Research Article

A Holling Type II Pest and Natural Enemy Model with Density Dependent IPM Strategy

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Resource limitations and density dependent releasing of natural enemies during the pest control and integrated pest management will undoubtedly result in nonlinear impulsive control. In order to investigate the effects of those nonlinear control strategies on the successful pest control, we have proposed a pest-natural enemy system concerning integrated pest management with density dependent instant killing rate and releasing rate. In particular, the releasing rate depicts how the number of natural enemy populations released was guided by their current density at the fixed moment. The threshold condition which ensures the existence and global stability of pest-free periodic solution has been discussed first, and the effects of key parameters on the threshold condition reveal that reducing the pulse period does not always benefit pest control; that is, frequent releasing of natural enemies may not be beneficial to the eradication of pests when the density dependent releasing method has been implemented. Moreover, the forward and backward bifurcations could occur once the pest-free periodic solution becomes unstable, and the system could exist with very complex dynamics. All those results confirm that the control actions should be carefully designed once the nonlinear impulsive control measures have been taken for pest management.

1. Introduction

Integrated pest management (IPM), also known as integrated pest control (IPC), is a wide approach that integrates multiple control measures for economic control of pests. IPM aims to maintain the pest population below the economic injury level (EIL) by taking biological, chemical, and physical control tactics. During the pest control, regular monitor is critically important which provides some cues on how and when to apply IPM strategies [1–5].

Note that the outbreaks of pest often cause serious ecological problems and bring large economic losses. The traditional methods to reduce the harm to agriculture due to insect pests include chemical pesticides and biological control measures [6–8]. Chemical pesticide is a measure directly acting on the pest. Biological control measure releasing natural enemies is a measure to increase the number of predators to control the population of the pest. There are several different releasing methods including inoculative release and inundative release which are different tactics of biological control that affect the target pest in different ways. Augmentative control includes the periodic introduction of natural enemies [3, 9]. With inundative release, predators are collected, mass-reared, and periodically released in large numbers into the pest area, which can result in an immediate reduction in the pest populations.

However, pesticide pollution has been recognized as a major health hazard to human, it can seriously change the ecological system, and it may cause damage to nontarget organisms. So, to combine these control measures together and to maximize control effects at minimum levels of control resources, a number of impulsive differential equations have been developed to investigate the dynamics of pest and enemies and to analyze the optimal control schemes [3, 4, 10–13].

In 2003, Liu and Chen studied the dynamics of the Holling type 2 predator-prey model by introducing releasing predators at fixed moments [14], and the model is as follows:

$$\frac{dx(t)}{dt} = rx(t)\left[1 - \frac{x(t)}{K}\right] - \frac{ax(t)\,y(t)}{1 + \omega x(t)},$$

$$\frac{dy(t)}{dt} = \frac{cx(t)y(t)}{1 + \omega x(t)} - dy(t),$$

$$t \neq nT,$$

$$x(t^+) = x(t),$$

$$y(t^+) = y(t) + \sigma,$$

$$t = nT,$$

$$(1)$$

where x and y are densities of the prey and predator populations at time t, respectively; r denotes the intrinsic growth rate of prey; K represents the environmental capacity of prey; d is the death rate of the predator; $ax(t)/(1 + \omega x(t))$ is the Holling type 2 functional response; c/a is the rate of conversing prey into predator; T is the pulse period; σ is the release amount of predator at time points $t = nT$. From then on, the predator-prey model considering both pulse spraying pesticides and releasing natural enemies have been extensively studied by many researchers [3, 4, 13, 15–21]. Note that for all those studies the pulse functions introduced to the models could be summarized as the following generalized linear functions:

$$x(t^+) = (1 - p_1)x(t),$$

$$y(t^+) = (1 - p_2)y(t) + \sigma, \qquad (2)$$

$$t = nT,$$

where p_1 and p_2 are the fraction of pests and predators which have been killed instantaneously after pesticide applications.

The key assumptions from the linear pulse functions shown in (2) are as follows: (a) the instant killing rate p_1 is a constant, which shows that no matter how large the number of pest populations is, the total pest killed linearly depends on its current density [22, 23]; (b) the adverse effect of pesticides on the natural enemies is also linear dependence on its current density; (c) the releasing number of natural enemies is a constant no matter how large the natural enemies remain in the field; that is, the density of natural enemies is not carefully monitored before IPM strategy is applied. However, the killing rate and releasing constant should strictly depend on a number of factors including current densities of both populations, resources limitation, and management purposes. For example, in reality, due to the resource limitation, the more proper and usual way of releasing of enemies may be the way that is guided by the density of the current natural enemies. Therefore, taking those factors into account, we propose the following model with density dependent pulse control actions:

$$\frac{dx(t)}{dt} = rx(t)\left[1 - \frac{x(t)}{K}\right] - \frac{ax(t)y(t)}{1 + \omega x(t)},$$

$$\frac{dy(t)}{dt} = \frac{cx(t)y(t)}{1 + \omega x(t)} - dy(t),$$

$$t \neq nT,$$

$$x(t^+) = \left(1 - \frac{\delta x(t)}{x(t) + h}\right)x(t),$$

$$y(t^+) = y(t) + \frac{\tau}{1 + \theta y(t)},$$

$$t = nT,$$

$$(3)$$

where $\delta > 0$ and $h > 0$ represent the maximal fatality rate and the half-saturation constant for the prey, $\tau > 0$ is the maximal release amount of the predator, and θ is a shape parameter. We assume that the control measures are conducted every T time, and thus when $t = nT$ the amounts of the prey and the predator are updated to $(1 - \delta x(t)/(x(t) + h))x(t)$ and $y(t) + \tau/(1 + \theta y(t))$ instantaneously. Note that the function $\tau/(1 + \theta y(t))$ representing the amount of predators releasing at $t = nT$ is a decreasing function with respect to $y(t)$. Note that the amount of natural enemies releasing at $t = nT$ depends on its density $y(nT)$, which indicates that the larger the density of natural enemy is, the smaller the natural enemy has been released at time $t = nT$, and obviously this strategy is more effective than the constant releasing method [3, 4, 16, 24].

The main purpose of the present paper is to focus on how the density dependent killing rate and density dependent releasing strategy affect the dynamics of the system and the effectiveness of the control measures. Based on this purpose, we first investigate the threshold condition which guarantees the existence and global stability of pest-free periodical solution. The sensitivity analyses for the key parameters on the threshold condition depict that reducing the pulse period may not be of benefit to the eradication of pests when the density dependent releasing method has been implemented, and this interesting result revealed by the proposed model shows that incorrect control methods may result in pest outbreak and resurgence. Furthermore, bifurcation analyses indicate that the forward and backward bifurcations could occur once the pest-free periodic solution becomes unstable, and the system could exist with two stable positive periodic solutions and even more complex dynamics. The important results found in this work confirm that the control actions should be carefully designed once the nonlinear impulsive control measures have been taken for pest management.

2. Mathematical Analysis and Threshold Conditions

In this section, we first demonstrate the existence of the pest-free periodic solution of model (3) and focus on its stability.

