porous medium are [31]

$$\begin{aligned} &\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\ &\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=mx^{2m-1}+(1+\Delta)\frac{\partial^2 u}{\partial y^2}+\Omega\theta+\Delta\frac{\partial N}{\partial y}-\left[M+\frac{1}{Da}\right](u-x^m),\\ &\frac{\partial N}{\partial t}+u\frac{\partial N}{\partial x}+v\frac{\partial N}{\partial y}=\lambda\frac{\partial^2 N}{\partial y^2}-\Delta B\left[\frac{\partial u}{\partial y}+2N\right],\\ &\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{P_r}\frac{\partial^2 \theta}{\partial y^2}. \end{aligned} \tag{165}$$

In the above equations, P_r is the Prandtl number, Ω is the mixed convection parameter, λ is the microrotation parameter, M is the magnetic parameter, Δ is the vortex-viscosity parameter, and Da is the permeability parameter.

The relevant boundary conditions are

$$\begin{aligned} &u=v=0,\\ &N=-n\frac{\partial u}{\partial y},\\ &T=x^{2m-1}\\ &\qquad\text{at } y=0,\\ &u=x^m,\\ &N\longrightarrow 0,\\ &T\longrightarrow T_\infty\\ &\qquad\text{as } y\longrightarrow\infty. \end{aligned} \tag{166}$$

The system of PDEs (165) is invariant under the algebra spanned by the following generators [31]:

$$\begin{aligned}
X_1 &= x\frac{\partial}{\partial x},\\
X_2 &= x\frac{\partial}{\partial y} + u\frac{\partial}{\partial v},\\
X_3 &= y\frac{\partial}{\partial y},\\
X_4 &= t\frac{\partial}{\partial t},\\
X_5 &= \frac{\partial}{\partial t},\\
X_6 &= u\frac{\partial}{\partial u},\\
X_7 &= v\frac{\partial}{\partial v},\\
X_8 &= N\frac{\partial}{\partial N},\\
X_9 &= \theta\frac{\partial}{\partial \theta},\\
X_{h_1(t)} &= h_1(t)\frac{\partial}{\partial x} + h_1'(t)\frac{\partial}{\partial u}.
\end{aligned} \tag{167}$$

The operators given in (167) are used to reduce the above system of PDEs into a system of nonlinear ODEs. The reduced ordinary differential system was solved numerically using a fourth-order Runge-Kutta algorithm and the shooting technique.

11.4. Group Properties and Invariant Solutions of Plane Micropolar Flows [32]. The study dealing with the plane micropolar flows from the viewpoint of Lie groups was made by Saccomandi [32]. An exact steady state solution for the plane flow of an incompressible micropolar fluid was deduced by employing the theory of Lie groups.

The equations governing the motion of an incompressible two-dimensional micropolar fluid are

$$\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\\
&\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - (\mu + k)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\\
&\quad - k\frac{\partial b}{\partial y} + \frac{\partial p}{\partial x} = 0,\\
&\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) - (\mu + k)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\\
&\quad + k\frac{\partial b}{\partial x} + \frac{\partial p}{\partial y} = 0,\\
&pI\left(\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}\right) + 2kb - k\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\\
&\quad - \alpha\left(\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2}\right) = 0.
\end{aligned} \tag{168}$$

The infinitesimals for the system of PDEs (168) are [32]

$$\begin{aligned}
\xi_1 &= \alpha,\\
\xi_2 &= -\gamma y + f(t),\\
\xi_3 &= \gamma x + g(t),\\
\eta_1 &= -\gamma v + f'(t),\\
\eta_2 &= \gamma u + g'(t),\\
\eta_3 &= j(t) - xf''(t) - yg''(t),\\
\eta_4 &= \lambda \exp\left(-\frac{2k}{pI}t\right),
\end{aligned} \tag{169}$$

where , γ, and λ are three arbitrary parameters and $f(t)$, $g(t)$, and $j(t)$ are arbitrary smooth functions of t.

With the use of symmetries given in (169), the group invariant solutions for the above system of PDEs are [32]

$$\begin{aligned}
u &= f(t) - (y - G(t))\left(c_1\varsigma^{-1} + c_7\ln(\varsigma) + c_8\right),\\
v &= g(t) + (x - F(t))\left(c_1\varsigma^{-1} + c_7\ln(\varsigma) + c_8\right),\\
p &= p\Big(-xf'(t) - yg'(t) - \frac{1}{2}c_1\varsigma^{-1}\\
&\quad + \left(c_7^2 - c_7c_8 + \frac{1}{2}c_8^2\right)\varsigma + c_1c_6\ln(\varsigma)\\
&\quad - \left(c_7^2 - c_7c_8\right)\varsigma\ln(\varsigma) + \frac{1}{2}c_1c_7[\ln(\varsigma)]^2\\
&\quad + \frac{1}{2}c_7^2\varsigma[\ln(\varsigma)]^2\Big) - 4(\mu + k)\tan^{-1}\left[\frac{x - F(t)}{y - G(t)}\right]\\
&\quad + 2kc_5\left(\tan^{-1}\left[\frac{x - F(t)}{y - G(t)}\right] + \tan^{-1}\left[\frac{y - G(t)}{x - F(t)}\right]\right)\\
&\quad + K(t),
\end{aligned} \tag{170}$$

where

$$\varsigma = [x - F(t)]^2 + [y - G(t)]^2. \tag{171}$$

12. Eyring-Powell Fluid Flow Problems

Here, we review the flow models of Eyring-Powell fluid which are solved using the Lie symmetry method.

The stress tensor of the Eyring-Powell non-Newtonian fluid model is

$$\boldsymbol{\tau}_{ij} = \mu\frac{\partial V_i}{\partial x_j} + \frac{1}{\beta}\sinh^{-1}\left[\frac{1}{c}\left(\frac{\partial V_i}{\partial x_j}\right)\right], \tag{172}$$

where

$$\sinh^{-1}\left[\frac{1}{c}\left(\frac{\partial V_i}{\partial x_j}\right)\right] \approx \frac{1}{c}\left(\frac{\partial V_i}{\partial x_j}\right) - \frac{1}{6}\left(\frac{1}{c}\frac{\partial V_i}{\partial x_j}\right)^3; \quad \left|\frac{1}{c}\frac{\partial V_i}{\partial x_j}\right| \ll 1. \tag{173}$$

In the above equations, μ is the dynamic viscosity, x_j is the number of space variables on which the velocity components depend, and β and c are the Eyring-Powell fluid parameters.

12.1. Similarity Solutions for Boundary Layer Equations of a Powell-Eyring Fluid [33]. A study is available in the literature dealing with the flow of Eyring-Powell fluid and solved by employing the Lie symmetry approach which is very recent by Hayat et al. [33]. They derived the boundary layer equations for the first time for the Eyring-Powell fluid model. Using a scaling symmetry of the equations, the partial differential system was transformed to an ordinary differential system. The resulting equations were numerically solved using a finite difference algorithm.

The dimensionless form of the boundary layer equations for Eyring-Powell fluid is [33]

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + (\varepsilon_1 + \varepsilon_2\varepsilon_3)\frac{\partial^2 u}{\partial y^2} \\ &\qquad - \frac{1}{2}\varepsilon_2\varepsilon_3^2\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}. \end{aligned} \tag{174}$$

The classical boundary conditions for the problem are

$$\begin{aligned} u(x,0) &= 0, \\ v(x,0) &= 0, \\ u(x,\infty) &= U(x). \end{aligned} \tag{175}$$

By the use of the Lie group method, the similarity transformation used for the reduction of the above system of PDEs is [33]

$$\begin{aligned} \xi &= \frac{y}{x^{1/3}}, \\ u &= x^{1/3} f(\xi), \\ v &= \frac{g(\xi)}{x^{1/3}}, \\ U &= x^{1/3}. \end{aligned} \tag{176}$$

The transformation given in (176) transforms the two-dimensional unsteady boundary layer equation problem of Eyring-Powell fluid to ordinary differential equations. The reduced ordinary differential equations were solved numerically by using a finite difference method.

12.2. Flow and Heat Transfer of Powell-Eyring Fluid over a Stretching Surface: A Lie Group Analysis [34]. The flow and heat transfer analysis of Powell-Eyring fluid over a permeable stretching surface was studied by Jalil and Asghar [34]. By using the Lie group analysis, the symmetries of the equations were obtained. Four finite parameter and one infinite parameter Lie group of transformations were found. Similarity transformations for the model were derived with the help of these symmetries. The governing system of partial differential equations was transformed to a system of ordinary differential equations by using the similarity transformations. The reduced equations were solved numerically using the Keller-box method.

The two-dimensional laminar flow of a steady, incompressible Powell-Eyring fluid over a semi-infinite surface stretching is governed by [34]

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (1+\varepsilon)\frac{\partial^2 u}{\partial y^2} - \varepsilon\delta\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}, \\ &u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{P_r}\frac{\partial^2 T}{\partial y^2}. \end{aligned} \tag{177}$$

The corresponding boundary conditions are

$$\begin{aligned} u &= u_w(x), \\ v &= v_w(x), \\ T &= T_w(x) \\ &\qquad \text{at } y = 0, \\ u &= 0, \\ T &= 0 \\ &\qquad \text{as } y \longrightarrow \infty. \end{aligned} \tag{178}$$

The infinitesimals for the system of PDEs (177) are [34]

$$\begin{aligned} \xi_1 &= a + bx, \\ \xi_2 &= \frac{b}{3}y + \gamma(x), \\ \eta_1 &= \frac{b}{3}u, \\ \eta_2 &= -\frac{b}{3}v + u\gamma'(x), \\ \eta_3 &= c + mT. \end{aligned} \tag{179}$$

With the use of the symmetries given in (179), the similarity transformations used for the reduction of above system of PDEs are [34]

$$\begin{aligned} \eta &= \frac{yx^{-1/3}}{(a+bx)^{1/3}}, \\ u &= u_w(x) f'(\eta), \\ v &= v_w(x) h(\eta), \\ T &= T_w(x) \theta(\eta). \end{aligned} \tag{180}$$

The similarity transformation given in (180) transforms the above system of PDEs into nonlinear system of ODEs. The reduced ordinary differential equations were solved numerically by using a Keller-box method.

12.3. Self-Similar Solutions for the Flow and Heat Transfer of Powell-Eyring Fluid over a Moving Surface in a Parallel Free Stream [35]. The boundary layer flow and heat transfer of Powell-Eyring fluid over a continuously moving permeable surface were studied by Jalil et al. [35]. The boundary layer equations were transformed to self-similar nonlinear ordinary differential equations using group of transformations. Numerical results of the resulting equations were obtained using the Keller-box method.

The dimensionless form of the boundary layer equations is [35]

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e \frac{du_e}{dx} + (1+\varepsilon)\frac{\partial^2 u}{\partial y^2} \\ &\qquad - \varepsilon\delta\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2}, \\ &u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{P_r}\frac{\partial^2 T}{\partial y^2}. \end{aligned} \tag{181}$$

The boundary conditions for the problem are

$$\begin{aligned} &u = \gamma u_w(x), \\ &v = s v_w(x), \\ &T = T_w(x) \\ &\qquad\qquad \text{at } y = 0, \\ &u = u_e(x), \\ &T = 0 \\ &\qquad\qquad \text{as } y \longrightarrow \infty, \end{aligned} \tag{182}$$

where ε and δ are the fluid parameters, γ is the velocity ratio, and s is a nondimensional constant. The symmetries for the system of PDEs (181) are [35]

$$\begin{aligned} \xi_1 &= ax, \\ \xi_2 &= \frac{a}{3}y, \\ \eta_1 &= \frac{a}{3}u, \\ \eta_2 &= -\frac{a}{3}v, \\ \eta_3 &= \frac{a}{3}u_e + mT. \end{aligned} \tag{183}$$

Using the symmetries given in (183), the transformations used for the reduction of above system of PDEs are [35]

$$\begin{aligned} \eta &= yx^{-1/3}, \\ u &= x^{1/3} f'(\eta), \\ v &= x^{-1/3} h(\eta), \\ T &= x^m \theta(\eta), \\ u_e &= kx^{1/3}. \end{aligned} \tag{184}$$

The above system of PDEs is transformed into nonlinear system of ODEs under the transformation given in (184). The reduced ODEs system was solved numerically by using a Keller-box method.

13. Oldroyd-B Fluid Flow Problems

In this section, we discuss the studies related to flow of an Oldroyd-B fluid that are solved with the aid of the Lie group method.

The constitutive equation of Oldroyd-B fluid is written as

$$\mathbf{S} + \lambda_1 \frac{\delta \mathbf{S}}{\delta t} = \mu\left(1 + \lambda_1 \frac{\delta}{\delta t}\right)\mathbf{A}_1, \tag{185}$$

where μ is the viscosity and λ_1 and λ_2 are material time constants referred to the characteristic relaxation and characteristic retardation times. It is assumed that $\lambda_1 \geq \lambda_2 \geq 0$. The tensor $\mathbf{A}_1$ and L are defined as

$$\begin{aligned} \mathbf{A}_1 &= L + L^T, \\ L &= \operatorname{grad} \mathbf{V}. \end{aligned} \tag{186}$$

The operator $\delta/\delta t$ operating on any tensor $\mathbf{S}$ is defined by

$$\frac{\delta \mathbf{S}}{\delta t} = \frac{\partial \mathbf{S}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{S} - \mathbf{S}L - L^T\mathbf{S}. \tag{187}$$

13.1. Similarity Solutions and Conservation Laws for Rotating Flows of an Oldroyd-B Fluid [36]. The similarity reduction arising from the classical Lie point symmetries of the

unsteady hydromagnetic flows of a rotating Oldroyd-B fluid under influence of Hall currents is carried out by Fakhar et al. [36]. They employed different combinations of translation and rotational symmetries and obtained a class of new exact solutions under certain initial and boundary conditions.

