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Research Article

The Extended Fractional $(D_\xi^\alpha G/G)$-Expansion Method and Its Applications to a Space-Time Fractional Fokas Equation

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Based on a fractional subequation and the properties of the modified Riemann-Liouville fractional derivative, we propose a new analytical method named extended fractional $(D_\xi^\alpha G/G)$-expansion method for seeking traveling wave solutions of fractional partial differential equations. To illustrate the effectiveness of the method, we discuss a space-time fractional Fokas equation, many types of exact analytical solutions are obtained, and the solutions include hyperbolic function and trigonometric and negative exponential solutions.

1. Introduction

The nonlinear fractional partial differential equations (FPDEs) have attracted much attention because of their potential applications in various fields of science, such as fluid mechanics (especially in viscoelastic flow and theory of viscoplasticity), medical (human tissue under mechanical load model), electrical engineering (ultrasound transmission), biochemistry (polymer and protein model), material diffusion (including normal diffusion and anomalous diffusion), signal processing, and control systems.

The exact solutions of FPDEs can facilitate illustrating the structural information about the complex physics phenomena and help better understand the physical interpretation. Thus, it is an important and significant task to find more exact solutions of different forms for the FPDEs. In recent decades, many mathematicians and physicists have made significant achievements and also presented some effective methods, for example, the fractional subequation method [1–3], the Jacobi elliptic equation method [4], the fractional mapping method [5], the (G'/G)-expansion method [6], the extended fractional Riccati expansion method [7], the first integral method [8], and the fractional complex transform [9]. Due to these methods, various exact solutions or numerical solutions of FPDEs have been established successfully.

Searching for exact solutions of nonlinear ODEs plays an important role in the study of physical phenomena and gradually becomes one of the most important and significant tasks. In the past several decades, both mathematicians and physicists have made many significant works in this direction and presented some effective methods, such as the global error minimization method [10].

Recently, Feng [11] introduced a new method which is called the improved fractional $(D_\xi^\alpha G/G)$ method to show traveling wave solutions of nonlinear FDEs. The method is based on the homogeneous balance principle and Jumarie's modified Riemann-Liouville derivative. By using the fractional Riccati equation $AGD_\xi^{2\alpha}G - BGD_\xi^\alpha G - C(D_\xi^\alpha G)^2 - EG^2 = 0$, Feng obtained traveling wave solutions of the (2 + 1)-dimensional space-time fractional Nizhnik-Novikov-Veselov system and the space-time fractional KP-BBM equation.

In order to get as many results as possible, we propose a new analytical method named extended fractional $(D_\xi^\alpha G/G)$-expansion method which adds in negative power exponent to seek more general traveling wave solutions. In this paper, using the proposed method, we present some new exact solutions of a space-time fractional Fokas equation. The

results suggest that the method not only is simple, effective, and straightforward, but also can be used for many other nonlinear FPDEs.

The organization of the paper is as follows. In Section 2, some basic properties of Jumarie's modified Riemann-Liouville derivative are shown. In Section 3, we describe the extended fractional $(D_\xi^\alpha G/G)$-expansion method for finding traveling wave solutions of the proposed FPDEs. In Section 4, we apply this method to obtain exact traveling wave solutions for a space-time fractional Fokas equation. The conclusion part is in Section 5.

2. Preliminaries

Jumarie's modified Riemann-Liouville derivative of order α is defined by the expression [1]

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1} \left(f(\xi) - f(0)\right) d\xi, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha} \left(f(\xi) - f(0)\right) d\xi, & 0 < \alpha < 1, \\ \left(f^{(\alpha-n)}(x)\right)^{(n)}, & n \le \alpha < n+1,\ n \ge 1. \end{cases} \tag{1}$$

The property for the proposed modified Riemann-Liouville derivatives is listed in [1] as follows:

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0. \tag{2}$$

The above equations play an important role in fractional calculus in the following sections.

3. Description of the Extended Fractional $(D_\xi^\alpha G/G)$-Expansion Method

In this section, the main steps of the extended fractional $(D_\xi^\alpha G/G)$-expansion method are described as follows.