2.1. Existence and Stability of the Pest-Free Periodic Solution. In order to investigate the existence of the pest-free periodic solution, we first focus on the pest-free set $\{(x, y) \in \mathfrak{R}_+^2, \ x = 0\}$, which is invariantly related to model (3). Within this set, model (3) becomes

$$\frac{dy(t)}{dt} = -dy(t), \quad t \neq nT,$$

$$y(t^+) = y(t) + \frac{\tau}{1 + \theta y(t)}, \quad t = nT,$$

$$y(0^+) = y_0. \tag{4}$$

Solving the system in interval $(nT, (n+1)T]$, one yields

$$y((n+1)T^+) = y(nT^+) e^{-dT} + \frac{\tau}{1 + \theta y(nT^+) e^{-dT}}. \tag{5}$$

Denote $y(nT^+) = y_n$; then the equation can be rewritten as follows:

$$y_{n+1} = y_n e^{-dT} + \frac{\tau}{1 + \theta y_n e^{-dT}} \doteq f(y_n). \tag{6}$$

This equation is the so-called stroboscopic map of model (4). It describes the relations of the number of natural enemies between any two successive pulse points. So, the existence of the positive steady state of this equation implies the existence of a positive periodic solution of model (4). Thus, the steady state of (6) is discussed first. The derivative of $f(y_n)$ with respect to y_n can be given as follows:

$$f'(y_n) = e^{-dT} \left(1 - \frac{\tau\theta}{(1 + \theta y_n e^{-dT})^2} \right). \tag{7}$$

Denote y^* to be the fixed point of the stroboscopic map (6); then y^* satisfies the equation $f(y^*) = y^* e^{-dT} + \tau/(1 + \theta y^* e^{-dT}) = y^*$. So, y^* satisfies the following equation:

$$A_1 (y^*)^2 + A_2 y^* + A_3 = 0, \tag{8}$$

where $A_1 = \theta e^{-dT}(1 - e^{-dT})$, $A_2 = 1 - e^{-dT}$, and $A_3 = -\tau$. Denote $\Delta = A_2^2 - 4A_1 A_3$; then we have

$$\Delta = \left(1 - e^{-dT} \right)^2 + 4\tau\theta e^{-dT} \left(1 - e^{-dT} \right) > 0. \tag{9}$$

So, (6) has a unique positive root:

$$y^* = \frac{-\left(1 - e^{-dT} \right) + \sqrt{\Delta}}{2\theta e^{-dT}}, \tag{10}$$

which is stable provided

$$\left| f'(y_n) \right|_{y_n = y^*} = \left| e^{-dT} \left(1 - \frac{\tau\theta}{(1 + \theta y^* e^{-dT})^2} \right) \right| < 1. \tag{11}$$

Obviously, $e^{-dT}(1 - \tau\theta/(1 + \theta y^* e^{-dT})^2) < 1$. Moreover, $(1 + \theta y^* e^{-dT})^2 = \tau\theta e^{-dT} + (1/2)(e^{-dT} - 1)^2 + (1/2)(e^{-dT} - 1)\sqrt{\Delta} + e^{-dT} + \sqrt{\Delta}$, which indicates that

$$\frac{\tau\theta}{(1 + \theta y^* e^{-dT})^2}$$

$$= \frac{\tau\theta}{\tau\theta e^{-dT} + (1/2)(e^{-dT} - 1)^2 + (1/2)(e^{-dT} - 1)\sqrt{\Delta} + e^{-dT} + \sqrt{\Delta}} \tag{12}$$

$$< e^{dT}.$$

Therefore,

$$e^{-dT} \left(1 - \frac{\tau\theta}{(1 + \theta y^* e^{-dT})^2} \right) > e^{-dT} - 1 > -1 \tag{13}$$

and consequently y^* is stable. Then, we can conclude that model (4) has a unique nontrivial positive periodic solution, denoted by $y_p(t)$, which can be expressed as follows:

$$y_p(t) = y^* e^{-d(t-nT)}, \quad \text{for } nT < t \le (n+1)T. \tag{14}$$

Further, $y_p(t)$ is locally stable because y^* is stable [25]. Then, we can obtain the following theorem by proving the global attractivity of y^*.

Theorem 1. *Model (4) has a positive periodic solution $y_p(t)$ with period T and for every solution $y(t)$ of (4) one has $|y(t) - y_p(t)| \to 0$ as $t \to +\infty$, where $y_p(t) = y^* e^{-d(t-nT)}$ ($nT < t \le (n+1)T$) and $y^* = (-(1 - e^{-dT}) + \sqrt{\Delta})/2\theta e^{-dT}$.*

Proof. Let $f(y) = ye^{-dT} + \tau/(1 + \theta ye^{-dT})$. To study the properties of the function $f(y)$, we first solve the equation $df(y)/dy = e^{-dT}(1 - \tau\theta/(1 + \theta ye^{-dT})^2) = 0$, with respect to y, and get two stationary points

$$y_1 = \frac{-\sqrt{\tau\theta} - 1}{\theta} e^{dT} < 0,$$

$$y_2 = \frac{\sqrt{\tau\theta} - 1}{\theta} e^{dT}. \tag{15}$$

The second derivative of the function $f(y)$ is $d^2 f/dy^2 = 2\tau(\theta e^{-dT})^2/(1 + \theta ye^{-dT})^3$. It is obvious that $(d^2 f/dy^2)(y_1) < 0$ and $(d^2 f/dy^2)(y_2) > 0$. So, y_1 is the local maximum and y_2 is the local minimum of the function $f(y)$ with $y_2 > y_1$ and $y_1 < 0$.

Thus, if $y_2 > 0$, the function f must be a convex function and if $y_2 < 0$ f must be monotonous in the first quadrant. So, based on the sign of y_2 and the positional relations between point y_2 and y^*, we consider the following three possible cases.

Case A ($y_2 < 0 < y^*$). For this case the function $f(y)$ is a monotonically increasing function for $y > y_2$. For any $y \in [0, y^*)$, according to $y < f(y) < y^*$ we know that $f^k(y)$ is monotonically increasing as k increases, and $f^k(y) \to y^*$ ($k \to \infty$).

For any $y > y^*$, we have $f^k(y) > y^*$ for all k, and then according to $f(y) < y$ we conclude that $f^k(y)$ is monotonically decreasing as k increases, which means that $f^k(y) \to y^*$ ($k \to \infty$).

Case B ($0 < y_2 < y^*$). For this case, if $y \in [y_2, y^*)$ or $y > y^*$, then the conclusions corresponding to the Case A are true; that is, we have $f^k(y) \to y^*$ ($k \to \infty$).

Moreover, for any $y \in [0, y_2)$, there must be $f(y) > y_2$, and according to the conclusion of Case A we obtain $f^{1+k}(y) \to y^*$ $(k \to \infty)$.

Case C $(y_2 > y^* > 0)$. For this case, it is easy to know that the function $f(y)$ is a monotonically decreasing function for $y \in [0, y_2]$, and it is an increasing function for $y \in (y_2, \infty)$.

It is easy to see that there must exist a positive integer l such that $f^l(y) \in [y^*, y_2]$ for $y \in [0, y^*)$ or $y \in (y_2, \infty)$. Thus if $f^k(y) \to y^*$ $(k \to \infty)$ for any $y \in (y^*, y_2]$, then $f^n(y) \to y^*$ $(n \to \infty)$ for any $y \in [0, \infty)$ and the results follow. Hence in the following we will focus on $f^k(y) \to y^*$ $(k \to \infty)$ for any $y \in (y^*, y_2]$.