The unsteady hydromagnetic flow of a rotating Oldroyd-B fluid under influence of a Hall current is governed by the following system [36]:

$$\begin{aligned} u_{zt} &= 2\Omega v_z - \lambda_1 \left(u_{ztt} - 2\Omega v_{zt}\right) \\ &\quad - \frac{M}{1+m^2}\left(u_z + \lambda_1 u_{zt}\right) \\ &\quad + \frac{mM}{1+m^2}\left(v_z + \lambda_1 v_{zt}\right) + \nu u_{zzz} + \nu\lambda_2 u_{zzzt}, \\ v_{zt} &= -2\Omega u_z - \lambda_1 \left(v_{ztt} + 2\Omega u_{zt}\right) \\ &\quad - \frac{M}{1+m^2}\left(v_z + \lambda_1 v_{zt}\right) \\ &\quad - \frac{mM}{1+m^2}\left(u_z + \lambda_1 u_{zt}\right) + \nu v_{zzz} + \nu\lambda_2 v_{zzzt}. \end{aligned} \tag{188}$$

The symmetry Lie algebra for the above system of PDEs is spanned by the following operators [36]:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial z}, \\ X_3 &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \\ X_4 &= v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \\ X_5\left(f_1\right) &= f_1\left(z,t\right)\frac{\partial}{\partial u}, \\ X_6\left(f_2\right) &= f_2\left(z,t\right)\frac{\partial}{\partial v}. \end{aligned} \tag{189}$$

The symmetry Lie operators given in (189) were employed to construct the group invariant solutions given by [36]

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \gamma_{01} \\ \gamma_{02} \end{pmatrix} + \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \end{pmatrix} e^{(1/b\lambda_1)(az-bt)} + \begin{bmatrix} \cos\frac{a\beta}{b}\left(z - \frac{b}{a}t\right) & -\sin\frac{a\beta}{b}\left(z - \frac{b}{a}t\right) \\ \sin\frac{a\beta}{b}\left(z - \frac{b}{a}t\right) & \cos\frac{a\beta}{b}\left(z - \frac{b}{a}t\right) \end{bmatrix} \begin{pmatrix} \gamma_{21} \\ \gamma_{22} \end{pmatrix} e^{(a\alpha/b)(z-(b/a)t)}, \tag{190}$$

where $c_j = \gamma_{j1} + i\gamma_{j2}$ $(j = 0, 1, 2)$ and

$$\begin{aligned} \lambda_1 &= \frac{1}{bk_1}, \\ b &= \frac{\alpha + i\beta}{k_2}, \\ \alpha &= \frac{M}{1+m^2}, \\ \beta &= m\alpha + 2\Omega. \end{aligned} \tag{191}$$

14. Third-Grade Fluid Flow Problems

In this section, we provide a review of those problems which are related to the flow of a third-grade fluid model.

The constitutive relation for an incompressible third-grade fluid is

$$\begin{aligned} \mathbf{T} &= -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 \\ &\quad + \beta_2\left(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1\right) + \beta_3\left(\operatorname{tr}\mathbf{A}_1^2\right)\mathbf{A}_1. \end{aligned} \tag{192}$$

In the above relation, p is the pressure, $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity, α_i $(i = 1, 2)$ and β_i $(i = 1\text{–}3)$ are the material constants, and $\mathbf{A}_i$ $(i = 1\text{–}3)$ are the Rivlin-Ericksen tensors.

Moreover, the Clausius-Duhem inequality and the result that the Helmholtz free energy is minimum in equilibrium provide the following restrictions:

$$\begin{aligned} \mu &\geq 0, \\ \alpha_1 &\geq 0, \\ \left|\alpha_1 + \alpha_2\right| &\leq \sqrt{24\mu\beta_3}, \\ \beta_1 &= \beta_2 = 0, \\ \beta_3 &\geq 0. \end{aligned} \tag{193}$$

Therefore, the constitutive relation for a thermodynamically compatible fluid of grade three becomes

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3\left(\operatorname{tr}\mathbf{A}_1^2\right)\mathbf{A}_1. \tag{194}$$

14.1. Exact Flow of a Third-Grade Fluid on a Porous Wall [37]. The flow of a third-grade fluid occupying the space over a wall was studied by Hayat et al. [37]. The governing nonlinear partial differential equation was solved analytically using the Lie group method.

The equation governing the flow model [37] is

$$\rho\left[\frac{\partial u}{\partial t}-W_0\frac{\partial u}{\partial y}\right]=\mu\frac{\partial^2 u}{\partial y^2}+\alpha_1\frac{\partial^3 u}{\partial y^2\partial t}-\alpha_1 W_0\frac{\partial^3 u}{\partial y^3}+6\beta_3\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}. \tag{195}$$

We know that from basics of Lie symmetry methods that if a differential equation is explicitly independent of any dependent or independent variable, then this differential equation remains invariant under the translation symmetry corresponding to that particular variable. The above PDE (195) admits Lie point symmetry generators, $\partial/\partial t$ (time translation) and $\partial/\partial y$ (space-translation in y). Thus, a travelling wave solution was obtained in [37] corresponding to the generator $\partial/\partial t + M\partial/\partial y$. After performing the successive symmetry reductions, the governing PDE (195) admits exact solution of the form [37]

$$\begin{aligned}u(y,t)&=\frac{1}{\Delta}2\alpha_1\left[(M+W_0)\exp\left(\frac{\Delta(Mt-y)}{2\alpha_1(M+W_0)}\right)\right]\\&\times\left[-A_1\right.\\&+\frac{\varepsilon(2\Delta+3\mu)A_1^3\exp(\Delta(Mt-y)/2\alpha_1(M+W_0))}{3\mu^2+16\alpha_1\rho(M+W_0)^2}\\&\left.-\varepsilon A_3\right],\end{aligned} \tag{196}$$

where

$$\Delta=\mu+\sqrt{\mu^2+4\alpha_1\rho(M+W_0)^2}. \tag{197}$$

The same problem was investigated later by Fakhar [38]. In [38], translational type symmetries were utilized to perform the travelling wave reduction on the governing model and the reduced model was solved using a power series method. Fakhar et al. [39] investigated the same model again. They made use of translational symmetries to reduce the governing model and presented some numerical solutions for a particular choice of boundary conditions.

14.2. Similarity Solutions of Boundary Layer Equations for a Special Non-Newtonian Fluid in a Special Coordinate System [40]. Yürüsoy [40] derived the boundary layer equations for a third-grade non-Newtonian fluid. By using Lie group methods, infinitesimal generators of the boundary layer equations were calculated. The equations were transformed into an ordinary differential system and then the numerical solutions of the resultant nonlinear differential equations were found by using a combination of a Runge-Kutta algorithm and a shooting technique.

The governing problem describing the flow model [40] is

$$\begin{aligned}&\frac{\partial W_\phi}{\partial\phi}+\frac{\partial W_\psi}{\partial\psi}=0,\\&W_\phi\frac{\partial W_\phi}{\partial\phi}+W_\psi\frac{\partial W_\phi}{\partial\psi}+\left(W_\phi^2-1\right)\frac{dQ_B}{d\phi}\\&\quad=\frac{\partial^2 W_\phi}{\partial\psi^2}+6k_{q_B}^4\phi\frac{\partial^2 W_\phi}{\partial\psi^2}\left(\frac{\partial W_\phi}{\partial\psi}\right)^2,\end{aligned} \tag{198}$$

with

$$\begin{aligned}&W_\phi(\phi,0)=W_\psi(\psi,0)=0,\\&W_\phi(\phi,\infty)=1,\end{aligned} \tag{199}$$

where $Q_B'=q_B'/q_B$ and k is a third-grade fluid parameter.

The infinitesimals for the system of PDEs (198) are [40]

$$\begin{aligned}\xi_1&=2a\phi+b,\\\xi_2&=a\psi,\\\eta_1&=0,\\\eta_2&=-aW_\psi'.\end{aligned} \tag{200}$$

The infinitesimals given in (200) were used to reduce the above system of PDEs to a system of nonlinear ODEs. The reduced ordinary differential system was solved by using a numerical approach for the classical boundary conditions given in (199).

14.3. Couette Flow of a Third-Grade Fluid with Variable Magnetic Field [41]. The study dealing with the analytic solution for the time-dependent flow of an incompressible third-grade fluid under the influence of a magnetic field of variable strength was made by Jalil et al. [35]. Group theoretic methods were employed to analyze the nonlinear problem and a solution for the velocity field was obtained analytically.

The governing equation of the problem [41] is

$$\frac{\partial u}{\partial t}=\frac{\nu}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\alpha}{r}\frac{\partial}{\partial r}(ru_{tr})+\frac{2\Gamma}{r}\frac{\partial}{\partial r}\left(ru_r^3\right)-H^2(t)+k, \tag{201}$$

where $\alpha=\alpha_1/\rho$, $\Gamma=\beta_3/\rho, M=\sigma\mu^2/\rho$, and ν is the kinematic viscosity.

The Lie point symmetries for the PDE (201) are [41]

$$\begin{aligned}X_1&=\frac{\partial}{\partial u},\\X_2&=\frac{\partial}{\partial t}-H^2(t)\frac{\partial}{\partial u}.\end{aligned} \tag{202}$$

With the use of the symmetries in (202), the group invariant solution for the PDE (201) was calculated in [41] and is given by

$$u(x,t) = \int \frac{54cx + 6\sqrt[2/3]{(48+81c^2x^2)} - 12}{6\left(54cx + 6\sqrt[1/3]{(48+81c^2x^2)}\right)} dx + c1 - 2ct - \int H^2(t)\,dt. \tag{203}$$

14.4. Hall Effects on Unsteady Magnetohydrodynamic Flow of a Third-Grade Fluid [42]. The model of unsteady MHD flow of an incompressible third-grade fluid bounded by an infinite porous plate in the presence of Hall current was investigated by Fakhar et al. [42]. Similarity transformations were employed to reduce the governing partial differential equation into two nonlinear ordinary differential equations. The numerical solutions of the reduced equations were presented by the use of finite difference schemes.

The equation governing the unsteady MHD flow of an incompressible electrically conducting third-grade fluid in the presence of Hall current is written as [42]

$$\frac{\partial F}{\partial t} - \frac{\partial F}{\partial y} = \frac{\partial^2 F}{\partial y^2} + \alpha\left(\frac{\partial^3 u}{\partial y^2 \partial t} - \frac{\partial^3 F}{\partial y^3}\right) + \varepsilon\left(\frac{\partial F}{\partial y}\right)^2 \frac{\partial^2 F}{\partial y^2} - \kappa F, \tag{204}$$

where $\kappa = (k/(1-i\psi))$ and $F = u + iw$.

In [42], the travelling wave symmetries were utilized to reduce the above PDE into two nonlinear ODEs. The reduced ODEs were solved for the particular choice of boundary conditions using the finite difference method.

Fakhar et al. [43] recently revisited the above problem. In [43], they performed the complete Lie group analysis of (204). The Lie point symmetries of PDE (204) found in [43] are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= e^{(-\kappa/(1+\psi^2))t}\left[e^{-i((\kappa\psi/(1+\psi^2))t)t}\frac{\partial}{\partial F} + e^{i((\kappa\psi/(1+\psi^2))t)}\frac{\partial}{\partial \overline{F}}\right], \\ X_4 &= -e^{(-\kappa/(1+\psi^2))t}\left[e^{-i((\kappa\psi/(1+\psi^2))t)}\frac{\partial}{\partial F} + e^{i((\kappa\psi/(1+\psi^2))t)}\frac{\partial}{\partial \overline{F}}\right]. \end{aligned} \tag{205}$$

The symmetry Lie generators given in (205) were used to reduce the above PDE into two nonlinear ODEs. The reduced ODEs were solved by making use of the homotopy analysis method.

14.5. Unsteady Solutions in a Third-Grade Fluid Filling the Porous Space [44]. An analysis was made of the unsteady flow of a third-grade fluid in a porous medium by Hayat et al. [44]. Reduction and exact solutions of the governing model were obtained by employing the Lie group theoretic approach.

The problem governing the model [44] is

$$\frac{\partial u}{\partial t} = \mu_* \frac{\partial^2 u}{\partial y^2} + \alpha \frac{\partial^3 u}{\partial y^2 \partial t} + \gamma_1 \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \gamma_2 u \left(\frac{\partial u}{\partial y}\right)^2 - \phi_1 u. \tag{206}$$

The relevant boundary and initial conditions are

$$\begin{aligned} u(0,t) &= u_0 V(t), \quad t > 0, \\ u(\infty,t) &= 0, \quad t > 0, \\ u(y,0) &= g(y), \quad y > 0, \end{aligned} \tag{207}$$

where

$$\begin{aligned} \mu_* &= \frac{\mu}{\rho + \alpha_1(\phi/k)}, \\ \alpha &= \frac{\alpha_1}{\rho + \alpha_1(\phi/k)}, \\ \gamma_1 &= \frac{6\beta_3}{\rho + \alpha_1(\phi/k)}, \\ \beta_* &= \frac{2\beta_3(\phi/k)}{\rho + \alpha_1(\phi/k)}, \\ \phi_1 &= \frac{\mu(\phi/k)}{\rho + \alpha_1(\phi/k)}. \end{aligned} \tag{208}$$

The complete Lie symmetry analysis of (206) resulted in [44].

Case 1 ($\phi_1 \neq \mu_*/\alpha$). For this case, a two-dimensional Lie algebra is generated by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial y}. \end{aligned} \tag{209}$$

Case 2 ($\phi_1 = \mu_*/\alpha$). Here a three-dimensional Lie algebra is generated by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= e^{2(\mu_*/\alpha)t}\frac{\partial}{\partial t} - \frac{\mu_*}{\alpha} u e^{2(\mu_*/\alpha)t}\frac{\partial}{\partial u}. \end{aligned} \tag{210}$$

With the use of the symmetries given above, two types of group invariant solutions have been found in [44] and are

$$u(y,t) = u_0 \exp\left(\frac{\sqrt{\gamma_2}(y+ct)}{-\sqrt{\gamma_1}}\right).$$
$$u(y,t) = \exp\left[-\left\{\left(\frac{\mu_*}{\alpha}\right)t + \left(\sqrt{\frac{\gamma_2}{\gamma_1}}\right)y\right\}\right]. \tag{211}$$

14.6. The Rayleigh Problem for a Third-Grade Electrically Conducting Fluid in a Magnetic Field [45]. The influence of a magnetic field on the flow of an incompressible third-grade electrically conducting fluid bounded by a rigid plate was investigated by Hayat et al. [45]. The Lie group approach was employed to perform the reduction of the model equation and thereafter numerical solutions were obtained.

