Hindawi
Mathematical Problems in Engineering
Volume 2017, Article ID 8082738, 8 pages
https://doi.org/10.1155/2017/8082738

Research Article

Game Approach for H_∞ Robust Control Strategy to Follow the Production in the Singularly Perturbed Bilinear Dynamic Input-Output Systems

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Received 29 March 2017; Accepted 7 June 2017; Published 11 July 2017

Academic Editor: Maria L. Gandarias

For simulating and analyzing the input and output problem of national economy more accurately, this paper considers the fast and slow production processes during the course of social production development, takes stochastic economic risks into consideration, and constructs a H_∞ robust control model to follow the production in the singularly perturbed dynamic input-output systems. Further introducing ideas of noncooperative differential game theory, the H_∞ robust control model is transformed into a saddle-point equilibrium game model, and a new method for solving dynamic input-output problem by using saddle-point equilibrium strategies is obtained. A numerical result is presented in the end to illustrate the effectiveness of the method.

1. Introduction

Since 1953, the static input-output model was developed into the dynamic model by the economist Wassily Leontief; the dynamic input-output model, which can better describe the actual economy, has been widely used in the field of information economy, population, education, income distribution, national economic accounting, and so on for decades. However, the dynamic input-output model presented by Leontief is based on the assumption that the input factors should be fixed and could not be changed, and as it is pointed out by Chander and Tokao, the model of real economy should take into account proportions' changing of various input factors, as well as other nonlinear production functions, which means the nonlinear dynamic input-output analysis should be studied. So during the recent years, the nonlinear dynamic input-output analysis has become a hot issue by many researchers.

In [1, 2], Zhang systematically studied the structural stability of closed-loop dynamic input-output system, the robust stratagem of production to follow the consumption, and the robust stratagem of consumption to follow the production in the dynamic input-output system. Jiang and Zhang constructed a tax included computable nonlinear dynamic I-O model and estimated its parameters using practical data of China [3]. Zhang and Hao extended the linear Leontief dynamic I-O model to a computable nonlinear dynamic I-O model and used this model to create a benefit possibility curve [4]. Zhang further gave the multiple embedded computable nonlinear dynamic I-O model and its balanced growth solutions about prices, output level, growth rate, and profit rate [5], as well as the optimal growth path of China's six sectors' economy based on the nonlinear dynamic I-O model [6]. Both Zhang and Hao in [4, 7] and Chander and Tokao in [8, 9] used nonlinear dynamic I-O model. Yong and Jin-Qing constructed a computable nonlinear dynamic I-O model according to the data of statistical yearbooks and I-O table from 20 sectors in 19 provinces during 1995–2005 and used the model to make a quantitative simulation study to test how China's environmental regulations influence industry output and its economic growth [10]. In [11], Jiamo put forward a simple nonlinear dynamic equation on the basis of macroscopic concept, gave the proof of the existence, uniqueness, and stability of the equation's solution in theory, and further obtained the feature vector model of unbalanced development. Liu and Chen obtained the nonlinear function forms of important coefficient with nonlinear regression analysis and set the nonlinear important coefficient input-occupancy-output model [12].

For better simulating and analyzing the input and output problem of national economy, the paper presents study of dynamic input-output problems of singularly perturbed systems based on noncooperative differential game theory. This paper builds continuous multisectoral dynamic input-output analysis model and discusses optimal strategy design problems on the basis of Leontief dynamic input-output model. Considering that there exist fast production process and slow production process during the course of social production development, it constructs singularly perturbed bilinear dynamic input-output analysis model. Taking uncertainties of environment, namely, economic risks, into consideration, this paper studies how to design control strategies on worst case, that is, a H_∞ robust control problem. With ideas of game theory, it transforms dynamic input-output system into a saddle-point equilibrium game model and obtains a new method for solving dynamic input-output problem by using saddle-point equilibrium strategies.

2. Model Construction

Leontief's input-output model is a powerful tool for analyzing and forecasting short-time economy, and to ensure the monotony of input structure and the stability of input coefficients of each sector, the model oversimplified and abstracted the complexity of the economic phenomenon, which made the input-output model become a linear model, very simple, and easy to be used. However, it is quite difficult to use this model in long-term forecasting and planning, for the linear model can only roughly approximate the economic operation process, which is quite different from the actual situation. So in order to better describe the economic reality, it is necessary to study the nonlinear input-output model.

Due to the complexity of realistic economic activities, the linear dynamic input-output model has some limitations, which is constructed on the assumption that the input factors' proportions should be fixed and could not be changed, but in the actual production process, the input factors' proportions are always changing. For example, in the backward, the developing countries usually invest more labor hours and use less fixed capital, but with the development of economy, they will use more and more fixed capital and at the same time, the investment of labor hours will change little. So with the passage of time, the input proportion of fixed capital and labor hours will change constantly. Therefore, it is necessary to study the nonlinear input-output problems. Chander and Tokao studied the nonlinear input-output model as the form of [13]

$$x(k) = A(x)\,x(k) + B\,[x(k+1) - x(k)] + y(k), \quad (1)$$

where $A(x)$ is the nonlinear functions of the output $x_1(k), \ldots, x_n(k)$. In the nonlinear system structures, the bilinear system is the simplest, and it is most close to the linear system, so the bilinear system is usually considered as a transition from the linear system to the nonlinear system. In addition to its wide existence and applications, the bilinear system also has good approximation properties. Tie et al. pointed out that the bilinear system could approximate any nonlinear system theoretically, and the accuracy was much higher than traditional linear approximation [14]. A bilinear system is acquired by truncating higher order terms of the Taylor series expansion for nonlinear system state at the steady point. It means that by Taylor expanding $A(x)$ to the second term, a bilinear system model can be obtained as formula (2):

$$\begin{aligned} x(k) = {} & [A + \{x(k)\,M(k)\}]\,x(k) \\ & + B\,[x(k+1) - x(k)] + y(k). \end{aligned} \quad (2)$$