Step 1. Suppose that a fractional partial differential equation in the variables x, y, t is given by

$$P\left(u, u_t, u_x, u_y, \frac{\partial^\alpha u}{\partial x^\alpha}, \frac{\partial^\alpha u}{\partial y^\alpha}, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}, \frac{\partial^{2\alpha} u}{\partial y^{2\alpha}}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \ldots\right) = 0, \quad 0 < \alpha \le 1, \tag{3}$$

where $\partial^\alpha u/\partial x^\alpha$, $\partial^\alpha u/\partial y^\alpha$, and $\partial^\alpha u/\partial t^\alpha$ are Jumarie's modified Riemann-Liouville derivatives of u with respect to x, y, and t, $u = u(x, y, t)$ is an unknown function, and P is a polynomial in u and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved.

Step 2. By using the traveling wave transformation,

$$u(x, y, t) = u(\xi), \quad \xi = kx + ly + \omega t, \tag{4}$$

where k, l, and ω are constants to be determined later, the FDE (3) is reduced to the following nonlinear fractional ordinary differential equation (ODE) for $u = u(\xi)$:

$$P\left(u, \omega u', k u', l u', k^\alpha D_\xi^\alpha u, l^\alpha D_\xi^\alpha u, \omega^\alpha D_\xi^\alpha u, \ldots\right) = 0. \tag{5}$$

Step 3. We suppose the solution u of (5) can be expressed in the following form:

$$u = \sum_{i=0}^{n} a_i \left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)^i + \sum_{j=1}^{n} b_j \left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)^{-j}, \tag{6}$$

where a_i, b_j $(i = 0, 1, 2, \ldots, n;\ j = 1, 2, \ldots, n)$ are arbitrary constants to be determined later, the positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (5), and $G = G(\xi)$ satisfies the following fractional ordinary differential equation:

$$AG(\xi) D_\xi^{2\alpha} G(\xi) - BG(\xi) D_\xi^\alpha G(\xi) - C\left(D_\xi^\alpha G(\xi)\right)^2 - EG^2(\xi) = 0, \tag{7}$$

where $D_\xi^\alpha G(\xi)$ denotes the modified Riemann-Liouville derivative of order α for $G(\xi)$ with respect to ξ.

Step 4. Substituting (6) into (5) and using (7) and collecting all terms with the same order of $(D_\xi^\alpha G/G)$ together, the left-hand side of (5) is converted into another polynomial in $(D_\xi^\alpha G/G)$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for a_i, b_j, $(i = 0, 1, \ldots, n,\ j = 1, 2, \ldots, n)$, k, l, and ω.

Step 5. Solving the equations system in Step 4 and using the general expressions for $(D_\xi^\alpha G/G)$, we can construct a variety of exact solutions for (3).

In order to obtain the general expressions for $(D_\xi^\alpha G/G)$ in (7), we suppose

$$G(\xi) = H(\eta), \quad \eta = \frac{\xi^\alpha}{\Gamma(1+\alpha)}. \tag{8}$$

Then by use of (2) one has $D_\xi^\alpha \eta = 1$ and furthermore, $D_\xi^\alpha G(\xi) = D_\xi^\alpha H(\eta) = H'(\eta) D_\xi^\alpha \eta = H'(\eta)$, so (7) can be turned into the following second ordinary differential equation:

$$AH(\eta) H''(\eta) - BH(\eta) H'(\eta) - C\left(H'(\eta)\right)^2 - EH^2(\eta) = 0. \tag{9}$$

By the general expressions for $H'(\eta)/H(\eta)$ in [12], one can obtain the following expressions for $D_\xi^\alpha G(\xi)/G(\xi)$:

(1) When $B \neq 0$, $\Psi = A - C$, and $\Omega = B^2 + 4E\Psi > 0$,

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)} = \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi}\frac{C_1 \sinh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \cosh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)}{C_1 \cosh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \sinh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)}. \tag{10}$$

(2) When $B \neq 0$, $\Psi = A - C$, and $\Omega = B^2 + 4E\Psi < 0$,

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)} = \frac{B}{2\Psi} + \frac{\sqrt{-\Omega}}{2\Psi}\frac{-C_1 \sin\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \cos\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)}{C_1 \cos\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \sin\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)}. \tag{11}$$

(3) When $B \neq 0$, $\Psi = A - C$, and $\Omega = B^2 + 4E\Psi = 0$,

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)} = \frac{B}{2\Psi} - \frac{AC_2\Gamma(1+\alpha)}{\left(C_1\Gamma(1+\alpha) - C_2\xi^\alpha\right)\Psi}. \tag{12}$$