According to the definition of the function f and by simple calculations, we have

$$f(y) = ye^{-dT} + \frac{\tau}{1 + \theta ye^{-dT}},$$

$$f(f(y)) = f(y)e^{-dT} + \frac{\tau}{1 + \theta f(y)e^{-dT}},$$

$$f'(y) = e^{-dT}\left(1 - \frac{\tau\theta}{(1 + \theta ye^{-dT})^2}\right), \quad (16)$$

$$f'(f(y)) = e^{-dT}\left(1 - \frac{\tau\theta}{(1 + \theta f(y)e^{-dT})^2}\right).$$

Now we first show that $f'(f(y)) > -f(f(y))/f(y)$ for any $y \in (y^*, y_2]$. In fact, for $y \in (y^*, y_2]$, $f'(f(y)) > -f(f(y))/f(y) \Leftrightarrow f'(f(y))f(y) > -f(f(y)))$, since

$$f'(f(y)) f(y)$$

$$= \left[e^{-dT}\left(1 - \frac{\tau\theta}{(1 + \theta f(y)e^{-dT})^2}\right)\right] f(y),$$

$$- f(f(y)) = -f(y)e^{-dT} - \frac{\tau}{1 + \theta f(y)e^{-dT}}, \quad (17)$$

$$2f(y)e^{-dT} > 0 > \frac{-\tau}{(1 + \theta f(y)e^{-dT})^2},$$

which indicates that

$$f'(f(y)) f(y) > -f(f(y)). \quad (18)$$

Next, we show that $f'(y) > -f(y)/y$ for any $y \in (y^*, y_2]$. In fact, for $y \in (y^*, y_2]$ we have $f'(y) > -f(y)/y \Leftrightarrow f'(y)y > -f(y)$. Since

$$f'(y) y = \left[e^{-dT}\left(1 - \frac{\tau\theta}{(1 + \theta ye^{-dT})^2}\right)\right] y,$$

$$- f(y) = -ye^{-dT} - \frac{\tau}{1 + \theta ye^{-dT}}, \quad (19)$$

$$2q_2 e^{-dT} y > 0 > \frac{-\tau}{[1 + \theta ye^{-dT}]^2},$$

thus we have

$$f'(f(y)) y > -f(y). \quad (20)$$

It follows from $f'(f(y)) > -f(f(y))/f(y)$ that we know

$$f'(f(y)) f'(y) y < -\frac{f'(y) y}{f(y)} f(f(y)) \quad (21)$$

and $f'(y)y > -f(y)$ for $f'(y) > -f(y)/y$; that is, $-f'(y)y/f(y) < 1$ holds. Furthermore, we have $0 < -f'(y)y/f(y) < 1$ due to $-f'(y)y/f(y) > 0$. All those confirm that $f'(f(y))f'(y)y < -(f'(y)y/f(y))f(f(y)) < f(f(y))$; namely,

$$f'(f(y)) f'(y) y < f(f(y)). \quad (22)$$

Denote $g(y) = f(f(y))/y$; then we have $g'(y) = f'(f(y))f'(y)y - f(f(y))/y^2$. Then, $g'(y) < 0$ and $g(y)$ is a monotonically decreasing function at interval $y \in (y^*, y_2]$ due to (22).

Therefore, $g(y^*) = f(f(y^*))/y^* = f(y^*)/y^* = y^*/y^* = 1$, and $g(y) < 1$ for $y \in (y^*, y_2]$; that is, $f(f(y)) < y$.

Based on the above discussions we conclude that $f^{2k}(y)$ is a monotonically decreasing function as k increases with $f^{2k}(y) \to y^*$ $(k \to \infty)$. Similarly, we can prove that $f^{2k+1}(y)$ is a monotonically decreasing function as k increases with $f^{2k}(y) \to y^*$ $(k \to \infty)$. Thus, $f^n(y) \to y^*$ $(n \to \infty)$ for any $y \in [0, \infty)$ holds.

Combined with all cases together we know that the unique equilibrium y^* is globally stable if it exists, and consequently the periodic solution $y_p(t)$ of model (4) is globally stable. This completes the proof. □

The stability of the positive periodic solution is determined by the stability of the positive equilibrium of the difference equation made by sequence of impulsive points; thus the positive periodic solution $y_p(t)$ of model (4) is globally stable according to Theorem 1. Therefore, a general expression of unique pest-free periodic solution of model (3) over the interval $nT < t \leq (n+1)T$ for all $n \in N$ can be denoted by

$$(0, y_p(t)) = (0, y^* e^{-d(t-nT)}). \quad (23)$$

In the following, we focus on the stability of the pest-free periodic solution for model (3). Firstly, define

$$(\overline{x}(t), \overline{y}(t)) = (x(t), y(t)) - (0, y_p(t)), \quad (24)$$

and then $(\overline{x}(t), \overline{y}(t))$ represents the deviations form of the pest-free solution. According to model (3), the derivations of $(\overline{x}(t), \overline{y}(t))$ are

$$\dot{\overline{x}} = r\overline{x} - \frac{r}{K}\overline{x}^2 - \frac{a\overline{x}}{1 + \omega\overline{x}}(\overline{y} + y_p(t)),$$

$$\dot{\overline{y}} = \frac{c\overline{x}}{1 + \omega\overline{x}}(\overline{y} + y_p(t)) - d\overline{y}. \quad (25)$$

Assuming that $(\overline{x}(t), \overline{y}(t))$ are small enough, then the linear approximation of the deviation system around the periodic solution $(0, y_p(t))$ can be obtained as follows:

$$\dot{\overline{x}} = \left(r - ay_p(t)\right)\overline{x},$$
$$\dot{\overline{y}} = cy_p(t)\overline{x} - d\overline{y}. \tag{26}$$

Then, we have the following main result.

Theorem 2. *The pest-free periodic solution* $(0, y_p(t))$ *of model (3) is locally stable in the first quadrant provided that*

$$R_1 \doteq \frac{rdT}{ay^* \left(1 - e^{-dT}\right)} < 1. \tag{27}$$

Proof. Let $\Phi(t)$ be the fundamental matrix of (26); then

$$\Phi(T) = \begin{pmatrix} e^{\int_0^T (r - ay_p(t))dt} & 0 \\ * & e^{-dT} \end{pmatrix}, \tag{28}$$

and $\Phi(0) = I$ is the identity matrix. The term $*$ is not required in the following analysis.

The resetting impulsive conditions of (3) become

$$\begin{pmatrix} \overline{x}(nT^+) \\ \overline{y}(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \dfrac{\tau\theta}{\left(1 + \theta y_p(nT)\right)^2} \end{pmatrix} \begin{pmatrix} \overline{x}(nT) \\ \overline{y}(nT) \end{pmatrix}$$
$$= B(nT) \begin{pmatrix} \overline{x}(nT) \\ \overline{y}(nT) \end{pmatrix}. \tag{29}$$

Thus, the Floquet multipliers of the matrix

$$M = B(T)\Phi(T) \tag{30}$$

are

$$\lambda_1 = e^{\int_0^T (r - ay_p(t))dt},$$
$$\lambda_2 = \left(1 - \frac{\tau\theta}{\left(1 + \theta y_p(T)\right)^2}\right)e^{-dT}. \tag{31}$$

It follows from the literature [26] that the pest-free solution is locally stable if and only if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. The former inequality is equal to $R_1 \doteq rdT/ay^*(1 - e^{-dT}) < 1$ and the latter inequality is true if and only if $R_2 \doteq |(1 - \tau\theta/(1 + \theta y^* e^{-dT})^2)e^{-dT}| < 1$, which is true naturally based on the analysis in the previous section. Therefore, the pest-free periodic solution is locally stable when $R_1 < 1$. $\square$

2.2. Boundness and Permanence of System (3). According to the proof of Theorem 2, the pest-free periodic solution is unstable when $R_1 > 1$. So, next we aim to prove the permanence of the system if $R_1 > 1$. First we prove the boundness of system (3).

