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Research Article

Uniqueness of the Minimal l_1-Norm Solution to the Monotone Linear Complementarity Problem

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The linear complementarity problem (LCP) has wide applications in economic equilibrium, operations research, and so on, which attracted a lot of interest of experts. Finding the sparsest solution to the LCP has real applications in the field of portfolio selection and bimatrix game. Motivated by the approach developed in compressive sensing, we may try to solve an l_1-minimization problem to obtain the sparsest solution to the LCP, where an important theoretical problem is to investigate uniqueness of the solution to the concerned l_1-minimization problem. In this paper, we investigate the problem of finding the minimal l_1-norm solution to the monotone LCP and propose a sufficient and necessary condition for the uniqueness of the minimal l_1-norm solution to the monotone LCP, which provides an important theoretical basis for finding the sparsest solution to the monotone LCP via solving the corresponding l_1-minimization problem. Furthermore, several examples are given to confirm our theoretical finding.

1. Introduction

The linear complementarity problem (LCP) can be described as follows; we want to find a vector $x \in \mathbb{R}^n$ such that

$$\begin{aligned} Ax + b &\geq 0, \\ x &\geq 0, \\ x^\top (Ax + b) &= 0, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are given. If A is positive semidefinite, then problem (1) is called a monotone LCP. Instances of the LCP can date back to 1940, but it became a scientific direction of study until the mid-1960s. This class of problems has many applications in economics and engineering [1–4]. In this paper, we denote (1) by the LCP(b, A) and the solution set of the LCP(b, A) by SOL(b, A).

Compressive sensing (CS) has attracted plenty of attention in the fields of signal processing and machine learning [5, 6]. The basic model of CS can be described as the problem of seeking the sparsest solution under linear equality constraints; that is,

$$\begin{aligned} \min \quad & \|x\|_0 \\ \text{s.t.} \quad & Qx = q, \end{aligned} \tag{2}$$

where $Q \in \mathbb{R}^{m\times n}$ $(m \ll n)$, $q \in \mathbb{R}^n$, and $\|x\|_0$ denotes the number of nonzero components of the vector $x \in \mathbb{R}^n$. Since (2) is known to be an NP-hard problem, a tractable approximation is the commonly considered l_1-minimization problem:

$$\begin{aligned} \min \quad & \|x\|_1 \\ \text{s.t.} \quad & Qx = q, \end{aligned} \tag{3}$$

where $\|x\|_1$ is the l_1-norm of x, that is, the sum of each component's absolute value.

Problem (3) is a convex program which can be reformulated as a linear program [7]. In most cases, l_1-minimization problem can find the sparsest solution to the underdetermined linear system effectively from a lot of empirical results. Recently, the theoretical results with respect to the equivalence of l_1-minimization problem and l_0-minimization problem have made significant progress and led to a sharp increase of research in CS [8–10]. In detail, the mutual-coherence-based analysis [11], the null space property (NSP) [12], the restricted isometry property (RIP) [10], and the range space property (RSP) [13] can guarantee the equivalence of problems (2) and (3), where uniqueness of problem (3) plays an important role in the theoretical analysis.

Problem (2) is to find the sparest solution to the system of equalities. However, only the equalities may be not enough for practical problems sometimes. As the deepening of the research, the nonnegative sparsest solution to the underdetermined linear system of equations has attracted people's interesting, and some theoretical results have been obtained, such as the RIP [14], the NSP [15], and the RSP [16]. Furthermore, Zhang et al. [17] proposed the conditions to guarantee the equivalence between the l_0-norm solution to the system of absolute value equations and its l_1-norm solution, where a key is to characterize uniqueness of the solution to the corresponding l_1-minimization problem.

Seeking the minimal l_0-norm solution of the LCP has many applications in real world, such as portfolio selection [18] and bimatrix game [1]. Chen and Xiang [19] investigated the characterization and computation of sparse solutions and least-p-norm ($p \in (0,1)$) solutions to the LCP and provided some conditions on the involved matrix such that a sparse solution can be found by solving a convex minimization problem. Shang et al. [20] studied minimal l_0-norm solutions of the LCP. They investigated the approximation of the concerned problem by using p-norm ($p \in (0,1)$) and proposed a sequential smoothing gradient method to solve it.

