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Research Article

The Cascadic Multigrid Method of the Weak Galerkin Method for Second-Order Elliptic Equation

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This paper is devoted to the analysis of the cascadic multigrid algorithm for solving the linear system arising from the weak Galerkin finite element method. The proposed cascadic multigrid method is optimal for conjugate gradient iteration and quasi-optimal for Jacobi, Gauss-Seidel, and Richardson iterations. Numerical results are also provided to validate our theoretical analysis.

1. Introduction

The weak Galerkin (WG) finite element method (FEM) is a recently developed numerical method for solving various types of partial differential equations. A new concept of the discrete weak gradient is introduced, which is the most significant feature of the weak Galerkin method. Due to the definition of weak gradient, the weak Galerkin finite element method is flexible in numerical approximation.

There have been some studies and applications of the weak Galerkin finite element method. The method was first introduced by Wang and Ye in [1] for second-order elliptic problems. The corresponding numerical analysis of the weak Galerkin method based on Raviart-Thomas (RT) elements and Brezzi-Douglas-Marini (BDM) elements is given in [2]. A stabilization technique was presented and applied to the weak Galerkin finite element method, and the resulting weak Galerkin finite element method is no longer limited to RT and BDM elements [3]. In [4], the weak Galerkin mixed finite element method for biharmonic equations has been developed. For the applications of the weak Galerkin finite element method for other types of partial differential equations, the readers are referred to [5–8].

In this paper, we consider the cascadic multigrid method for solving the linear system generated by the weak Galerkin finite element method for second-order elliptic problems. Multigrid methods [9] have been shown to be very effective in solving large scale system theoretically and numerically. The cascadic multigrid method [10, 11] is a one-way multigrid method and easy to be implemented since it requires no coarse grid corrections at all. Much effort has been made to the analysis of cascadic multigrid method (see, e.g., [12, 13]). Following the idea of [13], we can establish the error estimate in energy norm and the computational complexity estimate of the proposed cascadic multigrid method.

The rest of this paper is organized as follows. In Section 2, we introduce the weak Galerkin finite element method for second-order elliptic problems. In Section 3, a cascadic multigrid algorithm based on the weak Galerkin finite element discretization is proposed and analyzed, and the error estimates in energy norm and computational complexity are obtained. Numerical experiments are conducted to confirm our theoretical results in Section 4. Finally, we give the conclusion in Section 5.

2. Model Problem and Its WG Finite Element Approximation

Consider the following second-order elliptic problem:

$$\begin{aligned} -\nabla \cdot (\mathbb{A}(x)\nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a convex polygonal domain with boundary $\partial\Omega$ in $\mathbb{R}^2$, $\mathbb{A} \in [L^\infty(\Omega)]^{2\times 2}$, $f \in L^2(\Omega)$. Furthermore, assume that $\mathbb{A}$ is symmetric uniformly positive definite and uniformly bounded-above diffusion; namely, there exist positive constants α and β such that

$$\alpha\xi^T\xi \le \xi^T \mathbb{A}(x)\xi \le \beta\xi^T\xi, \quad \forall \xi \in \mathbb{R}^2,\ x \in \Omega. \tag{2}$$

Here and thereafter, for any subset $D \subseteq \mathbb{R}^2$, we use the standard notations for the Sobolev spaces $H^s(D)$ and $H^s_0(D)$ with $s \ge 0$. The inner-product, norm, and seminorm in $H^s(D)$ are denoted by $(\cdot,\cdot)_{s,D}$, $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively, and we skip the subscript D when $D = \Omega$.

Since the domain Ω is convex, the unique solution u of problem (1) exists and satisfies the full regularity assumption [14]

$$\|u\|_2 \le C\|f\|_0. \tag{3}$$

Let K be a polygonal domain with interior K^0 and boundary ∂K. Denote by $W(K)$ the space of weak functions on K; that is,

$$W(K) := \left\{v = \{v_0, v_b\} : v_0 \in L^2(K),\ v_b \in H^{1/2}(\partial K)\right\}. \tag{4}$$

For any $v \in W(K)$, the weak gradient of v is defined as $\nabla_w v$,

$$(\nabla_w v, \mathbf{q})_K = -(v_0, \nabla\cdot\mathbf{q})_K + \langle v_b, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial K}, \quad \forall \mathbf{q} \in H(\mathrm{div}, K), \tag{5}$$

where $H(\mathrm{div}, K) = \{\mathbf{q} : \mathbf{q} \in [L^2(K)]^2, \nabla\cdot\mathbf{q} \in L^2(K)\}$.