(4) When $B = 0$, $\Psi = A - C$, and $\Delta = \Psi E > 0$,

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)} = \frac{\sqrt{\Delta}}{\Psi}\frac{C_1 \sinh\left(\left(\sqrt{\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \cosh\left(\left(\sqrt{\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right)}{C_1 \cosh\left(\left(\sqrt{\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \sinh\left(\left(\sqrt{\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right)}. \tag{13}$$

(5) When $B = 0$, $\Psi = A - C$, and $\Delta = \Psi E < 0$,

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)} = \frac{\sqrt{-\Delta}}{\Psi}\frac{-C_1 \sin\left(\left(\sqrt{-\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \cos\left(\left(\sqrt{-\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right)}{C_1 \cos\left(\left(\sqrt{-\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \sin\left(\left(\sqrt{-\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right)}. \tag{14}$$

4. Applications of the Extended Fractional $(D_\xi^\alpha G/G)$ Method to the Space-Time Fractional Fokas Equation

We consider a space-time fractional Fokas equation [13]:

$$\begin{aligned} &4\frac{\partial^{2\alpha} q}{\partial t^\alpha \partial x_1^\alpha} - \frac{\partial^{4\alpha} q}{\partial x_1^{3\alpha}\partial x_2^\alpha} + \frac{\partial^{4\alpha} q}{\partial x_2^{3\alpha}\partial x_1^\alpha} + 12\frac{\partial^\alpha q}{\partial x_1^\alpha}\frac{\partial^\alpha q}{\partial x_2^\alpha} \\ &\quad + 12q\frac{\partial^{2\alpha} q}{\partial x_1^\alpha \partial x_2^\alpha} - 6\frac{\partial^{2\alpha} q}{\partial y_1^\alpha \partial y_2^\alpha} = 0, \quad 0 < \alpha \leq 1, \end{aligned} \tag{15}$$

which is a transformed generalization of the (4 + 1)-dimensional Fokas equation

$$\begin{aligned} \frac{\partial^2 q}{\partial t \partial x_1} &= \frac{1}{4}\frac{\partial^4 q}{\partial x_1^3 \partial x_2} - \frac{1}{4}\frac{\partial^4 q}{\partial x_1^3 \partial x_2} - \frac{3}{2}\frac{\partial^2 \left(q^2\right)}{\partial x_1 \partial x_2} \\ &\quad + \frac{3}{2}\frac{\partial^2 q}{\partial y_1 \partial y_2}. \end{aligned} \tag{16}$$

Equation (16) is one of the new high-dimensional nonlinear wave Fokas equations recently obtained by extending the integrable KP equation and DS equation.

To solve (15), we take the following traveling wave transformation:

$$q = q(\xi), \quad \xi = k_1 x_1 + k_2 x_2 + l_1 y_1 + l_2 y_2 + \omega t; \tag{17}$$

then (15) is reduced into a nonlinear fractional ODE in the form

$$\begin{aligned} &\left(k_1^\alpha k_2^{3\alpha} - k_1^{3\alpha} k_2^\alpha\right) D_\xi^{4\alpha} q + \left(4\omega^\alpha k_1^\alpha - 6l_1^\alpha l_2^\alpha\right) D_\xi^{2\alpha} q \\ &\quad + 12k_1^\alpha k_2^\alpha \left(D_\xi^\alpha q\right)^2 + 12k_1^\alpha k_2^\alpha q D_\xi^{2\alpha} q = 0. \end{aligned} \tag{18}$$

Balancing $D_\xi^{4\alpha} q$ and $qD_\xi^{2\alpha} q$, we have $n + 4 = 2n + 2 \Rightarrow n = 2$. So we have

$$q(\xi) = a_0 + a_1 \frac{D_\xi^\alpha G(\xi)}{G(\xi)} + a_2 \left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)^2 + b_1 \left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)^{-1} + b_2 \left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)^{-2}, \tag{19}$$

where a_0, a_1, a_2, b_1, and b_2 are constants to be determined later, and function $G(\xi)$ satisfies (7).

