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Research Article

H_2/H_∞ Control for MJLS with Infinite Markov Chain

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With the help of a stochastic bounded real lemma, we deal with finite horizon H_2/H_∞ control problem for discrete-time MJLS, whose Markov chain takes values in an infinite set. Besides, a unified control design for H_2, H_∞, and H_2/H_∞ is given.

1. Introduction

As we know, H_∞ control is one of the most important robust control designs, usually used to eliminate the effect of disturbance v_t. In particular, lots of results have been contributed to the stochastic H_∞ theory; see [1–7], among many others. For *Itô* systems and discrete-time systems with multiplicative noise, stochastic H_∞-type control problems have been considered in [2, 5], respectively. Reference [3] has designed a state feedback H_∞ controller for nonlinear stochastic systems. References [4, 7] have dealt with H_∞ control in the presence of stochastic uncertainty.

On the other hand, after Kalman presented the question "When is a linear control system optimal?" which preserves the quadratic form in the performance index, [8] has discussed H_2 or Linear Quadratic Gaussian (LQG) control, while [9] has retained the original weights and sought out a producer which can also achieve the desired degree of stability, and others have contributed to the stochastic LQ controller design [10–12]. Because of the popularity of the H_2 performance for engineering, more and more researchers have been attracted by the mixed H_2/H_∞ control topic; see [13–16]. For example, [14, 15] have investigated the finite horizon and infinite horizon H_2/H_∞ control problem for discrete-time Markov jump systems.

It should be pointed out that mostly researches on Markov jump systems assume that Markov chain takes values in a finite set, such as [6, 13–15, 17–20]. However, infinite Markov jump systems, where Markov process has an infinite state space, can be used to describe more plants in real world. Recently, infinite Markov jump systems have aroused more and more concern [1, 21, 22]. Specially, for discrete-time case, [21] has explored exponential stability and l_2 input-state stability which is strongly detectable; [1] has established the finite horizon stochastic bounded real lemma to attain a prescribed disturbance attenuation level. Based on [1], this note seeks to set up H_2/H_∞ result for infinite Markov jump systems, unlike [15] which considered the finite jump case.

This paper aims to handle the mixed H_2/H_∞ control problem through the solvability of four coupled difference matrix-valued recursions (CDMRs) for discrete-time infinite Markov jump systems. The rest of the paper is organized as follows: Section 2 provides some useful definitions and lemmas. In Section 3, we get the necessary and sufficient condition for the finite horizon H_2/H_∞ control problem based on the stochastic bounded real lemma. A unified control design for H_2, H_∞, and H_2/H_∞ control is given in Section 4. And Section 5 concludes the paper.

For convenience, we adopt the following notations. $\mathscr{R}$ is the set of all real numbers; $\mathscr{R}^n$ is n-dimensional real vector space; $\mathscr{R}^{m\times n}$ is the vector space of all $m \times n$ matrices with entries in $\mathscr{R}$; A' is the transpose of a matrix A; $A \geq 0$ ($A > 0$): A is a positive semidefinite (positive definite) symmetric matrix; I is the identity matrix; $\|\cdot\|$ is the operator norm of $\mathscr{R}^{m\times n}$ or the Euclidean norm of $\mathscr{R}^n$; Z^+ is the set of all nonnegative integers; Z_1^+ is the set of all positive integers; $\mathscr{S}_n$ is the set of all $n \times n$ symmetric matrices.

2. Preliminaries

Consider the following discrete-time infinite Markov jump systems defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$:

$$\begin{aligned} x_{t+1} &= A_0(\eta_t) x_t + B_0(\eta_t) v_t \\ &\quad + \sum_{k=1}^{d} [A_k(\eta_t) x_t + B_k(\eta_t) v_t] w_t^k, \\ z_t &= C(\eta_t) x_t, \quad \eta_t \in Z_1^+, \end{aligned} \tag{1}$$

where $x_t \in \mathcal{R}^n$, $v_t \in \mathcal{R}^{n_v}$, and $z_t \in \mathcal{R}^{n_z}$ represent the system state, disturbance signal, and measurement output, respectively. Markov chain $\{\eta_t\}_{t \in Z^+}$ takes values in Z_1^+ with switching governed by a stationary transposition probability matrix $\mathcal{P} = [p_{ij}]$, where $p_{ij} = P(\eta_{t+1} = j \mid \eta_t = i)$. Let $w_t = \{w_t = (w_t^1, \ldots, w_t^d)\}$ be a sequence of real random variables which satisfies $E(w_t) = 0$ and $E(w_t w_s') = \delta_{t,s}$ (Kronecker function). Make $\mathcal{F}_t$ be the σ-algebra generated by $\{\eta_k, w_s \mid 0 \le k \le t,\ 0 \le s \le t-1\}$. With the case $t = 0$, set $F_0 = \sigma\{\eta_0\}$. For any given $t \in Z^+$, the σ-algebras $\sigma\{w_0, \ldots, w_t\}$ and $\sigma\{\eta_0, \ldots, \eta_t\}$ are independent of each other. $l^2(0, T; \mathcal{R}^m)$ denotes the set of $\mathcal{R}^m$–valued processes $\{y(t, w) : \mathbf{Z}^+ \times \Omega \to \mathcal{R}^m\}$, which is $\mathcal{F}_t$-measurable and $\sum_{t=0}^{T} E\|y(t)\|^2 < \infty$. Apparently, $l_2(0, T; \mathcal{R}^m)$ is a real Hilbert space with the norm being $\|y\|_{l_2(0,T;\mathcal{R}^m)} = (\sum_{t=0}^{T} E\|y_t\|^2)^{1/2} < \infty$.