The discrete weak gradient, denoted by $\nabla_{w,r,K} v \in [P_r(K)]^2$, is defined as follows:

$$(\nabla_{w,r,K} v, \mathbf{q})_K = -(v_0, \nabla\cdot\mathbf{q})_K + \langle v_b, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial K}, \quad \forall \mathbf{q} \in [P_r(K)]^2. \tag{6}$$

Let T_h be a shape-regular, quasi-uniform triangular mesh of the domain Ω, with the mesh size h. C denotes a generic positive constant independent of the mesh size h throughout this paper. Denote the weak function space on T_h by V; that is,

$$V := \{v = \{v_0, v_b\} : \{v_0, v_b\}|_T \in W(T),\ \forall T \in T_h\}. \tag{7}$$

For any given integer $k \ge 1$, define $W_k(T)$ as the discrete weak function space consisting of polynomials of degree k in T and piecewise polynomials of degree k on ∂T; that is,

$$W_k(T) := \{v = \{v_0, v_b\} : v_0|_T \in P_k(T),\ v_b|_e \in P_k(e),\ e \in \partial T\}. \tag{8}$$

The weak Galerkin finite element spaces are defined as follows:

$$\begin{aligned} V_h &:= \{v = \{v_0, v_b\} : \{v_0, v_b\}|_T \in W_k(T),\ \forall T \in T_h\}, \\ V_h^0 &:= \{v : v \in V_h,\ v_b = 0 \text{ on } \partial\Omega\}. \end{aligned} \tag{9}$$

It follows from [3] that

$$\nabla_{w,k-1} v|_T = \nabla_{w,k-1}(v|_T), \quad \forall v \in V_h,\ T \in T_h. \tag{10}$$

For the discrete weak gradient, we will drop the subscript $k-1$ in the notation $\nabla_{w,k-1}$ for simplicity.

The weak Galerkin finite element method can be written as to find $u_h = \{u_0, u_b\} \in V_h^0$ such that

$$a_h(u_h, v) = (f, v_0), \quad \forall v = \{v_0, v_b\} \in V_h^0, \tag{11}$$

where

$$a_h(w, v) = \sum_{T\in T_h} (\mathbb{A}(x)\nabla_w w, \nabla_w v)_T + h_T^{-1}\langle w_0 - w_b, v_0 - v_b\rangle_{\partial T}, \tag{12}$$

for any $w = \{w_0, w_b\}$, $v = \{v_0, v_b\} \in V$.

For each element $T \in T_h$, denote by Q_0 the L^2 projection from $L^2(T)$ onto $P_k(T)$. E_h is the set of all edges in T_h. For each edge $e \in E_h$, let Q_b be the L^2 projection from $L^2(e)$ onto $P_k(e)$. Define $Q_h : V \to V_h$

$$Q_h v := \{Q_0 v_0, Q_b v_b\}, \quad \forall v = \{v_0, v_b\} \in V. \tag{13}$$

Denote by $\mathbb{Q}_h$ the L^2 projection onto the local discrete gradient space $[P_{k-1}(T)]^2$. Lemma 5.1 in [3] shows that, on each $T \in T_h$,

$$\nabla_w(Q_h\phi) = \mathbb{Q}_h(\nabla\phi), \quad \forall \phi \in H^1(\Omega). \tag{14}$$

For Q_h and $\mathbb{Q}_h$ defined above, the following lemmas provide some estimates.

Lemma 1 (see [3], Lemma 5.2). *For any $\phi \in H^{k+1}(\Omega)$, we have*

$$\sum_{T\in T_h} \|\phi - Q_0\phi\|_{0,T}^2 + \sum_{T\in T_h} h_T^2 \|\nabla(\phi - Q_0\phi)\|_{0,T}^2 \le Ch^{2(k+1)}\|\phi\|_{k+1}^2, \tag{15}$$

$$\sum_{T\in T_h} \|\mathbb{A}(\nabla\phi - \mathbb{Q}_h(\nabla\phi))\|_{0,T}^2 \le Ch^{2k}\|\phi\|_{k+1}^2. \tag{16}$$

Lemma 2 (see [3], Lemma 5.3). *For any $w \in H^{k+1}(\Omega)$ and $v = \{v_0, v_b\} \in V_h$, we have*

$$\left|\sum_{T\in T_h} h_T^{-1}\langle Q_0 w - Q_b w, v_0 - v_b\rangle_{\partial T}\right| \le Ch^k \|w\|_{k+1} |||v|||, \tag{17}$$

$$\left|\sum_{T\in T_h} \langle \mathbb{A}(\nabla w - \mathbb{Q}_h\nabla w)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T}\right| \le Ch^k \|w\|_{k+1} |||v|||, \tag{18}$$

where $|||\cdot|||$ is the energy norm; that is, for any $v \in V_h$, $|||v|||^2 = a_h(v, v)$.