Substituting (19) together with (7) into (18), the left-hand side of (18) is converted into polynomials in $(D_\xi^\alpha G/G)^i$, $(i = \ldots, -2, -1, 0, 1, 2, \ldots)$. We collect each coefficient of these resulting polynomials to zero, which yields a set of simultaneous algebraic equations for a_0, a_1, a_2, b_1, b_2, k_1, k_2, l_1, l_2, and ω. Solving this system of algebraic equations, with the aid of Maple, we obtain

$$\begin{aligned}
(1)\ a_0 &= \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha}, \\
a_1 &= \frac{\left(k_2^{2\alpha} - k_1^{2\alpha}\right)\Psi B}{A^2}, \\
a_2 &= \frac{\Psi^2\left(k_1^{2\alpha} - k_2^{2\alpha}\right)}{A^2}, \\
b_1 &= 0, \\
b_2 &= 0.
\end{aligned} \tag{20}$$

$$\begin{aligned}
(2)\ a_0 &= \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha}, \\
a_1 &= 0, \\
a_2 &= 0, \\
b_1 &= \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)BE}{A^2}, \\
b_2 &= \frac{E^2\left(k_1^{2\alpha} - k_2^{2\alpha}\right)}{A^2},
\end{aligned} \tag{21}$$

where $\Psi = A - C$, A, B, C, E, k_1, k_2, l_1, l_2, and ω are free parameters.

Substituting (20) into (19) and combining with (10)–(14), we can obtain the following exact traveling wave solutions to (15).

(1.1) When $B \neq 0$, $\Psi = A - C$, and $\Omega = B^2 + 4E\Psi > 0$,

$$\begin{aligned}
q_1(\xi) = {} & \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\
& + \frac{\left(k_2^{2\alpha} - k_1^{2\alpha}\right)B}{A^2}\left(\frac{B}{2} + \frac{\sqrt{\Omega}}{2}\frac{C_1 \sinh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \cosh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)}{C_1 \cosh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \sinh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)}\right) \\
& + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)}{A^2}\left(\frac{B}{2} + \frac{\sqrt{\Omega}}{2}\frac{C_1 \sinh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \cosh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)}{C_1 \cosh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \sinh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)}\right)^2.
\end{aligned} \tag{22}$$

(1.2) When $B \neq 0$, $\Psi = A - C$, and $\Omega = B^2 + 4E\Psi < 0$,

$$\begin{aligned}
q_2(\xi) = {} & \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\
& + \frac{\left(k_2^{2\alpha} - k_1^{2\alpha}\right)B}{A^2}\left(\frac{B}{2} + \frac{\sqrt{-\Omega}}{2}\frac{-C_1 \sin\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \cos\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)}{C_1 \cos\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \sin\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)}\right) \\
& + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)}{A^2}\left(\frac{B}{2} + \frac{\sqrt{-\Omega}}{2}\frac{-C_1 \sin\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \cos\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)}{C_1 \cos\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2 \sin\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)}\right)^2.
\end{aligned} \tag{23}$$

(1.3) When $B \neq 0$, $\Psi = A - C$, and $\Omega = B^2 + 4E\Psi = 0$,

$$q_3(\xi) = \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} + \frac{C_2^2\left(k_1^{2\alpha} - k_2^{2\alpha}\right)(\Gamma(1+\alpha))^2}{\left(C_1\Gamma(1+\alpha) - C_2\xi^\alpha\right)^2}. \tag{24}$$

(1.4) When $B = 0$, $\Psi = A - C$, and $\Delta = \Psi E > 0$,

$$q_4(\xi) = -\frac{2\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\Delta}{3A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\Delta}{A^2}\left(\frac{C_1\sinh\left(\left(\sqrt{\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\cosh\left(\left(\sqrt{\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right)}{C_1\cosh\left(\left(\sqrt{\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\sinh\left(\left(\sqrt{\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right)}\right)^2. \tag{25}$$

(1.5) When $B = 0$, $\Psi = A - C$, and $\Delta = \Psi E < 0$,

$$q_5(\xi) = -\frac{2\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\Delta}{3A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} - \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\Delta}{A^2}\left(\frac{-C_1\sin\left(\left(\sqrt{-\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\cos\left(\left(\sqrt{-\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right)}{C_1\cos\left(\left(\sqrt{-\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\sin\left(\left(\sqrt{-\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right)}\right)^2, \tag{26}$$

where $\xi = k_1x_1 + k_2x_2 + l_1y_1 + l_2y_2 + \omega t$, k_1, k_2, l_1, l_2, and ω are arbitrary constants.

Substituting (21) into (19) and combining with (10)–(14), we can obtain the following exact traveling wave solutions to (15).