Make $H_1^{m\times n}$ be the set $\{M \mid M = (M(1), M(2), \ldots),\ M(i) \in \mathcal{R}^{m\times n}\}$ which satisfies $\sum_{i=1}^{\infty} M(i) < \infty$. We can easily verify that $H_1^{m\times n}$ is a Banach space with $\|M(i)\|_1 = \sum_{i=1}^{\infty} M(i)$. Define another Banach space $H_\infty^{m\times n}$ with $\|M(i)\|_\infty = \sup_{i \in Z_1^+} M(i)$. $H_1^{m\times n}$ will be written as H_1^n with the case $m = n$, so does $H_\infty^{m\times n}$. For $i \in Z_1^+$, when $M(i) \in \mathcal{S}_n$ and $M(i) \ge 0$, $H_1^n(H_\infty^n)$ will be written as $H_1^{n+}(H_\infty^{n+})$.

For $M, N \in H_1^n$, if $M(i) \le N(i)$ for all $i \in Z_1^+$, we say $M \le N$. And it is easy to know that $\|M\|_1 \le \|N\|_1$. What is more, for a given real Banach space $\mathbb{X}$, let $\mathcal{B}(\mathbb{X})$ represent a Banach space including all bounded linear operators mapping $\mathbb{X}$ to $\mathbb{X}$. For $\Gamma \in \mathcal{B}(\mathbb{X})$, the induced norm is denoted by $\|\Gamma\|_{\mathbb{X}}$.

Next, we introduce the linear perturbation operator as follows.

Definition 1. Define a linear perturbation operator $L_T : l^2(0, T; \mathcal{R}^{n_v}) \to l^2(0, T; \mathcal{R}^{n_z})$ of system (1) as follows:

$$L_T(v) = C(\eta_t) x(t; 0, v), \tag{2}$$

where $x(t; 0, v)$ satisfies system (1) corresponding to $x_0 = 0$ and $v(\cdot)$. When $v \ne 0$, the H_∞ norm of L_T is determined by

$$\|L_T\|_\infty := \sup_{\substack{v \in l^2(0,T;\mathcal{R}^{n_v}),\\ v \ne 0, \eta_0 \in Z_1^+, x_0 = 0}} \frac{\|L_T(v)\|_{l^2(0,T;\mathcal{R}^{n_z})}}{\|v\|_{l^2(0,T;\mathcal{R}^{n_v})}}. \tag{3}$$

When $v = 0$, which means the system is unperturbed, the problem is trivial (in this case, $\|L_T\|_\infty = 0$).

Define the functional:

$$J_1^T(x_0, \eta_0, u_t, v_t) = \sum_{t=0}^{T} E\left[\|z_t\|^2 - \gamma^2 \|v_t\|^2\right], \tag{4}$$

which is connected with the H_∞ performance.

To make the formulae more brief, we adopt the following notations. Denote P as a series of symmetric matrices derived by the time t and the mode i; that is $P = \{P(\eta_t) \in S_n : \eta_t \in Z_1^+, t \in Z^+\}$, $P(t) = (P(\eta_1), P(\eta_2), P(\eta_3), \ldots)$. In subsequent analysis, we will use the notations as follows with $U \in H_\infty^n$:

$$\begin{aligned} \epsilon_i(U) &= \sum_{j=1}^{\infty} p_{ij}(t) U(j), \\ \Phi_i^1(U) &= \sum_{k=0}^{d} A_k(i)' \epsilon_i(U) A_k(i) + C(i)' C(i), \\ \Phi_i^2(U) &= \sum_{k=0}^{d} A_k(i)' \epsilon_i(U) B_k(i), \\ \Phi_i^3(U) &= \sum_{k=0}^{d} B_k(i)' \epsilon_i(U) B_k(i) - \gamma^2 I. \end{aligned} \tag{5}$$

Lemma 2. *Given $x_0 \in \mathcal{R}^n$, $v \in l^2(0, T; \mathcal{R}^{n_v})$, $\eta_0 \in Z_1^+$, and $x_t := x(t; x_0, \eta_0, v)$ being the solution of system (1), for any given $T \in Z^+$, we have*

$$\begin{aligned} J_1^T(x_0, \eta_0, u_t, v_t) &= \sum_{t=0}^{T} E\begin{pmatrix} x_t \\ v_t \end{pmatrix}' \prod_{\eta_t}(t, P) \begin{pmatrix} x_t \\ v_t \end{pmatrix} \\ &\quad + E x_0' P(0, \eta_0) x_0 \\ &\quad - E x_{T+1}' P(T+1, \eta_{T+1}) x_{T+1}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \prod_{\eta_t}(t, P) &= \prod_i(t, P) \\ &= \begin{pmatrix} \Phi_i^1(P(t+1)) - P(t, i) & \Phi_i^2(P(t+1)) \\ \Phi_i^2(P(t+1))' & \Phi_i^3(P(t+1)) \end{pmatrix} \end{aligned} \tag{7}$$

for $\eta_t = i$.