It is well-known that the existence and uniqueness of the solution to the LCP(b, A) play an important role in theory and algorithms for solving the LCP(b, A) [1–3]. Motivated by the researches mentioned above, in this paper, we consider the following l_1-minimization problem:

$$\begin{aligned} \min \quad & \|x\|_1 \\ \text{s.t.} \quad & Ax + b \geq 0, \\ & x \geq 0, \\ & x^\top (Ax + b) = 0, \end{aligned} \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$ is positive semidefinite and $b \in \mathbb{R}^n$. It is a relaxation of the problem to find the sparsest solution to the LCP:

$$\begin{aligned} \min \quad & \|x\|_0 , \\ \text{s.t.} \quad & Ax + b \geq 0, \\ & x \geq 0, \\ & x^\top (Ax + b) = 0. \end{aligned} \tag{5}$$

We will propose a sufficient and necessary condition to guarantee uniqueness of the solution to problem (4), which provides an important theoretical basis in studying the equivalence between problems (4) and (5) so that the sparest solution to the monotone LCP (1) can be found by solving problem (4).

This paper is organized as follows. Some basic concepts and results for the LCP are stated in Section 2. In Section 3, we investigate the necessary conditions for uniqueness of the solution to problem (4) if it has at least a solution. In Section 4, we propose a sufficient and necessary condition for uniqueness of the minimal l_1-norm solution to the monotone LCP. In Section 5, we give two specific examples to confirm the theoretical result. Conclusions and further work are arrived at the last section.

Now, we outline the notations that will be used in this paper. We denote $[n] := \{1, 2, \ldots, n\}$ throughout this paper. For any matrix $M \in \mathbb{R}^{m\times n}$, index sets $I \subseteq [m]$ and $J \subseteq [n]$, $M^I \in \mathbb{R}^{|I|\times n}$ denotes the submatrix of matrix M with rows in I, $M_J \in \mathbb{R}^{m\times|J|}$ denotes the submatrix of M with columns in J, and $M_{I\times J} \in \mathbb{R}^{|I|\times|J|}$ is the submatrix with rows in I and columns in J, where $|I|(|J|)$ denotes the number of the elements in $I(J)$. For any vector $x \in \mathbb{R}^n$ and an index set $I \subseteq [n]$, $x_I \in \mathbb{R}^{|I|}$ denotes the vector with components of x in set I. For the vector $x^* \in \mathbb{R}^n$ with $x^* \geq 0$, we define index sets I_+ and I_0 by $I_+ := \{i \in [n] \mid x_i^* > 0\}$ and $I_0 := \{i \in [n] \mid x_i^* = 0\}$, respectively. $e \in \mathbb{R}^n$ denotes the vector whose components are all one. For simplicity, we denote $(x^\top, y^\top)^\top$ as (x, y) for any vectors $x, y \in \mathbb{R}^n$.

2. Preliminaries

For the LCP(b, A) (1), the following results are significant in our sequential analysis.

Theorem 1 (Cottle et al. [1]). *Let $A \in \mathbb{R}^{n\times n}$ be positive semidefinite and $b \in \mathbb{R}^n$. Then,*

(a) *if the LCP(b, A) (1) is feasible, that is, there exists a vector $\overline{x} \in \mathbb{R}^n$ such that $\overline{x} \geq 0$ and $A\overline{x} \geq b$ hold, then the LCP(b, A) (1) is solvable; that is,* SOL$(b, A) \neq \emptyset$*;*

(b) *if the LCP(b, A) has a solution, then SOL(b, A) is polyhedral and*

$$\begin{aligned} SOL(b, A) = \{ & x \in \mathbb{R}^n \mid x \geq 0,\ b + Ax \geq 0,\ b^\top (x - \overline{x}) \\ & = 0,\ (A + A^\top)(x - \overline{x}) = 0\}, \end{aligned} \tag{6}$$

where $\overline{x} \in \mathbb{R}^n$ is an arbitrary solution.

Suppose that $A \in \mathbb{R}^{n\times n}$ is positive semidefinite and the LCP(b, A) has a solution, denoted by $\overline{x} \in \mathbb{R}^n$; then by using Theorem 1(b), we have

$$\begin{aligned} & \mathrm{SOL}(b, A) \\ & \quad = \{x \mid Ax + b \geq 0,\ x \geq 0,\ Ax = u,\ b^\top x = v\}, \end{aligned} \tag{7}$$

where $u = A\overline{x}$ and $v = b^\top \overline{x}$.