Formula (2) is a discrete multisector dynamic input-output model described by difference equation. Further extending it to the continuous form described by differential equation, we can have formula (3):

$$x(t) = [A + \{x(t)\,M\}]\,x(t) + B\dot{x}(t) + y(t). \quad (3)$$

Model (3) is a definite dynamic input-output model; however, there exist lots of uncertainties in the real economic activities, which prevents model (3) from accurately describing economic laws in reality. It is stated by Jiamo in [11] that there are many uncertain factors affecting economic development, which can be divided into two categories as follows. (1) Gross errors: influence factors such as the world economic crisis, natural disasters, wars, and political movements are called gross errors influence. The output curve of sectors affected by gross errors will show a large decline in volatility, but a few years later, the national economy will resume its regular growth, which indicates that gross errors can change the inherent operation state of economic systems. (2) Errors: influence factors such as construction, retirement and abandonment of fixed assets, the influence of the improvement of technology and management on economic development, domestic and foreign markets, and statistical errors caused by statistical techniques are called errors. This type of errors is caused by the reason that the state of economic operation has not really been reflected, and it does not change the inherent operation state of economic systems [11].

For the complexity of realistic economic activities, it is inappropriate to adopt traditional methods (that is omitting the influence of uncertain factors) to deal with these uncertainties, and we should study the actual operation state of economic systems in recognition of the influence of uncertainties. So considering various stochastic factors in real economic activities, a parameter $w(t)$ is introduced to describe various stochastic factors in social economic system; then the model (3) is revised and a new model (4) can be obtained as follows:

$$x(t) = [A + \{x(t)\,M\}]\,x(t) + B\dot{x}(t) + y(t) + w(t). \quad (4)$$

Presently, the research on the dynamic input-output problems mainly focuses on the situation that the investment coefficient matrix B is deterministic, which means that all sectors have investment on all products, and the direct occupancy coefficient of production funds is a nonzero constant. However, in actual economic systems, not all industrial sectors will invest on products, and so some researchers established dynamic input-output models with singular investment coefficient matrices. But as Renquan and Hongye stated in [15], "singular system is a theoretical simplification of singularly perturbed system, and some of its properties are very similar to those of singularly perturbed system. But in the reality, singular system generally does not exist, and it is often assumed by researchers, for the convenience of research, that singular system exists in theory and by analyzing some characteristics of singular system, properties of singularly perturbed systems can be obtained." Therefore, it cannot accurately describe the real economic system by simply regarding investment coefficient matrices as singular matrices.

Moreover, Zhang pointed out in [1] that social production processes can be divided into the fast production process and the slow production process. Instead of being immediately consumed during the production process, output vectors of the slow production process, such as factory buildings and manufacturing equipment, depreciate gradually and usually have longer production processing delay time. For example, it often takes 2-3 or even more years to build a factory. However, output vectors of the fast production process, such as raw materials, usually have much shorter production processing delay time, and products can be produced at the same period of the production factors being input [1]. Furthermore, with the development of economic globalization, and with the rapid growth of international trade and the expansion of capital markets, the influence of international interest rates and exchange rates on the world economy is becoming more and more evident, which leads to the volatility of investment coefficients of the input-output model [16]. And for economic production process, the fast production process, especially, such as consumables, changes in the international economic environment, the influence of international interest rates and exchange rates, and the volatility of investment coefficients usually leads to a sharp increase or decrease of the output in a short time.

Based on the above analysis, following the methods in [17] of Ning et al. and considering the slow and fast processes during economic production activities, we can introduce the output vector $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ in the dynamic input-output equation (4), where $x_1(t)$ is the output vector for the slow production process, such as factory buildings and manufacturing equipment, and $x_2(t)$ is the output vector for the fast production process, such as raw materials. Furthermore, taking into account that the output vector during the fast production process is usually vulnerable to international interest rates and exchange rates, which will bring about perturbation to the investment coefficient, then we can let the investment coefficient $B = \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix}$, in which B_1, B_2 denote, respectively, the investment coefficient matrices of the slow and fast processes; then $B_\varepsilon = \begin{bmatrix} I & \\ & \varepsilon \end{bmatrix} B = \begin{bmatrix} B_1 & \\ & \varepsilon B_2 \end{bmatrix}$ represents the singularly perturbed investment coefficient matrix, where the small singular perturbation parameter $\varepsilon > 0$ represents small fluctuation of international economic environment in fast dynamic changing processes. Then the dynamic input-output model (4) can be written as

$$\begin{aligned} x(t) = {} & [A + \{x(t) M\}] x(t) + B_\varepsilon \dot{x}(t) + Cy(t) \\ & + Ew(t). \end{aligned} \tag{5}$$

Involving the slow and fast processes, two matrices C and E are introduced in model (5), where $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ and C_1, C_2 denote, respectively, the contribution ratios of slow and fast production processes to the final consumption product vector; $E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$, and E_1, E_2 denote, respectively, the weighted values of stochastic factors' influence during the two types of production processes. And C_1, C_2, E_1, and E_2 are constant matrices of appropriate dimensions.