Proof. Due to the assumption, we know that w_t^k is independent of the Markov chain $\{\eta_t\}$ and is uncorrelated with v_t, so $A_0(\eta_t)x_t + B_0(\eta_t)v_t$ and $A_k(\eta_t)x_t + B_k(\eta_t)v_t$ are uncorrelated with w_t^k, $k = 1, \ldots, d$. Besides, $A_0(\eta_t)x_t + B_0(\eta_t)v_t$ and $A_k(\eta_t)x_t + B_k(\eta_t)v_t$ are $\mathcal{F}_t$-measurable, $k = 1, \ldots, d$; thus

$$\begin{aligned} &E\left\{[A_0(\eta_t) x_t + B_0(\eta_t) v_t]' P(t+1, \eta_{t+1}) \right. \\ &\quad \left. \cdot [A_k(\eta_t) x_t + B_k(\eta_t) v_t] \mid \mathcal{F}_t\right\} = 0. \end{aligned} \tag{8}$$

When $\eta_t = i$, we have

$$\begin{aligned} &E\left[x_{t+1}' P(t+1, \eta_{t+1}) x_{t+1} - x_t' P(t, \eta_t) x_t \mid \mathcal{F}_t, \eta_t = i\right] \\ &= E\begin{pmatrix} x_t \\ v_t \end{pmatrix}' \prod_{\eta_t}^{*}(t, P) \begin{pmatrix} x_t \\ v_t \end{pmatrix}, \end{aligned} \tag{9}$$

where

$$\prod_{\eta_t}^{*}(t,P)=\prod_{i}^{*}(t,P)$$
$$=\begin{pmatrix}\Phi_i^1(P(t+1))-P(t,i)-C(i)'C(i) & \Phi_i^2(P(t+1))\\ \Phi_i^2(P(t+1))' & \Phi_i^3(P(t+1))+\gamma^2 I\end{pmatrix}. \tag{10}$$

Taking $t=0,\dots,T$ above and summarizing together, we get that

$$E\left[x_{T+1}'P(T+1,\eta_T+1)x_{T+1}-x_0'P(0,\eta_0)x_0\mid\mathscr{F}_t,\eta_t\right.$$
$$\left.=i\right]=\sum_{t=0}^{T}E\begin{pmatrix}x_t\\ v_t\end{pmatrix}'\prod_{\eta_t}^{*}(t,P)\begin{pmatrix}x_t\\ v_t\end{pmatrix}. \tag{11}$$

According to the definition of $J_1^T(x_0,\eta_0,u_t,v_t)$, we obtain

$$J_1^T(x_0,\eta_0,u_t,v_t)=\sum_{t=0}^{T}E\left[\|z_t\|^2-\gamma^2\|v_t\|^2\right]$$
$$+\sum_{t=0}^{T}E\begin{pmatrix}x_t\\ v_t\end{pmatrix}'\prod_{\eta_t}^{*}(t,P)\begin{pmatrix}x_t\\ v_t\end{pmatrix} \tag{12}$$
$$+Ex_0'P(0,\eta_0)x_0$$
$$-Ex_{T+1}'P(T+1,\eta_{T+1})x_{T+1}.$$

By simple computation, we can get the desired result. □

Lemma 3 (see [1] (stochastic bounded real lemma)). *For system (1), given $\gamma>0$ and $T\in Z_1^+$, $\|L_T\|_\infty<\gamma$ if and only if the following difference Riccati recursions are solvable with $P(t)\in H_\infty^{n+}$:*

$$P(t,i)$$
$$=\Phi_1^i(P(t+1))$$
$$-\Phi_i^2(P(t+1))\Phi_i^3(P(t+1))^{-1}\Phi_i^2(P(t+1))', \tag{13}$$
$$P(T+1,i)=0,\quad i\in Z_1^+,\ t\in[0,T].$$

And $P(t)$ satisfies

$$\Phi_i^3(P(t+1))<-\epsilon_0 I_{n_v},\quad\forall\varepsilon_0\in\left(0,\gamma^2-\|L_T\|_\infty^2\right). \tag{14}$$

3. H_2/H_∞ Control

In this subsection, we attempt to discuss the finite horizon H_2/H_∞ control problem. Consider the following linear systems with infinite Markov jump:

$$x_{t+1}$$
$$=A_0(\eta_t)x_t+\overline{B}_0(\eta_t)u_t+B_0(\eta_t)v_t$$
$$+\sum_{k=1}^{d}\left[A_k(\eta_t)x_t+\overline{B}_k(\eta_t)u_t+B_k(\eta_t)v_t\right]w_t^k, \tag{15}$$
$$z_t=\begin{pmatrix}C(\eta_t)x_t\\ D(\eta_t)u_t\end{pmatrix},$$
$$D(\eta_t)'D(\eta_t)=I,\quad\eta_t\in Z_1^+.$$

Given $T\in Z_1^+$ and $\gamma>0$, our objective is to find $u^*(\cdot)\in l_2(0,T;\mathscr{R}^{n_u})$ such that

(i) $\|L_T\|_\infty<\gamma$, for all $v(\cdot)\in l_2(0,T;\mathscr{R}^{n_v})$ and $\eta_0\in Z_1^+$;

(ii) when the worst case disturbance v_t^* is enforced on system (15), $u^*(\cdot)$ will minimize the energy functional $J_2^T(x_0,\eta_0,u_t,v_t^*):=\sum_{t=0}^{T}E\|z_t\|^2$, $\forall x_0\in\mathscr{R}^n,\eta_0\in Z_1^+$.