Thus, in order to investigate uniqueness of the sparsest solution to the monotone LCP, we consider the following problem:

$$\begin{aligned} \min \quad & \|x\|_0 \\ \text{s.t.} \quad & Ax = c, \\ & b^\top x = d, \\ & Ax + b \geq 0, \\ & x \geq 0, \end{aligned} \tag{8}$$

where $A \in \mathbb{R}^{n\times n}$ is positive semidefinite, $b, c \in \mathbb{R}^n$, and $d \in \mathbb{R}$. Since problem (8) is NP-hard, we consider its relaxed problem:

$$\begin{aligned} \min \quad & \|x\|_1 \\ \text{s.t.} \quad & Ax = c, \\ & b^\top x = d, \\ & Ax + b \geq 0, \\ & x \geq 0, \end{aligned} \tag{9}$$

which can be rewritten as follows:

$$\begin{aligned} \min \quad & \|x\|_1 \\ \text{s.t.} \quad & \begin{pmatrix} A \\ b^\top \end{pmatrix} x = \begin{pmatrix} c \\ d \end{pmatrix}, \\ & Ax \geq -b, \\ & x \geq 0. \end{aligned} \tag{10}$$

Suppose that x^* is an optimal point of problem (10). Then, for inequality constraint $Ax \geq -b$, we denote the index set of the active constraints at point x^* by I^* and the index set of the inactive constraints by $\widehat{I}^*$, which means

$$\begin{aligned} A^{I^*} x^* &= -b_{I^*}, \\ A^{\widehat{I}^*} x^* &> -b_{\widehat{I}^*}. \end{aligned} \tag{11}$$

These notations will be used throughout this paper.

At the end of this section, we recall a classical theory for linear programming (LP). Consider the LP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Mx = p, \\ & x \geq 0 \end{aligned} \tag{12}$$

and its dual

$$\begin{aligned} \max \quad & p^\top y \\ \text{s.t.} \quad & M^\top y + z = c, \\ & z \geq 0, \end{aligned} \tag{13}$$

where M is a given matrix and p and c are two given vectors. The following result is significant for our analysis.

Theorem 2. *If the LP (12) and its dual (13) are both feasible, then there exists a pair of strictly complementary solutions $(x^*, (y^*, z^*))$ satisfying $x^* + z^* > 0$.*

In the following, we will discuss some conditions for uniqueness of the solution to problem (10), which further leads to a sufficient and necessary conditions for uniqueness of the minimal l_1-norm solution to the monotone LCP.

3. Necessary Conditions

In this section, we give two necessary conditions for uniqueness of the solution to problem (10) under the assumption that it has at least a solution.

Suppose that x^* is an optimal point to problem (10); then problem (10) has a unique optimal solution if and only if

$$\begin{aligned} &\{w \mid Aw = c,\ b^\top w = d,\ Aw \geq -b,\ w \geq 0,\ \|w\|_1 \\ &\quad \leq \|x^*\|_1\} = \{x^*\}. \end{aligned} \tag{14}$$

Since $x^* \geq 0$ and $w \geq 0$, it follows that $\|x^*\|_1 = e^\top x^*$ and $\|w\|_1 = e^\top w$. Thus the above relationship can be written as

$$\begin{aligned} &\{w \mid Aw = Ax^*,\ b^\top w = b^\top x^*,\ Aw \geq -b,\ w \\ &\quad \geq 0,\ e^\top w \leq e^\top x^*\} = \{x^*\}. \end{aligned} \tag{15}$$

So we have the following lemma.

Lemma 3. *Suppose that x^* is an optimal point to problem (10); then problem (10) has a unique optimal solution if and only if $(w, s, t) = (x^*, Ax^* + b, 0)$ is the unique optimal point of the following problem:*

$$\begin{aligned} \min \quad & 0^\top w \\ \text{s.t.} \quad & \begin{pmatrix} A \\ b^\top \end{pmatrix} w = \begin{pmatrix} A \\ b^\top \end{pmatrix} x^*, \\ & -Aw + s = b, \\ & e^\top w + t = e^\top x^*, \\ & w, s, t \geq 0. \end{aligned} \tag{16}$$

The dual problem of problem (16) is given by

$$\begin{aligned} \max \quad & \left[\begin{pmatrix} A \\ b^\top \end{pmatrix} x^*\right]^\top y_1 + b^\top y_2 + \left(e^\top x^*\right)\sigma \\ \text{s.t.} \quad & \begin{pmatrix} A^\top & b \end{pmatrix} y_1 + (-A)^\top y_2 + \sigma e \leq 0, \\ & y_2 \leq 0, \\ & \sigma \leq 0. \end{aligned} \tag{17}$$