Supposing B is reversible, model (5) can be converted into the following form:

$$\begin{aligned} \begin{bmatrix} I & \\ & \varepsilon \end{bmatrix} \dot{x}(t) = {} & B^{-1} [I - A - \{x(t) M\}] x(t) \\ & - B^{-1} Cy(t) - B^{-1} Ew(t). \end{aligned} \tag{6}$$

Apparently, model (6) is a dynamic input-output model described by a singularly perturbed bilinear system. Furthermore, controlling the final consumer product vector $y(t)$, that is, choosing the control variable $u(t) = y(t)$, we can construct a dynamic input-output model system model as follows:

$$\begin{aligned} \begin{bmatrix} I & \\ & \varepsilon \end{bmatrix} \dot{x}(t) = {} & B^{-1} [I - A - \{x(t) M\}] x(t) \\ & - B^{-1} Cu(t) - B^{-1} Ew(t). \end{aligned} \tag{7}$$

It is supposed that the n dimensional column vector $h(t)$ represents the ideal output, which is a known continuous function vector. And if the national economy is dynamically stable, during the planning period, the actual output $x(t)$ should be closer to the ideal output $h(t)$ [18]. However, the supply and demand imbalance is very difficult to completely avoid. So we hope that the performance index $J(u, w)$ can attain minimum by adjusting the control variable $u(t)$, and at the same time, $J(u, w)$ can attain maximum in spite of the worst disturbance of the stochastic factor $w(t)$, in which

$$\begin{aligned} J(u, w) = {} & \frac{1}{2} \int_{t_0}^{t_f} \Big\{ [x(t) - h(t)]^T Q [x(t) - h(t)] \\ & + u^T(t) Ru(t) - \gamma^2 w^T(t) w(t) \Big\} dt, \end{aligned} \tag{8}$$

where $h(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}$, $h_1(t)$ represents the target vector of products for slow production system in planning period, and $h_2(t)$ represents the target vector of products for fast production system in planning period. $\gamma > 0$ is the disturbance attenuation level and Q and R are positive definite matrices of order n, whose actual significance is weights for separating the difference between the final consumer products provided by each sector and the social demand and the difference of output capacity of each sector.

Then models (7) and (8) construct H_∞ robust control problem to follow the production in the dynamic system of input and output. The economic explanation of the problem is as follows: when establishing policies for future economic development, we design the control strategy $u(t)$ to ensure the minimum gap (that is, to realize balance of the supply and demand) between the optimal amount of output (or the consumer goods) and the target amount of output (or the social demand) with policies' fluctuation and stochastic factors' disturbance.

3. Model Resolution

Apparently, in model (7), the performance index $J(u,w)$ is required to attain minimum by adjusting the control variable $u(t)$, and at the same time, $J(u,w)$ is required to attain maximum in spite of the worst disturbance of the stochastic factor $w(t)$. According to the basic method of robust control designing based on the noncooperative differential game theory, we can regard the control strategy designer as one player, that is, Player 1, and the stochastic disturbance as another player. Consequently, the robust control problem is converted into a problem of two players' game, that is, when anticipating the possible disturbance, how Player 1 should design his strategy to achieve the equilibrium with Player 2 and optimize his goal at the same time [13]. So models (7) and (8) build a H_∞ robust control strategy designing problem for a singularly perturbed Markov system based on differential game theory. The system state equation is

$$\begin{bmatrix} I & \\ & \varepsilon \end{bmatrix} \dot{x}(t) = B^{-1}[I - A - \{x(t)M\}]x(t) - B^{-1}Cu(t) - B^{-1}Ew(t) \tag{9}$$

with initial condition $x_0 = [x_{10}, x_{20}]^T$.

And $x(t) = [x_1(t), x_2(t)]^T \in R^n$ are state vector, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$, $x_1(t) \in R^{n_1}$, $x_2(t) \in R^{n_2}$ are, respectively, slow and fast state variable and $n_1 + n_2 = n$, $u(t) \in R^m$ is a control vector, $w(t) \in R^l$ denotes the disturbance input, the small singular perturbation parameter $\varepsilon > 0$ represents small time constants, inertias, masses, and so on, and A_{ij}, B_i, E_i, M_i, and C $(i, j = 1, 2)$ are constant matrices of appropriate dimensions, with

$$\begin{aligned} \{x(t)M\} &= \left\{ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \begin{bmatrix} M_s \\ M_f \end{bmatrix} \right\} \\ &= \sum_{j=1}^{n_1} x_{1j} \begin{bmatrix} M_{sj} \\ M_{fj} \end{bmatrix} + \sum_{j=n_1+1}^{n_1+n_2} x_{2j} \begin{bmatrix} M_{sj} \\ M_{fj} \end{bmatrix}. \end{aligned} \tag{10}$$