If there exists (u_t^*,v_t^*) such that (i) and (ii) hold simultaneously, we say that the mixed H_2/H_∞ control problem is solvable. Before discussing, we provide the following coupled matrix recursions which is defined on $i\in Z_1^+$:

$$X^1(t,i)=\sum_{k=0}^{d}\left[A_k(i)+\overline{B}_k(i)K_i^2(X^2)\right]'\varepsilon_i\left(X^1(t+1)\right)$$
$$\cdot\left[A_k(i)+\overline{B}_kK_i^2(X^2)\right]+C(i)'C(i)-K_i^3(X^1)$$
$$\cdot H_i^1(X^1)^{-1}K_i^3(X^1)'+K_i^2(X^2)'K_i^2(X^2), \tag{16}$$
$$X^1(t)\geq 0,$$
$$X^1(T+1)=0,$$
$$H_i^1(X^1)<0,$$
$$K_i^1(X^1)=-H_i^1(X^1)^{-1}K_i^3(X^1)', \tag{17}$$
$$X^2(t,i)=\sum_{k=0}^{d}\left[A_k(i)+B_k(i)K_i^1(X^1)\right]'\varepsilon_i\left(X^2(t+1)\right)$$
$$\cdot\left[A_k(i)+B_k(i)K_i^1(X^1)\right]+C(i)'C(i)$$
$$-K_i^4(X^2)H_i^2(X^2)^{-1}K_i^4(X^2)', \tag{18}$$
$$X^2(t,i)\geq 0,$$
$$X^2(T+1)=0,$$
$$H_i^2(X^2)>0,$$

$$K_i^2\left(X^2\right) = -H_i^2\left(X^2\right)^{-1} K_i^4\left(X^2\right)', \tag{19}$$

where

$$\begin{aligned}
H_i^1\left(X^1\right) &= \sum_{k=0}^{d} B_k(i)' \varepsilon_i\left(X^1(t+1)\right) B_k(i) - \gamma^2 I, \\
H_i^2\left(X^2\right) &= I + \sum_{k=0}^{d} \overline{B}_k(i)' \varepsilon_i\left(X^2(t+1)\right) G_k(i), \\
K_i^3\left(X^1\right) &= \sum_{k=0}^{d} \left[A_k(i) + \overline{B}_k(i) K_i^2\left(X^2\right)\right]' \\
&\quad \cdot \varepsilon_i\left(X^1(t+1)\right) B_k(i), \\
K_i^4\left(X^2\right) &= \sum_{k=0}^{d} \left[A_k(i) + B_k(i) K_i^1\left(X^1\right)\right]' \\
&\quad \cdot \varepsilon_i\left(X^2(t+1)\right) \overline{B}_k(i).
\end{aligned} \tag{20}$$

Theorem 4. *The finite horizon H_2/H_∞ control problem is solvable with the solution $u_t^* = K_{\eta_t}^2(X^2)x_t$, $v_t^* = K_{\eta_t}^1(X^1)x_t$ if and only if CDMRs (16)–(19) have a pair of solutions $(X^1(t,i), K_i^1(X^1)); (X^2(t,i), K_i^2(X^2))$ for $\forall(t,i) \in [0,T] \times Z_1^+$.*

Proof. $\Leftarrow$: Assume CDMRs (16)–(19) have a pair of solutions $(X^1(t,i), K_i^1(X^1)); (X^2(t,i), K_i^2(X^2))$, $\forall(t,i) \in [0,T] \times Z_1^+$. Constructing $u_t^* = K_{\eta_t}^2(X^2)x_t$ and putting u into system (15), we get $\|L_T\|_\infty < \gamma$ for all $v \in l^2(0,T;\mathscr{R}^{n_v})$ and $\eta_0 \in Z_1^+$. By Lemma 3, together with the completing squares technique, it yields from (16) that

$$\begin{aligned}
&J_1^T\left(x_0, \eta_0, u_t^*, v_t\right) \\
&\quad = \sum_{t=0}^{T} E\left\{E\left\{\left[v_t - v_t^*\right]' H_i^1\left(X^1\right)\left[v_t - v_t^*\right] \mid \eta_t = i\right\}\right\} \\
&\qquad + x_0' X^1\left(0, \eta_0\right) x_0 \geq J_1^T\left(x_0, \eta_0, u_t^*, v_t\right) \\
&\quad = x_0' X^1\left(0, \eta_0\right) x_0,
\end{aligned} \tag{21}$$

where $v_t^* = K_{\eta_t}^1(X^1)x_t$ and $K_{\eta_t}^1(X^1)$ is defined by (17). As shown above, $J_1^T(x_0, \eta_0, u_t^*, v)$ is minimized by $v_t^* = K_{\eta_t}^1(X^1)x_t, \forall x_0 \in \mathscr{R}^n$, and v_t^* is the worst case disturbance. By Lemma 2 and the technique of completing squares, we obtain

$$\begin{aligned}
&J_2^T\left(x_0, \eta_0, u_t, v_t^*\right) \\
&\quad = \sum_{t=0}^{T} E\left\{E\left\{\left[u_t - u_t^*\right]' H_i^2\left(X^2\right)\left[u_t - u_t^*\right] \mid \eta_t = i\right\}\right\} \\
&\qquad + x_0' X^2\left(0, \eta_0\right) x_0 \geq J_2^T\left(x_0, \eta_0, u_t^*, v_t^*\right) \\
&\quad = x_0' X^2\left(0, \eta_0\right) x_0,
\end{aligned} \tag{22}$$

where $u_t^* = K_{\eta_t}^2(X^2)x_t$, and $K_{\eta_t}^2(X^2)$ is defined by (19). By reasoning as above, it is revealed that u_t^* is the controller to minimize the $J_2^T(x_0, \eta_0, u_t, v_t^*)$. Thus, we conclude that (u_t^*, v_t^*) is a pair of solutions to the finite horizon H_2/H_∞ control problem.