Furthermore, by introducing slack variables r_1, r_2, r, problem (17) can be converted to the following problem:

$$\begin{aligned} \max \quad & \left[\begin{pmatrix} A \\ b^\top \end{pmatrix} x^*\right]^\top y_1 + b^\top y_2 + (e^\top x^*)\sigma \\ \text{s.t.} \quad & (A^\top \ \ b)\, y_1 + (-A)^\top y_2 + \sigma e + r_1 = 0, \\ & y_2 + r_2 = 0, \\ & \sigma + r = 0, \\ & r_1, r_2, r \ge 0. \end{aligned} \tag{18}$$

Based on problems (16) and (18), by using the dual theory of the LP, we will show the following result.

Theorem 4. *If x^* is the unique optimal point of problem (10) with I^* being defined by (11), then there exist $\zeta \in \mathbb{R}^{|I^*|}$ and $\eta \in \mathbb{R}^n$ such that*

$$\begin{aligned} & A_{I^*}^\top \zeta + \eta \in \mathit{Range}\ \left((A^\top \ \ b)\right), \\ & \zeta < 0, \\ & \eta_i = 1, \quad \forall i \in I_+, \\ & \eta_i < 1, \quad \forall i \in I_0. \end{aligned} \tag{19}$$

Proof. Since problems (16) and (18) are both feasible, there exists a pair of strictly complementary solutions of (16) and (18) by Theorem 2. Denote the strictly complementary solution pair by (w, s, t) and $(r_1, r_2, r) = (-[(A^\top \ b) y_1 + (-A)^\top y_2 + \sigma e], -y_2, -\sigma)$. Then, we have

$$\begin{aligned} w^\top r_1 &= 0, \\ s^\top r_2 &= 0, \\ tr &= 0, \\ w + r_1 &> 0, \\ s + r_2 &> 0, \\ t + r &> 0. \end{aligned} \tag{20}$$

Since x^* is the unique optimal point of the problem (10), it follows from Lemma 3 that $(w, s, t) = (x^*, Ax^* + b, 0)$. Thus, $w_i > 0$ if $i \in I_+$ and $w_i = 0$ if $i \in I_0$ and $t = 0$. These imply

$$(r_1)_i \begin{cases} = 0, & \text{if } i \in I_+, \\ > 0, & \text{if } i \in I_0, \end{cases} \qquad r > 0. \tag{21}$$

That is,

$$[(A^\top \ \ b)\, y_1 + (-A)^\top y_2 + \sigma e]_i \begin{cases} = 0, & \text{if } i \in I_+, \\ < 0, & \text{if } i \in I_0, \end{cases} \qquad \sigma < 0. \tag{22}$$

Furthermore, since $s = Ax^* + b$, it follows that $(s_{I^*})_i = 0$ if $i \in I^*$ and $(s_{\widehat{I}^*})_i > 0$ if $i \in \widehat{I}^*$, which imply that

$$(y_2)_i \begin{cases} = 0, & \text{if } i \in \widehat{I}^*, \\ < 0, & \text{if } i \in I^*. \end{cases} \tag{23}$$

Substituting (23) into (22), we have

$$\begin{aligned} & [(A^\top \ \ b)\, y_1 + (-A)_{I^*}^\top (y_2)_{I^*} + \sigma e]_i = 0, \\ & \qquad\qquad (y_2)_{I^*} < 0 \text{ when } i \in I_+, \\ & [(A^\top \ \ b)\, y_1 + (-A)_{I^*}^\top (y_2)_{I^*} + \sigma e]_i < 0, \\ & \qquad\qquad (y_2)_{I^*} < 0 \text{ when } i \in I_0. \end{aligned} \tag{24}$$

Let $\zeta = -(y_2)_{I^*}/\sigma$; then $\zeta < 0$. So we have

$$\begin{aligned} & \left[(A^\top \ \ b)\, \frac{y_1}{-\sigma} - A_{I^*}^\top \zeta - e\right]_i = 0 \quad \text{when } i \in I_+, \\ & \left[(A^\top \ \ b)\, \frac{y_1}{-\sigma} - A_{I^*}^\top \zeta - e\right]_i < 0 \quad \text{when } i \in I_0. \end{aligned} \tag{25}$$