Lemma 3 (see [3], Theorem 8.1, Theorem 8.2). *Assume the exact solution $u \in H^{k+1}(\Omega)$; then we have*

$$|||u_h - Q_h u||| \leq Ch^k \|u\|_{k+1}, \tag{19}$$

$$\|Q_0 u - u_0\|_0 \leq Ch^{k+1} \|u\|_{k+1}. \tag{20}$$

For any $v = \{v_0, v_b\}$, $w = \{w_0, w_b\} \in V$, we define an inner-product by

$$((v, w)) = \sum_{T \in T_h} (v_0, w_0)_T + \sum_{T \in T_h} h_T \langle v_0 - v_b, w_0 - w_b \rangle_{\partial T}. \tag{21}$$

Define the following discrete norms:

$$\|v\|_{0,h,T}^2 = \|v_0\|_{0,T}^2 + h_T \|v_0 - v_b\|_{0,\partial K}^2, \quad \forall T \in T_h,$$
$$\|v\|_{0,h}^2 = \left(\sum_{T \in T_h} \|v\|_{0,h,T}^2 \right)^{1/2}. \tag{22}$$

It is clear that $\|v\|_{0,h}^2 = ((v, v))$.

With the above estimates, the following lemma can be proved, which is needed in Section 3.

Lemma 4. *Let $u \in H^{k+1}(\Omega)$ be the exact solution of problem (1), and let $u_h = \{u_0, u_b\}$ be the weak Galerkin finite element solution of problem (11); then we have*

$$|||u - u_h||| \leq Ch^k \|u\|_{k+1}, \tag{23}$$

$$\|u - u_h\|_{0,h} \leq Ch^{k+1} \|u\|_{k+1}. \tag{24}$$

Proof. Apparently, for any edge $e \in E_h$, we have

$$\|Q_0 u - Q_b u\|_e \leq \|Q_0 u - u\|_e. \tag{25}$$

T is an element with e as an edge. For any function $\varphi \in H^1(T)$, the following trace inequality is well known

$$\|\varphi\|_e^2 \leq C \left(h_T^{-1} \|\varphi\|_{0,T}^2 + h_T \|\nabla \varphi\|_{0,T}^2 \right). \tag{26}$$

Using the trace inequalities (26) and (15), we have

$$\begin{aligned} \sum_{T \in T_h} \|Q_0 u - Q_b u\|_{0,\partial T}^2 &\leq \|Q_0 u - u\|_{0,\partial T}^2 \\ &\leq \sum_{T \in T_h} C h_{0,T}^{-1} \left(\|Q_0 u - u\|_T^2 + h_T^2 \|\nabla (Q_0 u - u)\|_{0,T}^2 \right) \\ &\leq Ch^{2k+1} \|u\|_{k+1}^2. \end{aligned} \tag{27}$$

Then it follows from (14) and (16) that

$$\sum_{T \in T_h} (\mathbb{A} (\nabla u - \nabla_w (Q_h u)), \nabla u - \nabla_w (Q_h u)) \leq Ch^{2k} \|u\|_{k+1}^2. \tag{28}$$

Thus, we have

$$\begin{aligned} |||u - Q_h u|||^2 &= \sum_{T \in T_h} (\mathbb{A} (\nabla u - \nabla_w (Q_h u)), \nabla u - \nabla_w (Q_h u))_T \\ &\quad + h_T^{-1} \|Q_0 u - Q_b u\|_{0,\partial T}^2 \leq Ch^{2k} \|u\|_{k+1}^2. \end{aligned} \tag{29}$$

By using the triangle inequality, we get from (19) that

$$\begin{aligned} |||u_h - u||| &\leq |||u_h - Q_h u||| + |||Q_h u - u||| \\ &\leq Ch^k \|u\|_{k+1}. \end{aligned} \tag{30}$$

According to the definition of $\|\cdot\|_{0,h}$ and $|||\cdot|||$, we have

$$\begin{aligned} \|u - u_h\|_{0,h}^2 &= \|u - u_0\|_0^2 \\ &\quad + \sum_{T \in T_h} h_T^2 h_T^{-1} \|(u - u_h)_0 - (u - u_h)_b\|_{0,\partial T}^2 \\ &\leq \|u - u_0\|_0^2 + h^2 |||u - u_h|||^2. \end{aligned} \tag{31}$$

It follows from (15) and (20) that

$$\begin{aligned} \|u - u_0\|_0 &\leq \|u - Q_0 u\|_0 + \|Q_0 u - u_0\|_0 \\ &\leq Ch^{k+1} \|u\|_{k+1}. \end{aligned} \tag{32}$$

Combining the above three inequalities, we obtain the aimed result (24). □

3. Cascadic Multigrid Algorithm

In this section, the error estimate and computational complexity of the cascadic multigrid method are analyzed.