$\Rightarrow$: Suppose that system (15) has a pair of solutions to the finite horizon H_2/H_∞ control problem with $u_t^* = K_{\eta_t}^2(X^2)x_t$, $v_t^* = K_{\eta_t}^1(X^1)x_t$. Putting u_t^* into (15), we come to the following equations:

$$\begin{aligned}
x_{t+1} &= \left[A_0(\eta_t) + \overline{B}_0(\eta_t) K_{\eta_t}^2\left(X^2\right)\right] x_t + B_0(\eta_t) v_t \\
&\quad + \sum_{k=1}^{d} \left\{\left[A_k(\eta_t) + \overline{B}_k(\eta_t) K_{\eta_t}^2\left(X^2\right)\right] x_t + B_k(\eta_t) v_t\right\} \\
&\quad \cdot w_t^k, \\
z_t &= \begin{pmatrix} C(\eta_t) \\ D(\eta_t) K_{\eta_t}^2\left(X^2\right) \end{pmatrix} x_t, \\
D(\eta_t)' D(\eta_t) &= I, \quad \eta_t \in Z_1^+,\ t \in Z^+.
\end{aligned} \tag{23}$$

Applying Lemma 3 to (23), we justify that $X^1(t,i)$ satisfies (16) on $[0,T]$, also $X^1(t,i) \geq 0$. From the proof of the sufficiency, we confirm that the worst case disturbance $v_t^* = K_{\eta_t}^1(X^1)x_t$ with $K_{\eta_t}^1(X^1)$ given by (17). Enforcing v_t^* on system (15), we obtain

$$\begin{aligned}
x_{t+1} &= \left[A_0(\eta_t) + B_0(\eta_t) K_{\eta_t}^1\left(X^1\right)\right] x_t + \overline{B}_0(\eta_t) u_t \\
&\quad + \sum_{k=1}^{d} \left\{\left[A_k(\eta_t) + B_k(\eta_t) K_{\eta_t}^1\left(X^1\right)\right] x_t + \overline{B}_k(\eta_t) u_t\right\} \\
&\quad \cdot w_t^k, \\
z_t &= \begin{pmatrix} C(\eta_t) x_t \\ D(\eta_t) u_t \end{pmatrix}, \\
D(\eta_t)' D(\eta_t) &= I, \quad \eta_t \in Z_1^+,\ t \in Z^+.
\end{aligned} \tag{24}$$

According to the assumption, we deduce that u^* is the optimal solution of the following problem:

$$\begin{aligned}&\min_{u_t\in l_2(0,T;\mathscr{R}^{n_u})}\left\{J_2^T\left(x_0,\eta_0,u_t,v_t^*\right)=\sum_{t=0}^{T}E\left[u_t'u_t+x_t'C\left(\eta_t\right)'C\left(\eta_t\right)x_t\right]\right\},\\&\text{subject to}\quad(24).\end{aligned}\tag{25}$$

This is a standard LQ control for Markov jump systems defined on a finite horizon. Similar to the proof of Theorem 1 in [7], it is not difficult to prove that (18) is solvable with $X^2(t,i)\geq 0$. The proof is ended. □

Remark 5. Compared to finite horizon H_2/H_∞ control problem considered in [15], whose Markov chain takes values in a finite set, the dynamical model taken into account in this note is more general.

Example 6. Consider the following one-dimensional discrete-time infinite Markov jump system:

$$\begin{aligned}x_{t+1}&=\frac{1}{2\left(\eta_t+1\right)}x_t+u_t+v_t\\&\quad+\left[\frac{1}{\eta_t+1}x_t+u_t+v_t\right]w_t,\quad\eta_t\in Z_1^+,\\z_t&=\begin{pmatrix}\frac{3}{10}x_t\\u_t\end{pmatrix}.\end{aligned}\tag{26}$$

In (26), the transition probability is defined by $p(i,i)=3/4$, $p(i,i+1)=1/4$, $p(i,j)=0$, $j\neq i$, $i+1$, $i,j\in Z_1^+$. Set $T=2$, $\gamma=0.5$, the solutions of the four coupled matrix recursions (16)–(19) are given by $X^1(0,i)=0.14/(i+1)^2+0.09$, $X^1(0,i)=6.66/(i+1)^2+0.09$, $K_i^1(X^1)=5.36/(i+1)$, $K_i^2(X^2)=-0.17/(i+1)$.