Theorem 3. *There exists a constant $M_0 > 0$ such that $x(t) \leq M_0$ and $y(t) \leq M_0$ for each solution of system (3) with all t large enough.*

Proof. Suppose $x(t)$ and $y(t)$ are any solution of system (3). Let $V(t) = (c/a)x(t) + y(t)$. Then, when $t \neq nT$,

$$\frac{dV(t)}{dt} + \lambda V(t) = \frac{c}{a}(r + \lambda)x - \frac{cr}{aK}x^2 - (\lambda - d)y. \tag{32}$$

When $0 < \lambda \leq d$, the right hand of the equation is bounded; namely,

$$\frac{dV(t)}{dt} + \lambda V(t) \leq M_1, \tag{33}$$

with $M_1 > 0$. When $t = nT$,

$$V(t^+) = V(t) + \frac{\tau}{1 + \theta y(t)} - \frac{c}{a}\frac{\delta x(t)}{h + x(t)} \leq V(t) + \tau. \tag{34}$$

Considering the system

$$\frac{dV(t)}{dt} \leq -\lambda V(t) + M_1, \quad t \neq nT,$$
$$V(t^+) \leq V(t) + \tau, \quad t = nT, \tag{35}$$

we have

$$V(t) \leq \left(V(0)^+ - \frac{M_1}{\lambda}\right)e^{-\lambda t} + \frac{\tau\left(1 - e^{-n\lambda T}\right)}{1 - e^{-\lambda T}}e^{-\lambda(t - nT)}$$
$$+ \frac{M_1}{\lambda}. \tag{36}$$

So, $V(t)$ is uniformly ultimately bounded. Hence, there exists a constant $M_0 > 0$ such that $x(t) \leq M_0$ and $y(t) \leq M_0$ for each solution of system (3) with all t large enough. This completes the proof. $\square$

Theorem 4. *System (3) is permanent provided $R_1 > 1$.*

Proof. It follows from Theorem 1 that there exists an $m_1 > 0$ such that $y \geq m_1$ for t large enough. Thus, in order to show the permanence of model (3) we shall next find a constant $m_2 > 0$, such that $x \geq m_2$ for t large enough.

Considering the second equation of system (3), we have

$$\frac{dy(t)}{dt} \leq -dy(t), \quad t \neq nT,$$
$$y(t^+) = y(t) + \frac{\tau}{1 + \theta y(t)}, \quad t = nT. \tag{37}$$

Therefore, according to the comparison theorem and the analyses in the previous subsection, we have

$$y(t) \leq y_p(t) + \varepsilon_1 \tag{38}$$

for $t > t_1$ and $\varepsilon_1 > 0$. Then, by the first equation of system (3) we have

$$
\frac{dx(t)}{dt} \geq rx(t)\left(1 - \frac{x(t)}{K}\right) - a\left(y_p(t) + \varepsilon_1\right)x(t),
$$

$$
t \neq nT, \quad (39)
$$

$$
x(t^+) = \left(1 - \frac{\delta x(t)}{x(t) + h}\right)x(t), \quad t = nT.
$$

Considering the auxiliary system

$$
\frac{du(t)}{dt} = ru(t)\left(1 - \frac{u(t)}{K}\right) - a\left(y_p(t) + \varepsilon_1\right)u(t),
$$

$$
t \neq nT, \quad (40)
$$

$$
u(t^+) = \left(1 - \frac{\delta u(t)}{u(t) + h}\right)u(t), \quad t = nT
$$

and using the same method as system (4), we have

$$
u_{n+1}^+ = u_{n+1}\left(1 - \frac{\delta u_{n+1}}{u_{n+1} + h}\right) \qquad (41)
$$

with

$$
u_{n+1} = \frac{u_n^+ e^{\int_0^T (r - a(y_p(t) + \varepsilon_1))dt}}{1 + u_n^+ \int_0^T (r/K)e^{\int_0^t (r - a(y_p(\xi) + \varepsilon_1))d\xi}dt}. \qquad (42)
$$

Let $M_1 = e^{\int_0^T (r - a(y_p(t) + \varepsilon_1))dt}$ and $M_2 = \int_0^T (r/K)e^{\int_0^t (r - a(y_p(\xi) + \varepsilon_1))d\xi}dt$; then $u_{n+1} = u_n^+ M_1/(1 + M_2 u_n^+)$. Therefore, we have the following different equation with respect to u_n^+:

$$
u_{n+1}^+ = \frac{u_n^+ M_1}{1 + M_2 u_n^+}\left(1 - \frac{\delta\left(u_n^+ M_1/\left(1 + M_2 u_n^+\right)\right)}{h + u_n^+ M_1/\left(1 + M_2 u_n^+\right)}\right)
$$

$$
\doteq F\left(u_n^+\right). \qquad (43)
$$

Let u^* be the steady state of the above difference equation, and by simple calculation we have

$$
a_1\left(u^*\right)^2 + a_2 u^* + a_3 = 0, \qquad (44)
$$

where $a_1 = M_2^2 + M_2 M_1$, $a_2 = M_2 + M_1 + M_2 h - M_2 M_1 - M_1^2 + \delta M_1^2$, and $a_3 = h(1 - M_1)$. Obviously, $a_1 > 0$. Moreover, $a_3 < 0$ when $R_1 > 1$ for ε_1 sufficiently small, which shows that the above difference equation exists with a unique positive equilibrium u^*. Therefore, system (40) has only one positive periodic solution when $R_1 > 1$, denoted by $\bar{u}(t)$.

To study the stability of this periodic solution, we first investigate the properties of the function $F(u) = (M_1 u/(1 + M_2 u))(1 - \delta M_1 u/(h(1 + M_2 u) + M_1 u))$. Taking the derivation of $F(u)$ with respect to u, one yields

$$
F'(u) = \frac{M_1\left[(hM_2 + M_1)u + h\right]^2 - \left(2\delta M_1^2 hM_2 + \delta M_2^3\right)u^2 - 2\delta M_1^2 hu}{(1 + M_2 u)^2\left[(hM_2 + M_1)u + h\right]^2}. \qquad (45)
$$

Let $F'(u) = 0$; then we get the following equation:

$$
b_1 u^2 + b_2 u + b_3 = 0, \qquad (46)
$$

where $b_1 = M_1 M_2^2 h^2 + 2(1 - \delta)hM_1^2 M_2 + (1 - \delta)M_1^3$, $b_2 = 2h^2 M_1 M_2 + 2(1 - \delta)hM_1^2$, and $b_3 = h^2 M_1$. It follows from $0 < \delta < 1$ that $b_1 > 0$, $b_2 > 0$, and $b_3 > 0$. Moreover, we can prove that $b_2^2 - 4b_1 b_3 = 4M_1^4 h^2 \delta(1 - \delta) > 0$. Therefore, the equation $F'(u) = 0$ does not exist with any positive root. Thus, $F'(u) > 0$ for all $u > 0$; that is, the function $F(u)$ is a monotonically increasing function for all $u > 0$. Further, it is easy to show that $F'(0) = M_1$ according to the expression of $F'(u)$ with $M_1 > 1$ for $R_1 > 1$ and ε_1 sufficiently small. All those confirm that the intersection angle of the function $F(u)$ and the horizontal coordinate is larger than $\pi/4$. Therefore, the function image is as shown in Figure 1. Then, according to $u < F(u) < u^*$ we have the fact that $F^k(u)$ is monotonically increasing as k increases for any $u \in [0, u^*)$, and $F^k(u) \to u^*(k \to \infty)$; according to $F(u) < u$ we have the fact that $F^k(u)$ is monotonically decreasing as k increases for any $u > u^*$,

and $F^k(u) \to u^*(k \to \infty)$. Therefore, the equilibrium u^* is globally stable.

Thus, the positive periodic solution $\bar{u}(t)$ is globally stable. Then, according to the comparison theory, $x(t) \geq u(t) \geq \bar{u}(t) - \varepsilon_2$ for $t \geq t_2$. Let $m_2 = \min(\bar{u}(t)) - \varepsilon_2$, $\varepsilon_2 > 0$; then we have $x \geq m_2$ for t large enough. Thus, system (3) is permanent provided $R_1 > 1$. This completes the proof. $\qquad \square$

2.3. Bifurcation of Positive Periodic Solution. In this section, we investigate the existence and stability of positive periodic solution of system (3) by employing the bifurcation analysis related to the pest-free periodic solution $X_0 = (0, y_p(t))$ [26].