Assume that T_{h_l} $(l \geq 0)$ is a triangular partition of Ω with the mesh size h_l, E_{h_l} is the set of all edges in T_{h_l}, and V_{h_l} is the corresponding weak discrete space on mesh T_{h_l}. Noting that T_{l+1} is obtained by connecting the midpoints of three edges of all triangles in T_l, we have $h_l = 2^{-l} h_0$, where h_0 is the mesh size of T_0. For simplicity, define $T_l := T_{h_l}$, $E_l := E_{h_l}$, $V_l := V_{h_l}^0$, $\|\cdot\|_{0,h_l} := \|\cdot\|_{0,l}$, and $|||\cdot|||_{h_l} := |||\cdot|||_l$. The weak Galerkin finite element approximation of problem (11) on level l can be rewritten as to find $u_l = \{u_0, u_b\} \in V_l$ such that

$$a_l(u_l, v_l) := a_{h_l}(u_l, v_l) = (f, v_0), \quad \forall v_l = \{v_0, v_b\} \in V_l. \tag{33}$$

Define an intergrid transfer operator $I_l : V_{l-1} \to V_l$, for any $v \in V_{l-1}$

(1) If $x \in K_l^0$ and the element $K_l \in T_l$ is obtained by refining $K_{l-1} \in T_{l-1}$, then

$$(I_l v)_0 (x) = v_0 (x). \tag{34}$$

(2) If $x \in e_l$ and edge $e_l \in E_l$ locates in the interior of $K_{l-1} \in T_{l-1}$, then

$$(I_l v)_b (x) = v_0 (x). \tag{35}$$

(3) If $x \in e_l$ and edge $e_l \in E_l$ is part of edge $e_{l-1} \in E_{l-1}$, then

$$(I_l v)_b(x) = v_b(x). \tag{36}$$

Then the cascadic multigrid method can be written as follows.

Step 1. Set $u_0^0 = u_0^* = u_0$.

Step 2. For $l = 1, \ldots, L$, set $u_l^0 = I_l u_{l-1}^*$, and do iterations

$$u_l^{m_l} = C_l^{m_l} u_l^0. \tag{37}$$

Step 3. Set $u_l^* = u_l^{m_l}$.

The notation C_l in Step 2 denotes the iterative operator on level l. For the operator C_l, we assume that there exists a linear operator $R_l : V_l \to V_l$ such that

$$\begin{gathered} u_l - C_l^{m_l} u_l^0 = R_l^{m_l}\left(u_l - u_l^0\right), \\ \left|\left|\left|R_l^{m_l} v\right|\right|\right|_l \le C \frac{h_l^{-1}}{m_l^{\gamma}} \|v\|_{0,l}, \quad \forall v \in V_l, \\ \left|\left|\left|R_l^{m_l} v\right|\right|\right|_l \le |||v|||_l, \quad \forall v \in V_l, \end{gathered} \tag{38}$$

where m_l represents the number of iteration steps on level l and the parameter $0 \le \gamma \le 1$. As a matter of fact, the assumptions above hold for the Richardson, Jacobi, and Gauss-Seidel iterations with $\gamma = 1/2$ and for conjugate gradient iteration with $\gamma = 1$. We refer to [9, 13] for details on these results.

The following two lemmas can be proved based on the definition of I_l.

Lemma 5. *I_l is the intergrid transfer operator. For any $v = \{v_0, v_b\} \in V_{l-1}$, we have*

$$\|I_l v\|_{0,l} \le \|v\|_{0,l-1}, \tag{39}$$

$$|||I_l v|||_l \le C\,|||v|||_{l-1}. \tag{40}$$

Proof. For any element $K_{l-1} \in T_{l-1}$, let E_l^0 be the collection of edges e_l located in K_{l-1}. It follows from the definition of I_l that

$$\left\|(I_l v)_0 - (I_l v)_b\right\|_{0,e_l} = 0, \quad \forall e_l \in E_l^0. \tag{41}$$

Thus, we have

$$\begin{aligned} \|I_l v\|_{0,l}^2 &= \sum_{K_l \in T_l} \left(\left\|(I_l v)_0\right\|_{0,K_l}^2 + \sum_{e_l \in \partial K_l} h_l \left\|(I_l v)_0 - (I_l v)_b\right\|_{0,e_l}^2 \right) \\ &= \sum_{K_{l-1} \in T_{l-1}} \left(\left\|(I_l v)_0\right\|_{0,K_{l-1}}^2 + \sum_{e_{l-1} \in \partial K_{l-1}} h_l \left\|(I_l v)_0 - (I_l v)_b\right\|_{0,e_{l-1}}^2 \right) \\ &\le \sum_{K_{l-1} \in T_{l-1}} \left\|(I_l v)_0\right\|_{0,K_{l-1}}^2 + \sum_{K_{l-1} \in T_{l-1}} h_{l-1} \left\|(I_l v)_0 - (I_l v)_b\right\|_{0,\partial K_{l-1}}^2 = \|v\|_{0,l-1}^2. \end{aligned} \tag{42}$$

For any $K \in T_{l-1}$, from the definition of weak gradient (5), we have

$$(\nabla v_0, \mathbf{q})_K = (\nabla_w v, \mathbf{q})_K + \langle v_0 - v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial K}, \quad \forall \mathbf{q} \in H(\mathrm{div}, \Omega). \tag{43}$$