4. H_2, H_∞, and H_2/H_∞ Control

In this section, we will develop a unified control design of H_2, H_∞, and H_2/H_∞ control for system (15). Give the following indices:

$$\begin{aligned}J_1^T\left(x_0,\eta_0,u_t,v_t\right)&=\sum_{t=0}^{T}E\left[\gamma^2\left\|v_t\right\|^2-\left\|z_t\right\|^2\right],\\J_2^T\left(x_0,\eta_0,u_t,v_t\right)&=\sum_{t=0}^{T}E\left[\left\|z_t\right\|^2-\rho^2\left\|v_t\right\|^2\right],\end{aligned}\tag{27}$$

where $\gamma\geq 0$ and $\rho\geq 0$ are defined on $\mathscr{R}$.

Definition 7. We call an admissible set $(u^*(\cdot),v^*(\cdot))\in l_2(0,T;\mathscr{R}^{n_u})\times l_2(0,T;\mathscr{R}^{n_v})$ a Nash equilibrium, if the following inequalities hold simultaneously for all admissible $(u(\cdot),v(\cdot))\in l_2(0,T;\mathscr{R}^{n_u})\times l_2(0,T;\mathscr{R}^{n_v})$,

$$\begin{aligned}J_1^T\left(x_0,\eta_0,u_t^*,v_t^*\right)&\leq J_1^T\left(x_0,\eta_0,u_t^*,v_t\right),\\J_2^T\left(x_0,\eta_0,u_t^*,v_t^*\right)&\leq J_1^T\left(x_0,\eta_0,u_t,v_t^*\right).\end{aligned}\tag{28}$$

For convenient, we give the following coupled matrix recursions on $(t,i)\in Z^+\times Z_1^+$:

$$\begin{aligned}X^1(t,i)&=\sum_{k=0}^{d}\left[A_k(i)+\overline{B}_k(i)K_i^2\left(X^2\right)\right]'\\&\quad\cdot\varepsilon_i\left(X^1(t+1)\right)\left[A_k(i)+\overline{B}_k(i)K_i^2\left(X^2\right)\right]-C(i)'\\&\quad\cdot C(i)-K_i^3\left(X^1\right)H_i^1\left(X^1\right)^+K_i^3\left(X^1\right)'-K_i^2\left(X^2\right)'\\&\quad\cdot K_i^2\left(X^2\right),\end{aligned}\tag{29}$$

$$\begin{aligned}&X^1(T+1)=0,\\&H_i^1\left(X^1\right)\geq 0,\\&K_i^3\left(X^1\right)=K_i^3\left(X^1\right)H_i^1\left(X^1\right)^+H_i^1\left(X^1\right),\end{aligned}\tag{30}$$

$$\begin{aligned}X^2(t,i)&=\sum_{k=0}^{d}\left[A_k(i)+B_k(i)K_i^1\left(X^1\right)\right]'\\&\quad\cdot\varepsilon_i\left(X^2(t+1)\right)\left[A_k(i)+B_k(i)K_i^1\left(X^1\right)\right]+C(i)'\\&\quad\cdot C(i)-K_i^4\left(X^2\right)H_i^2\left(X^2\right)^+K_i^4\left(X^2\right)'\\&\quad-\rho^2K_i^1\left(X^1\right)'K_i^1\left(X^1\right),\end{aligned}\tag{31}$$

$$\begin{aligned}&H_i^2\left(X^2\right)\geq 0,\\&X^2(T+1)=0,\\&K_i^4\left(X^1\right)=K_i^4\left(X^2\right)H_i^2\left(X^2\right)^+H_i^2\left(X^2\right),\end{aligned}\tag{32}$$

where

$$H_i^1\left(X^1\right)=\sum_{k=0}^{d}B_k(i)'\,\varepsilon_i\left(X^1(t+1)\right)B_k(i)+\gamma^2 I,$$

$$H_i^2\left(X^2\right)=I+\sum_{k=0}^{d}\overline{B}_k(i)'\,\varepsilon_i\left(X^2(t+1)\right)\overline{B}_k(i),$$

$$\begin{aligned}&K_i^3\left(X^1\right)\\&=\sum_{k=0}^{d}\left[A_k(i)+\overline{B}_k(i)K_i^2\left(X^2\right)\right]'\varepsilon_i\left(X^1(t+1)\right)B_k(i),\end{aligned}$$

$$
\begin{aligned}
&K_i^4\left(X^2\right)\\
&=\sum_{k=0}^{d}\left[A_k(i)+B_k(i)K_i^1\left(X^1\right)\right]'\varepsilon_i\left(X^2(t+1)\right)\overline{B}_k(i),\\
&K_i^1\left(X^1\right)=-H_i^1\left(X^1\right)^{+}K_i^3\left(X^1\right)',\\
&K_i^2\left(X^2\right)=-H_i^2\left(X^2\right)^{+}K_i^4\left(X^2\right)'.
\end{aligned}
\tag{33}
$$