Furthermore, the system state equation (7) can be written as

$$\begin{bmatrix} \dot{x}_1(t) \\ \varepsilon \dot{x}_2(t) \end{bmatrix} = N \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \widetilde{C}u(t) - \widetilde{E}w(t) \tag{11}$$

in which

$$\begin{aligned} N &= \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = B^{-1}[I - A - \{x(t)M\}], \\ \widetilde{C} &= B^{-1}C = B^{-1}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \end{bmatrix}, \\ \widetilde{E} &= B^{-1}E = B^{-1}\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \widetilde{E}_1 \\ \widetilde{E}_2 \end{bmatrix}. \end{aligned} \tag{12}$$

Then (11) can be written as

$$\dot{x}_1(t) = N_{11}x_1(t) + N_{12}x_2(t) - \widetilde{C}_1u(t) - \widetilde{E}_1w(t) \tag{13a}$$

$$\varepsilon\dot{x}_2(t) = N_{21}x_1(t) + N_{22}x_2(t) - \widetilde{C}_2u(t) - \widetilde{E}_2w(t). \tag{13b}$$

3.1. Decomposition of Slow and Fast Systems. Using the singularly perturbed technology in system analysis and design can reduce the system order and the ill-condition caused by the cross-coupled interaction of the slow and fast dynamic modes. To avoid considering the problem of the whole singularly perturbed system, a fact is often used that when the little parameter ε converges to zero, the fast and slow phenomenon of the system will be separated. Based on this fact, Chow and Kokotovic presented a decomposition method for systems with slow and fast modes, which can separate the original system into two subsystems, the boundary layer system (the fast subsystem) and the reduced system (the slow subsystem), then design, respectively, quadratic optimal control to the fast and slow subsystems (thus having to solve two Riccati equations), and further obtain the composite control of the original system [19].

Using the decomposition method of slow and fast systems, the state equation of the slow subsystem can be written as

$$\begin{aligned} \dot{x}_{1s} &= N_{11}x_{1s} + N_{12}x_{2s} - \widetilde{C}_1u_s - \widetilde{E}_1w_s \\ 0 &= N_{21}x_{1s} + N_{22}x_{2s} - \widetilde{C}_2u_s - \widetilde{E}_2w_s \end{aligned} \tag{14}$$

with $x_{1s} = x_{10}$

Assuming that N_{22} is nonsingular, we have

$$\begin{aligned} \dot{x}_{1s} &= N_0x_{1s} + \widetilde{C}_0u_s + \widetilde{E}_0w_s, \\ x_{2s} &= -N_{22}^{-1}\left(N_{21}x_{1s} - \widetilde{C}_2u_s - \widetilde{E}_2w_s\right), \end{aligned} \tag{15}$$

where $N_0 = N_{11} - N_{12}N_{22}^{-1}N_{21}$, $\widetilde{C}_0 = N_{12}N_{22}^{-1}\widetilde{C}_2 - \widetilde{C}_1$, and $\widetilde{E}_0 = N_{12}N_{22}^{-1}\widetilde{E}_2 - \widetilde{E}_1$.

Then the performance index of the slow subsystem is

$$\begin{aligned}
J_s(u,w) &= \frac{1}{2}\int_{t_0}^{t_f}\left\{[x(t)-h(t)]^T Q[x(t)-h(t)]\right. \\
&\quad \left.+u^T(t)Ru(t)-\gamma^2 w^T(t)w(t)\right\}dt=\frac{1}{2} \\
&\quad \cdot\int_{t_0}^{t_f}\left[(x_{1s}-h_1)^T Q_{11}(x_{1s}-h_1)\right. \\
&\quad +(x_{2s}-h_2)^T Q_{22}(x_{2s}-h_2)+u_s{}^T Ru_s \\
&\quad \left.-\gamma^2 w^T w\right]dt=\frac{1}{2}\int_{t_0}^{t_f}\left(x_{1s}{}^T Q_1 x_{1s}\right. \\
&\quad +2x_{1s}{}^T D_1 u_s+2x_{1s}{}^T D_2 w_s+2u_s{}^T D_3 w_s+2x_{1s}{}^T D_4 \\
&\quad +2u_s{}^T D_5+2w_s{}^T D_6+u_s{}^T R_{1s}u_s+w_s{}^T R_{2s}w_s \\
&\quad \left.+Q_2\right)dt
\end{aligned} \tag{16}$$

in which

$$\begin{aligned}
Q_1 &= Q_{11}+N_{21}^T N_{22}^{-T}Q_{22}N_{22}^{-1}N_{21}, \\
D_1 &= -N_{21}^T N_{22}^{-T}Q_{22}N_{22}^{-1}\widetilde{C}_2, \\
D_2 &= -N_{21}^T N_{22}^{-T}Q_{22}N_{22}^{-1}\widetilde{E}_2, \\
D_3 &= \widetilde{C}_2{}^T N_{22}^{-T}Q_{22}N_{22}^{-1}\widetilde{E}_2, \\
D_4 &= N_{21}^T N_{22}^{-T}Q_{22}h_2-Q_{11}h_1, \\
D_5 &= -\widetilde{C}_2 N_{22}^{-T}Q_{22}h_2, \\
D_6 &= -\widetilde{E}_2 N_{22}^{-T}Q_{22}h_2, \\
R_{1s} &= R+\widetilde{C}_2{}^T N_{22}^{-T}Q_{22}N_{22}^{-1}\widetilde{C}_2, \\
R_{2s} &= \widetilde{E}_2{}^T N_{22}^{-T}Q_{22}N_{22}^{-1}\widetilde{E}_2-\gamma^2, \\
Q_2 &= h_1{}^T Q_{11}h_1+h_2{}^T Q_{22}h_2.
\end{aligned} \tag{17}$$