By setting $\eta = (A^\top \ \ b)(y_1/-\sigma) + A_{I^*}^\top((y_2)_{I^*}/\sigma)$, then we have

$$\eta + A_{I^*}^\top \zeta = (A^\top \ \ b)\, \frac{y_1}{-\sigma} \in \text{Range}\ \left((A^\top \ \ b)\right), \tag{26}$$

and by (25),

$$\begin{aligned} (\eta - e)_i &= 0 \quad \text{if } i \in I_+, \\ (\eta - e)_i &< 0 \quad \text{if } i \in I_0. \end{aligned} \tag{27}$$

These, together with $\zeta < 0$, imply the desired result. □

In this paper, condition (19) given in Theorem 4 is called the range space property (RSP).

Now we establish the other necessary condition.

Theorem 5. *If x^* is the unique optimal point of problem (10) with I^* being defined by (11), then*

$$H = \begin{pmatrix} A_{I_+} \\ b_{I_+}^\top \\ A_{I^* \times I_+} \end{pmatrix} \tag{28}$$

has column full rank.

Proof. Define

$$\widehat{H} = \begin{pmatrix} A_{I_+} \\ b_{I_+}^\top \\ A_{I^* \times I_+} \\ e_{I_+}^\top \end{pmatrix}; \tag{29}$$

we first show that the matrix $\widehat{H}$ has column full rank. Suppose that $\widehat{H}$ does not have column full rank; then there must exist a nonzero vector ν such that $\widehat{H}\nu = 0$. On the one hand, let (w, s, t) be given by $w = (w_{I_+}, w_{I_0}) = (x^*_{I_+}, 0)$, $s = Aw + b$, and $t = 0$; it is easy to see that (w, s, t) is an optimal point to problem (16). On the other hand, let $(\overline{w}, \overline{s}, \overline{t})$ be given by $\overline{w} = (\overline{w}_{I_+}, \overline{w}_{I_0}) = (w_{I_+} + \lambda\nu, 0)$, $\overline{s} = A\overline{w} + b$, and $\overline{t} = 0$. Since $w_{I_+} = x^*_{I_+} > 0$, there must exist a small $\lambda \neq 0$ satisfying $\overline{w}_{I_+} = w_{I_+} + \lambda\nu \geq 0$ and $\overline{s} = A\overline{w} + b \geq 0$. Thus $(\overline{w}, \overline{s}, \overline{t})$ is also an optimal point to problem (16). However, since $\overline{w} \neq w$, we obtain two different optimal points of problem (16), which contradicts Lemma 3. Thus, the matrix $\widehat{H}$ defined by (29) has column full rank.

Since x^* is the unique optimal point of problem (10), it follows from Theorem 4 that condition (19) holds, which further implies that there exist θ and ζ such that

$$e^\top_{I_+} = \theta^\top \begin{pmatrix} A \\ b^\top \end{pmatrix}_{I_+} - \zeta^\top A_{I^* \times I_+}. \tag{30}$$

Thus, we have $\operatorname{rank}(H) = \operatorname{rank}(\widehat{H})$, which leads to the desired result. □

4. Sufficient and Necessary Condition

In this section, we first discuss the sufficient and necessary condition for uniqueness of the solution to problem (10), which is given as follows.

Theorem 6. *Suppose that x^* is an optimal point to problem (10) with I^* being defined by (11). Then, x^* is the unique optimal point of problem (10) if and only if the matrix H defined by (28) has column full rank and there exist $\zeta \in \mathbb{R}^{|I^*|}$ and $\eta \in \mathbb{R}^n$ such that RSP condition (19) holds.*

Proof. By Theorems 4 and 5, we only need to prove the sufficiency of Theorem 6. Suppose that there exist $\zeta \in \mathbb{R}^{|I^*|}$ and $\eta \in \mathbb{R}^n$ such that RSP condition (19) holds. Then, there exists θ such that $\eta = (A^\top \;\; b)\theta - A^\top_{I^*}\zeta$. Since $\eta_i = 1$ for any $i \in I_+$ and $\eta_i < 1$ for any $i \in I_0$, by letting $\sigma = -1$, we have

$$\begin{aligned} &[(A^\top \;\; b)\,\theta - A^\top_{I^*}\zeta]_i + \sigma = 0, \quad \text{if } i \in I_+, \\ &[(A^\top \;\; b)\,\theta - A^\top_{I^*}\zeta]_i + \sigma < 0, \quad \text{if } i \in I_0. \end{aligned} \tag{31}$$