The governing problem describing the flow model [45] is

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + \epsilon\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - MH^2(t)u, \tag{212}$$

with

$$u(0,t) = u_0 V(t), \quad t > 0,$$
$$u(y,t) \longrightarrow 0 \quad \text{as } y \longrightarrow \infty,\ t > 0, \tag{213}$$
$$u(y,0) = g(y), \quad y > 0.$$

The symmetry Lie algebra of the PDE (212) is spanned by the following operators [45]:

$$X_1 = \frac{\partial}{\partial y},$$
$$X_2 = e^{2\beta t}\frac{\partial}{\partial t} - \beta u e^{2\beta t}\frac{\partial}{\partial u}, \tag{214}$$
$$X_3 = L(t)\frac{\partial}{\partial u},$$

where

$$L(t) = \exp\left(-M\int_o^t H^2(s)\,ds\right). \tag{215}$$

The point symmetries given in (214) were used to reduce the PDE (212) into a nonlinear ODE. The reduced ODE was solved numerically with suitable invariant boundary conditions.

14.7. A Note on the Interplay between Symmetries, Reduction, and Conservation Laws of Stokes' First Problem for Third-Grade Rotating Fluids [46]. Fakhar et al. [46] studied the equations governing the Stokes' first problem for a third-grade rotating fluid. Lie symmetry formulation was used to perform various reductions of the governing system.

The system of equations describing the Stokes' first problem for a third-grade rotating fluid is [46]

$$u_t - 2Cv = u_{xx} + au_{txx} + 2b\frac{\partial}{\partial x}\left(u_x^3 + u_x v_x^2\right),$$
$$v_t + 2Cu = v_{xx} + av_{txx} + 2b\frac{\partial}{\partial x}\left(v_x^3 + v_x u_x^2\right). \tag{216}$$

The symmetries for the above system of PDEs were calculated in [46] and are given by

$$X_1 = \frac{\partial}{\partial x},$$
$$X_2 = \frac{\partial}{\partial t},$$
$$X_3 = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}, \tag{217}$$
$$X_4 = -\sin(2Ct)\frac{\partial}{\partial v} + \cos(2Ct)\frac{\partial}{\partial u},$$
$$X_5 = \cos(2Ct)\frac{\partial}{\partial v} + \sin(2Ct)\frac{\partial}{\partial u}.$$

The five Lie symmetries given in (217) were used to reduce the PDE system (216) to a system of ODEs.

14.8. Group Invariant Solutions for the Unsteady MHD Flow of a Third-Grade Fluid in a Porous Medium [47]. Aziz et al. [47] investigated the governing nonlinear partial differential equation for the unidirectional flow of a third-grade fluid by using the symmetry approach. Three types of analytical solutions were obtained for the governing model by employing the Lie symmetry method.

The problem governing the unsteady magnetohydrodynamic flow of a third-grade fluid in a porous medium is [47]

$$\frac{\partial u}{\partial t} = \mu_*\frac{\partial^2 u}{\partial y^2} + \alpha_*\frac{\partial^3 u}{\partial y^2 \partial t} + \beta\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}$$
$$- \beta_* u\left(\frac{\partial u}{\partial y}\right)^2 - \phi_* u - M_*^2 u. \tag{218}$$

The relevant time and space dependent velocity boundary conditions are

$$u(y,0) = g(y), \quad y > 0,$$
$$u(0,t) = V(t), \quad t > 0, \tag{219}$$
$$u(\infty,t) = 0, \quad t > 0,$$

where

$$\mu_* = \frac{\mu}{1+\alpha_1\phi},$$
$$\alpha_* = \frac{\alpha_1}{1+\alpha_1\phi},$$
$$\beta = \frac{3\beta_3}{1+\alpha_1\phi},$$
$$\beta_* = \frac{\beta_3\phi}{1+\alpha_1\phi}, \tag{220}$$
$$\phi_* = \frac{\phi}{1+\alpha_1\phi},$$
$$M_*^2 = \frac{M^2}{1+\alpha_1\phi}.$$

The Lie point symmetries for the PDE (218) are [47]

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial y},$$
$$X_3 = -\left(\frac{1}{\phi_* + M_*^2}\right) e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial t} + u e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial u}, \tag{221}$$
$$\text{where } \phi_* + M_*^2 = \frac{\mu_*}{\alpha_*}.$$

With the use of the symmetries given in (221), three types of invariant solutions were obtained in [47] for PDE (218) and are given by

$$u(y,t) = \exp\left(\frac{\sqrt{\beta_*}\,(y - ct)}{-\sqrt{\beta}}\right),$$
$$u(y,t) = \exp\left[-\left\{\left(\phi_* + M_*^2\right)t + \sqrt{\frac{\beta_*}{\beta}}\,y\right\}\right], \tag{222}$$
$$u(y) = v_0 \exp\left(-\frac{\sqrt{\beta_*}}{\sqrt{\beta}}\,y\right).$$

14.9. MHD Flow of a Third-Grade Fluid in a Porous Half Space with Plate Suction or Injection [48]. The modelling and solution of the unsteady flow of an incompressible third-grade fluid over a porous plate within a porous medium were performed by A. Aziz and T. Aziz [48]. Lie group theory was employed to find the symmetries of the model equation. These symmetries were applied to transform the original third-order partial differential equation to third-order ordinary differential equations. These third-order ordinary differential equations were solved analytically.

The governing problem describing the flow model [48] is

$$\begin{aligned}\frac{\partial u}{\partial t} &= \mu_* \frac{\partial^2 u}{\partial y^2} + \alpha_* \frac{\partial^3 u}{\partial y^2 \partial t} - \alpha_* W_0 \frac{\partial^3 u}{\partial y^3} \\ &\quad + \gamma \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \gamma_* u \left(\frac{\partial u}{\partial y}\right)^2 \\ &\quad + W_0 \frac{\partial u}{\partial y} - \phi_* u - M_*^2 u,\end{aligned} \tag{223}$$

$$u(0,t) = V(t), \quad t > 0, \tag{224}$$

$$u(y,t) \longrightarrow 0 \quad \text{as } y \longrightarrow \infty,\ t > 0, \tag{225}$$

$$u(y,0) = f(y), \quad y > 0, \tag{226}$$

where

$$\begin{aligned}\mu_* &= \frac{1}{(1 + \alpha\phi)}, \\ \alpha_* &= \frac{\alpha}{(1 + \alpha\phi)}, \\ \gamma &= \frac{3\beta}{(1 + \alpha\phi)}, \\ \gamma_* &= \frac{\beta\phi}{(1 + \alpha\phi)}, \\ \phi_* &= \frac{\phi}{(1 + \alpha\phi)}, \\ M_*^2 &= \frac{M^2}{(1 + \alpha\phi)}.\end{aligned} \tag{227}$$

The symmetry Lie algebra of the PDE (225) is spanned by the following generators [48].

Case 1 (when $(\phi_* + M_*^2) \neq \mu_*/\alpha_*$). For this case, we find a two-dimensional Lie algebra generated by

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial y}. \tag{228}$$

Case 2 (when $(\phi_* + M_*^2) = \mu_*/\alpha_*$). Here we obtain a three-dimensional Lie algebra generated by

$$\begin{aligned}X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= -e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial t} + W_0 e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial y} \\ &\quad + u\left(\phi_* + M_*^2\right) e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial u}.\end{aligned} \tag{229}$$

The generators given above were utilized to obtain three different types of group invariant solutions [48]

$$u(y,t) = \exp\left(-\frac{\sqrt{\gamma_*}\,(y - ct)}{\sqrt{\gamma}}\right).$$
$$u(y,t) = \exp\left(-\frac{\sqrt{\gamma_*}\,(y + ct)}{\sqrt{\gamma}}\right).$$
$$u(y,t) = u_o \cdot \exp\left[-\left\{\left(\sqrt{\frac{\gamma_*}{\gamma}} W_0 + \phi_* + M_*^2\right)t + \sqrt{\frac{\gamma_*}{\gamma}}\,y\right\}\right]. \tag{230}$$

Remark 1. A revisit has been made by Aziz et al. [49] for three flow problems [44, 47, 48] related to a third-grade fluid model and discussed earlier. A conditional symmetry approach was employed in [49] to construct some new exact solutions of these models. All possible nonclassical symmetry generators were calculated in [49] for these problems. The concept of conditional/nonclassical symmetry was not previously used to find conditionally invariant solutions of non-Newtonian fluid flow problems. This is the first time that a complete nonclassical symmetry analysis was performed to tackle a nonlinear problem dealing with the flow models of non-Newtonian fluids.

For the flow model [44], the following nonclassical symmetry generators were found in [49]

$$\begin{aligned} X_1 &= \left[\exp\left(-\frac{2}{K_*}t\right) - K_*\right]\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \\ X_2 &= -K_*\frac{\partial}{\partial t} + \exp\left(-\frac{2}{K_*}t\right)\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}, \\ X_3 &= \frac{\partial}{\partial t} + \theta u\frac{\partial}{\partial u}, \\ X_4 &= \frac{\partial}{\partial t} + \lambda(t)\, u\frac{\partial}{\partial u}, \end{aligned} \tag{231}$$

where $\lambda(t)$ is any function of time t. Thus, infinitely many nonclassical symmetry generators were obtained for any arbitrary value of the function $\lambda(t)$. With the use of the nonclassical symmetries given in (231), the following new invariant solutions were obtained in [49] for the model [44]

$$\begin{aligned} u(t) &= u_0 \exp\left[-\frac{t}{(\kappa/\phi\nu + \alpha_1/\mu)}\right], \\ u(y,t) &= \exp\left[-\left\{\left(\frac{\beta - \mu_*\beta_*K_*}{K_*\beta - \alpha_*K_*\beta_*}\right)t + \sqrt{\frac{\beta_*}{\beta}}\,y\right\}\right], \\ u(y,t) &= \exp\left[\gamma(t) - \sqrt{\frac{1}{K_*\mu_*}}\,y\right] \\ &\qquad \text{where } \gamma(t) = \int \lambda(t)\,dt, \\ u(y,t) &= \left[a_3 \exp\left(\frac{2}{K_*}t\right)\right]^{-1/2} \exp\left(-\frac{\sqrt{\beta_*}}{\sqrt{\beta}}\,y\right). \end{aligned} \tag{232}$$

For the model [46], the following nonclassical symmetry generators were found in [49]

$$\begin{aligned} X_1 &= \left[\exp\left(-\left(\frac{1}{K_*} + M_*^2\right)2t\right) - \left(K_* + M_*^2\right)\right]\frac{\partial}{\partial t} \\ &\quad + u\frac{\partial}{\partial u}, \\ X_2 &= -\left(K_* + M_*^2\right)\frac{\partial}{\partial t} \\ &\quad + \exp\left(-\left(\frac{1}{K_*} + M_*^2\right)2t\right)\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}, \\ X_3 &= \frac{\partial}{\partial t} + \lambda u\frac{\partial}{\partial u}, \\ X_4 &= \frac{\partial}{\partial t} + \delta(t)\, u\frac{\partial}{\partial u}, \end{aligned} \tag{233}$$

where $\delta(t)$ is any arbitrary function of time. With the use of the symmetries given in (233), the following new exact solutions were found in [49] for the model [46]

$$\begin{aligned} u(y,t) &= \exp\left[-\left(\frac{1 + KM^2}{K + \alpha}\right)t\right], \\ u(y,t) &= \exp\left[-\left\{\left(\frac{\beta + K_*\beta M_*^2 - \mu_*\beta_*K_*}{K_*\beta - \alpha_*K_*\beta_*}\right)t \right.\right. \\ &\quad \left.\left. + \left(\sqrt{\frac{\beta_*}{\beta}}\right)y\right\}\right], \\ u(y,t) &= \exp\left[\zeta(t) - \sqrt{\left(\frac{1}{K_*\mu_*} + \frac{M_*^2}{\mu_*}\right)y}\right] \\ &\qquad \text{where } \zeta(t) = \int \delta(t)\,dt, \\ u(y,t) &= \left[b_3 \exp\left(\frac{1}{K_*} + M_*^2\right)2t\right]^{-1/2} \exp\left(-\frac{\sqrt{\beta_*}}{\sqrt{\beta}} \right. \\ &\quad \left. \cdot\, y\right). \end{aligned} \tag{234}$$

For the flow model [48], the only nonclassical symmetry operator found in [49] is

$$\chi = \frac{\partial}{\partial t} + \epsilon u\frac{\partial}{\partial u}. \tag{235}$$

With the use of the symmetry given above, the new group invariant solution of the flow model [48] found in [49] is given by

$$\begin{aligned} u(y,t) &= \exp\left[-\left\{\left(\frac{1}{\beta - \alpha_*\beta_*}\right)\right.\right. \\ &\quad \cdot\left(\frac{\beta}{K_*} + \beta M_*^2 - \mu_*\beta_* + W\sqrt{\frac{\beta_*}{\beta}}\,(\beta - \alpha_*\beta_*)\right)t \\ &\quad \left.\left. + \sqrt{\frac{\beta_*}{\beta}}\,y\right\}\right]. \end{aligned} \tag{236}$$

15. Fourth-Grade Fluid Flow Problems

In this section, all those problems dealing with the flow of a fourth-grade non-Newtonian fluid model investigated by using Lie symmetry approach are revisited.