We introduce the following notations:

$$\begin{aligned}
T_1 &= \left(1-R_{1s}{}^{-1}D_3R_{2s}{}^{-1}D_3{}^T\right)^{-1} \\
&\quad \cdot\left(-R_{1s}{}^{-1}D_1{}^T+R_{1s}{}^{-1}D_3R_{2s}{}^{-1}D_2{}^T\right), \\
T_2 &= \left(1-R_{1s}{}^{-1}D_3R_{2s}{}^{-1}D_3{}^T\right)^{-1} \\
&\quad \cdot\left(R_{1s}{}^{-1}D_3R_{2s}{}^{-1}\widetilde{E}_0{}^T-R_{1s}{}^{-1}\widetilde{C}_0{}^T\right), \\
T_3 &= \left(1-R_{1s}{}^{-1}D_3R_{2s}{}^{-1}D_3{}^T\right)^{-1} \\
&\quad \cdot\left(R_{1s}{}^{-1}D_3R_{2s}{}^{-1}D_6-R_{1s}{}^{-1}D_5\right), \\
T_4 &= -R_{2s}{}^{-1}\left(D_2{}^T+D_3{}^T T_1\right), \\
T_5 &= -R_{2s}{}^{-1}\left(D_3{}^T T_2+\widetilde{E}_0{}^T\right), \\
T_6 &= -R_{2s}{}^{-1}\left(D_3{}^T T_3+D_6\right) \\
A_S &= N_0+\widetilde{C}_0T_1+\widetilde{E}_0T_4, \\
B_s &= \widetilde{C}_0T_2+\widetilde{E}_0T_5, \\
Q_s &= Q_1+D_1T_1+D_2T_4.
\end{aligned} \tag{18}$$

Assumption 1. The triplet $(A_s, B_s, \sqrt{Q_s})$ is stabilizable and detectable.

Theorem 2. *Under Assumption 1, the saddle-point equilibrium strategy of the slow subsystem exists and has the following form:*

$$\begin{aligned}
u_s{}^* &= (T_1+T_2p_s)x_{1s}+T_2r_s+T_3 \\
w_s{}^* &= (T_4+T_5p_s)x_{1s}+T_5r_s+T_6.
\end{aligned} \tag{19}$$

in which p_s satisfies the following Riccati equation (20):

$$-\dot{p}_s = A_s{}^T p_s+p_sA_s+p_sB_sp_s+Q_s. \tag{20}$$

Proof. To obtain the equilibrium strategy of the slow subsystem, the Hamiltonian function $H(t)$ is constructed as

$$\begin{aligned}
H &= \frac{1}{2}\left(x_{1s}{}^T Q_1x_{1s}+2x_{1s}{}^T D_1u_s+2x_{1s}{}^T D_2w_s\right. \\
&\quad +2u_s{}^T D_3w_s+2x_{1s}{}^T D_4+2u_s{}^T D_5+2w_s{}^T D_6 \\
&\quad \left.+u_s{}^T R_{1s}u_s+w_s{}^T R_{2s}w_s+Q_2\right)+\lambda^T\left(N_0x_{1s}+\widetilde{C}_0u_s\right. \\
&\quad \left.+\widetilde{E}_0w_s\right).
\end{aligned} \tag{21}$$

For the control equation $\partial H/\partial u_s = 0$, we can get the optimal control strategy of the slow subsystem

$$u_s = -R_{1s}{}^{-1}\left(D_1{}^T x_{1s}+D_3w_s+\widetilde{C}_0{}^T\lambda+D_5\right). \tag{22}$$

For the control equation $\partial H/\partial w_s = 0$, we can get the worst disturbance input of the slow subsystem

$$w_s = -R_{2s}{}^{-1}\left(D_2{}^T x_{1s}+D_3{}^T u_s+\widetilde{E}_0{}^T\lambda+D_6\right). \tag{23}$$

From (22) and (23), we then obtain

$$\begin{aligned}
u_s &= T_1x_{1s}+T_2\lambda+T_3 \\
w_s &= T_4x_{1s}+T_5\lambda+T_6
\end{aligned} \tag{24}$$

in which

$$\begin{aligned}
T_1 &= \left(1 - R_{1s}^{-1} D_3 R_{2s}^{-1} D_3^{T}\right)^{-1} \left(-R_{1s}^{-1} D_1^{T} + R_{1s}^{-1} D_3 R_{2s}^{-1} D_2^{T}\right), \\
T_2 &= \left(1 - R_{1s}^{-1} D_3 R_{2s}^{-1} D_3^{T}\right)^{-1} \left(R_{1s}^{-1} D_3 R_{2s}^{-1} \widetilde{E}_0^{T} - R_{1s}^{-1} \widetilde{C}_0^{T}\right), \\
T_3 &= \left(1 - R_{1s}^{-1} D_3 R_{2s}^{-1} D_3^{T}\right)^{-1} \left(R_{1s}^{-1} D_3 R_{2s}^{-1} D_6 - R_{1s}^{-1} D_5\right), \\
T_4 &= -R_{2s}^{-1} \left(D_2^{T} + D_3^{T} T_1\right), \\
T_5 &= -R_{2s}^{-1} \left(D_3^{T} T_2 + \widetilde{E}_0^{T}\right), \\
T_6 &= -R_{2s}^{-1} \left(D_3^{T} T_3 + D_6\right).
\end{aligned} \tag{25}$$