(2.1) When $B \neq 0$, $\Psi = A - C$, and $\Omega = B^2 + 4E\Psi > 0$,

$$\begin{aligned} q_6(\xi) = {} & \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\ & + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)BE}{A^2\left(B/2\Psi + \left(\sqrt{\Omega}/2\Psi\right)\left(\left(C_1\sinh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\cosh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)\right)/\left(C_1\cosh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\sinh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)\right)\right)} \\ & + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)E^2}{A^2\left(B/2\Psi + \left(\sqrt{\Omega}/2\Psi\right)\left(\left(C_1\sinh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\cosh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)\right)/\left(C_1\cosh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\sinh\left(\left(\sqrt{\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)\right)\right)^2}. \end{aligned} \tag{27}$$

(2.2) When $B \neq 0$, $\Psi = A - C$, and $\Omega = B^2 + 4E\Psi < 0$,

$$\begin{aligned} q_7(\xi) = {} & \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\ & + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)EB}{A^2\left(B/2\Psi + \left(\sqrt{-\Omega}/2\Psi\right)\left(\left(-C_1\sin\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\cos\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)\right)/\left(C_1\cos\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\sin\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)\right)\right)} \\ & + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)E^2}{A^2\left(B/2\Psi + \left(\sqrt{-\Omega}/2\Psi\right)\left(\left(-C_1\sin\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\cos\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)\right)/\left(C_1\cos\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\sin\left(\left(\sqrt{-\Omega}/2A\Gamma(1+\alpha)\right)\xi^\alpha\right)\right)\right)^2}. \end{aligned} \tag{28}$$

(2.3) When $B \neq 0$, $\Psi = A - C$, and $\Omega = B^2 + 4E\Psi = 0$,

$$q_8(\xi) = \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} + \frac{B^2 C_2^2 \left(k_1^{2\alpha} - k_2^{2\alpha}\right)(\Gamma(1+\alpha))^2}{\left((2AC_2 - BC_1)\Gamma(1+\alpha) + BC_2\xi^\alpha\right)^2}. \tag{29}$$

(2.4) When $B = 0$, $\Psi = A - C$, and $\Delta = \Psi E > 0$,

$$q_9(\xi) = -\frac{2\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\Delta}{3A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\Delta}{A^2}\left(\frac{C_1\cosh\left(\left(\sqrt{\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\sinh\left(\left(\sqrt{\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right)}{C_1\sinh\left(\left(\sqrt{\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\cosh\left(\left(\sqrt{\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right)}\right)^2. \tag{30}$$

(2.5) When $B = 0$, $\Psi = A - C$, and $\Delta = \Psi E < 0$,

$$q_{10}(\xi) = -\frac{2\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\Delta}{3A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} - \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\Delta}{A^2}\left(\frac{C_1\cos\left(\left(\sqrt{-\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right) + C_2\sin\left(\left(\sqrt{-\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right)}{C_2\cos\left(\left(\sqrt{-\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right) - C_1\sin\left(\left(\sqrt{-\Delta}/A\Gamma(1+\alpha)\right)\xi^\alpha\right)}\right)^2, \tag{31}$$

where $\xi = k_1x_1 + k_2x_2 + l_1y_1 + l_2y_2 + \omega t$, k_1, k_2, l_1, l_2, and ω are arbitrary constants.

In particular, if $C_1 = 0$, but $C_2 \neq 0$, or $C_2 = 0$, but $C_1 \neq 0$, then $q(\xi)$ changes as follows:

(3.1) When $B \neq 0$, $\Psi = A - C$, and $\Omega = B^2 + 4E\Psi > 0$,

$$\begin{aligned} q_{11}(\xi) = {} & \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\ & + \frac{\left(k_2^{2\alpha} - k_1^{2\alpha}\right)B}{A^2}\left(\frac{B}{2} + \frac{\sqrt{\Omega}}{2}\tanh\left(\frac{\sqrt{\Omega}\xi^\alpha}{2A\Gamma(1+\alpha)}\right)\right) \\ & + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)}{A^2}\left(\frac{B}{2} + \frac{\sqrt{\Omega}}{2}\tanh\left(\frac{\sqrt{\Omega}\xi^\alpha}{2A\Gamma(1+\alpha)}\right)\right)^2, \end{aligned} \tag{32}$$