For a fourth-grade fluid model, the Cauchy stress tensor satisfies the constitutive equations

$$\mathbf{T} = -p\mathbf{I} + \sum_{j=1}^{n} \mathbf{S}_j \quad \text{with } n = 4, \tag{237}$$

where p is the pressure, $\mathbf{I}$ is the identity tensor, and $\mathbf{S}_j$ is the extra stress tensor with components

$$\begin{aligned}
\mathbf{S}_1 &= \mu \mathbf{A}_1, \\
\mathbf{S}_2 &= \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \\
\mathbf{S}_3 &= \beta_1 \mathbf{A}_3 + \beta_2 (\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3 (\operatorname{tr} \mathbf{A}_1^2) \mathbf{A}_1, \\
\mathbf{S}_4 &= \gamma_1 \mathbf{A}_4 + \gamma_2 (\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3) + \gamma_3 \mathbf{A}_2^2 \\
&\quad + \gamma_4 (\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2) \gamma_5 (\operatorname{tr} \mathbf{A}_2) \mathbf{A}_2 \\
&\quad + \gamma_6 (\operatorname{tr} \mathbf{A}_2) \mathbf{A}_1^2 \\
&\quad + [\gamma_7 \operatorname{tr} \mathbf{A}_3 + \gamma_8 \operatorname{tr} (\mathbf{A}_2\mathbf{A}_1)] \mathbf{A}_1.
\end{aligned} \tag{238}$$

Here μ is the dynamic viscosity, α_i $(i = 1, 2)$, β_i $(i = 1, 2, 3)$, and γ_i $(i = 1, 2, \ldots, 8)$ are the material constants, and $\mathbf{A}_1$ to $\mathbf{A}_4$ are the Rivlin-Ericksen tensors.

15.1. The Unsteady Flow of a Fourth-Grade Fluid Past a Porous Plate [50]. Hayat et al. [50] examined the unsteady flow of a hydrodynamic fluid past a porous plate. The solution of the governing nonlinear problem was obtained by the implementation of the Lie group method.

The problem governing the unsteady flow model of a fourth-grade fluid past a porous plate is [50]

$$\begin{aligned}
\rho \left[\frac{\partial u}{\partial t} - W \frac{\partial u}{\partial y} \right] &= \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \left[\frac{\partial^3 u}{\partial y^2 \partial t} - W \frac{\partial^3 u}{\partial y^3} \right] \\
&\quad + \beta_1 \left[\frac{\partial^4 u}{\partial y^2 \partial t^2} - 2W \frac{\partial^4 u}{\partial y^3 \partial t} + W_0^2 \frac{\partial^4 u}{\partial y^4} \right] + 6 (\beta_2 \\
&\quad + \beta_3) \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_1 \left[\frac{\partial^5 u}{\partial y^2 \partial t^3} - 3W \frac{\partial^5 u}{\partial y^3 \partial t^2} \right. \\
&\quad \left. + 3W^2 \frac{\partial^5 u}{\partial y^4 \partial t} - W^3 \frac{\partial^5 u}{\partial y^5} \right] + \Gamma \frac{\partial}{\partial y} \left[\left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y \partial t} \right] \\
&\quad - \Gamma W \frac{\partial}{\partial y} \left[\left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} \right],
\end{aligned} \tag{239}$$

where $\Gamma = 2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)$. The relevant initial and boundary conditions are

$$\begin{aligned}
&u(0, t) = 0, \\
&u(y, t) \longrightarrow U \quad \text{as } y \longrightarrow \infty, \\
&\frac{\partial u^n}{\partial y^n} \longrightarrow 0 \quad \text{as } y \longrightarrow \infty \ (n = 1, 2, 3).
\end{aligned} \tag{240}$$

Equation (239) admits the Lie point symmetry generators $G = \partial/\partial t$ (time translation) and $X = \partial/\partial y$ (translation in y). The invariant solution corresponding to the generator $G + cX$ which represents a travelling wave solution with constant wave speed c has been considered in [50]. The final form of the travelling wave solution found in [50] is

$$\begin{aligned}
u(y, t) &= -U \exp \left(\frac{-2c(\beta_2 + \beta_3)(t - T)}{\Gamma(c + W)} \right) \\
&\quad \cdot \left[\exp \left(\frac{2(\beta_2 + \beta_3) y}{\Gamma(c + W)} \right) - 1 \right].
\end{aligned} \tag{241}$$

The same model was recently discussed by Aziz and Mahomed [51] by considering a different set of boundary and initial conditions and they found some more physically meaningful exact solutions. The closed-form travelling wave and the steady state solutions found in [51] are

$$\begin{aligned}
u(y, t) &= \exp \left[\frac{-\beta(y + ct)}{6\Gamma(c - W_0)} \right], \quad \text{with } c > W_0, \\
u &= H(y) = \exp \left(\frac{\beta y}{3\Gamma W_0} \right).
\end{aligned} \tag{242}$$

15.2. Travelling Wave Solutions to Stokes' Problem for a Fourth-Grade Fluid [52]. A nonlinear partial differential equation modelling the flow of a fourth-grade fluid was derived by Hayat et al. [52]. Travelling wave solutions admitted by the model equation were deduced by employing the Lie symmetry approach.

The nonlinear governing PDE for the unsteady flow of a fourth-grade fluid over a flat rigid plate is [52]

$$\begin{aligned}
\rho \frac{\partial u}{\partial t} &= \mu \frac{\partial^2 u}{\partial y^2} + \alpha \frac{\partial^3 u}{\partial y^2 \partial t} + \beta \frac{\partial^4 u}{\partial y^2 \partial t^2} \\
&\quad + \beta_* \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma \frac{\partial^5 u}{\partial y^2 \partial t^3} \\
&\quad + \Gamma \frac{\partial}{\partial y} \left[\left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y \partial t} \right],
\end{aligned} \tag{243}$$

where $\beta_* = (\beta_2 + \beta_3)$ and $\Gamma = 2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)$.

It can be seen that the PDE (243) admits Lie point symmetry generators

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= \frac{\partial}{\partial y}.
\end{aligned} \tag{244}$$

With the use of symmetry generators given in (244), the exact travelling wave solutions found in [52] are given by

$$u(y, t) = \exp \left[\frac{\beta_* (y - ct)}{6c\Gamma} \right]. \tag{245}$$

The same PDE (243) was investigated recently by Aziz and Mahomed [53] with a suitable choice of physically realistic

boundary conditions. Both forward and the backward wave front type travelling wave solutions were found in [53] with the aid of the Lie group approach and are given by

$$u(y,t) = \exp\left[\frac{-\beta(y+ct)}{6c\gamma}\right] \quad \text{with } c > 0.$$
$$u(y,t) = \exp\left[\frac{\beta(y-ct)}{6c\gamma}\right] \quad \text{with } c > 0. \tag{246}$$

A conditional/nonclassical symmetry solution of PDE (243) was also found by Aziz and Mahomed [53] and is given by

$$u(y,t) = \exp\left[-\left\{\left(\frac{\beta}{6\gamma}\right)t + \sqrt{\left(\frac{(-\beta/6\gamma)}{1-\alpha(\beta/6\gamma)+\beta_1(\beta/6\gamma)^2-\gamma_1(\beta/6\gamma)^3}\right)}y\right\}\right]. \tag{247}$$

15.3. Effect of Magnetic Field on the Flow of a Fourth-Order Fluid [54]. A study was conducted by Hayat et al. [54] to examine the flow engendered in a semi-infinite expanse of an incompressible non-Newtonian fluid by an infinite rigid plate moving with an arbitrary velocity in its own plane. The fluid was considered to be fourth-order and electrically conducting. A magnetic field was applied in the transverse direction to the flow. The nonlinear problem was solved for a constant magnetic field analytically using Lie reduction methods.

The governing problem describing the flow model [54] is given by

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2\partial t} + \beta_1\frac{\partial^4 u}{\partial y^2\partial t^2} + \beta\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}\\ &\quad + \gamma_1\frac{\partial^5 u}{\partial y^2\partial t^3} + \gamma\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right]\\ &\quad - \rho M H^2 u,\end{aligned} \tag{248}$$

with

$$\begin{aligned}u(0,t) &= u_0 V(t), \quad t > 0,\\ u(\infty,t) &= 0, \quad t > 0,\\ u(y,0) &= g(y), \quad y > 0,\end{aligned} \tag{249}$$

where u_0 is the reference velocity.

Note that (248) has two translation symmetry generators: namely.

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial y}. \tag{250}$$

Thus, a travelling wave solution was obtained in [54] for the above problem using the translational symmetries and is given by

$$u(y,t) = u_0 \exp\left[\frac{\beta y}{3c\gamma} - \frac{\beta t}{3\gamma}\right], \tag{251}$$

provided that

$$\begin{aligned}-c\rho\left(\frac{\beta}{3c\gamma}\right) &= \mu\left(\frac{\beta}{3c\gamma}\right)^2 - \alpha_1 c\left(\frac{\beta}{3c\gamma}\right)^3\\ &\quad + \beta_1 c^2\left(\frac{\beta}{3c\gamma}\right)^4 - \gamma_1 c^3\left(\frac{\beta}{3c\gamma}\right)^5\\ &\quad - \rho M H^2 = 0.\end{aligned} \tag{252}$$

15.4. A Note on Some Solutions for the Flow of a Fourth-Grade Fluid in a Porous Space [55]. Hayat et al. [55] investigated the time-independent unidirectional flow of a fourth-grade fluid filling the porous half space. Flow modelling was based upon a modified Darcy's law. Travelling wave and conditional symmetry solutions were developed for the governing model.

The flow problem governing the time-dependent flow of a fourth-grade fluid in a porous space is given by [55]

$$\begin{aligned}\frac{\partial u}{\partial t} &= \mu_*\frac{\partial^2 u}{\partial y^2} + \alpha_*\frac{\partial^3 u}{\partial y^2\partial t} + \beta\frac{\partial^4 u}{\partial y^2\partial t^2}\\ &\quad + \beta_*\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \gamma\frac{\partial^5 u}{\partial y^2\partial t^3}\\ &\quad + \gamma_*\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right] - \phi_1 u - \phi_2 u\frac{\partial^2 u}{\partial t^2}\\ &\quad - \phi_3 u\left(\frac{\partial u}{\partial y}\right)^2 - \phi_4 u\frac{\partial^3 u}{\partial t^3}\\ &\quad - \phi_5 u\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y\partial t}\right),\end{aligned} \tag{253}$$

where

$$\begin{aligned}\mu_* &= \frac{\mu}{(\rho+\alpha(\phi/\kappa))},\\ \alpha_* &= \frac{\alpha_1}{(\rho+\alpha(\phi/\kappa))},\\ \beta &= \frac{\beta_1}{(\rho+\alpha(\phi/\kappa))},\\ \beta_* &= \frac{6(\beta_2+\beta_3)}{(\rho+\alpha(\phi/\kappa))},\\ \gamma &= \frac{\gamma_1(\phi/\kappa)}{(\rho+\alpha(\phi/\kappa))},\\ \gamma_* &= \frac{\Gamma}{(\rho+\alpha(\phi/\kappa))},\\ \gamma_* &= \frac{2(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)}{(\rho+\alpha(\phi/\kappa))},\end{aligned}$$

$$\phi_1 = \frac{\mu\varphi(\phi/\kappa)}{(\rho + \alpha(\phi/\kappa))},$$
$$\phi_2 = \frac{\beta_1(\phi/\kappa)}{(\rho + \alpha(\phi/\kappa))},$$
$$\phi_3 = \frac{2(\beta_2 + \beta_3)(\phi/\kappa)}{(\rho + \alpha(\phi/\kappa))},$$
$$\phi_4 = \frac{\gamma_1(\phi/\kappa)}{(\rho + \alpha(\phi/\kappa))},$$
$$\phi_5 = \frac{2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8)(\phi/\kappa)}{(\rho + \alpha(\phi/\kappa))}. \tag{254}$$

The relevant initial and boundary conditions are

$$\begin{aligned} u(0,t) &= u_0 V(t), \quad t > 0, \\ u(\infty,t) &= 0, \quad t > 0, \\ u(y,0) &= f(y), \quad y > 0, \\ \frac{\partial u(y,0)}{\partial t} &= g(y), \quad y > 0, \\ \frac{\partial^2 u(y,0)}{\partial t^2} &= h(y), \quad y > 0. \end{aligned} \tag{255}$$

With the use of translational symmetry generators, the travelling wave solution for the PDE (253) found in [55] is given by

$$u(y,t) = u_0 \exp\left[-\frac{\phi_2}{c\phi_4}(y + ct)\right]. \tag{256}$$

The nonclassical symmetry generator for the PDE (253) found in [55] is given by

$$X = \frac{\partial}{\partial t} - \frac{\phi_2}{\phi_4} u \frac{\partial}{\partial u}. \tag{257}$$

With the use of nonclassical symmetry operator (257), the conditionally invariant solution found in [55] is given by

$$u(y,t) = \exp\left(-\frac{\phi_2}{\phi_4} t\right) B(y), \tag{258}$$

where

$$B(y)\left(\phi_1 - \frac{\phi_2}{\phi_4}\right) = \left(\frac{-\gamma_1\phi_2^3}{\phi_4^3} + \frac{\beta_2\phi_2^2}{\phi_4^2} - \frac{\alpha\phi_2}{\phi_4^2}\right), \tag{259}$$

with

$$\begin{aligned} B(0) &= 1, \\ B(l) &= 0, \quad l > 0. \end{aligned} \tag{260}$$

15.5. Invariant Solutions for the Unsteady Magnetohydrodynamics (MHD) Flow of a Fourth-Grade Fluid Induced due to the Impulsive Motion of a Flat Porous Plate [56]. An analysis is carried out recently by Aziz et al. [56] to investigate the time-dependent flow of an incompressible electrically conducting fourth-grade fluid over an infinite porous plate. The governing nonlinear problem was solved by invoking the Lie group theoretic approach and a numerical technique. Travelling wave solutions of the forward and backward type, together with a steady state solution, were obtained in [56].