If φ is a polynomial in K, from trace inequality (26) and the standard inverse inequality, we have

$$\|\varphi\|_e^2 \le C h_K^{-1} \|\varphi\|_{0,K}^2. \tag{44}$$

Setting $\mathbf{q} = \nabla v_0$, with (44), we have

$$\begin{aligned} \|\nabla v_0\|_K^2 &= (\nabla_w v, \nabla v_0)_K + \langle v_0 - v_b, \nabla v_0 \cdot \mathbf{n} \rangle_{\partial K} \\ &\le \|\nabla_w v\|_{0,K} \|\nabla v_0\|_{0,K} + \left(h_K^{-1/2} \|v_0 - v_b\|_{0,\partial K}\right)\left(h_K^{1/2} \|\nabla v_0 \cdot \mathbf{n}\|_{0,\partial K}\right) \\ &\le C\,|||v|||_K \|\nabla v_0\|_{0,K}. \end{aligned} \tag{45}$$

This implies

$$\|\nabla v_0\|_{0,K} \le C\,|||v|||_K. \tag{46}$$

Note that K consisted of some elements in T_l; that is, there exist $K_i \in T_l$, $i = 1, \ldots, m$, such that $K = \cup_{i=1}^m K_i$.

Application of (44) yields

$$\|\nabla_w (I_l v) \cdot \mathbf{n}\|_{0,\partial K_i} \le C h_{K_i}^{-1/2} \|\nabla_w (I_l v)\|_{0,K_i}, \quad \forall v \in V_{l-1}. \tag{47}$$

By the definition of $\nabla_w v$, (46) and (47), we have

$$\begin{aligned} \|\nabla_w (I_l v)\|_{0,K_i}^2 &= (\nabla_w (I_l v), \nabla_w (I_l v))_{K_i} \\ &= (\nabla (I_l v)_0, \nabla_w (I_l v))_{K_i} - \langle (I_l v)_0 - (I_l v)_b, \nabla_w (I_l v) \cdot \mathbf{n} \rangle_{\partial K_i} \\ &\le \|\nabla v_0\|_{0,K_i} \|\nabla_w (I_l v)\|_{0,K_i} + \|(I_l v)_0 - (I_l v)_b\|_{0,\partial K_i} \|\nabla_w (I_l v) \cdot \mathbf{n}\|_{0,\partial K_i} \\ &\le \|\nabla v_0\|_{0,K} \|\nabla_w (I_l v)\|_{0,K_i} + \|v_0 - v_b\|_{0,\partial K} \|\nabla_w (I_l v) \cdot \mathbf{n}\|_{0,\partial K_i} \\ &\le |||v|||_K \|\nabla_w (I_l v)\|_{0,K_i} + h_K^{1/2} |||v|||_K h_{K_i}^{-1/2} \|\nabla_w (I_l v)\|_{0,K_i} \\ &\le C\,|||v|||_K \|\nabla_w (I_l v)\|_{0,K_i}. \end{aligned} \tag{48}$$

Since

$$\begin{aligned} h_{K_i}^{-1}\left\|(I_l v)_0-(I_l v)_b\right\|_{0,\partial K_i}^2 &\le h_{K_i}^{-1}\left\|v_0-v_b\right\|_{0,\partial K}^2 \\ &\le C h_K^{-1}\left\|v_0-v_b\right\|_{0,\partial K}^2 \le C|||v|||_K, \end{aligned} \tag{49}$$

we obtain

$$\begin{aligned} &|||I_l v|||_l^2 \\ &= \sum_{T\in T_l}\left(\mathbb{A}\nabla_w(I_l v),\nabla_w(I_l v)\right)_T \\ &\quad + h_T^{-1}\left\langle (I_l v)_0-(I_l v)_b,(I_l v)_0-(I_l v)_b\right\rangle_{\partial T} \\ &\le C\sum_{T\in T_l}\left(\left\|\nabla_w(I_l v)\right\|_{0,T}^2+h_T^{-1}\left\|(I_l v)_0-(I_l v)_b\right\|_{0,\partial T}^2\right) \\ &\le C|||v|||_{l-1}^2, \end{aligned} \tag{50}$$

which completes the proof of this lemma. □

Lemma 6. *Assume that u_l and u_{l-1} are the weak Galerkin finite element solutions associated with V_l and V_{l-1}, respectively; then*

$$\left\|u_l-I_l u_{l-1}\right\|_{0,l}\le Ch_l^2\|f\|_0. \tag{51}$$

Proof. Let u_l^F and u_{l-1}^F be the finite element solutions associated with T_l and T_{l-1}, respectively. By the triangle inequality, (39), (24), and the regularity (3), we have