For the accompany equation

$$-\dot{\lambda} = Q_1 x_{1s} + D_1 u_s + D_2 w_s + D_4 + N_0^{T} \lambda \tag{26}$$

letting $\lambda = p_s x_{1s} + r_s$, we have

$$\begin{aligned}
-\dot{p}_s = {} & p_s \left(N_0 + \widetilde{C}_0 T_1 + \widetilde{E}_0 T_4\right) \\
& + \left(N_0^{T} + D_1 T_2 + D_2 T_5\right) p_s \\
& + p_s \left(\widetilde{C}_0 T_2 + \widetilde{E}_0 T_5\right) p_s \left(Q_1 + D_1 T_1 + D_2 T_4\right)
\end{aligned} \tag{27}$$

$$\begin{aligned}
-\dot{r}_s = {} & \left(p_s \widetilde{C}_0 T_2 + p_s \widetilde{E}_0 T_5 + D_1 T_2 + D_2 T_5 + N_0^{T}\right) r_s \\
& + \left(p_s \widetilde{C}_0 T_3 + p_s \widetilde{E}_0 T_6 + D_1 T_3 + D_2 T_6 + D_4\right)
\end{aligned} \tag{28}$$

because

$$\left(N_0 + \widetilde{C}_0 T_1 + \widetilde{E}_0 T_3\right)^{T} = N_0^{T} + D_1 T_2 + D_2 T. \tag{29}$$

Then (27) can be written as the following algebraic Riccati equation:

$$-\dot{p}_s = A_s^{T} p_s + p_s A_s + p_s B_s p_s + Q_s. \tag{30}$$

Under Assumption 1, the saddle-point equilibrium strategy of the slow subsystem is

$$\begin{aligned}
u_s^{*} &= \left(T_1 + T_2 p_s\right) x_{1s} + T_2 r_s + T_3 \\
w_s^{*} &= \left(T_4 + T_5 p_s\right) x_{1s} + T_5 r_s + T_6
\end{aligned} \tag{31}$$

in which p_s satisfies the above Riccati equation (20).

Theorem 2 is proved. □

In the fast subsystem, we assume that the slow variables are constant in the boundary layer. Redefining the fast variables $x_{2f}(t) = x_2(t) - x_{2s}(t)$ and the fast control strategies $u_f(t) = u(t) - u_s(t)$, $w_f(t) = w(t) - w_s(t)$, the fast subsystem is formulated as

$$\begin{aligned}
\dot{x}_{2f} &= \frac{1}{\varepsilon} N_{22} x_{2f} - \frac{1}{\varepsilon} \widetilde{C}_2 u_f - \frac{1}{\varepsilon} \widetilde{E}_2 w_f \\
x_{2f}(0) &= x_{20} - x_{2s}(0).
\end{aligned} \tag{32}$$

Then we can obtain the quadratic cost function for the fast subsystem

$$\begin{aligned}
J_f = \frac{1}{2} \int_{t_0}^{t_f} & \left(x_{2f}^{T} Q_{22} x_{2f} + h_2^{T} Q_{22} h_2 + u_f^{T} R u_f \right. \\
& \left. - \gamma^2 w_f^{T} w_f\right) dt.
\end{aligned} \tag{33}$$

Assumption 3. The triplets $(N_{22}, \widetilde{C}_2, \sqrt{Q_{22}})$ and $(A_{22}, \widetilde{E}_2, \sqrt{Q_{22}})$ are stabilizable and detectable.

Under Assumption 3, the equilibrium strategies of the fast subsystem are given by

$$\begin{aligned}
u_f^{*} &= R^{-1} \widetilde{C}_2^{T} p_f x_{2f} \\
w_f^{*} &= \frac{\widetilde{E}_2^{T}}{\gamma^2} p_f x_{2f}.
\end{aligned} \tag{34}$$

And p_f satisfies the following algebraic Riccati equation (35):

$$\begin{aligned}
& p_f N_{22} + N_{22}^{T} p_f - p_f \left(\widetilde{C}_2 R^{-1} \widetilde{C}_2^{T} + \widetilde{E}_2 \frac{\widetilde{E}_2^{T}}{\gamma^2}\right) p_f \\
& \quad + Q_{22} = 0.
\end{aligned} \tag{35}$$

3.2. The Robust Control Strategy. The composite strategy pair of the full-order singularly perturbed dynamic input-output system (7) is constructed as follows [20]:

$$\begin{aligned}
u_c &= u_s^{*} + u_f^{*} \\
&= \left[\left(T_1 + T_2 p_s\right) x_{1s} + T_2 r_s + T_3\right] + R^{-1} \widetilde{C}_2^{T} p_f x_{2f} \\
w_c &= w_s^{*} + w_f^{*} \\
&= \left[\left(T_4 + T_5 p_s\right) x_{1s} + T_5 r_s + T_6\right] + \frac{\widetilde{E}_2^{T}}{\gamma^2} p_f x_{2f}
\end{aligned} \tag{36}$$

with x_1 replacing x_{1s} and x_2 replacing $x_{2s} + x_{2f}$; for $x_{2s} = -N_{22}^{-1}(N_{21} x_{1s} - \widetilde{C}_2 u_s - \widetilde{E}_2 w_s)$, we obtain

$$\begin{aligned}
u_c &= G_1 x_1 + G_2 x_2 + G_3 \\
w_c &= G_4 x_1 + G_5 x_2 + G_6,
\end{aligned} \tag{37}$$