Theorem 5. *As the parameter T passes through the critical value T_0 satisfying $e^{\int_0^{T_0}[r - ay_p(\xi)]d\xi} = 1$, a nontrivial positive periodic solution appears near X_0. The bifurcation is supercritical provided $BC < 0$, while it is subcritical if $BC > 0$, where*

$$
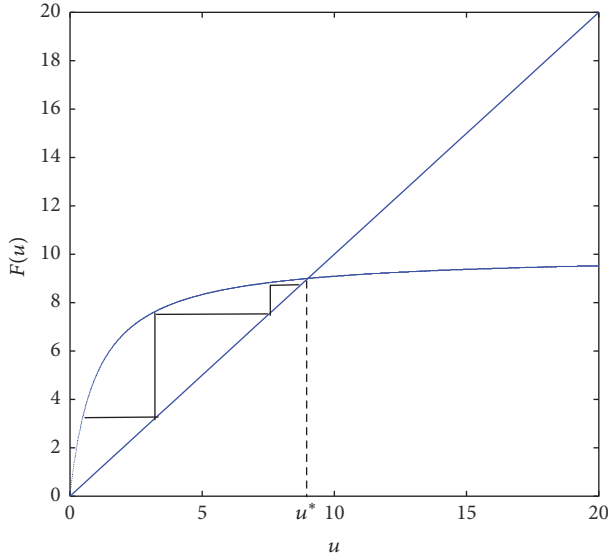B = -\left[\left(r - ay_p(T_0)\right) + adT_0 y^*(T_0)\frac{C_2 e^{-dT_0}}{1 - C_2 e^{-dT_0}}\right]
$$

FIGURE 1: Illustration of the global stability of u^*.

$$\cdot e^{\int_0^{T_0}[r-ay_p(\xi)]d\xi},$$

(47)

$$C = 2\delta h^{-1}e^{2\int_0^{T_0}[r-ay_p(\xi)]d\xi} - 2aT_0\frac{1-C_1C_2}{1-C_2e^{-dT_0}}$$

(48)

$$\cdot e^{\int_0^{T_0}[r-ay_p(\xi)]d\xi} - C_3.$$

Proof. To use the theory of [26], we employ the same notations as those in [26]; then

$$F_1(x,y) = \frac{cxy}{1+\omega x} - dy,$$

$$F_2(x,y) = rx\left[1 - \frac{x}{K}\right] - \frac{axy}{1+\omega x},$$

$$\Theta_1(x,y) = y + \frac{\tau}{1+\theta y},$$

(49)

$$\Theta_2(x,y) = \left(1 - \frac{\delta x}{x+h}\right)x.$$

By simple calculations, we get

$$\frac{\partial \Phi_1(T_0, X_0)}{\partial y} = e^{-dT_0} > 0,$$

$$\frac{\partial \Phi_2(T_0, X_0)}{\partial x} = e^{\int_0^{T_0}[r-ay_p(\xi)]d\xi} > 0,$$

$$\frac{\partial \Phi_1(T_0, X_0)}{\partial x} = c\int_0^{T_0} y_p(u)e^{-d(T_0-u)}e^{\int_0^u[r-ay_p(\xi)]d\xi}du$$

(50)

$$\doteq C_1 > 0,$$

$$a_0' = 1 - C_2e^{-dT_0},$$

$$d_0' = 1 - e^{\int_0^{T_0}[r-ay_p(\xi)]d\xi},$$

$$b_0' = 1 - C_1C_2,$$

where $C_2 = 1 - \tau\theta/(1+\theta y_p(T_0))^2$ and T_0 is the root of $d_0' = 0$, satisfying the condition $cy^*(e^{-d(T_0-nT)} - 1)/adT_0 = 1$. Obviously, we have the fact that $C_2 < 1$, which shows that $a_0' > 0$. Further, we have

$$\frac{\partial \Phi_1(T_0, X_0)}{\partial \overline{T}} = -dy^*(T_0)e^{-dT_0},$$

$$\frac{\partial^2 \Phi_2(T_0, X_0)}{\partial x \partial y} = -aT_0 e^{\int_0^{T_0}[r-ay_p(\xi)]d\xi} < 0,$$

$$\frac{\partial^2 \Phi_2(T_0, X_0)}{\partial x \partial \overline{T}} = (r - ay_p(T_0))e^{\int_0^{T_0}[r-ay_p(\xi)]d\xi},$$

$$\frac{\partial^2 \Phi_2(T_0, X_0)}{\partial x^2} = T_0\left(-\frac{2r}{K} + 2a\omega y_p(T_0)\right)$$

(51)

$$\cdot e^{\int_0^{T_0}[r-ay_p(\xi)]d\xi}$$

$$- \int_0^{T_0} ae^{\int_u^{T_0}[r-ay_p(\xi)]d\xi}\left[\int_0^u e^{-d(u-v)}cy_p(v)\right.$$

$$\left.\cdot e^{\int_0^{T_0}[r-ay_p(\xi)]d\xi}dv\right]du \doteq C_3.$$

$\partial\Theta_1(T_0, X_0)/\partial y = C_2$, $\partial\Theta_1(T_0, X_0)/\partial x = 0$, $\partial\Theta_2/\partial x = 1$, $\partial\Theta_2(T_0, X_0)/\partial y = 0$, $\partial^2\Theta_1(T_0, X_0)/\partial x\partial y = -2\delta h^{-1}$, and $\partial^2\Theta_1(T_0, X_0)/\partial x\partial y = 0$. Thus,

$$B = -\left[(r - ay_p(T_0)) + adT_0y^*(T_0)\frac{C_2e^{-dT_0}}{1-C_2e^{-dT_0}}\right]$$

$$\cdot e^{\int_0^{T_0}[r-ay_p(\xi)]d\xi},$$

(52)

$$C = 2\delta h^{-1}e^{2\int_0^{T_0}[r-ay_p(\xi)]d\xi} - 2aT_0\frac{1-C_1C_2}{1-C_2e^{-dT_0}}$$

$$\cdot e^{\int_0^{T_0}[r-ay_p(\xi)]d\xi} - C_3.$$

Therefore, according to Theorem 2 in literature [26], the supercritical branch occurs at the point T_0 if $BC < 0$ and the subcritical branch occurs if $BC > 0$. This completes the proof. □

3. Numerical Investigations

Bifurcation analyses shown above indicate that system (3) could exist with complex dynamics once the pest-free periodic solution becomes unstable, which is a challenge for theoretical studies. Therefore, in this section we will focus on the complex dynamics by using numerical techniques.

3.1. Backward and Forward Bifurcation. Our theoretical analysis shows that when $R_1 > 1$ system (3) is permanent and when $R_1 < 1$ the pest-free periodic solution is locally stable in the first quadrant. Moreover, Theorem 5 indicates the existence of nontrivial positive periodic solution under certain conditions. Thus, to verify these theoretical results, we
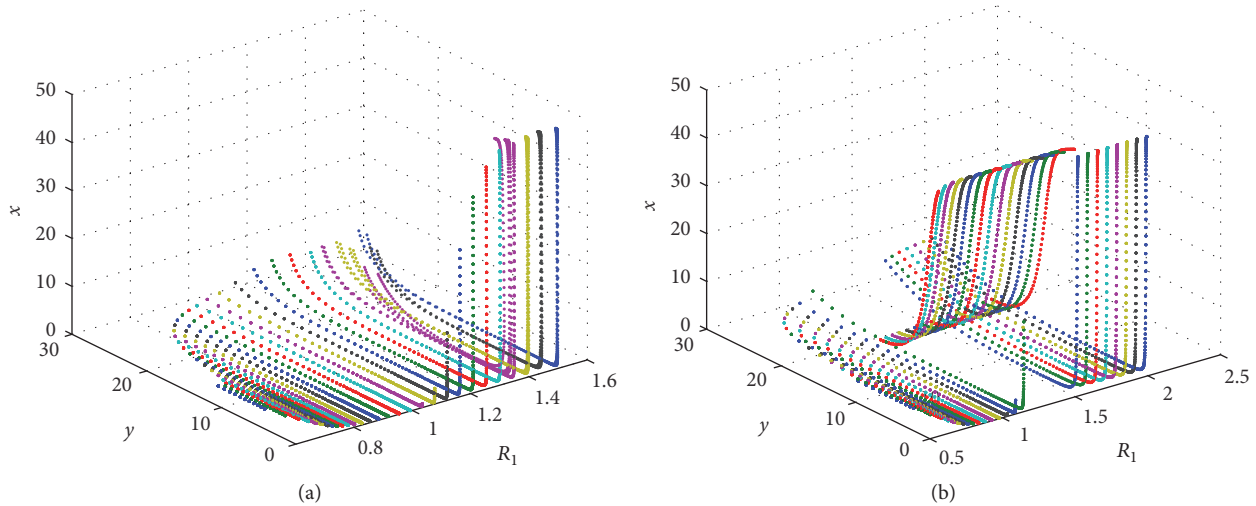
(a)