$$\begin{aligned} \left\|u_l-I_l u_{l-1}\right\|_{0,l} &\le \left\|u_l-u\right\|_{0,l}+\left\|u-u_{l-1}^F\right\|_{0,l} \\ &\quad+\left\|I_l u_{l-1}^F-I_l u_{l-1}\right\|_{0,l} \\ &\le \left\|u_l-u\right\|_{0,l}+\left\|u-u_{l-1}^F\right\|_0 \\ &\quad+\left\|u_{l-1}^F-u_{l-1}\right\|_{0,l-1}\le Ch_l^2\|u\|_2 \\ &\le Ch_l^2\|f\|_0. \end{aligned} \tag{52}$$

which completes the proof of this lemma. □

Let V_{l-1}^F be the standard finite element space associated with T_{l-1}. Define the projection operator $P_{l-1}: V_{l-1}\to V_{l-1}^F$ by

$$a_l\left(P_{l-1}u,v\right)=a_l\left(I_l u,v\right),\quad \forall v\in V_{l-1}^F. \tag{53}$$

It is easy to check that

$$a_l\left(P_{l-1}u,v\right)=a_{l-1}\left(P_{l-1}u,v\right),\quad v\in V_{l-1}^F, \tag{54}$$

$$a_l\left(I_l u,v\right)=a_{l-1}(u,v),\quad v\in V_{l-1}^F, \tag{55}$$

$$|||P_{l-1}v|||_l\le |||v|||_{l-1},\quad \forall v\in V_{l-1}. \tag{56}$$

The following lemma is needed in the convergence analysis.

Lemma 7. *For the projection operator P_l, we have*

$$\left\|I_l v-P_{l-1}v\right\|_{0,l}\le Ch_l|||v|||_{l-1},\quad \forall v\in V_{l-1}. \tag{57}$$

Proof. Since Ω is a convex polygonal domain, for a given $v\in V_{l-1}$, we introduce an auxiliary problem, that is, to find $\xi\in H^2(\Omega)$ such that

$$-\nabla\cdot(\mathbb{A}\nabla\xi)=\left(I_l v-P_{l-1}v\right)_0\quad\text{in }\Omega, \tag{58}$$

$$\xi=0\quad\text{on }\partial\Omega. \tag{59}$$

The solution ξ satisfies the following inequality:

$$\|\xi\|_2\le\left\|\left(I_l v-P_{l-1}v\right)_0\right\|_0. \tag{60}$$

Let $\eta=\{\eta_0,\eta_b\}=I_l v-P_{l-1}v$. By the definition of P_l, we have

$$a_l\left(I_l v-P_{l-1}v,w\right)=a_l(\eta,w)=\sum_{T\in T_l}\left(\nabla_w\eta,\nabla w\right)_T,\quad \forall w\in V_{l-1}^F. \tag{61}$$

Then (58), the definition of weak gradient, and (14) give

$$\begin{aligned} \left\|\eta_0\right\|_0^2 &= -\sum_{T\in T_l}\left(\nabla\cdot(\mathbb{A}\nabla\xi),\eta_0\right)_T \\ &= \sum_{T\in T_l}\left(\left(\mathbb{A}\mathbb{Q}_l(\nabla\xi),\nabla\eta_0\right)-\left\langle\mathbb{A}\nabla\xi\cdot\mathbf{n},\eta_0\right\rangle_{\partial T}\right) \\ &= \sum_{T\in T_l}\left(\left(-\nabla\cdot\mathbb{A}\mathbb{Q}_l(\nabla\xi),\eta_0\right)+\left\langle\mathbb{A}\mathbb{Q}_l(\nabla\xi)\cdot\mathbf{n},\eta_0\right\rangle_{\partial T}\right. \\ &\quad \left.-\left\langle\mathbb{A}\nabla\xi\cdot\mathbf{n},\eta_0\right\rangle_{\partial T}\right)=\sum_{T\in T_l}\left(\left(\mathbb{A}\nabla_w(Q_l\xi),\nabla_w\eta\right)\right. \\ &\quad +\left\langle\mathbb{A}\mathbb{Q}_l(\nabla\xi)\cdot\mathbf{n}-\mathbb{A}\nabla\xi\cdot\mathbf{n},\eta_0-\eta_b\right\rangle_{\partial T} \\ &\quad \left.-\left\langle\mathbb{A}\nabla\xi\cdot\mathbf{n},\eta_b\right\rangle_{\partial T}\right). \end{aligned} \tag{62}$$

Since

$$\sum_{T\in T_l}\left\langle\mathbb{A}\nabla\xi\cdot\mathbf{n},\eta_b\right\rangle_{\partial T}=0, \tag{63}$$

we get from (18) that

$$\left\|\eta_0\right\|_0^2\le a_l\left(Q_l\xi,\eta\right)+Ch_l\|\xi\|_2|||\eta|||_l. \tag{64}$$