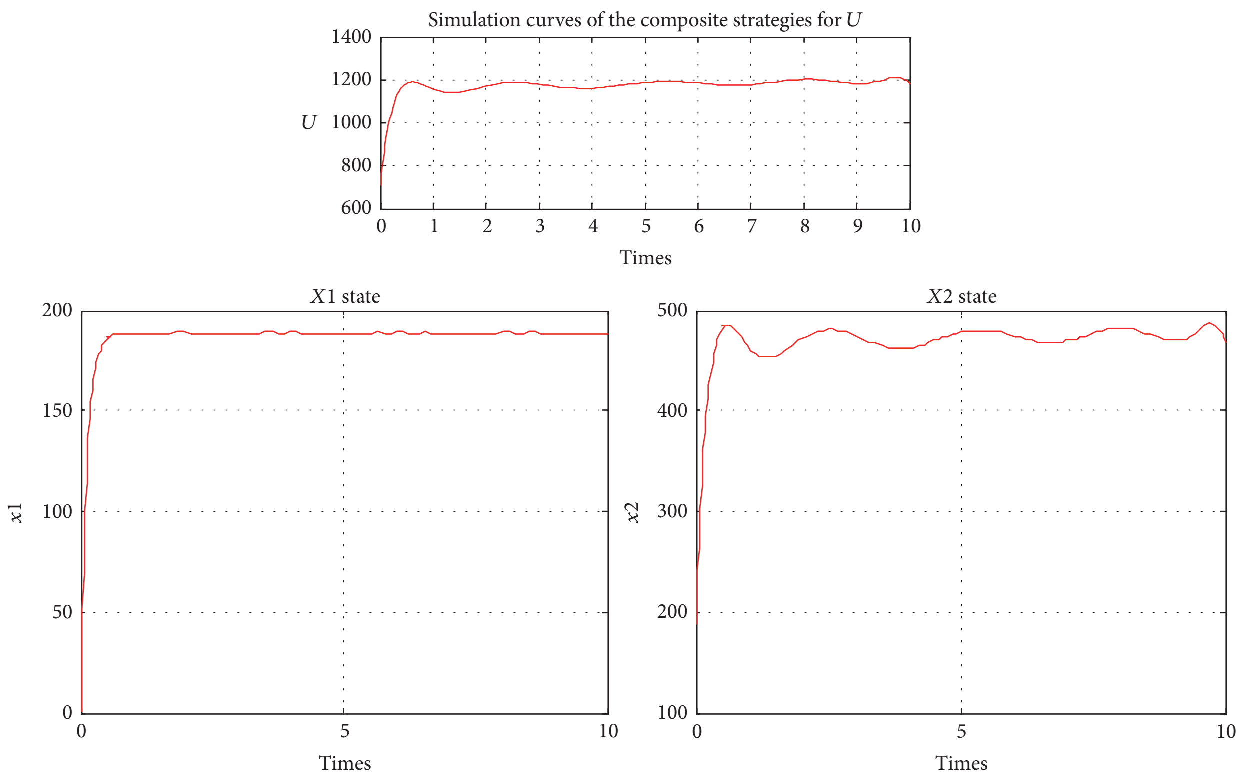

FIGURE 1: The simulation curve of the control law and the trajectories of x_1 and x_2 for the dynamic input-output problem.

where

$$\begin{aligned}
G_1 &= (T_1 + T_2 p_s) + R^{-1}\widetilde{C}_2^{\,T} p_f N_{22}^{-1} \left[N_{21} \right.\\
&\quad \left. - \widetilde{C}_2 (T_1 + T_2 p_s) - \widetilde{E}_2 (T_4 + T_5 p_s)\right]\\
G_2 &= R^{-1}\widetilde{C}_2^{\,T} p_f\\
G_3 &= T_2 r_s + T_3 - R^{-1}\widetilde{C}_2^{\,T} p_f N_{22}^{-1} \left[\widetilde{C}_2 (T_2 r_s + T_3) \right.\\
&\quad \left. + \widetilde{E}_2 (T_5 r_s + T_6)\right]\\
G_4 &= (T_4 + T_5 p_s) + \frac{\widetilde{E}_2^{\,T}}{\gamma^2} p_f N_{22}^{-1} \left[N_{21} \right.\\
&\quad \left. - \widetilde{C}_2 (T_1 + T_2 p_s) - \widetilde{E}_2 (T_4 + T_5 p_s)\right]\\
G_5 &= \frac{\widetilde{E}_2^{\,T}}{\gamma^2} p_f\\
G_6 &= T_5 r_s + T_6 - \frac{\widetilde{E}_2^{\,T}}{\gamma^2} p_f N_{22}^{-1} \left[\widetilde{C}_2 (T_2 r_s + T_3) \right.\\
&\quad \left. + \widetilde{E}_2 (T_5 r_s + T_6)\right].
\end{aligned} \tag{38}$$

The composite strategy pair constitutes an $o(\varepsilon)$ (near) saddle-point equilibrium of the full-order game. The proof can be found in [21].