The governing PDE for the flow model [56] is given by

$$\begin{aligned} \frac{\partial u}{\partial t} &= W\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} + \alpha_1\left[\frac{\partial^3 u}{\partial y^2 \partial t} - W\frac{\partial^3 u}{\partial y^3}\right] \\ &+ \beta_1\left[\frac{\partial^4 u}{\partial y^2 \partial t^2} - 2W\frac{\partial^4 u}{\partial y^3 \partial t} + W_0^2\frac{\partial^4 u}{\partial y^4}\right] \\ &+ \beta\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \gamma\left[\frac{\partial^5 u}{\partial y^2 \partial t^3} - 3W\frac{\partial^5 u}{\partial y^3 \partial t^2}\right. \\ &\left. + 3W^2\frac{\partial^5 u}{\partial y^4 \partial t} - W^3\frac{\partial^5 u}{\partial y^5}\right] \\ &+ 2\Gamma\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y \partial t}\right] \\ &- 2\Gamma W\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}\right] - M^2 u. \end{aligned} \tag{261}$$

The relevant boundary and initial conditions are [56]

$$\begin{aligned} u(0,t) &= V(t), \quad t > 0, \\ u(\infty,t) &= 0, \quad t > 0, \\ u(y,0) &= f(y), \quad y > 0, \\ \frac{\partial u}{\partial t}(y,0) &= g(y), \quad y > 0, \\ \frac{\partial^2 u}{\partial t^2}(y,0) &= h(y), \quad y > 0. \end{aligned} \tag{262}$$

With the use of translational symmetry generators, the backward type travelling wave solution for the PDE (261) found in [56] is given by

$$\begin{aligned} u(y,t) = \exp\Bigg[\Bigg(&-\frac{M^2}{(m-W)} + \frac{\gamma\beta^5}{(6\Gamma)^5(m-W)^3} \\ &- \frac{\beta_1\beta^4}{(6\Gamma)^4(m-W)^3} + \frac{\alpha\beta^3}{(6\Gamma)^3(m-W)^3} \\ &- \frac{\beta^2}{(6\Gamma)^2(m-W)^3}\Bigg)(y + mt)\Bigg]. \end{aligned} \tag{263}$$

The forward type travelling wave solution for the PDE (261) is given by

$$u(y,t) = \exp\left[\frac{\beta(y - mt)}{6\Gamma(m + W)}\right]. \tag{264}$$

Finally the stead-state solution of the problem found in [56] is

$$u = R(y) = \exp\left(\frac{\beta y}{6\Gamma W}\right). \tag{265}$$

15.6. Group Theoretical Analysis and Invariant Solutions for Unsteady Flow of a Fourth-Grade Fluid over an Infinite Plate Undergoing Impulsive Motion in a Darcy Porous Medium [57]. An incompressible time-dependent flow of a fourth-grade fluid in a porous half space has been investigated very recently by Aziz et al. [57]. The partial differential equation governing the motion was reduced to ordinary differential equations by means of Lie group theoretic analysis. Various new classes of group invariant solutions were developed for the model problem by employing the classical and nonclassical symmetry methods. Travelling wave solutions, steady state solution, and conditional symmetry solutions were obtained in [57] as closed-form exponential functions.

The governing PDE for the flow model [57] is given by

$$\begin{aligned}\frac{\partial u}{\partial t} &= \mu_* \frac{\partial^2 u}{\partial y^2} + \alpha_* \frac{\partial^3 u}{\partial y^2 \partial t} + \beta \frac{\partial^4 u}{\partial y^2 \partial t^2} \\ &\quad + \beta_* \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_* \frac{\partial^5 u}{\partial y^2 \partial t^3} \\ &\quad + 2\Gamma_* \frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y \partial t}\right] - \kappa_1 u - \kappa_2 \frac{\partial^2 u}{\partial t^2} \\ &\quad - \kappa_3 u \left(\frac{\partial u}{\partial y}\right)^2 - \kappa_4 \frac{\partial^3 u}{\partial t^3} \\ &\quad - \kappa_5 u \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y \partial t}\right).\end{aligned} \tag{266}$$

The relevant space and time-dependent velocity boundary and initial conditions are [57]

$$\begin{aligned} u(0,t) &= V(t), \quad t > 0, \\ u(\infty,t) &= 0, \quad t > 0, \\ u(y,0) &= f(y), \quad y > 0, \\ \frac{\partial u(y,0)}{\partial t} &= g(y), \quad y > 0, \\ \frac{\partial^2 u(y,0)}{\partial t^2} &= h(y), \quad y > 0, \end{aligned} \tag{267}$$

where $f(y) = I(y)/U_0$, $g(y) = J(y)/U_0$, and $h(y) = K(y)/U_0$.

The Lie point symmetries for the PDE (266) are [57]

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= -\exp\left(\frac{\kappa_3 t}{\kappa_5}\right)\frac{\partial}{\partial t} + \left(\frac{\kappa_3}{\kappa_5}\right)\exp\left(\frac{\kappa_3 t}{\kappa_5}\right) u \frac{\partial}{\partial u}, \end{aligned} \tag{268}$$

provided that

$$\begin{aligned} 3\Gamma_*\left(\frac{\kappa_3}{\kappa_5}\right) - \beta_* &= 0, \\ 6\kappa_4\left(\frac{\kappa_3}{\kappa_5}\right) - \kappa_2 &= 0, \\ 6\gamma_*\left(\frac{\kappa_3}{\kappa_5}\right) - \beta &= 0, \\ \kappa_4\left(\frac{\kappa_3}{\kappa_5}\right)^3 + \kappa_2\left(\frac{\kappa_3}{\kappa_5}\right)^2 + \left(\frac{\kappa_3}{\kappa_5}\right) - 3\kappa_1 &= 0, \\ 4\kappa_4\left(\frac{\kappa_3}{\kappa_5}\right)^2 + 3\kappa_2\left(\frac{\kappa_3}{\kappa_5}\right) - 2 &= 0, \\ 4\gamma_*\left(\frac{\kappa_3}{\kappa_5}\right)^2 + 3\beta\left(\frac{\kappa_3}{\kappa_5}\right) - 2\alpha_* &= 0, \\ \gamma_*\left(\frac{\kappa_3}{\kappa_5}\right)^3 + \beta\left(\frac{\kappa_3}{\kappa_5}\right)^2 + \alpha_*\left(\frac{\kappa_3}{\kappa_5}\right) - 3\mu_* &= 0. \end{aligned} \tag{269}$$

Here X_1 is translation in time, X_2 is translation in space, and X_3 has path curves which are equivalent to a combination of translations in t and scaling in u. With the use of the symmetries given in (268), the following exact solutions were found in [57] for the model [57]

$$\begin{aligned} u(y,t) &= \exp\left[-\left\{\left(\frac{\kappa_3}{\kappa_5}\right)t + \sqrt{\frac{\left(\kappa_1 - \kappa_2\beta_*^2/9\Gamma_*^2\right)}{\left(\mu_* - \beta^3/36\gamma_*^2\right)}}\, y\right\}\right], \\ u(y) &= \exp\left(-\sqrt{\frac{\kappa_3}{\beta_*}}\, y\right), \\ u(y,t) &= \exp\left[-\left\{\frac{\beta_* + \sqrt{\beta_*^2 + 12m\Gamma_*\left(\kappa_3 + \kappa_5 m\right)}}{6\Gamma_* m}\right\} \cdot (y + mt)\right], \quad m > 0. \end{aligned} \tag{270}$$

The nonclassical symmetry generator for the PDE (266) found in [57] is given by

$$\chi = \frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \tag{271}$$

With the use of nonclassical symmetry operator (271), the conditionally invariant solutions found in [57] are given by

$$u(y,t) = \exp\left[-\left\{t + \sqrt{\left(\frac{\kappa_3 - \kappa_5}{\beta_* - 6\Gamma_*}\right)}y\right\}\right],$$
$$u(y,t) = \exp\left[-\left\{t + \sqrt{\left(\frac{\kappa_1 + \kappa_2 - \kappa_4 - 1}{\mu_* - \alpha_* + \beta - \gamma_*}\right)}y\right\}\right]. \tag{272}$$

16. Couple Stress Fluid Flow Problems

In this section, we provide a review of those problems which deal with the flow of non-Newtonian couple stress fluid and solved using the classical Lie group approach.

The governing equations for the flow of an incompressible couple stress fluid are

$$\overline{\nabla} \cdot \overline{\mathbf{q}} = 0,$$
$$\rho\left[\frac{\partial}{\partial t} + \left(\overline{\mathbf{q}} \cdot \overline{\nabla}\right)\overline{\mathbf{q}}\right] = -\overline{\nabla} p - \mu\overline{\nabla} \times \overline{\nabla} \times \overline{\mathbf{q}} - \eta\overline{\nabla} \times \overline{\nabla} \times \overline{\nabla} \times \overline{\nabla} \times \overline{\mathbf{q}}, \tag{273}$$

where ρ is the density, $\overline{\mathbf{q}}$ is the velocity vector, p is the fluid pressure, μ is the fluid viscosity, and η is the couple stress fluid parameter.

The force stress tensor $\mathbf{T}$ and the couple stress tensor $\mathbf{M}$ that arise in the theory of couple stress fluids are given by

$$\mathbf{T} = \left(-p + \lambda_1 \operatorname{div} \overline{\mathbf{q}}\right)\mathbf{I} + \mu\left[\operatorname{grad} \overline{\mathbf{q}} + \left(\operatorname{grad} \overline{\mathbf{q}}\right)^T\right] + \frac{1}{2}\mathbf{I} \times \left[\operatorname{div} \mathbf{M} + \rho C\right],$$
$$\mathbf{M} = m\mathbf{I} + 2\eta \operatorname{grad}\left(\operatorname{curl} \overline{\mathbf{q}}\right) + 2\eta^l \left(\operatorname{grad}\left(\operatorname{curl} \overline{\mathbf{q}}\right)\right)^T, \tag{274}$$

where m is 1/3 trace of $\mathbf{M}$ and ρC is the body couple tensor. The quantity λ_1 is the material constant and η^l is the constant associated with couple stresses. These material constants satisfy the inequalities

$$\mu \geq 0,$$
$$3\lambda + 2\mu \geq 0,$$
$$\eta \geq 0,$$
$$\eta^l \leq \eta. \tag{275}$$

16.1. Flow and Heat Transfer of Couple Stress Fluid in a Porous Channel with Expanding and Contracting Walls [58]. The study related to an incompressible laminar flow of a couple stress fluid in a porous channel with expanding or contracting walls was considered by Srinivasacharya et al. [58]. With the use of similarity transformations, the governing equations were reduced to nonlinear ordinary differential equations. The resulting equations were solved numerically using a quasi-linearization technique.

The equations governing the flow model [58] are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$
$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\nabla^2 u - \eta\nabla^4 u,$$
$$\rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\nabla^2 v - \eta\nabla^4 v,$$
$$\rho C\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right] = \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right] + \eta\left[\left(\nabla^2 v\right)^2 + \left(\nabla^2 u\right)^2\right] + k\nabla^2 T. \tag{276}$$

The boundary conditions on the velocity profile and temperature are

$$\begin{aligned} u(x,y) &= 0, \\ v(x,y) &= v_w = -A\alpha, \\ \nabla \times q &= 0, \\ T(x,y) &= T_1, \\ &\qquad \text{at } y = a(t), \\ u(x,y) &= 0, \\ v(x,y) &= 0, \\ \nabla \times q &= 0, \\ T(x,y) &= T_2, \\ &\qquad \text{at } y = 0. \end{aligned} \tag{277}$$

With the use of classical Lie similarity approach, the solution for the above system of PDEs is written as [58]

$$u = \frac{ux}{\rho a^2} F^1(\eta, t),$$
$$v = -\frac{ux}{\rho a} F(\eta, t), \tag{278}$$
$$\text{where } \eta = \frac{y}{a}.$$

The similarity transformation given in (278) was used to reduce the above system of PDEs into a system of nonlinear ODEs. The reduced ordinary differential system was solved by using numerical methods for the boundary conditions given in (277).

17. Upper Convected Maxwell (UCM) Fluid Flow Problems

In this section, we discuss the studies related to flow of an upper convected Maxwell fluid and solved with the use of the Lie symmetry method.

The normalized momentum equation for unsteady flow is

$$\frac{\partial u}{\partial t} + v_w \frac{\partial u}{\partial y} = \frac{1}{R_e} \frac{\partial \tau_{xy,p}}{\partial y}, \tag{279}$$

with the polymer shear stress ($\tau_{xy} = \tau_{yx}$) given by the UCM differential constitutive equation having a relaxation time λ and a viscosity coefficient η_p, as mentioned above.

For a UCM fluid, the polymer extra stress tensor is written in index notation and in nondimensional form as

$$\begin{aligned} &\tau_{ij,p} + D_e \left[\frac{\partial \tau_{ij,p}}{\partial t} + u_k \frac{\partial \tau_{ij,p}}{\partial x_k} - \tau_{ik,p} \frac{\partial u_i}{\partial x_k} - \tau_{ik,p} \frac{\partial u_j}{\partial x_k} \right] \\ &\quad = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_k}. \end{aligned} \tag{280}$$

17.1. Group Theoretical Analysis and Similarity Solutions for Stress Boundary Layers in Viscoelastic Flows [59]. Atalık [59] employed the Lie group theory to obtain the point symmetries of the boundary layer equations derived in the literature for the high Weissenberg number flow of an upper convected Maxwell (UCM) fluid. The equations were reduced to ordinary differential equation systems with the use of scaling and spiral transformation groups.

The boundary layer equations for the UCM fluid in stream function formulation are [59]

$$\begin{aligned} &\frac{\partial^2 T_{xx}}{\partial x \partial y} + \frac{\partial^2 T_{xy}}{\partial y^2} = 0, \\ &\frac{\partial \psi}{\partial y} \frac{\partial T_{xx}}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T_{xx}}{\partial y} - 2 \frac{\partial^2 \psi}{\partial x \partial y} T_{xx} - 2 \frac{\partial^2 \psi}{\partial y^2} T_{xy} \\ &\quad + T_{xx} = 0, \\ &\frac{\partial \psi}{\partial y} \frac{\partial T_{xy}}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T_{xy}}{\partial y} - \frac{\partial^2 \psi}{\partial y^2} T_{yy} + \frac{\partial^2 \psi}{\partial x^2} T_{xx} + T_{xy} \\ &\quad - \frac{\partial^2 \psi}{\partial y^2} = 0, \\ &\frac{\partial \psi}{\partial y} \frac{\partial T_{yy}}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T_{yy}}{\partial y} + 2 \frac{\partial^2 \psi}{\partial x^2} T_{xy} + 2 \frac{\partial^2 \psi}{\partial x \partial y} T_{yy} + T_{yy} \\ &\quad + 2 \frac{\partial^2 \psi}{\partial x \partial y} = 0. \end{aligned} \tag{281}$$

The symmetry Lie algebra for the above system of PDEs is spanned by the following generators [59]:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial \psi}, \\ X_3 &= f(x) \frac{\partial}{\partial y} + T_{xx} f'(x) \frac{\partial}{\partial T_{xy}} \\ &\quad + 2 T_{xy} f'(x) \frac{\partial}{\partial T_{yy}}, \\ X_4 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2\psi \frac{\partial}{\partial \psi}, \\ X_5 &= -\frac{y}{2} \frac{\partial}{\partial y} - \frac{\psi}{2} \frac{\partial}{\partial \psi} + T_{xx} \frac{\partial}{\partial T_{xx}} + \frac{T_{xy}}{2} \frac{\partial}{\partial T_{xy}}, \end{aligned} \tag{282}$$

where $f(x)$ is an arbitrary function.