Let

$$\begin{aligned} y_1 &= \theta, \\ (y_2)_{I^*} &= \zeta, \\ (y_2)_{\widehat{I}^*} &= 0, \\ \sigma &= -1. \end{aligned} \tag{32}$$

Clearly, (y_1, y_2, σ) is a feasible point of problem (17); then we prove that (y_1, y_2, σ) is an optimal point of problem (17). In fact, the objective function value of problem (17) at (y_1, y_2, σ) can be calculated as follows:

$$\begin{aligned} &\left[\begin{pmatrix} A \\ b^\top \end{pmatrix} x^*\right]^\top y_1 + b^\top y_2 + (e^\top x^*)\,\sigma \\ &= x^{*\top}(A^\top \;\; b)\,y_1 + b^\top y_2 + (e^\top x^*)\,\sigma \\ &= \sum_{i \in I_+} x^*_i\,[(A^\top \;\; b)\,y_1]_i + \sigma \sum_{i \in I_+} x^*_i + b^\top y_2 \\ &= \sum_{i \in I_+} x^*_i\left([(A^\top \;\; b)\,y_1]_i + \sigma\right) + b^\top y_2 \\ &= \sum_{i \in I_+} x^*_i\,(A^\top_{I^*}\zeta)_i + b^\top y_2 = x^{*\top}_{I_+} A^\top_{I_+ \times I^*}\zeta + b^\top y_2 \\ &= x^{*\top}_{I_+} A^\top_{I_+ \times I^*}\zeta + b^\top_{I^*}\zeta = \left(A_{I^* \times I_+} x^*_{I_+} + b_{I^*}\right)^\top \zeta = 0. \end{aligned} \tag{33}$$

By weak duality, the maximal value of the dual problem (17) is 0. Therefore, the point (y_1, y_2, β) is an optimal point to problem (17). Then the slack variables in (18) have the properties which are described in the following:

$$\begin{aligned} (r_1)_i &\begin{cases} = 0, & \forall i \in I_+, \\ > 0, & \forall i \in I_0, \end{cases} \\ (r_2)_i &\begin{cases} = 0, & \forall i \in \widehat{I}^*, \\ > 0, & \forall i \in I^*, \end{cases} \\ r &= 1. \end{aligned} \tag{34}$$

Suppose that (w, s, t) is an arbitrary optimal point of problem (16), in order to prove the uniqueness of the optimal point x^* for the problem (10); we just need to verify $(w, s, t) = (x^*, Ax^* + b, 0)$ by Lemma 3. Using Theorem 2, we have

$$\begin{aligned} w^\top[-(A^\top \;\; b)\,y_1 + A^\top y_2 - \sigma e] &= 0, \\ s^\top(-y_2) &= 0, \\ t\,(-\sigma) &= 0, \\ w + [-(A^\top \;\; b)\,y_1 + A^\top y_2 - \sigma e] &> 0, \\ s + (-y_2) &> 0, \\ t + (-\sigma) &> 0. \end{aligned} \tag{35}$$

Using (34), we obtain that

$$\begin{aligned} w_i &= 0, \quad \forall i \in I_0; \\ s_i &= 0, \quad \forall i \in I^*; \\ t &= 0. \end{aligned} \tag{36}$$

According to the constraints of problem (16), we have

$$\begin{aligned} \begin{pmatrix} A \\ b^\top \end{pmatrix}_{I_+} w_{I_+} &= \begin{pmatrix} A \\ b^\top \end{pmatrix}_{I_+} x^*_{I_+}, \\ A_{I^* \times I_+} w_{I_+} &= A_{I^* \times I_+} x^*_{I_+}. \end{aligned} \tag{37}$$

Let $d = (w - x^*)_{I_+}$, then we have

$$\begin{gathered}\begin{pmatrix} A \\ b^\top \end{pmatrix}_{I_+} d = 0, \\ A_{I^* \times I_+} d = 0,\end{gathered} \tag{38}$$

which mean $Hd = 0$. Since H has column full rank, we have $d = 0$; that is, $w_{I_+} = x^*_{I_+}$. Furthermore, we can obtain $w = x^*$ and $s = Ax^* + b$. □

By using Theorems 1(a) and 6, it is easy to obtain the sufficient and necessary condition for uniqueness of the solution to the monotone LCP, which is given as follows.