(I) H_2 Control. Letting $\rho = 0$ and $\gamma \to +\infty$ in (27), it is easy to get that the performance index $J_1^T(x_0,\eta_0,u_t,v_t)$ holds naturally, and

$$
\begin{aligned}
J_2^T\left(x_0,\eta_0,u_t,v_t\right)&:=\sum_{t=0}^{T}E\left\|z_t\right\|^2\\
&=\sum_{t=0}^{T}E\left[x_t'C(i)'C(i)x_t+u_t'u_t\right]
\end{aligned}
\tag{34}
$$

which turns into a stochastic LQ optimal control problem. Because of $\rho = 0$ in $J_2^T(x_0,\eta_0,u_t,v_t)$, taking $B_k(i)=0$ in (15), we get

$$
\begin{aligned}
x_{t+1}&=A_0\left(\eta_t\right)x_t+\overline{B}_0\left(\eta_t\right)u_t\\
&\quad+\sum_{k=1}^{d}\left[A_k\left(\eta_t\right)x_t+\overline{B}_k\left(\eta_t\right)u_t\right]w_t^k,\\
z_t&=\begin{pmatrix}C\left(\eta_t\right)x_t\\ D\left(\eta_t\right)u_t\end{pmatrix},\\
&D\left(\eta_t\right)'D\left(\eta_t\right)=I,\quad \eta_t\in Z_1^+,\ t\in Z^+.
\end{aligned}
\tag{35}
$$

By Theorem 4, we know $\min_{u_t\in l^2(0,T;\mathscr{R}^{n_u})}J_2^T(x_0,\eta_0,u_t,v_t) = J_2^T(x_0,\eta_0,u_t^*,v_t)$, where $u_t^* = K_{\eta_t}^2(X^2)x_t$ with $K_i^2(X^2) = -H_i^2(X^2)^{-1}\{\sum_{k=1}^{d}[A_k(i)'\varepsilon_i(X^2(t+1))\overline{B}_k(i)]\}'$ and X^2 is the solution of the following equations:

$$
\begin{aligned}
X^2(t,i)&=\sum_{k=0}^{d}\left[A_k(i)'\varepsilon_i\left(X^2(t+1)\right)A_k(i)\right]+C(i)'\\
&\quad\cdot C(i)-\left[\sum_{k=0}^{d}A_k(i)'\varepsilon_i\left(X^1(t+1)\right)\cdot\overline{B}_k(i)\right]\\
&\quad\cdot H_i^2\left(X^2\right)^{-1}\left[\sum_{k=0}^{d}A_k(i)'\varepsilon_i\left(X^1(t+1)\right)\overline{B}_k(i)\right]',\\
X^2(T+1)&=0,\\
H_i^2\left(X^2\right)&=I+\sum_{k=0}^{d}\overline{B}_k(i)'\varepsilon_i\left(X^1(t+1)\right)\overline{B}_k(i)>0,\\
&\qquad i\in Z_1^+.
\end{aligned}
\tag{36}
$$

Also, the optimal value is obtained by

$$
\min_{u_t\in l^2(0,T;\mathscr{R}^{n_u})}J_2^T\left(x_0,\eta_0,u_t,0\right)=x_0'X^2\left(0,\eta_0\right)x_0.
\tag{37}
$$

Remark 8. Via a series of calculation, we can rewrite the first equality of (36) under the case of positiveness of $H_i^2(X^2)$ as

$$
\begin{aligned}
&X^2(t,i)\\
&=\sum_{k=0}^{d}\left[A_k(i)+\overline{B}_k(i)K_i^2\left(X^2\right)\right]'\varepsilon_i\left(X^2(t+1)\right)\\
&\quad\cdot\left[A_k(i)+\overline{B}_k(i)K_i^2\left(X^2\right)\right]+C(i)'C(i)\\
&\quad+K_i^2\left(X^2\right)'K_i^2\left(X^2\right).
\end{aligned}
\tag{38}
$$

Obviously, we have $X^2(t,i)\geq 0$; thus $H_i^2(X^2)>0$.

(II) H_∞ Control. Letting $\rho=\gamma$ in (27), it follows that

$$
J_1^T\left(x_0,\eta_0,u_t,v_t\right)+J_2^T\left(x_0,\eta_0,u_t,v_t\right)=0.
\tag{39}
$$

In this case, the Nash game (28) turns into a zero-sum game. We know from Theorem 4 that there exists a Nash equilibrium if and only if (29)–(32) is solvable. By computing, we obtain

$$
\begin{aligned}
&X^1(t,i)+X^2(t,i)\\
&=\sum_{k=0}^{d}\left[A_k(i)+B_k(i)K_i^1\left(X^1\right)+\overline{B}_k(i)K_i^2\left(X^2\right)\right]'\\
&\quad\cdot\varepsilon_i\left(X^1(t+1)+X^2(t+1)\right)\\
&\quad\cdot\left[A_k(i)+B_k(i)K_i^1\left(X^1\right)+\overline{B}_k(i)K_i^2\left(X^2\right)\right].
\end{aligned}
\tag{40}
$$

Noticing that $X^1(T+1,i)+X^2(T+1,i)=0$, we conclude the above equation has a unique solution such that $X^1(t,i)+X^2(t,i)=0$. Showing (31) in another way,