The generators given in (282) were utilized to reduce the PDE system (281) to a nonlinear system of ODEs which was then solved numerically.

18. Phan-Tien-Tanner (PTT) Fluid Flow Problems

In this section, we provide a review of those studies which deal with the flow of a Phan-Tien-Tanner fluid and solved using the classical Lie similarity approach.

The constitutive equation for Phan-Tien-Tanner fluid model is

$$\begin{aligned} &W_e \left[(\mathbf{V} \cdot \nabla) \mathbf{T} - (\nabla \mathbf{V}) \mathbf{T} - \mathbf{T} (\nabla \mathbf{V})^T + k \operatorname{trace}(\mathbf{T}) \mathbf{T} \right] \\ &\quad + \mathbf{T} = \nabla \mathbf{V} + (\nabla \mathbf{V})^T, \end{aligned} \tag{283}$$

where W_e is the Weissenberg number characterizing elastic effects.

Atalık [59] also used Lie group theory to unveil point symmetry groups of the equations of stress boundary layers for creeping flow of a PTT type non-Newtonian fluid. Similarity transformations were obtained for the boundary layer equations to reduce the governing partial differential equations into nonlinear system of ordinary differential equations. Numerical integration of the reduced ordinary differential equation systems was performed thereafter.

The boundary layer equations for the PTT fluid in the form of stream function are [59]

$$\begin{aligned} &\frac{\partial^2 T_{xx}}{\partial x \partial y} + \frac{\partial^2 T_{xy}}{\partial y^2} = 0, \\ &\frac{\partial \psi}{\partial y} \frac{\partial T_{xx}}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T_{xx}}{\partial y} - 2 \frac{\partial^2 \psi}{\partial x \partial y} T_{xx} - 2 \frac{\partial^2 \psi}{\partial y^2} T_{xy} \\ &\quad + T_{xx}^2 = 0, \end{aligned}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T_{xy}}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T_{xy}}{\partial y}-\frac{\partial^2\psi}{\partial y^2}T_{yy}+\frac{\partial^2\psi}{\partial x^2}T_{xx}+T_{xx}T_{xy}-\frac{\partial^2\psi}{\partial y^2}=0,$$
$$\frac{\partial\psi}{\partial y}\frac{\partial T_{yy}}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T_{yy}}{\partial y}+2\frac{\partial^2\psi}{\partial x^2}T_{xy}+2\frac{\partial^2\psi}{\partial x\partial y}T_{yy}+T_{xx}T_{yy}+2\frac{\partial^2\psi}{\partial x\partial y}=0. \tag{284}$$

The Lie point symmetries for the PDE system (284) are [59]

$$\begin{aligned}X_1&=\frac{\partial}{\partial x},\\X_2&=\frac{\partial}{\partial\psi},\\X_3&=f(x)\frac{\partial}{\partial y}+T_{xx}f'(x)\frac{\partial}{\partial T_{xy}}+2T_{xy}f'(x)\frac{\partial}{\partial T_{yy}},\\X_4&=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+2\psi\frac{\partial}{\partial\psi},\\X_5&=-\frac{y}{2}\frac{\partial}{\partial y}+\frac{\psi}{2}\frac{\partial}{\partial\psi}+T_{xx}\frac{\partial}{\partial T_{xx}}+\frac{T_{xy}}{2}\frac{\partial}{\partial T_{xy}}.\end{aligned} \tag{285}$$

The operators given in (285) were employed to reduce the above nonlinear system of PDEs to a nonlinear system of ODEs. The reduced nonlinear ODE system was solved by using a numerical approach.

19. Nanofluid Flow Problems

In this section, we present the problems related to the flow of a non-Newtonian nanofluid and solved by employing the Lie group approach.

The governing equations of the flow of an incompressible nanofluid are

$$\nabla\cdot\mathbf{V}=0, \tag{286}$$

$$\rho_f\left[\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right]=-\nabla p+\mu\nabla^2\mathbf{V}+\left[C\rho_p+(1-C)\left\{\rho_f\left(1-\beta\left(T-T_\infty\right)\right)\right\}\right]\mathbf{g}, \tag{287}$$

$$(C\rho)_f\left[\frac{\partial T}{\partial t}+\mathbf{V}\cdot\nabla T\right]=k\nabla^2T+(C\rho)_pD_B\nabla C\cdot\nabla T+\left(\frac{\nabla_T}{T_\infty}\right)\nabla T\cdot\nabla T, \tag{288}$$

$$\frac{\partial C}{\partial t}+\mathbf{V}\cdot\nabla C=D_B\nabla^2C+\left(\frac{\nabla_T}{T_\infty}\right)\nabla^2T. \tag{289}$$

Equation (286) is the conservation of total mass, (287) is the conservation of momentum, (288) is the conservation of thermal energy, and (289) is the equation of nanoparticles, respectively. In the above equations, ρ_f is the density of the base fluid, and μ, k, and β are the viscosity, thermal conductivity, and volumetric volume expansion coefficient of the nanofluid, while ρ_f is the density of the particles. The gravitational acceleration is denoted by $\mathbf{g}$.

19.1. Scaling Transformations for Boundary Layer Flow Near the Stagnation-Point on a Heated Permeable Stretching Surface with a Nanofluid [60]. Hamad and Pop [60] obtained the similarity solution of the steady boundary layer flow near the stagnation-point flow on a permeable stretching sheet in a porous medium saturated with a nanofluid and in the presence of internal heat generation/absorption. The governing partial differential equations with the corresponding boundary conditions were reduced to a set of ordinary differential equations with the appropriate boundary conditions via Lie group analysis and thereafter a numerical approach was utilized.

The problem governing the boundary layer flow near the stagnation-point on a heated permeable stretching surface in a porous medium saturated with a nanofluid and heat generation/absorption effects in the form of stream function is [60]

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=U\frac{dU}{dx}+\frac{1}{(1-\phi)^{2.5}\left[1-\phi+\phi\left(\rho_s/\rho_f\right)\right]}\frac{\partial^3\psi}{\partial y^3}+\frac{1}{(1-\phi)^{2.5}\left[1-\phi+\phi\left(\rho_s/\rho_f\right)\right]}K_1\left(U-\frac{\partial\psi}{\partial y}\right), \tag{290}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}=\frac{1}{P_r}\left(\frac{\alpha_n}{\alpha_f}\right)+\left[\frac{\lambda}{1-\phi+\phi\left(\rho C_p\right)_s/\left(\rho C_p\right)_f}\right]\theta,$$

with the boundary conditions

$$\begin{aligned}&\frac{\partial\psi}{\partial y}=x,\\&\frac{\partial\psi}{\partial x}=S,\\&\theta=1\\&\qquad\text{at } y=0,\\&\frac{\partial\psi}{\partial y}\longrightarrow U=\frac{a}{c}x,\\&\theta\longrightarrow 0\\&\qquad\text{as } y\longrightarrow\infty.\end{aligned} \tag{291}$$

With the use of the Lie symmetry method, the scaling transformation for the above system of PDEs found in [60] is given by

$$\begin{aligned} y &= \eta, \\ \psi &= xF(\eta), \\ \theta &= \theta(\eta). \end{aligned} \tag{292}$$

The transformation given in (292) reduces the above PDE into ordinary differential equations. The reduced ordinary differential equations were solved numerically.

19.2. Scaling Group Transformation for MHD Boundary Layer Flow of a Nanofluid Past a Vertical Stretching Surface in the Presence of Suction/Injection [61]. The problem of laminar fluid flow which results from the stretching of a vertical surface with variable stream conditions in a nanofluid was investigated numerically by Kandasamy et al. [61]. The symmetry groups admitted by the corresponding boundary value problem were obtained by using a special form of Lie group transformations, namely. the scaling group of transformations.

The governing problem describing the flow model [61] is given by

$$\begin{aligned} &\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \nu\frac{\partial^3\psi}{\partial y^3} \\ &\quad = -\frac{\sigma B_0^2}{\rho_f}\frac{\partial\psi}{\partial y} + (1-\phi_\infty)\rho f_\infty \beta g\theta\Delta\theta \\ &\qquad - (\rho_p - \rho f_\infty) g\phi\Delta\phi, \\ &\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} \\ &\quad = \alpha\frac{\partial^2\theta}{\partial y^2} + \tau D_B\frac{\partial\phi}{\partial y}\frac{\partial\theta}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial\theta}{\partial y}\right)^2, \\ &\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y} = D_B\frac{\partial^2\phi}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2\theta}{\partial y^2}, \end{aligned} \tag{293}$$

where $\Delta\theta = T_w - T_\infty$ $\Delta\phi = C_w - C_\infty$. The appropriate boundary conditions are

$$\begin{aligned} &\frac{\partial\psi}{\partial y} = x^m, \\ &\frac{\partial\psi}{\partial x} = -V_0 x^{(m-1)/2}, \\ &\theta = \phi = 1 \\ &\qquad\qquad \text{at } y = 0, \\ &\frac{\partial\psi}{\partial y} \longrightarrow 0, \\ &\theta \longrightarrow 0, \\ &\phi \longrightarrow 0 \\ &\qquad\qquad \text{as } y \longrightarrow \infty. \end{aligned} \tag{294}$$

The similarity transformation used to reduce the PDE system (293) of [61] is given by

$$\begin{aligned} yx^{-1/4} &= \eta, \\ \psi &= x^{3/4} f(\eta), \\ \theta &= \theta(\eta), \\ \phi &= \phi(\eta). \end{aligned} \tag{295}$$

With the use of transformations (295), the above system of PDEs was reduced to a nonlinear system of ODEs which was then solved using a numerical method.

19.3. Magnetic Field Effects on Free Convection Flow of a Nanofluid Past a Vertical Semi-Infinite Flat Plate [62]. The similarity reductions for problems of magnetic field effects on free convection flow of a nanofluid past a semi-infinite vertical flat plate were studied by Hamad et al. [62]. The application of a one-parameter group reduced the number of independent variables by one and consequently the governing partial differential equation with the auxiliary conditions to an ordinary differential equation with the appropriate corresponding conditions. The differential equations obtained were solved numerically.

The governing problem describing the flow model [62] is

$$\begin{aligned} &\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} \\ &\quad = \frac{1}{\left(1-\phi+\phi\left(\rho_s/\rho_f\right)\right)}\left[\frac{1}{(1-\phi)^{2.5}}\frac{\partial^3\psi}{\partial y^3}\right] \\ &\qquad + \frac{1}{\left(1-\phi+\phi\left(\rho_s/\rho_f\right)\right)}\left[(1-\phi)\right] \\ &\qquad + \frac{1}{\left(1-\phi+\phi\left(\rho_s/\rho_f\right)\right)}\left[\phi\frac{(\rho\beta)_s}{(\rho\beta)_f}\theta - M\frac{\partial\psi}{\partial y}\right], \\ &\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} \\ &\quad = \frac{1}{P_r}\left(\frac{k_{nf}}{k_f}\right) \\ &\qquad + \left[\frac{1}{1-\phi+\phi\left(\rho C_p\right)_s/\left(\rho C_p\right)_f}\right]\frac{\partial^2\theta}{\partial y^2}, \end{aligned} \tag{296}$$

with

$$\begin{aligned} &\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial x} = 0, \\ &\theta = 1 \\ &\qquad\qquad \text{at } y = 0, \\ &\frac{\partial\psi}{\partial y} \longrightarrow 0, \\ &\theta \longrightarrow 0 \\ &\qquad\qquad \text{as } y \longrightarrow \infty. \end{aligned} \tag{297}$$

With the use Lie symmetry method, the generators for the above PDE system obtained in [62] are given by

$$\begin{aligned} X_1 &= x\frac{\partial}{\partial x} + g(x)\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi} + \theta\frac{\partial}{\partial \theta}, \\ X_2 &= \frac{\partial}{\partial x} + g(x)\frac{\partial}{\partial y}, \\ X_3 &= g(x)\frac{\partial}{\partial y} + \frac{\partial}{\partial \psi}. \end{aligned} \tag{298}$$

The operators given in (298) are used to find the similarity transformations

$$\begin{aligned} y &= \eta, \\ \psi &= xF_1(\eta), \\ \theta &= xF_2(\eta). \end{aligned} \tag{299}$$

The similarity transformations (299) are utilized to reduce the above PDE system to a nonlinear ODE system which was solved numerically.

19.4. Unsteady Hiemenz Flow of Cu-Nanofluid over a Porous Wedge in the Presence of Thermal Stratification due to Solar Energy Radiation: Lie Group Transformation [63]. The unsteady Hiemenz flow of an incompressible viscous Cu-nanofluid past a porous wedge due to incident radiation was investigated by Kandasamy et al. [63]. The partial differential equations governing the model under investigation were transformed into a system of ordinary differential equations by utilizing one-parameter Lie group of transformation. The reduced ODE system was solved numerically using Runge-Kutta Gill based shooting method.