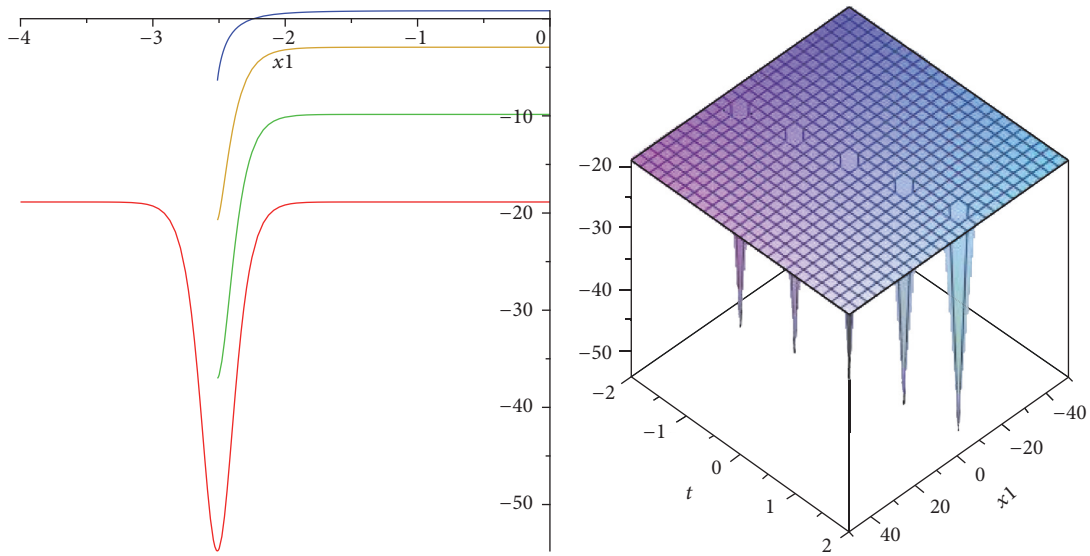
$$\begin{aligned} q_{12}(\xi) = {} & \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\ & + \frac{\left(k_2^{2\alpha} - k_1^{2\alpha}\right)B}{A^2}\left(\frac{B}{2} + \frac{\sqrt{\Omega}}{2}\coth\left(\frac{\sqrt{\Omega}\xi^\alpha}{2A\Gamma(1+\alpha)}\right)\right) \\ & + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)}{A^2}\left(\frac{B}{2} + \frac{\sqrt{\Omega}}{2}\coth\left(\frac{\sqrt{\Omega}\xi^\alpha}{2A\Gamma(1+\alpha)}\right)\right)^2, \end{aligned} \tag{33}$$

$$\begin{aligned} q_{13}(\xi) = {} & \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\ & + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)BE}{A^2\left(B/2\Psi + \left(\sqrt{\Omega}/2\Psi\right)\left(\tanh\left(\sqrt{\Omega}\xi^\alpha/2A\Gamma(1+\alpha)\right)\right)\right)} \\ & + \frac{E^2\left(k_1^{2\alpha} - k_2^{2\alpha}\right)}{A^2\left(B/2\Psi + \left(\sqrt{\Omega}/2\Psi\right)\left(\tanh\left(\sqrt{\Omega}\xi^\alpha/2A\Gamma(1+\alpha)\right)\right)\right)^2}, \end{aligned} \tag{34}$$

$$\begin{aligned} q_{14}(\xi) = {} & \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\ & + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)BE}{A^2\left(B/2\Psi + \left(\sqrt{\Omega}/2\Psi\right)\left(\coth\left(\sqrt{\Omega}\xi^\alpha/2A\Gamma(1+\alpha)\right)\right)\right)} \\ & + \frac{E^2\left(k_1^{2\alpha} - k_2^{2\alpha}\right)}{A^2\left(B/2\Psi + \left(\sqrt{\Omega}/2\Psi\right)\left(\coth\left(\sqrt{\Omega}\xi^\alpha/2A\Gamma(1+\alpha)\right)\right)\right)^2}. \end{aligned} \tag{35}$$