For any $\xi_{l-1}^F\in V_{l-1}^F$, we have

$$a_l\left(I_l v-P_{l-1}v,\xi_{l-1}^F\right)=0, \tag{65}$$

which implies

$$\begin{aligned} \sum_{T\in T_l}\left(\mathbb{A}\nabla_w\eta,\nabla\xi_{l-1}^F\right) &= \sum_{T\in T_l}\left(\mathbb{A}\nabla_w\left(I_l v-P_{l-1}v\right),\nabla\xi_{l-1}^F\right) \\ &= 0. \end{aligned} \tag{66}$$

Thus, combining (16) and (18), we have

$$\begin{aligned} a_l\left(Q_l\xi, I_l v - P_{l-1}v\right) &= a_l\left(Q_l\xi - \xi_{l-1}^F, I_l v - P_{l-1}v\right) \\ &\le \sum_{T\in T_l}\left(\mathbb{A}\nabla_w\left(Q_l\xi - \xi\right), \nabla_w\eta\right)_T \\ &\quad + \left(\mathbb{A}\nabla\left(\xi - \xi_{l-1}^F\right), \nabla_w\eta\right)_T \\ &\quad + h_T^{-1}\left\langle Q_0\xi - Q_b\xi, \eta_0 - \eta_b\right\rangle_{\partial T} \\ &\le Ch_l\left\|\xi\right\|_2 |||\eta|||_l . \end{aligned} \tag{67}$$

By using the regularity (60), we obtain

$$\left\|\eta_0\right\|_0 = \left\|\left(I_l v - P_{l-1}v\right)_0\right\|_0 \le Ch_l\, |||I_l v - P_{l-1}v|||_l . \tag{68}$$

Then it follows from (40) and (56) that

$$\left\|\eta_0\right\|_0 \le Ch_l\, |||v|||_{l-1} . \tag{69}$$

It is easy to show that

$$\begin{aligned} &\sum_{T\in T_l} h_T\left\langle \eta_0 - \eta_b, \eta_0 - \eta_b\right\rangle_{\partial T} \\ &\quad \le h_l^2\sum_{T\in T_l} h_T^{-1}\left\langle \left(I_l v\right)_0 - \left(I_l v\right)_b, \left(I_l v\right)_0 - \left(I_l v\right)_b\right\rangle_{\partial T} \\ &\quad \le Ch_l^2\sum_{T\in T_{l-1}} h_T^{-1}\left\langle v_0 - v_b, v_0 - v_b\right\rangle_{\partial T} \\ &\quad \le Ch_l^2\, |||v|||_{l-1}^2 . \end{aligned} \tag{70}$$

Combining (68) and (70), we obtain

$$\begin{aligned} \left\|I_l v - P_{l-1}v\right\|_{0,l}^2 &= \left\|\eta_0\right\|_0^2 + \sum_{T\in T_l} h_T\left\langle \eta_0 - \eta_b, \eta_0 - \eta_b\right\rangle_{\partial T} \\ &\le Ch_l^2\, |||v|||_{l-1}^2 , \end{aligned} \tag{71}$$

which completes the proof of this lemma. □

The following two theorems are the main results of this paper, which can be proved in a similar way of [13] based on the above lemmas.

Theorem 8. *If we take the CG iteration as the smoother, and the number of iterations on the level l is the minimum integer satisfying*

$$m_l \ge \beta^{L-1} m_L \tag{72}$$

with some fixed $\beta > 1$ and m_L, then the cascadic multigrid method is optimal: that is, the error

$$|||u_L - u_L^*|||_L \approx |||u - u_L|||_L \tag{73}$$

and the computational complexity

$$\text{amount of work} = O\left(n_L\right), \tag{74}$$

where n_l denotes the dimension of the space V_L on level l.

Theorem 9. *If we take the Richardson, Jacobi, and Gauss-Seidel iterations as the smoother, the number of iterations on the level l is the minimum integer satisfying*

$$m_l \ge \beta^{L-1} m_L, \tag{75}$$

while the number of iterations on the level l is the minimum integer satisfying

$$m_L \ge \left[m_* L^2\right], \tag{76}$$

with some fixed $m_* \ge 1$, then the cascadic multigrid method is quasi-optimal: that is, the error

$$|||u_L - u_L^*|||_L \le C\frac{h_L}{m_*^{1/2}}\left\|f\right\| \tag{77}$$

and the computational complexity

$$\sum_{k=1}^{L} m_k n_k \le Cm_* n_L\left(1 + \log n_L\right)^3 . \tag{78}$$

4. Numerical Examples

In this section, we give some numerical experiments of the cascadic multigrid algorithm based on the weak Galerkin finite element method to verify the theoretical results proved in Section 3.

For simplicity, we choose $k = 1$; that is, the weak Galerkin finite element space is

$$\begin{aligned} V_l = \{v = \{v_0, v_b\},\ v_0 \in P_1(T),\ v_b \in P_1(\partial T),\ v_b \\ = 0 \text{ on } \partial\Omega,\ \forall T \in T_l\}, \end{aligned} \tag{79}$$

and the weak gradient space is

$$G_l = \left\{\mathbf{q} \in \left[P_0(T)\right]^2,\ \forall T \in T_l\right\}. \tag{80}$$

For the model problem (1) in the domain $\Omega = (0,1)^2$, we give the following numerical examples.