Theorem 7. *Let $A \in \mathbb{R}^{n\times n}$ be positive semidefinite and $b \in \mathbb{R}^n$. Suppose that there exists a vector $\overline{x} \in \mathbb{R}^n$ such that $\overline{x} \geq 0$ and $A\overline{x} \geq b$ hold; then SOL$(b, A) \neq \emptyset$. Furthermore, let $x^* \in$ SOL(b, A) with I^* being defined by (11); then x^* is the unique optimal point of problem (4) if and only if the matrix H defined by (28) has column full rank, and there exist $\zeta \in \mathbb{R}^{|I^*|}$ and $\eta \in \mathbb{R}^n$ such that RSP condition (19) holds.*

5. Examples

In this section, we give two examples to verify the obtained sufficient and necessary condition for uniqueness of the minimal l_1-norm solution to the monotone LCP.

Example 8. Consider the LCP(b, A), where

$$\begin{gathered}A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \\ b = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.\end{gathered} \tag{39}$$

It is easy to find that

$$x^* = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \tag{40}$$

is the unique minimal l_1-norm solution of the LCP(b, A). In the following, we show that all conditions in Theorem 7 hold for this problem.

First, it is easy to see that the matrix A is positive semidefinite.

Second, it is easy to obtain that

$$\begin{gathered}I_+ = \{1, 3\}, \\ I_0 = \{2\}, \\ I^* = \{1, 3\}, \\ \widehat{I}^* = \{2\},\end{gathered} \tag{41}$$

and hence, the matrix H defined by (28) can be written as

$$H = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{42}$$

Obviously, H has column full rank.

Third, there exist

$$\begin{gathered}\zeta = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \\ \eta = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}, \\ \theta = \begin{pmatrix} \frac{5}{2} \\ 2 \\ \frac{3}{4} \\ \frac{3}{4} \end{pmatrix},\end{gathered} \tag{43}$$

such that $A_{I^*}^\top \zeta + \eta = (A^\top \;\; b)\theta \in \text{Range}((A^\top \;\; b))$, which shows that the RSP conditions (19) hold at x^*.

Example 9. Consider the LCP(b, A), where

$$\begin{gathered}A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ b = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.\end{gathered} \tag{44}$$

It is easy to find that

$$\begin{gathered}x^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ x^{(2)} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\end{gathered} \tag{45}$$

are two minimal l_1-norm solutions of the LCP(b, A), so the minimal l_1-norm solution is not unique. In the following, we show that some condition in Theorem 7 does not hold for this problem.

If we take $x^* := x^{(1)}$, then

$$I_+ = \{1\},\quad I_0 = \{2\},\quad I^* = \{1,2\},\quad \widehat{I}^* = \emptyset. \tag{46}$$

In this case, we show that the RSP conditions (19) do not hold. In fact, if the RSP conditions hold at $x^{(1)}$, then we have $\eta = \binom{1}{\epsilon}$ with $|\epsilon| < 1$. Denote $\zeta = (\zeta_1\ \ \zeta_2)^\top < 0$ and $\theta = (\theta_1\ \ \theta_2\ \ \theta_3)^\top$; then from $A_{I^*}^\top \zeta + \eta = (A^\top\ \ b)\theta \in \text{Range}((A^\top\ \ b))$, we can obtain

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} + \begin{pmatrix} 1 \\ \epsilon \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}; \tag{47}$$

that is,

$$\begin{aligned} \zeta_1 + \zeta_2 + 1 &= \theta_1 + \theta_2 - \theta_3, \\ \zeta_1 + \zeta_2 + \epsilon &= \theta_1 + \theta_2 - \theta_3, \end{aligned} \tag{48}$$

which is impossible. So the RSP conditions (19) do not hold in this case.

If we take $x^* := x^{(2)}$, then

$$I_+ = \{1,2\},\quad I_0 = \emptyset,\quad I^* = \{1,2\},\quad \widehat{I}^* = \emptyset. \tag{49}$$

In this case, the matrix H defined by (28) is

$$H = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{50}$$

Obviously, H does not have column full rank.

6. Conclusions

In this paper, we investigated the uniqueness of the minimal l_1-norm solution to the monotone LCP. By using an equivalence reformulation, we proposed a sufficient and necessary condition to guarantee uniqueness of the minimal l_1-norm solution to the monotone LCP, which provide an important theoretical basis for finding the sparsest solution to the monotone LCP.