(3.2) When $B \neq 0$, $\Psi = A - C$, and $\Omega = B^2 + 4E\Psi < 0$,

$$\begin{aligned} q_{15}(\xi) = {} & \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\ & + \frac{\left(k_2^{2\alpha} - k_1^{2\alpha}\right)B}{A^2}\left(\frac{B}{2} - \frac{\sqrt{-\Omega}}{2}\tan\left(\frac{\sqrt{-\Omega}\xi^\alpha}{2A\Gamma(1+\alpha)}\right)\right) \\ & + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)}{A^2}\left(\frac{B}{2} - \frac{\sqrt{-\Omega}}{2}\tan\left(\frac{\sqrt{-\Omega}\xi^\alpha}{2A\Gamma(1+\alpha)}\right)\right)^2, \end{aligned}$$



(a) $\alpha = 0.45; 0.75; 0.9; 1; t = 0.5$ (b) $\alpha = 1,\ -2 \le t \le 2$

FIGURE 1: 2D and 3D figures of solution $q_{11}(\xi)$ in (32) with $k_1 = 4, k_2 = 0.2, l_1 = 4, l_2 = 1, A = 1, C = -3, E = -1, \omega = 20, x_2 = 0.01, y_1 = 0.01, y_2 = 0.01, B = 5$.

$$\begin{aligned}
q_{16}(\xi) &= \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\
&\quad + \frac{\left(k_2^{2\alpha} - k_1^{2\alpha}\right)B}{A^2}\left(\frac{B}{2} + \frac{\sqrt{-\Omega}}{2}\cot\left(\frac{\sqrt{-\Omega}\xi^\alpha}{2A\Gamma(1+\alpha)}\right)\right) \\
&\quad + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)}{A^2}\left(\frac{B}{2} + \frac{\sqrt{-\Omega}}{2}\cot\left(\frac{\sqrt{-\Omega}\xi^\alpha}{2A\Gamma(1+\alpha)}\right)\right)^2, \\
q_{17}(\xi) &= \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\
&\quad + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)BE}{A^2\left(B/2\Psi - \left(\sqrt{-\Omega}/2\Psi\right)\left(\tan\left(\sqrt{-\Omega}\xi^\alpha/2A\Gamma(1+\alpha)\right)\right)\right)} \\
&\quad + \frac{E^2\left(k_1^{2\alpha} - k_2^{2\alpha}\right)}{A^2\left(B/2\Psi - \left(\sqrt{-\Omega}/2\Psi\right)\left(\tan\left(\sqrt{-\Omega}\xi^\alpha/2A\Gamma(1+\alpha)\right)\right)\right)^2}, \\
q_{18}(\xi) &= \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\left(B^2 - 8E\Psi\right)}{12A^2} + \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\
&\quad + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)BE}{A^2\left(B/2\Psi + \left(\sqrt{-\Omega}/2\Psi\right)\left(\cot\left(\sqrt{-\Omega}\xi^\alpha/2A\Gamma(1+\alpha)\right)\right)\right)} \\
&\quad + \frac{E^2\left(k_1^{2\alpha} - k_2^{2\alpha}\right)}{A^2\left(B/2\Psi + \left(\sqrt{-\Omega}/2\Psi\right)\left(\cot\left(\sqrt{-\Omega}\xi^\alpha/2A\Gamma(1+\alpha)\right)\right)\right)^2}.
\end{aligned} \tag{36}$$

(3.3) When $B = 0$, $\Psi = A - C$, and $\Delta = \Psi E > 0$,

$$\begin{aligned}
q_{19}(\xi) &= \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\
&\quad + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\Delta}{A^2}\left(-\frac{2}{3} + \tanh^2\left(\frac{\sqrt{\Delta}\xi^\alpha}{A\Gamma(1+\alpha)}\right)\right),
\end{aligned} \tag{37}$$

$$\begin{aligned}
q_{20}(\xi) &= \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\
&\quad + \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\Delta}{A^2}\left(-\frac{2}{3} + \coth^2\left(\frac{\sqrt{\Delta}\xi^\alpha}{A\Gamma(1+\alpha)}\right)\right).
\end{aligned} \tag{38}$$