(b)

FIGURE 2: Backward (forward) bifurcation of system (3) with respect to threshold value R_1. The parameter values are as follows: $T = 6$, $\tau = 30$, $\theta = 0.5$, $\delta = 0.85$, $h = 0.8$, $r = 1.5$, $a = 0.3$, $K = 50$, $\omega = 0.3$, $c = 0.23$, and $d = 0.6$ in (a) and $d = 0.8$ in (b).

first take some numerical simulations under certain parameter spaces and initial values. Figure 2 shows the bifurcation diagram of system (3) with respect to the threshold condition R_1 with $d = 0.6$ and $d = 0.8$.

It follows from Figure 2(a) that when $R_1 < 1$ there exists a stable pest-free periodic solution, and when $R_1 > 1$ the pest-free periodic solution becomes unstable and a stable positive periodic solution appears, which is called a forward bifurcation at $R_1 = 1$. Figure 2(b) indicates that when $R_1 < 0.78(T < 0.406)$ there exists only one pest-free periodic solution and it is stable. When $R_1 > 1$, there exists at least one stable positive periodic solution. Specifically, when $0.78 < R_1 < 1$, the pest-free periodic and the positive periodic solutions can coexist which are bistable; namely, when the value of R_1 passes through 1 the supercritical bifurcation occurs or backward bifurcation occurs. Both of these two figures confirm that when $R_1 < 1$ the pest-free periodic solution is locally stable and when $R_1 > 1$ system (3) is permanent. Moreover, there may also exist a locally stable positive periodic solution when $R_1 < 1$, and this interesting result could be produced by the density dependent pulse control strategies.

As shown in Figure 2(b), there may be two locally stable periodic solutions when $R_1 < 1$. Therefore, the initial values of both the pest and predator populations are crucial. To see how different initial values affect the successful pest control, we plot the basins of attraction of two locally stable periodic solutions of system (3), as shown in Figure 2(b). It follows from Figure 3 that the pest can be eradicated when the initial values of the predator are very small or large. However, when the initial values of the predator have been chosen between 2 and 30, the pest can be eradicated only if the initial values of the pest are very small.

The theoretical results only give the existence and global stability of the pest-free periodic solution or local stability of the positive periodic solution. To further study the dynamics of system (3), the bifurcation diagrams for the
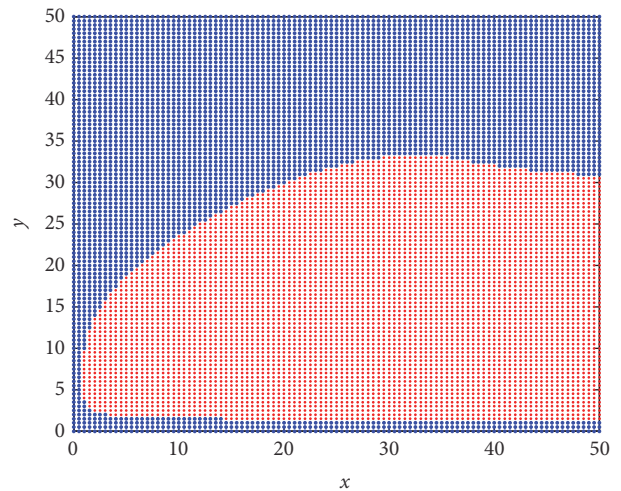

FIGURE 3: Basins of attraction of system (3). The parameter values are as follows: $T = 6$, $r = 1.5$; $a = 0.3$; $K = 50$; $\omega = 0.3$; $c = 0.23$; $d = 0.8$; $\tau = 30$; $h = 0.8$; $\delta = 0.85$; $\theta = 0.5$.

pest population with respect to bifurcation parameters T, τ, and δ are displayed in Figures 4, 5, and 6. The resulting bifurcation diagrams clearly show rich dynamics of system (3). Specifically, Figure 5 shows the existence of period-halving bifurcations, and Figure 4 depicts the existence of period-doubling bifurcations, while Figure 6 indicates that several types of attractor may coexist. All those complex dynamics may result in complexity of pest control.

3.2. Effects of Density Dependent Releasing Strategy. According to the main results shown in Theorem 4, system (3) is permanent provided $R_1 > 1$. Thus, to control the pest population, we need to know how the parameters affect the value of the threshold R_1. Obviously, the threshold condition is determined by the key parameters including the
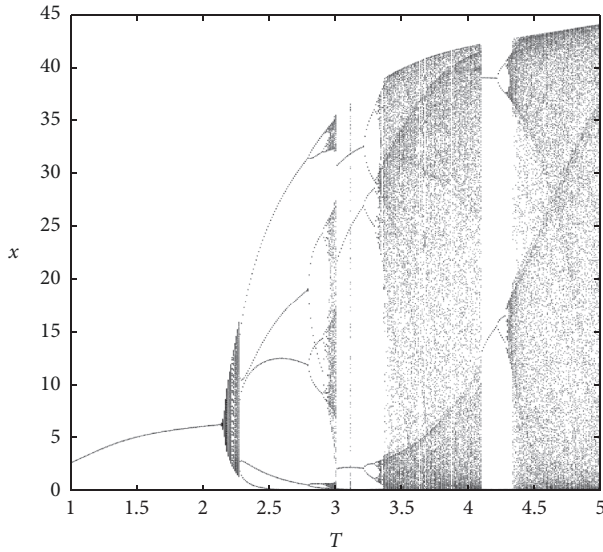
FIGURE 4: Bifurcation diagram of system (3) with respect to T. The parameter values are as follows: $\tau = 1$, $\theta = 0.5$, $\delta = 0.85$, $h = 0.8$, $r = 1.5$, $a = 0.3$, $K = 50$, $\omega = 0.3$, $d = 0.8$, $c = 0.5$.