$$
\begin{aligned}
X^2(t,i)&=\sum_{k=0}^{d}\left[A_k(i)+B_k(i)K_i^1\left(X^1\right)\right]'\varepsilon_i\left(X^2(t+1)\right)\\
&\quad\cdot\left[A_k(i)+B_k(i)K_i^1\left(X^1\right)\right]+C(i)'C(i)\\
&\quad-K_i^4\left(X^2\right)H_i^2\left(X^2\right)^{+}K_i^4\left(X^2\right)'-\gamma^2K_i^1\left(X^1\right)'\\
&\quad\cdot K_i^1\left(X^1\right)=\sum_{k=0}^{d}\left[A_k(i)+\overline{B}_k(i)K_i^2\left(X^2\right)\right]'\varepsilon_i\Big(X^2(t\\
&\quad+1)\Big)\left[A_k(i)+\overline{B}_k(i)K_i^2\left(X^2\right)\right]+K_i^1\left(X^1\right)'\Big\{-\gamma^2I
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=0}^{d}\left[B_k(i)'\varepsilon_i\left(X^2(t+1)\right)B_k(i)\right]\Bigg\}K_i^1\left(X^1\right)\\
&+K_i^1\left(X^1\right)'\Bigg\{\sum_{k=0}^{d}\left[B_k(i)'\varepsilon_i\left(X^2(t+1)\right)\right.\\
&\left.\cdot\left[A_k(i)+\overline{B}_k(i)K_i^2\left(X^2\right)\right]\right]\Bigg\}+\sum_{k=0}^{d}\Big\{\left[A_k(i)\right.\\
&\left.+\overline{B}_k(i)K_i^2\left(X^2\right)\right]'\varepsilon_i\left(X^2(t+1)\right)B_k(i)\Big\}K_i^1\left(X^1\right)\\
&+K_i^2\left(X^2\right)'K_i^2\left(X^2\right)+C(i)'C(i).
\end{aligned}\tag{41}
$$

Substituting $X^2(t,i) = -X^1(t,i)$ into (41), it is easy to recognize that

$$
\begin{aligned}
-X^1(t,i)={}&-\sum_{k=0}^{d}\left[A_k(i)+\overline{B}_k(i)K_i^2\left(X^2\right)\right]'\\
&\cdot\varepsilon_i\left(X^1(t+1)\right)\left[A_k(i)+\overline{B}_k(i)K_i^2\left(X^2\right)\right]\\
&-K_i^1\left(X^1\right)'H_i^1\left(X^1\right)K_i^1\left(X^1\right)-K_i^1\left(X^1\right)'K_i^3\left(X^1\right)'\\
&-K_i^3\left(X^1\right)K_i^1\left(X^1\right)+K_i^2\left(X^2\right)'K_i^2\left(X^2\right)+C(i)'\\
&\cdot C(i).
\end{aligned}\tag{42}
$$

Similar as Theorem 1 in [23], we can prove that $\|L_T\| < \gamma$, which concludes that $H_i^1(X) > 0$. Since $H_i^1(X^1)^+ = H_i^1(X^1)^{-1}$, $H_i^1(X^1) > 0$ indicates that (42) is consistent with (29). Thus, when $\rho = \gamma$, the solution X of (42) satisfies

$$
\begin{aligned}
X(i)={}&\sum_{k=0}^{d}\left[A_k(i)+\overline{B}_k(i)K_i^2(X)\right]'\varepsilon_i(X(t+1))\\
&\cdot\left[A_k(i)+\overline{B}_k(i)K_i^2(X)\right]-C(i)'C(i)-K_i^3(X)\\
&\cdot H_i^1(X)^{-1}K_i^3(X)'-K_i^2(X)'K_i^2(X),\\
X(i)\le{}&0,\\
X(T+1)={}&0,\\
H_i^1(X)\ge{}&0.
\end{aligned}\tag{43}
$$

Besides, according to Theorem 4, ($u_t^* = K_{\eta_t}^2(X)x_t$, $v_t^* = K_{\eta_t}^1(X)x_t$) follows that

$$
\begin{aligned}
J_1^T\left(x_0,\eta_0,u_t,v_t^*\right)&\le J_1^T\left(x_0,\eta_0,u_t^*,v_t^*\right)\\
&\le J_1^T\left(x_0,\eta_0,u_t^*,v_t\right),
\end{aligned}\tag{44}
$$

which shows that u_t^* is the optimal control design and v_t^* is the related worst case disturbance for $\gamma > 0$.

(III) Mixed H_2/H_∞ Control. Letting $\rho = 0$ in $J_2^T(x_0,\eta_0,u_t,v_t)$. For system (15), the indices (27) turn into

$$
\begin{aligned}
J_1^T\left(x_0,\eta_0,u_t,v_t\right)&\coloneqq\sum_{t=0}^{T}E\left[\gamma^2\|v_t\|^2-\|z_t\|^2\right],\\
J_2^T\left(x_0,\eta_0,u_t,v_t\right)&\coloneqq\sum_{t=0}^{T}E\left[\|z_t\|^2\right].
\end{aligned}\tag{45}
$$

When $x_0 = 0$, because of the linearity of system (15), we have $u_t^* = 0$, $v_t^* = 0$, and

$$
\begin{aligned}
J_1^T\left(0,\eta_0,u_t^*,v_t\right)&\ge J_1^T\left(0,\eta_0,u_t^*,v_t^*\right)=0,\\
J_2^T\left(0,\eta_0,u_t^*,v_t^*\right)&\le J_2^T\left(x_0,\eta_0,u_t,v_t^*\right);
\end{aligned}\tag{46}
$$

that is, (u_t^*, v_t^*) is the optimal solution to the mixed H_2/H_∞ control problem.

Remark 9. The discussion here expands the version to the infinite Markov jump case compared to [23].

5. Conclusion

This note has supplied a finite horizon state feedback H_2/H_∞ controller based on the solution of four CDMRs for discrete-time infinite Markov jump systems with multiplicative noise. And, a numerical example has been given to illustrate its efficiency. How to handle the infinite horizon case for the concerned systems is our next work in the future.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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