The nondimensional form of governing equations in terms of stream function is [63]

$$\begin{aligned} &\frac{\partial^2\psi}{\partial t\partial y} + \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} \\ &\quad = \frac{1}{\left(1-\varsigma+\varsigma\left(\rho_s/\rho_f\right)\right)}\left[\left\{\left(1-\varsigma+\varsigma\frac{(\rho\beta)_s}{(\rho\beta)_f}\right)\gamma\right.\right. \\ &\quad \left.\cdot\cos\frac{\Omega}{2}\theta\right\} + \frac{1}{(1-\varsigma)^{2.5}}\frac{\partial^3\psi}{\partial y^3} + \left\{\frac{\partial U}{\partial t} + U\frac{dU}{dx}\right\} \\ &\quad \left.\cdot\frac{\rho_{fn}}{\rho_f} - \frac{\nu_f}{K(1-\varsigma)^{2.5}}\left(\frac{\partial\psi}{\partial y} - U\right)\right], \\ &\frac{\partial T}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} \\ &\quad = \left[\frac{1}{1-\varsigma+\varsigma\left(\rho C_p\right)_s/\left(\rho C_p\right)_f}\right]\times\left[\frac{1}{P_r}\left\{\frac{k_{f_n}}{k_f}\right.\right. \\ &\quad \left.\left.\cdot\frac{\partial^2 T}{\partial y^2} + \frac{4}{3}\left(\left(c_T+T\right)^3 T'\right)T'\right\}\right], \end{aligned} \tag{300}$$

with the boundary conditions

$$\begin{aligned} &\frac{\partial\psi}{\partial y} = 0, \\ &\frac{\partial\psi}{\partial x} = -V_0, \\ &T = T_w \\ &\qquad\qquad \text{at } y = 0, \\ &\frac{\partial\psi}{\partial y} \longrightarrow \frac{\nu x^m}{\delta^{m+1}}, \\ &T \longrightarrow T_\infty \longrightarrow (1-n)T_0 + nT_w \\ &\qquad\qquad \text{as } y \longrightarrow \infty. \end{aligned} \tag{301}$$

Using Lie groups, the symmetry generators for the above PDE system obtained in [63] are

$$\begin{aligned} X_1 &= x\frac{\partial}{\partial x} + g(x)\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi} + T\frac{\partial}{\partial T}, \\ X_2 &= \frac{\partial}{\partial x} + g(x)\frac{\partial}{\partial y}, \\ X_3 &= g(x)\frac{\partial}{\partial y} + \frac{\partial}{\partial \psi}. \end{aligned} \tag{302}$$

The corresponding similarity transformations are given by [63]

$$y = \eta,$$
$$\psi = xf\eta, \tag{303}$$
$$T = xT(\eta).$$

Transformations (302) are employed to reduce the PDE system (300) to a nonlinear ODE system which was solved numerically by Runge-Kutta Gill based shooting technique.

Rosmila et al. [64] studied the magnetohydrodynamics convection flow and heat transfer of an incompressible viscous nanofluid past a semi-infinite vertical stretching. The partial differential equations governing the problem under consideration were transformed by a special form of the Lie symmetry group transformations, that is, a one-parameter group of transformations into a system of ordinary differential equations which was numerically solved using the Runge-Kutta-Gill-based shooting method.

Recently, Das [65] has numerically investigated the steady MHD boundary layer flow of an electrically conducting nanofluid past a vertical convectively heated permeable stretching surface with variable stream conditions. The symmetry groups admitted by the corresponding boundary value problem were obtained by using scaling group of transformations which have been utilized to reduce the governing PDE system to a nonlinear ODE system.

20. Casson Fluid Flow Problems

Here we discuss the problems dealing with the flow of a Casson fluid that are solved using the Lie group approach.

The constitutive equations for Casson fluid are presented as follows:

$$\tau^{1/n} = \tau_0^{1/n} + \mu\dot{\gamma}^{1/n}, \tag{304}$$

or

$$\tau_{ij} = \left[\mu_B + \left(\frac{P_y}{\sqrt{2\pi}}\right)^{1/n}\right]^n 2e_{ij}, \tag{305}$$

where μ is the dynamic viscosity, $\pi = e_{ij}e_{ij}$ is the product of the component of deformation rate with e_{ij} being the (i, j)th component of the deformation rate, μ_B is plastic dynamic viscosity of the non-Newtonian fluid, and P_y is the yield stress of fluid, with $n \gg 1$.

20.1. Heat Source/Sink Effects on Non-Newtonian MHD Fluid Flow and Heat Transfer over a Permeable Stretching Surface: Lie Group Analysis [66]. The only study available in the literature dealing with the flow of a Casson fluid and solved by using the Lie symmetry method was performed by Tufail et al. [66]. In [66], an analysis is carried out for flow and heat transfer of a Casson fluid over a permeable stretching surface through a porous medium. Lie symmetry analysis was used to reduce the governing partial differential equations to nonlinear ordinary differential equations. These reduced ordinary differential equations were solved exactly and solutions were obtained in terms of Kummer's function.

The dimensionless form of the equations governing for flow and heat transfer of Casson model is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} - \left(\frac{1}{K} + K\right)u, \tag{306}$$
$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial y^2} + Q\theta.$$

The boundary conditions on the velocity profile and temperature are

$$\begin{aligned} u(x, y) &= x, \\ v(x, y) &= \frac{v_w}{\sqrt{b\nu}}, \\ \theta(x, y) &= 1, \\ &\qquad \text{at } y = 0, \\ u(x, y) &= 0, \\ v(x, y) &= 0, \\ \theta(x, y) &= 1, \\ &\qquad \text{at } y \longrightarrow \infty. \end{aligned} \tag{307}$$

In the above equations, K is the porous medium permeability parameter, M is the magnetic field parameter, P_r is the Prandtl number, and Q is the dimensionless heat source or sink parameter.

The infinitesimals for the system of PDEs (306) are [66]

$$\begin{aligned} \xi_1 &= c_3 + c_4 x, \\ \xi_2 &= g(x), \\ \eta_1 &= c_1 + c_4\psi, \\ \eta_2 &= c_2\theta. \end{aligned} \tag{308}$$

The infinitesimals given in (308) were used to reduce the above system of PDEs to a system of nonlinear ODEs. The reduced ordinary differential system was solved exactly in the closed-form Kummer's function.

21. Concluding Remarks

In this topical survey, we have classified all those studies which are related to the flow models of different non-Newtonian fluids and solved using Lie symmetry and conditional symmetry approaches. The mathematical modelling, the symmetries found, and the solutions obtained for each of the studies under investigation were presented. We have

observed that the symmetry methods used to solve different non-Newtonian fluid flow models are useful for a wide range of nonlinear problems in fluids given the paucity of known exact solutions especially in non-Newtonian fluids. In the majority of the models that we have reviewed, the Lie symmetry methods have been successfully applied to obtain exact solutions. Another notable significant feature of the Lie theory analysis that we have observed is that it has been used to construct the similarity transformations which are used for the reductions of the governing equations into reduced (these reduced equations are sometimes solved numerically) or exactly solvable equations. The Lie approach is not only applicable to many nonlinear and complicated scalar ODEs and PDEs but also to a nonlinear system of ODEs and PDEs in the context of non-Newtonian fluid mechanics.

In recent years, the ansatz method has been used to construct exact solutions of nonlinear differential equations arising in the study of non-Newtonian fluids. In the ansatz method, different forms of the solution are assumed and different techniques are used to develop analytical results. We pinpoint one of the central points of this survey which is the usefulness of systematic group theoretic approach. Other methods such as homotopy approach, Adomian method, and similar iterative methods have been applied for number of times for similar class of problems discussed in this survey. However, their precise application is still unclear and the results they produce are not dictated by the physical problem at hand. The applications of Lie symmetry methods for non-Newtonian fluid flow models as discussed here are more systematic, rigorous, and general treatment for such type of problems. In other words, the group theoretical methods provide a unified treatment to classify exact solutions of models of Newtonian and non-Newtonian fluids which are solved in the literature using different techniques.

Another important aspect of this survey we have observed is that many researchers have employed Lie group methods to investigate different interesting features of non-Newtonian fluids in different situations but very limited studies have been reported in the literature in which researchers have utilized the nonclassical symmetry method to solve non-Newtonian fluid flow models. In particular, only a single study [49] has been found in which the authors have performed the nonclassical symmetry analysis of a particular model of a non-Newtonian third-grade fluid. Thus, the concept of nonclassical symmetry has not been widely used to find conditionally invariant solutions of non-Newtonian fluid flow problems. This also applies to the use of weak and higher symmetries in the context of fluids. We believe that these deserve further importance in tackling non-Newtonian fluid flow problems. Since the problems dealing with the flow of non-Newtonian fluids have received much attention in recent years, the present survey is intended to provide a platform for researchers to apply group methods to tackle nonlinear problems in the fertile field of fluid mechanics.

Moreover, we did not focus on symmetry associated with conservation law works such as that of [67–69]. This necessitated giving notions on conservation laws of partial differential equations and the group approaches related to these which are extensive.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M. B. Abd-el-Malek, N. A. Badran, and H. . Hassan, "Solution of the Rayleigh problem for a power law non-Newtonian conducting fluid via group method," *International Journal of Engineering Science*, vol. 40, no. 14, pp. 1599–1609, 2002.

[2] C. Wafo Soh, "Invariant solutions of the unidirectional flow of an electrically charged power-law non-Newtonian fluid over a flat plate in presence of a transverse magnetic field," *Communications in Nonlinear Science and Numerical Simulation*, vol. 10, no. 5, pp. 537–548, 2005.

[3] M. Yürüsoy, "Unsteady boundary layer flow of power-law fluid on stretching sheet surface," *International Journal of Engineering Science*, vol. 44, no. 5-6, pp. 325–332, 2006.

[4] N. N. Nguetchue and E. Momoniat, "Axisymmetric spreading of a thin power-law fluid under gravity on a horizontal plane," *Nonlinear Dynamics. An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems*, vol. 52, no. 4, pp. 361–366, 2008.

[5] R. J. Moitsheki, S. Abelman, and T. Hayat, "Symmetry reductions of a flow with power law fluid and contaminant-modified viscosity," *Mathematical and Computational Applications*, vol. 15, no. 4, pp. 685–696, 2010.

[6] A. A. Mutlag, M. J. Uddin, A. I. Md. Ismail, and M. A. A. Hamad, "Scaling group transformation under the effect of thermal radiation heat transfer of a non newtonian power-law fluid over a vertical stretching sheet with momentum slip boundary condition," *Applied Mathematical Sciences*, vol. 6, no. 121-124, pp. 6035–6052, 2012.

[7] M. B. Akgül and M. Pakdemirli, "Lie group analysis of a non-Newtonian fluid flow over a porous surface," *Scientia Iranica*, vol. 19, no. 6, pp. 1534–1540, 2012.

[8] M. Jalil and S. Asghar, "Flow of power-law fluid over a stretching surface: a Lie group analysis," *International Journal of Non-Linear Mechanics*, vol. 48, pp. 65–71, 2013.

[9] A. G. Fareo and D. P. Mason, "Group invariant solution for a pre-existing fracture driven by a power-law fluid in impermeable rock," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 12, pp. 3298–3316, 2013.

[10] M. Molati, T. Hayat, and F. Mahomed, "Rayleigh problem for a MHD Sisko fluid," *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, vol. 10, no. 6, pp. 3428–3434, 2009.

[11] H. M. Mamboundou, M. Khan, T. Hayat, and F. Mahomed, "Reduction and solutions for magnetohydrodynamic flow of a sisko fluid in a porous medium," *Journal of Porous Media*, vol. 12, no. 7, pp. 695–714, 2009.

[12] T. Hayat, R. J. Moitsheki, and S. Abelman, "Stokes' first problem for Sisko fluid over a porous wall," *Applied Mathematics and Computation*, vol. 217, no. 2, pp. 622–628, 2010.

[13] G. Sari, M. Pakdemirli, T. Hayat, and Y. Aksoy, "Boundary layer equations and lie group analysis of a sisko fluid," *Journal of Applied Mathematics*, vol. 2012, Article ID 259608, 2012.

[14] M. Khan, A. Shahzad, A. Anjum, and F. M. Mahomed, "Analytic approximate solutions for time-dependent flow and heat transfer of a Sisko fluid," *International Journal of Numerical Methods*

for Heat Fluid Flow, vol. 24, no. 5, Article ID 17111470, pp. 1005–1019, 2014.

[15] M. Khan, S. Abelman, D. P. Mason, and F. M. Mahomed, "Self-similar unsteady flow of a Sisko fluid in a cylindrical tube undergoing translation," *Mathematical Problems in Engineering*, Article ID 135424, Art. ID 135424, 14 pages, 2015.

[16] K. S. Mekheimer, S. Z.-A. Husseny, A. T. Ali, and A. E. Abo-Elkhair, "Lie point symmetries and similarity solutions for an electrically conducting Jeffrey fluid," *Physica Scripta*, vol. 83, no. 1, Article ID 015017, 2011.

[17] Y. Aksoy, T. Hayat, and M. Pakdemirli, "Boundary layer theory and symmetry analysis of a Williamson fluid," *Zeitschrift fur Naturforschung - Section A Journal of Physical Sciences*, vol. 67, no. 6-7, pp. 363–368, 2012.

[18] N. A. Khan, S. Khan, and F. Riaz, "Boundary Layer Flow of Williamson Fluid with Chemically Reactive Species using Scaling Transformation and Homotopy Analysis Method," *Mathematical Sciences Letters*, vol. 3, no. 3, pp. 199–205, 2014.

[19] M. Yürüsoy, M. Pakdemirli, and O. F. Noyan, "Lie group analysis of creeping flow of a second grade fluid," *International Journal of Non-Linear Mechanics*, vol. 36, no. 6, pp. 955–960, 2001.

[20] M. Yürüsoy, "Similarity solutions for creeping flow and heat transfer in second grade fluids," *International Journal of Non-Linear Mechanics*, vol. 39, no. 4, pp. 665–672, 2004.

[21] K. Fakhar, C. Yi, J. Xiaoda, and L. Xiaodong, "Lie symmetry analysis and some new exact solutions for rotating flow of a second-order fluid on a porous plate," *International Journal of Engineering Science*, vol. 44, no. 13-14, pp. 889–896, 2006.