Some issues are worth studying in our further research. It is meaningful to investigate various recovery conditions and design high-efficiency algorithms to seek the minimal l_1-norm solution to the monotone LCP. Moreover, since the nonconvex relaxation methods have been studied extensively in the last years (see, e.g., [19–23] and references therein), it is also worth investigating theory and algorithms for solving some nonconvex relaxation problems of finding the sparsest solution to the monotone LCP.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

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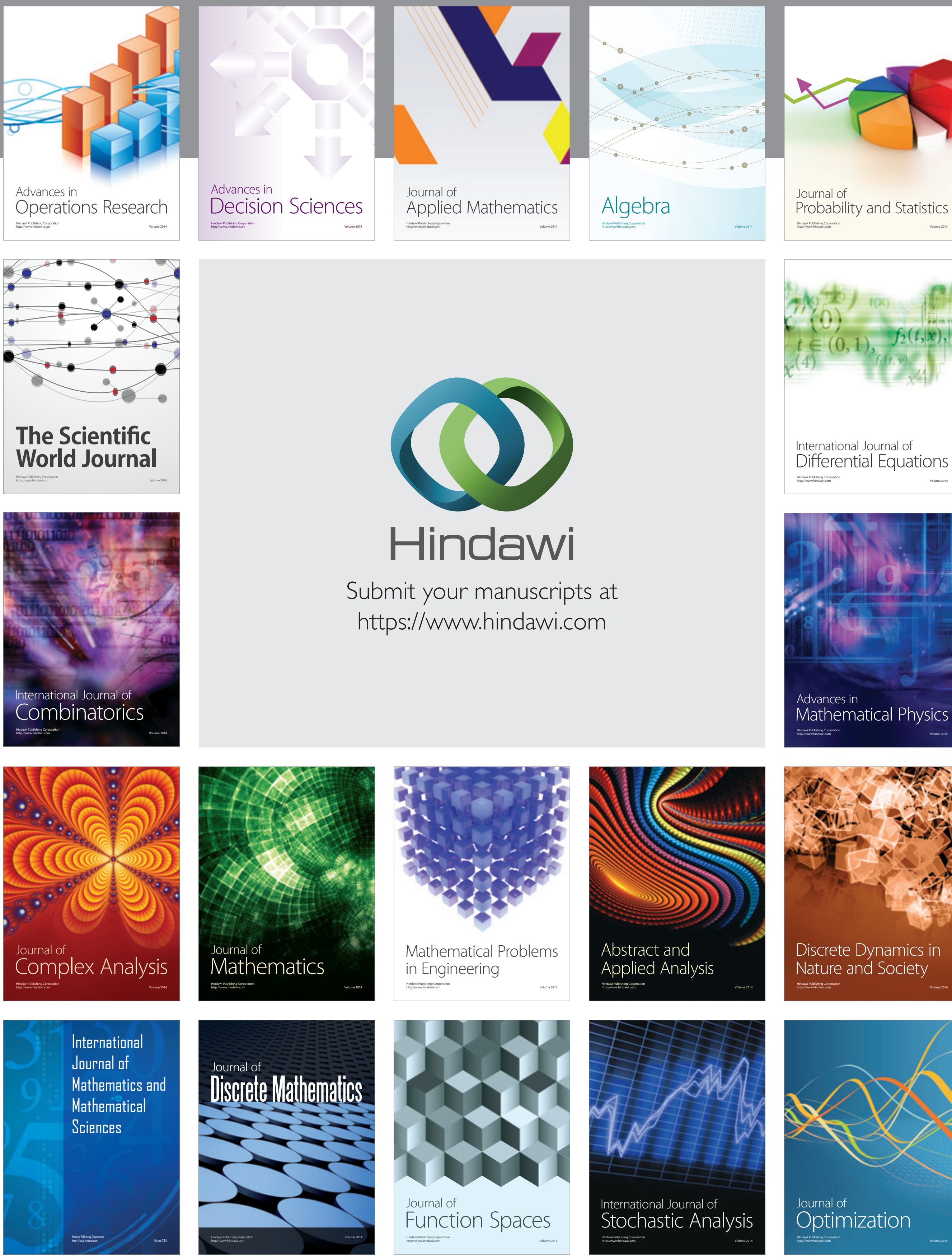

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