Case 1. Let the exact solution $u = \sin(\pi x)\sin(\pi y)$ and set $\mathbb{A}(x, y) = e^{x+y}$. The numerical results are reported in Table 1.

Case 2. Let the exact solution $u = x(1-x)y(1-y)$ and set $\mathbb{A}(x, y) = 1$. The numerical results are reported in Table 2.

Case 3. Let the exact solution $u = x(1-x)y(1-y)$ and set

$$\mathbb{A}(x, y) = \begin{bmatrix} x^2 + 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{81}$$

The numerical results are reported in Table 3.

The right-hand function f in the performed numerical examples is chosen to match the given u and $\mathbb{A}$. We take the parameters of the cascadic multigrid method as $\beta = 3$ and $m_L = 30$. The numerical experiments are conducted in the computer of Intel(R) Core(TM) i5-2500 CPU 3.30 GH,

Table 1: Results of Case 1: $m_L = 30$, $\beta = 3$.

h	n_L	L	$\|\|\|u_L^* - u\|\|\|_L$	Time (seconds)
$\frac{1}{64}$	49408	1	$5.5032e-02$	13.30
		2	$5.5067e-02$	0.92
		3	$5.5067e-02$	0.73
$\frac{1}{128}$	197120	1	$2.7503e-02$	109.7200
		2	$2.7530e-02$	5.41
		3	$2.7530e-02$	4.56
		4	$2.7530e-02$	3.73
$\frac{1}{256}$	787456	1	$1.3750e-02$	942.54
		2	$1.3765e-02$	30.45
		3	$1.3766e-02$	18.30
		4	$1.3766e-02$	17.10
		5	$1.3766e-02$	17.04
$\frac{1}{512}$	3147776	1	$6.8749e-03$	7816.06
		2	$6.8826e-03$	180.41
		3	$6.8830e-03$	84.72
		4	$6.8830e-03$	72.63
		5	$6.8830e-03$	71.39
		6	$6.8830e-03$	70.92

Table 2: Results of Case 2: $m_L = 30$, $\beta = 3$.

h	n_L	L	$\|\|\|u_L^* - u\|\|\|_L$	Time (seconds)
$\frac{1}{64}$	49408	1	$3.6360e-03$	4.49
		2	$3.6355e-03$	0.75
		3	$3.6355e-03$	0.72
$\frac{1}{128}$	197120	1	$1.8181e-03$	36.11
		2	$1.8181e-03$	4.10
		3	$1.8181e-03$	3.78
		4	$1.8181e-03$	3.76
$\frac{1}{256}$	787456	1	$9.0907e-04$	246.85
		2	$9.0906e-04$	21.19
		3	$9.0906e-04$	17.41
		4	$9.0906e-04$	16.90
		5	$9.0906e-04$	16.87
$\frac{1}{512}$	3147776	1	$4.5454e-04$	2125.11
		2	$4.5453e-04$	106.25
		3	$4.5457e-04$	75.06
		4	$4.5457e-04$	71.62
		5	$4.5457e-04$	70.23
		6	$4.5457e-04$	70.14

12.0 GB, Windows 7 (x64). The time each experiment takes is also listed in the tables.

The numerical results obtained by using the conjugate gradient iteration to solve these examples directly are also provided at the rows with $L = 1$. We can observe that the proposed cascadic multigrid method is optimal for both convergence rate and computation complexity, which confirms our theoretical results.

5. Conclusion

This paper gives error and complexity estimates of the cascadic multigrid algorithm of the weak Galerkin finite

TABLE 3: Results of Case 3: $m_L = 30$, $\beta = 3$.

h	n_L	L	$\|\|\|u_L^* - u\|\|\|_L$	Time (seconds)
$\frac{1}{64}$	49408	1	$3.6360e-03$	4.43
		2	$3.6355e-03$	0.94
		3	$3.6355e-03$	0.72
$\frac{1}{128}$	197120	1	$1.8181e-03$	36.08
		2	$1.8181e-03$	4.10
		3	$1.8181e-03$	3.68
		4	$1.8181e-03$	3.59
$\frac{1}{256}$	787456	1	$9.0907e-04$	264.49
		2	$9.0906e-04$	21.37
		3	$9.0906e-04$	17.39
		4	$9.0906e-04$	16.91
		5	$9.0906e-04$	16.92
$\frac{1}{512}$	3147776	1	$4.5454e-04$	2126.06
		2	$4.5453e-04$	106.97
		3	$4.5457e-04$	74.85
		4	$4.5457e-04$	71.07
		5	$4.5457e-04$	70.65
		6	$4.5457e-04$	70.53

element. The numerical solution is convergent with rate $O(h)$ and the computing time is proportional to the number of unknowns, which verify the theoretical results.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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