[22] A. A. Afify, "Some new exact solutions for MHD aligned creeping flow and heat transfer in second grade fluids by using Lie group analysis," *Nonlinear Analysis. Theory, Methods and Applications. An International Multidisciplinary Journal*, vol. 70, no. 9, pp. 3298–3306, 2009.

[23] N. A. Khan, A. Ara, M. Jamil, and A. Yildirim, "Travelling waves solution for MHD aligned flow of a second grade fluid with heat transfer: A symmetry independent approach," *Journal of King Saud University - Science*, vol. 24, no. 1, pp. 63–67, 2012.

[24] M. B. Abd-el-Malek and H. S. Hassan, "Symmetry analysis for steady boundary-layer stagnation-point flow of Rivlin-Ericksen fluid of second grade subject to suction," *Lithuanian Association of Nonlinear Analysts (LANA). Nonlinear Analysis. Modelling and Control*, vol. 15, no. 4, pp. 379–396, 2010.

[25] N. Islam, B. Islam, H. P. Mazumdar, and J. Lahiri, "Application of the Lie groups of transformations for an approximate solution of MHD flow of a visco-elastic second grade fluid," *APPS. Applied Sciences*, vol. 13, pp. 65–73, 2011.

[26] Y. Aksoy, M. Pakdemirli, and C. M. Khalique, "Boundary layer equations and stretching sheet solutions for the modified second grade fluid," *International Journal of Engineering Science*, vol. 45, no. 10, pp. 829–841, 2007.

[27] M. Pakdemirli, Y. g. Aksoy, M. Yürüsoy, and C. M. Khalique, "Symmetries of boundary layer equations of power-law fluids of second grade," *Acta Mechanica Sinica*, vol. 24, no. 6, pp. 661–670, 2008.

[28] K. S. Mekheimer, M. F. El-Sabbagh, and R. E. Abo-Elkhair, "Lie group analysis and similarity solutions for hydro-magnetic Maxwell fluid through a porous medium," *Boundary Value Problems*, 2012:15, 18 pages, 2012.

[29] C. Calmelet-Eluhu and V. Rosenhaus, "Symmetries and solution of a micropolar fluid flow through a cylinder," *Acta Mechanica*, vol. 147, no. 1-4, pp. 59–72, 2001.

[30] F. Shahzad, M. Sajid, T. Hayat, and M. Ayub, "Analytic solution for flow of a micropolar fluid," *Acta Mechanica*, vol. 188, no. 1-2, pp. 93–102, 2007.

[31] M. A. Mansour, R. A. Mohamed, M. M. Abd-Elaziz, and S. E. Ahmed, "Lie group Analysis of unsteady MHD mixed convection boundary layer flow of a micropolar fluid along a symmetric wedge with variable surface temperature saturated porous medium," *Int. J. Appl. Math and Mech*, vol. 5, no. 2, pp. 97–114, 2009.

[32] G. Saccomandi, "Group properties and invariant solutions of plane micropolar flows," *International Journal of Engineering Science*, vol. 29, no. 5, pp. 645–648, 1991.

[33] T. Hayat, M. Pakdemirli, and Y. g. Aksoy, "Similarity solutions for boundary layer equations of a Powel-Eyring fluid," *Mathematical and Computational Applications*, vol. 18, no. 1, pp. 62–70, 2013.

[34] M. Jalil and S. Asghar, "Flow and heat transfer of powell-eyring fluid over a stretching surface: A lie group analysis," *Journal of Fluids Engineering, Transactions of the ASME*, vol. 135, no. 12, Article ID 121201, 2013.

[35] M. Jalil, S. Asghar, and S. M. Imran, "Self similar solutions for the flow and heat transfer of Powell-Eyring fluid over a moving surface in a parallel free stream," *International Journal of Heat and Mass Transfer*, vol. 65, pp. 73–79, 2013.

[36] K. Fakhar, A. H. Kara, R. Morris, and T. Hayat, "Similarity solutions and conservation laws for rotating flows of an Oldroyd-B fluid," *Indian Journal of Physics*, vol. 87, no. 10, pp. 1035–1040, 2013.

[37] T. Hayat, A. H. Kara, and E. Momoniat, "Exact flow of a third-grade fluid on a porous wall," *International Journal of Non-Linear Mechanics*, vol. 38, no. 10, pp. 1533–1537, 2003.

[38] K. Fakhar, "Exact solutions for nonlinear partial differential equation arising in third grade fluid flows," *Southeast Asian Bulletin of Mathematics*, vol. 32, no. 1, pp. 65–70, 2008.

[39] K. Fakhar, Z. Xu, and C. Yi, "Exact solutions of a third grade fluid flow on a porous plate," *Applied Mathematics and Computation*, vol. 202, no. 1, pp. 376–382, 2008.

[40] M. Yürüsoy, "Similarity solutions of boundary layer equations for a special non-Newtonian fluid in a special coordinate system," *Applied Mathematics and Mechanics. English Edition*, vol. 25, no. 5, pp. 587–594, 2004.

[41] T. Hayat and A. H. Kara, "Couette flow of a third-grade fluid with variable magnetic field," *Mathematical and Computer Modelling*, vol. 43, no. 1-2, pp. 132–137, 2006.

[42] K. Fakhar, Z.-L. Xu, and Y. Cheng, "Hall effects on unsteady magnetohydrodynamic flow of a third grade fluid," *Chinese Physics Letters*, vol. 24, no. 5, article no. 001, pp. 1129–1132, 2007.

[43] K. Fakhar, A. H. Kara, I. Khan, and M. Sajid, "On the computation of analytical solutions of an unsteady magnetohydrodynamics flow of a third grade fluid with Hall effects," *Computers and Mathematics with Applications. An International Journal*, vol. 61, no. 4, pp. 980–987, 2011.

[44] T. Hayat, H. Mambili-Mamboundou, and F. M. Mahomed, "Unsteady solutions in a third-grade fluid filling the porous space," *Mathematical Problems in Engineering*, Article ID 139560, Art. ID 139560, 12 pages, 2008.

[45] T. Hayat, H. Mambili-Mamboundou, E. Momoniat, and F. M. Mahomed, "The Rayleigh problem for a third grade electrically conducting fluid in a magnetic field," *Journal of Nonlinear Mathematical Physics*, vol. 15, no. suppl. 1, pp. 77–90, 2008.

[46] K. Fakhar, A. A. Zainal, and A. H. Kara, "A note on the interplay between symmetries, reduction and conservation laws of Stokes' first problem for third-grade rotating fluids," *Pramana - Journal of Physics*, vol. 77, no. 3, pp. 439–445, 2011.

[47] T. Aziz, F. M. Mahomed, and A. Aziz, "Group invariant solutions for the unsteady MHD flow of a third grade fluid in a porous medium," *International Journal of Non-Linear Mechanics*, vol. 47, no. 7, pp. 792–798, 2012.

[48] A. Aziz and T. Aziz, "MHD flow of a third grade fluid in a porous half space with plate suction or injection: an analytical approach," *Applied Mathematics and Computation*, vol. 218, no. 21, pp. 10443–10453, 2012.

[49] T. Aziz, F. M. Mahomed, M. Ayub, and D. P. Mason, "Nonlinear time-dependent flow models of third grade fluids: a conditional symmetry approach," *International Journal of Non-Linear Mechanics*, vol. 54, pp. 55–65, 2013.

[50] T. Hayat, A. H. Kara, and E. Momoniat, "The unsteady flow of a fourth-grade fluid past a porous plate," *Mathematical and Computer Modelling*, vol. 41, no. 11-12, pp. 1347–1353, 2005.

[51] T. Aziz and F. M. Mahomed, "Reductions and solutions for the unsteady flow of a fourth grade fluid on a porous plate," *Applied Mathematics and Computation*, vol. 219, no. 17, pp. 9187–9195, 2013.

[52] T. Hayat, A. H. Kara, and E. Momoniat, "Travelling wave solutions to Stokes' problem for a fourth grade fluid," *Applied Mathematical Modelling. Simulation and Computation for Engineering and Environmental Systems*, vol. 33, no. 3, pp. 1613–1619, 2009.

[53] T. Aziz and F. M. Mahomed, "Closed-form solutions for a nonlinear partial differential equation arising in the study of a fourth grade fluid model," *Journal of Applied Mathematics*, vol. 2012, Article ID 931587, 2012.

[54] T. Hayat, H. Mambili-Mamboundou, C. M. Khalique, and F. M. Mahomed, "Effect of magnetic field on the flow of a fourth order fluid," *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, vol. 10, no. 6, pp. 3413–3419, 2009.

[55] T. Hayat, H. Mambili-Mamboundou, and F. M. Mahomed, "A note on some solutions for the flow of a fourth grade fluid in a porous space," *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, vol. 10, no. 1, pp. 368–374, 2009.

[56] T. Aziz, A. B. Magan, and F. M. Mahomed, "Invariant Solutions for the Unsteady Magnetohydrodynamics (MHD) Flow of a Fourth-Grade Fluid Induced Due to the Impulsive Motion of a Flat Porous Plate," *Brazilian Journal of Physics*, vol. 45, no. 1, pp. 120–131, 2014.

[57] T. Aziz, A. Fatima, A. Aziz, and F. M. Mahomed, "Group theoretical analysis and invariant solutions for unsteady flow of a fourth-grade fluid over an infinite plate undergoing impulsive motion in a darcy porous medium," *Zeitschrift fur Naturforschung - Section A Journal of Physical Sciences*, vol. 70, no. 7, pp. 483–498, 2015.

[58] D. Srinivasacharya, N. Srinivasacharyulu, and O. Odelu, "Flow and heat transfer of couple stress fluid in a porous channel with expanding and contracting walls," *International Communications in Heat and Mass Transfer*, vol. 36, no. 2, pp. 180–185, 2009.

[59] K. Atalık, "Group theoretical analysis and similarity solutions for stress boundary layers in viscoelastic flows," *Journal of Non-Newtonian Fluid Mechanics*, vol. 153, no. 1, pp. 62–71, 2008.

[60] M. A. Hamad and I. Pop, "Scaling transformations for boundary layer flow near the stagnation-point on a heated permeable stretching surface in a porous medium saturated with a nanofluid and heat generation/absorption effects," *Transport in Porous Media*, vol. 87, no. 1, pp. 25–39, 2011.

[61] R. Kandasamy, P. Loganathan, and P. P. P. Arasu, "Scaling group transformation for MHD boundary-layer flow of a nanofluid past a vertical stretching surface in the presence of suction/injection," *Nuclear Engineering and Design*, vol. 241, no. 6, pp. 2053–2059, 2011.

[62] M. A. Hamad, I. Pop, and A. I. Ismail, "Magnetic field effects on free convection flow of a nanofluid past a vertical semi-infinite flat plate," *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, vol. 12, no. 3, pp. 1338–1346, 2011.

[63] R. Kandasamy, I. Muhaimin, A. B. Khamis, and R. B. Roslan, "Unsteady Hiemenz flow of Cu-nanofluid over a porous wedge in the presence of thermal stratification due to solar energy radiation: Lie group transformation," *International Journal of Thermal Sciences*, vol. 65, pp. 196–205, 2013.

[64] A. B. Rosmila, R. Kandasamy, and I. Muhaimin, "Lie symmetry group transformation for MHD natural convection flow of nanofluid over linearly porous stretching sheet in presence of thermal stratification," *Applied Mathematics and Mechanics. English Edition*, vol. 33, no. 5, pp. 593–604, 2012.

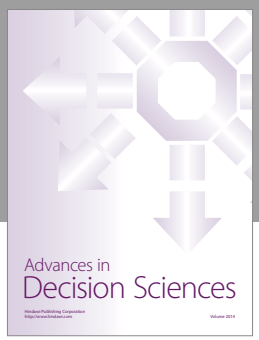
[65] K. Das, "Lie group analysis for nanofluid flow past a convectively heated stretching surface," *Applied Mathematics and Computation*, vol. 221, pp. 547–557, 2013.

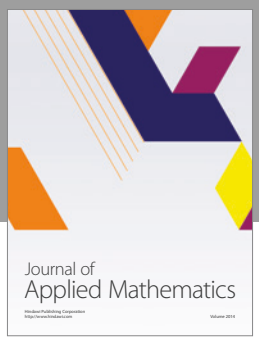
[66] M. N. Tufail, A. S. Butt, and A. Ali, "Heat source/sink effects on non-Newtonian MHD fluid flow and heat transfer over a permeable stretching surface: lie group analysis," *Indian Journal of Physics*, vol. 88, no. 1, pp. 75–82, 2014.

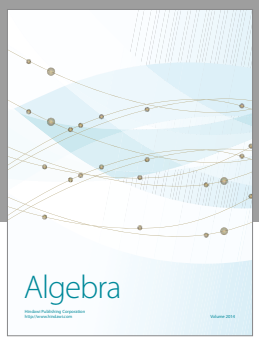
[67] R. Naz, F. M. Mahomed, and D. P. Mason, "Comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics," *Applied Mathematics and Computation*, vol. 205, no. 1, pp. 212–230, 2008.

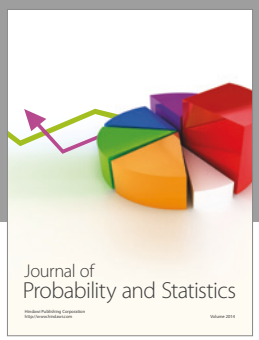
[68] R. Naz, D. P. Mason, and F. M. Mahomed, "Conservation laws and conserved quantities for laminar two-dimensional and radial jets," *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, vol. 10, no. 5, pp. 2641–2651, 2009.

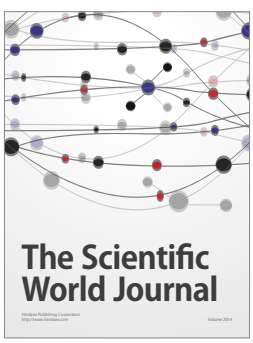
[69] R. Naz, F. M. Mahomed, and D. P. Mason, "Conservation laws via the partial Lagrangian and group invariant solutions for radial and two-dimensional free jets," *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, vol. 10, no. 6, pp. 3457–3465, 2009.











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