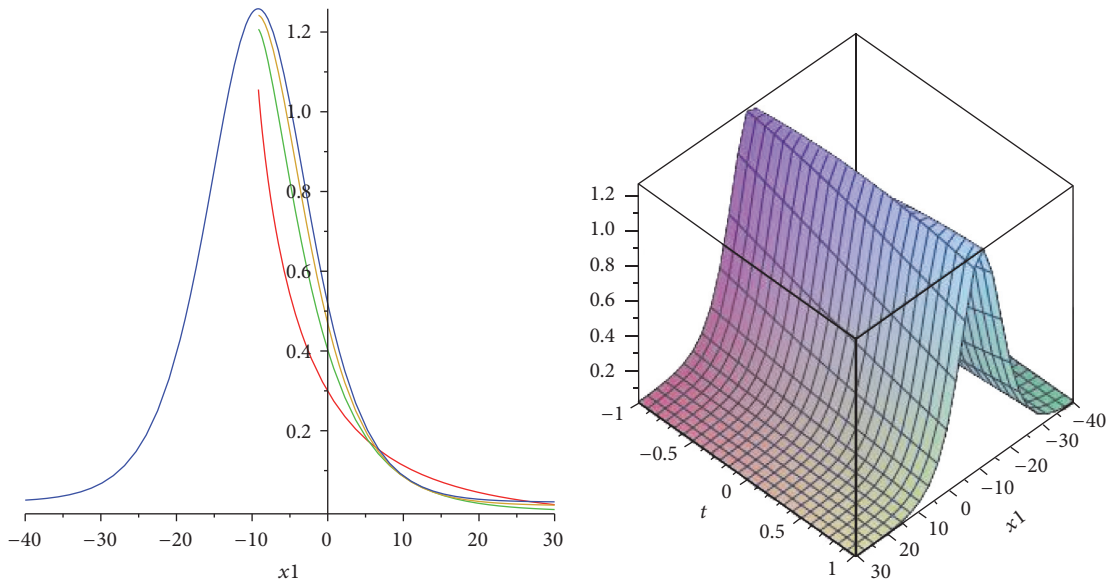
(3.4) When $B = 0$, $\Psi = A - C$, and $\Delta = \Psi E < 0$,

$$\begin{aligned}
q_{21}(\xi) &= \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\
&\quad - \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\Delta}{A^2}\left(\frac{2}{3} + \tan^2\left(\frac{\sqrt{-\Delta}\xi^\alpha}{A\Gamma(1+\alpha)}\right)\right), \\
q_{22}(\xi) &= \frac{3l_1^\alpha l_2^\alpha - 2\omega^\alpha k_1^\alpha}{6k_1^\alpha k_2^\alpha} \\
&\quad - \frac{\left(k_1^{2\alpha} - k_2^{2\alpha}\right)\Delta}{A^2}\left(\frac{2}{3} + \cot^2\left(\frac{\sqrt{-\Delta}\xi^\alpha}{A\Gamma(1+\alpha)}\right)\right),
\end{aligned} \tag{39}$$

where $\xi = k_1x_1 + k_2x_2 + l_1y_1 + l_2y_2 + \omega t$, k_1, k_2, l_1, l_2, and ω are arbitrary constants.

5. Figures of Some Exact Solutions

In this section, some typical wave figures are given as Figures 1 and 2.



(a) $\alpha = 0.45; 0.75; 0.9; 1; t = 0.5$ (b) $\alpha = 1,\ -1 \le t \le 1$

Figure 2: 2D and 3D figures of solution $q_{19}(\xi)$ in (37) with $k_1 = 0.1, k_2 = 1, l_1 = 0.1, l_2 = 1, A = 2, C = 1, E = 5, \omega = 0.2, x_2 = 0.3, y_1 = 0.2, y_2 = 0.5$.

6. Conclusions

Finding exact traveling wave solutions of FPDEs is an important and difficult work. Based on Feng's [11] work, by adding in negative power exponents, a new analytical method named extended fractional $(D_\xi^\alpha G/G)$-expansion method is proposed. It can help to seek more general traveling wave solutions. In this paper, using this new method, we obtain more new exact solutions of the space-time fractional Fokas equation. These exact solutions include hyperbolic function and trigonometric function which are useful to understand the mechanisms of the complicated nonlinear physical phenomena and fractional differential equations. To the best of our knowledge, the solutions obtained in this paper have not been reported in other literatures. The traveling wave transformation used for ξ here ensures that a certain fractional partial differential equation can be converted into another fractional ordinary differential equation, and the solutions of the latter are assumed to possess forms in one certain polynomial in $G(\xi)$, where $G(\xi)$ satisfies a given fractional ordinary differential equation denoted by (7), and the degree of the polynomial can be determined by the homogeneous balance principle. It is worthy of further study to extend other existing methods, like the generalized fractional Jacobi elliptic equation-based subequation method [4] used for differential equations, to fractional differential equations. This is our task in the future.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

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