4. Numerical Example

Similar to [22], parameters in the model are set as follows:

$$\begin{aligned}
x0 &= (2690.7, 2172.3)\,T,\\
t0 &= 0,\\
tf &= 10,\\
A &= \begin{bmatrix} 0.1500 & 0.2700 \\ 0.0500 & 0.2700 \end{bmatrix},\\
B &= \begin{bmatrix} 0.2700 & 0.2700 \\ 0.0200 & 0.0200 \end{bmatrix},\\
M_1 &= \begin{bmatrix} -1 \\ 0 \end{bmatrix},\\
M_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix},\\
C = E &= \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix},\\
Q &= 10 - 4I,\\
R &= 10 - 3I,\\
h &= (1510, 1805)\,T.
\end{aligned} \tag{39}$$

We choose $\gamma = 0.5$ and $\varepsilon = 0.001$ and obtain the simulation curves for the optimal control strategy and the state as Figure 1.

5. Conclusion

The above simulation curve indicates that compared with the slow production process of the dynamic input-output system, the fast production shows a greater fluctuation, which means that, in the real economic system, outputs of the fast production process are more vulnerable to the perturbation influence of external factors such as international interest rates and exchange. The control variable reflects the control strategy of the system with stochastic disturbance. If economic strategies are designed as the above control curve to control the final consumption vector level, the minimum gap between the optimal amount of output and the target amount of output with policies' fluctuation and stochastic factors' disturbance can be obtained, that is, to realize balance of the supply and demand.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work is supported by the National Natural Science Fund of China (71571053), the Natural Science Fund of Guangdong Province (2014A030310366 and 2015A030310218), a grant from Guangdong Province, Soft Science Research Project (2016A070705052).

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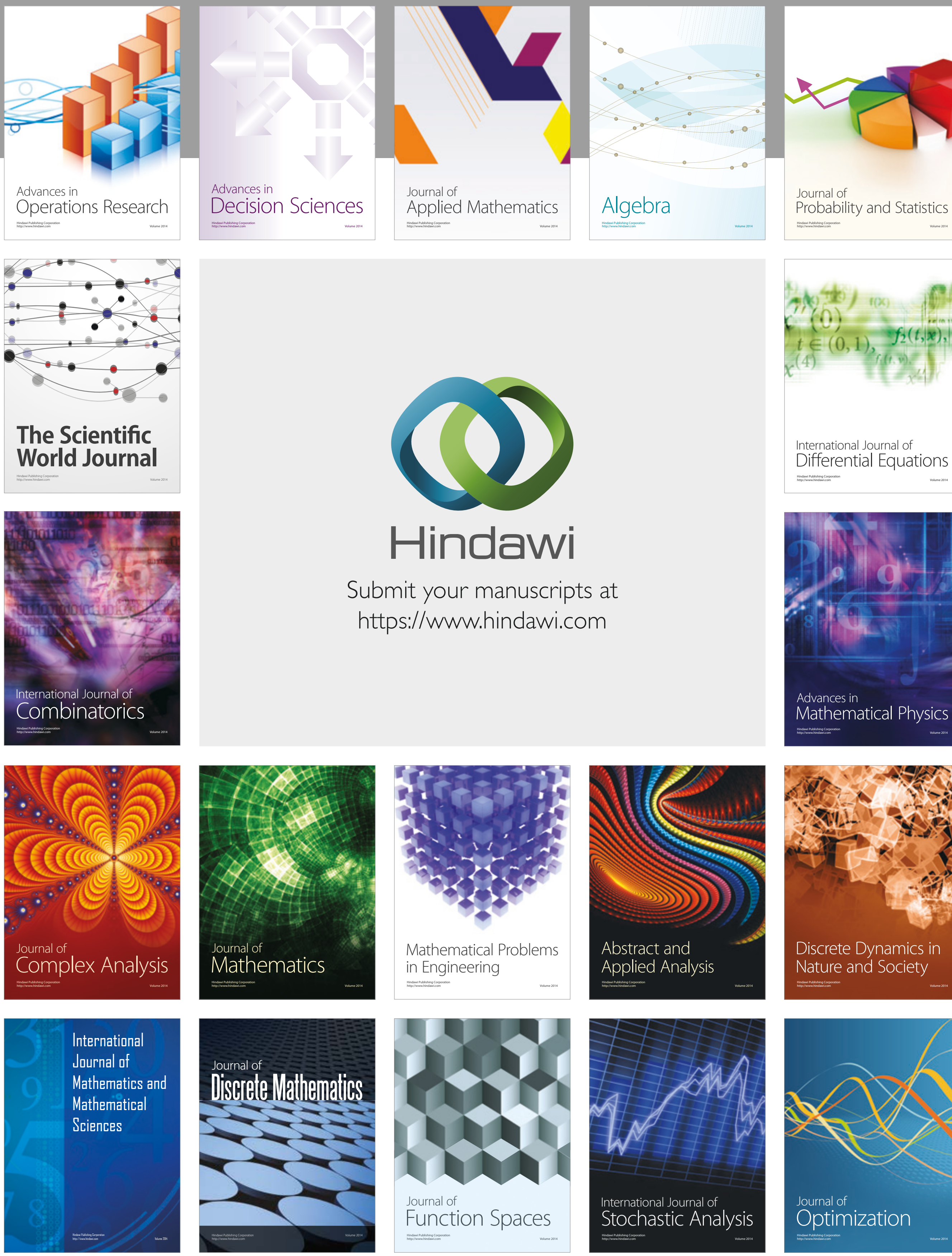
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