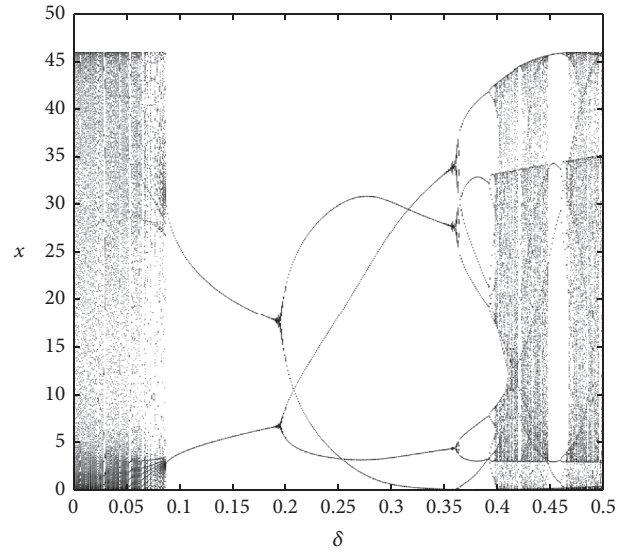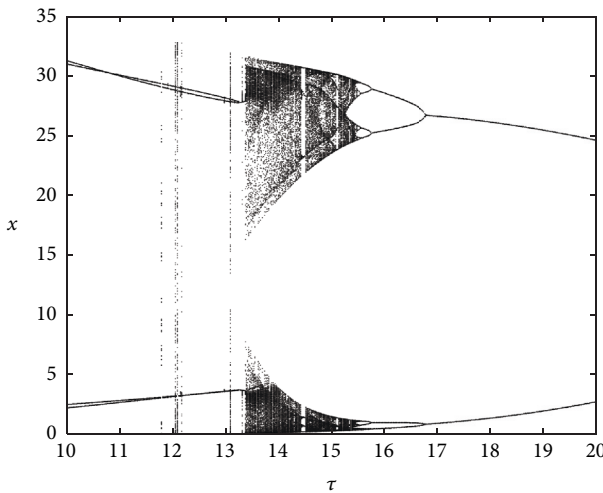FIGURE 5: Bifurcation diagram of system (3) with respect to parameter τ. The other parameter values are as follows: $T = 6$ $\theta = 0.5$, $\delta = 0.85$, $h = 0.8$, $r = 1.5$, $a = 0.3$, $K = 50$, $\omega = 0.3$, $d = 0.8$, $c = 0.5$.

FIGURE 6: Bifurcation diagram of system (3) with respect to parameter δ. The other parameter values are as follows: $T = 6$, $\tau = 1$, $\theta = 0.5$, $h = 0.8$, $r = 1.5$, $a = 0.3$, $K = 50$, $\omega = 0.3$, $d = 0.8$, $c = 0.5$.

pulse frequency, the maximum fatality rate, and the half-saturation constant. Figure 7 shows the relationship between the threshold R_1 and the pulse period T. It follows from this figure that the value of R_1 could increase as pulse period T increases, as shown in Figure 7(a). However, it also can decrease as T increases, as shown in Figure 7(b). All those indicate that the relations between the threshold R_1 and pulse period T are quite complex, which depicts the threshold condition R_1 is nonmonotonic with respect to T. Thus, reducing the pulse period T is not always beneficial for pest control. That is, frequent releasing of natural enemies may not be beneficial to the eradication of pests when the amount of predator released depends on the predator population, which is one of the novel results obtained in the present work

compared with the previous results obtained on this topic [3–5, 13, 27].

The releasing term in model (3) is formulated by $\tau/(1 + \theta y)$, which is a predator density dependent function and depicts how the number of natural enemy populations released was guided by its current density at the fixed moment. The parameter θ is a shape parameter which indicates the dependence of releasing amount and the predator population. Thus, to investigate the effects of the predator population guided releasing method on the pest control, the variations of the threshold R_1 with respect to parameters T, θ, and τ have been addressed, as shown in Figure 8.

Figure 8(a) shows the contour plots of R_1 with respect to θ and τ. It follows from this figure that increasing τ can lead to an obvious decrease of R_1, no matter how much the value of θ is, while decreasing θ can also lead to a slight decrease of R_1 and the effect is more evident when the value of τ is large. Figure 8(b) shows the contour plots of R_1 with respect to T and τ. This figure indicates that both T and τ together can affect the threshold condition R_1 significantly. Note that for relatively small τ the threshold condition R_1 is a decreasing function as T decreases. However, for a relatively large τ, decreasing period T may first lead to a complex effect on R_1; that is, R_1 is nonmonotonic with respect to T, which is similar to the important results shown in Figure 7.

Similarly, this type of interesting results can also be found in Figure 8(c) for the contour plots of R_1 with respect to θ and T. Figure 8(c) indicates that if T is small, decreasing θ can result in an obvious decrease of R_1, while if T is relatively large (larger than 8), decreasing θ almost has no effect on R_1. Thus, Figure 8 reveals that the key control factors including the pulse frequency, pulse amount, and the dependence of pulse amount and the current amount of the predator could affect each other. For example, when the control measure is implemented frequently (i.e., T is small) or the releasing
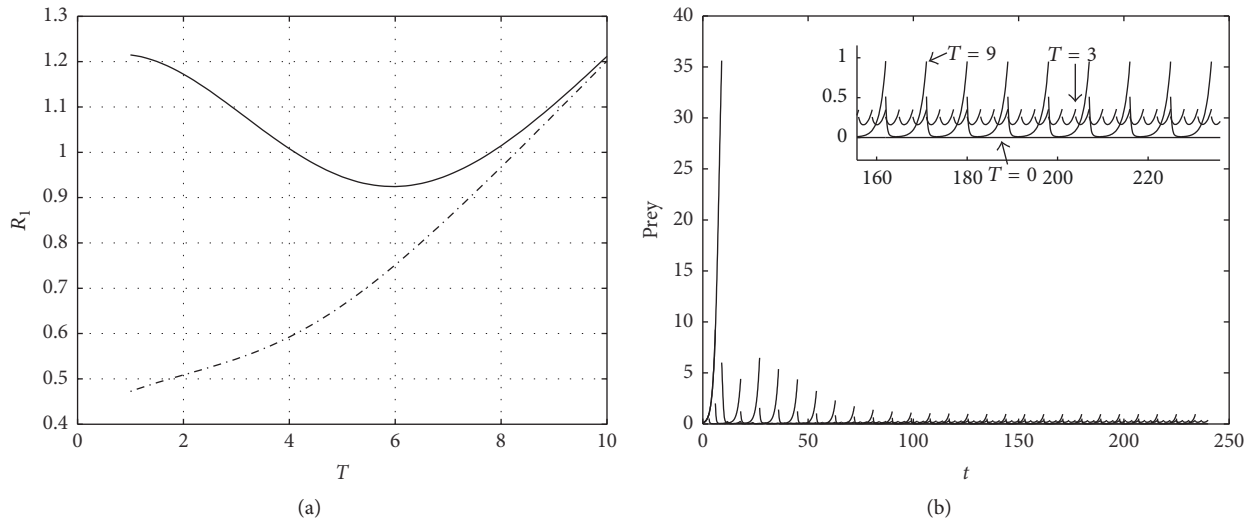
(a)

(b)

FIGURE 7: (a) Relationship between the threshold R_1 and the pulse period T with $\theta = 0.5$ (the dashed line) and $\theta = 4$ (the solid line). (b) Behavior of the prey when $T = 3, 6,$ and 9 for the case $\theta = 4$. The values of R_1 when $T = 3, 6,$ and 9 are 1.0928, 0.9245, and 1.1056, respectively. Other parameter values are $r = 0.8$, $a = 0.3$, $\tau = 20$, $d = 0.9$.

amount is large (i.e., τ is large), effect of the dependence between the pulse amount and the current amount of the predator is very small. Moreover, the complex relationships between T and R_1 usually appear when τ and θ are relatively large. All those confirm the control measures should be carefully designed in practice. Otherwise, the paradox and hormesis could occur when pesticides are applied in the field.

4. Discussion

Considering the effects of resources limitation and pest management purpose, we proposed a Holling type II pest-natural enemy model with density guided releasing strategy and killing rate, which can provide a more nature understanding and a better portrayal of the reality. However, the nonlinear pulse functions not only result in difficulties for dynamic analyses, but also can reveal some important results related to pest control compared with those obtained in the previous papers [3, 4, 9, 22, 23].

Theoretical results show that the pest-free periodic solution is locally stable when $R_1 < 1$ and the system is permanent when $R_1 > 1$. Besides, theoretical analysis also indicates that when T passes through the critical value T_0, a nontrivial positive periodic solution appears near the pest-free periodic solution. Further, extensive numerical investigations imply that there may be a locally stable positive periodic solution even if $R_1 < 1$. This results in a backward bifurcation and is quite different from the previous works [3, 17], in which the pulse functions are linear. This also indicates that the pest may not be eradicated even if R_1 is controlled to be below 1. All those results demonstrate that the dynamic behavior of model (3) may be dramatically affected by small changes in the value of initial densities of pest and natural enemy under resource limitation. Moreover, as shown in Figure 3, small or large initial values of the natural enemies may lead to the eradication of the pest. Medium level of the initial

values of the natural enemies usually can not suppress the pest population and often lead to the outbreak of the pest.

The threshold condition R_1 depends on all key parameters related to the pest control, such as releasing period, maximum releasing constant, and shape parameters of the nonlinear functions. To analyze the effects of those parameters, we first investigate the relationship between R_1 and the pulse period. Our results show that R_1 may be a nonmonotonic function with respect to the pulse period T. This indicates that shortening the pulse period may lead to an increase of R_1 and consequently may cause the outbreak of the pest, as shown in Figure 7. This result is not the same as the traditional results obtained in the previous works [3, 4, 9, 22, 23]. This may be because that frequent release leads to large number of current predators. As the number of predators released is guided by its current density, the large number of the current predators results in a small amount of releasing every time. So, under this circumstance, frequent release of the predator may be beneficial or detrimental to the eradication of pests.

The releasing number and killing rate guided by both the pest and natural enemy populations have been introduced into the model; one of the main purposes is to analyze the effect of this kind of nonlinear control methods in terms of reducing the pest population. So, it is essential to suggest how to choose the shape parameter θ and the releasing rate τ and to determine how the releasing period affects the control output. Numerical results as shown in Figure 8 reveal that when releasing rate is relatively large, reducing the dependence of releasing amount and the predator population, namely, reducing the value of θ, has little effect on the reduction of the pest population, while it has large effect on the reduction of the population of the pest when the releasing rate is small. Moreover, frequent releasing of the predator can only lead to decreasing of R_1 when the dependence of releasing amount and the predator population is small and frequent releasing of the predator may result in an increase
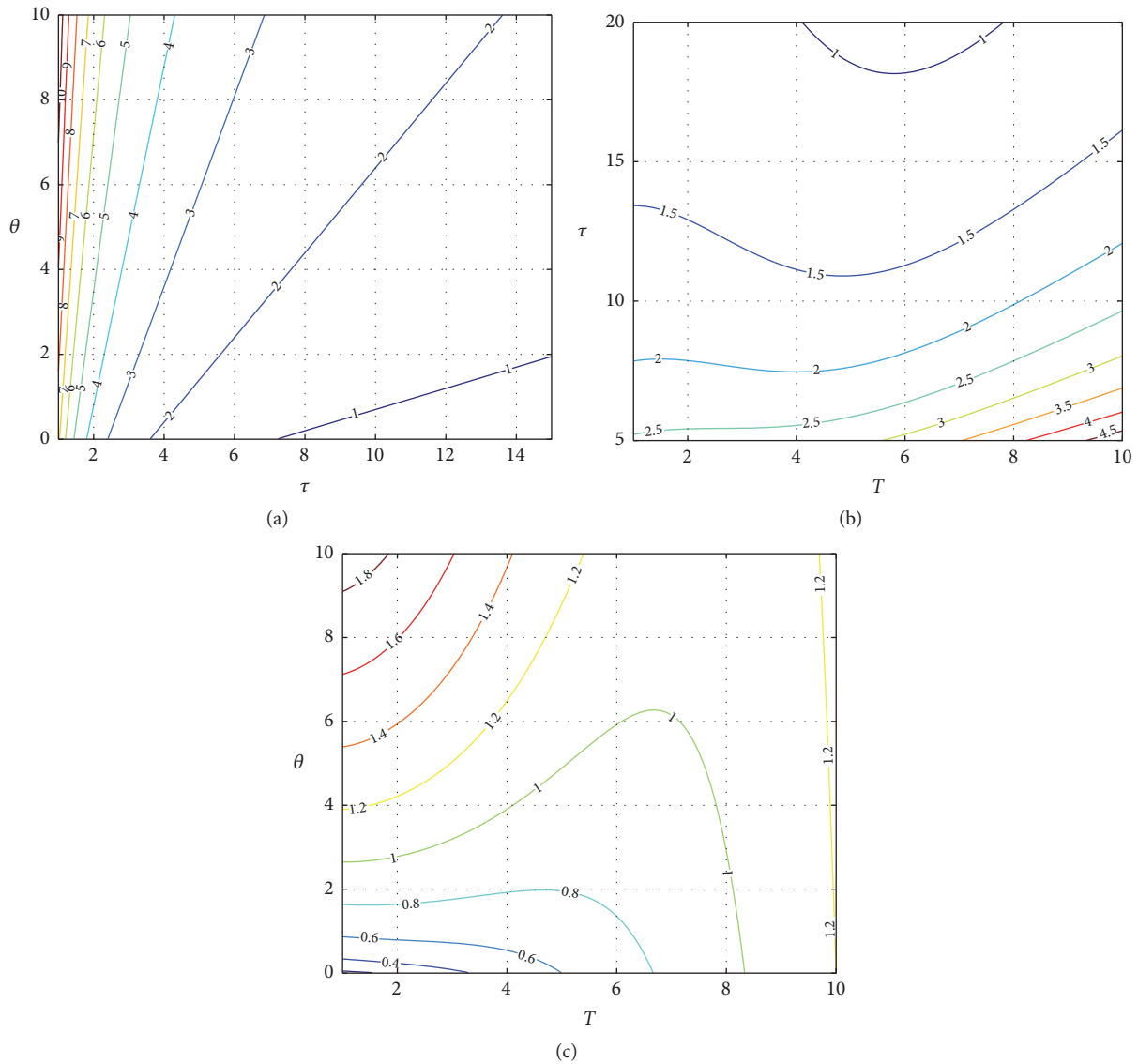
(a)

(b)

(c)

Figure 8: (a) Contour plots of R_1 with respect to θ and τ for $T = 3$. (b) Contour plots of R_1 with respect to τ and T when $\theta = 4$. (c) Contour plots of R_1 with respect to θ and T when $\tau = 20$. The other parameters are fixed as $r = 0.8$, $d = 0.9$, and $a = 0.3$.

of R_1 when the dependence of releasing amount and the predator population is large.

The results presented in this paper indicate that the successful biological control depends on the initial densities of the pest and predator and the reasonable implementing plan of control measures. The releasing of predators guided by the predator population is more sensitive to the pulse period and releasing amount every time. However, some simplifications of the real prey-predator interactions are neglected, such as predator hiding and effect of pesticide on the predator. In fact, the prey refuge can be broadly defined including spatial or temporal refuges, prey aggregations, or reduced prey activity [28–31]. It has been shown in many previous papers that the existence of refuges may have important effects on the coexistence of the prey and predator. Therefore, more realistic mathematical model and its dynamic analyses should be studied in the future which can help us to design control strategies and to make management decisions.
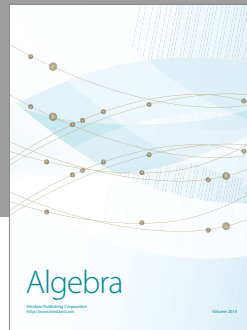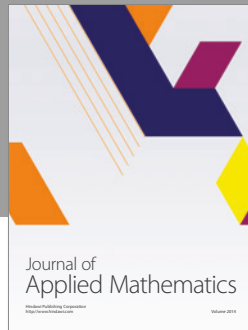
Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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