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*Research Article*

# Simultaneous Robust Fault and State Estimation for Linear Discrete-Time Uncertain Systems

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We consider the problem of robust simultaneous fault and state estimation for linear uncertain discrete-time systems with unknown faults which affect both the state and the observation matrices. Using transformation of the original system, a new robust proportional integral filter (RPIF) having an error variance with an optimized guaranteed upper bound for any allowed uncertainty is proposed to improve robust estimation of unknown time-varying faults and to improve robustness against uncertainties. In this study, the minimization problem of the upper bound of the estimation error variance is formulated as a convex optimization problem subject to linear matrix inequalities (LMI) for all admissible uncertainties. The proportional and the integral gains are optimally chosen by solving the convex optimization problem. Simulation results are given in order to illustrate the performance of the proposed filter, in particular to solve the problem of joint fault and state estimation.

## 1. Introduction

This paper is concerned with the problem of joint fault and state estimation of linear discrete-time uncertain systems under convex bounded parametric uncertainty. This problem is solved by using a robust filtering approach to produce a robust fault and state estimation [1–6].

The proposed filter can play a significant role in several applications, for example, model based fault detection and isolation (FDI) problem [1–7] and fault tolerant control (FTC) problem [7, 8].

In the past three decades, the problem of robust state estimation in the presence of uncertainties has attracted the interests of many researchers. This problem is largely treated in the literature by different approaches: the guaranteed cost design [9–12], the $H_\infty$ filtering [13–15], and the set-valued estimation [16, 17]. One limitation of the different design approach for online filter operation is that they require continuous testing of a certain existence condition. When the condition fails at any particular iteration, the proposed filters can diverge.

From the point of view of minimizing the worst possible regularized residual norm over the class of admissible uncertainties, new robust filters are designed for linear uncertain systems by [18, 19]. Compared with the aforementioned robust formulations the developed filters perform data regularization rather than deregularization which represent an important property for online operation. The proposed filters in [19] are based in data regularization solution. This filter guarantees an error variance with an optimized guaranteed upper bound for any allowed uncertainty. To improve robustness against uncertainties such as disturbances and modeling errors, [20] introduced a proportional integral Kalman filter (PIKF). The proportional and the integral Kalman gains were obtained from the solution of Riccati equation leading to minimum error variance. Later, [21] developed a new robust proportional integral Kalman filter for stochastic linear uncertain systems. The filtering problem is converted into a convex optimization problem for continuous time systems with polytopic uncertainties and the filter parameters are optimally chosen by solving this problem.

The problem of robust Kalman filtering and optimal filtering in the presence of unknown inputs and unknown faults has received considerable attention in the last two decades due to its significations role in many applications, for example, geophysical and environmental applications, fault detection and isolation (FDI) problems, and fault tolerant control (FTC) problems.

The FDI (fault detection and isolation) problem for linear systems with unknown disturbances is largely studied in the literature by different approaches; see, for example, [2–7] and [22, 23]. By using the error innovation technique, a robust fault detection and isolation filter in continuous time is developed in [22] to generate unbiased white residuals signals. In [23] a new method is developed for linear time-invariant (LTI) stochastic discrete-time systems with unknown inputs. This method is important to detect and isolate multiple faults appearing simultaneously or sequentially in linear time-invariant (LTI) systems.

The optimal filtering and robust fault diagnosis problem has been treated for stochastic systems with unknown disturbances in [6, 7]. An optimal observer is proposed for linear time-varying systems. This observer can produce disturbances decoupled state estimation with minimum-variance. The output estimation error with disturbance decoupling [6, 7] is used as a residual signal. After that, a statistical testing procedure is applied to examine the residual and to diagnose faults. Nevertheless, the simultaneous actuator and sensor faults problem is not considered in [6, 7].

More recently, [3] presents a new recursive filter to joint fault and state estimation of linear time-varying discrete-time systems in the presence of unknown disturbances. The method is based on the assumption that no prior knowledge about the dynamical evolution of the fault and the unknown disturbances is available. The filter considers an arbitrary direct feedthrough matrix of the fault and it permits a multiple faults estimations. However, the obtained filter may in certain cases suffer from poor quality fault estimation.

Later, in [24] the problem of joint fault and state estimation of linear systems in the presence of unknown input with uncertain noise covariances was presented. This problem was solved by using the proportional integral three-stage Kalman filter (PI-ThSKF) to estimate the state and the fault of stochastic discrete-time systems with unknown inputs. However, this approach assumes that the models for the dynamical evolution of the fault and the unknown inputs are available.

Based on the assumption that no prior knowledge about the dynamical evolution of the fault is available, the same author [25] was proposed a new recursive optimal filter structure with transformation of the original system. A new recursive optimal unbiased minimum-variance filter has been developed when the direct feedthrough matrix of the fault has an arbitrary rank. However, the filtering algorithm requires the knowledge of a perfect dynamic model. Thus the developed filter may not be robust against modeling uncertainty in the state and the output matrices.

One limitation of the proposed design approach [3, 25] is that they require testing of a certain existence conditions. When the condition fails at any particular iteration, the desired performance is lost and the filter can diverge. In addition the disadvantages of the existing approaches [3, 4, 24, 25] are that the filter lost its optimality in the presence of uncertainties in the state and the output matrices.

In this paper, we consider the problem of robust joint fault and state estimation for linear discrete-time systems with norm bounded uncertainties in both the state and output matrices. The problem addressed is the design of robust linear filters that bound the state covariance matrix for all admissible uncertainties. It is shown that a robust proportional filter (RPF) is developed using transformation of the original system. This transformation is based on the singular value decomposition of the direct feedthrough matrix distribution of the fault which assumed to be arbitrary rank. The proposed filter guarantees that the variance of the estimation error is not more than an optimized upper bound for all admissible uncertainties. The minimization problem of the upper bound on the estimation error variance is formulated as a convex optimization problem subject to linear matrix inequalities and the filter parameters are optimally chosen by solving this problem. To improve robustness against uncertainties and to improve robust estimation of unknown time-varying fault, a new robust proportional integral filter (RPIF) is proposed. The proportional and the integral gains are optimally chosen by solving a convex optimization problem. So the resulted filter will be applied to solve a simultaneous actuator and sensor faults estimation problem.

The remainder of this paper is organized as follows. In Section 2 we set up the robust regularized least square problem for models with data uncertainties. Section 3 states the problem of interest. In Section 4 we design the robust proportional filter. Next in Section 5 we propose a design approach for the robust proportional integral filter (RPIF). Finally, in Section 6, the estimation performance of the proposed filters is demonstrated through an illustrative example.

## 2. Preliminaries

Consider the following optimization problem:

$$\min_{x} \max_{\{\delta A,\delta b\}} \left[\|x\|_{\Pi}^{2} + \|(A+\delta A)\,x - (b+\delta b)\|_{W}^{2}\right], \quad (1)$$

where $A$ is the data matrix, $b$ is the measurement vector which is assumed to be known, $x$ is the unknown vector, $\Pi = \Pi^{T} \geq 0$ and $W = W^{T} > 0$ are given weighting matrices, and $\{\delta A, \delta b\}$ are uncertainties assumed to satisfy a model of the form:

$$\left[\delta A \quad \delta b\right] = H\Delta \left[N_a \quad N_b\right], \quad (2)$$

where $\Delta$ is an arbitrary contraction, $\|\Delta\| \leq 1$, and $\{H, N_a, N_b\}$ are known quantities of appropriate dimension.

Problem (1) and (2) has a unique solution $\widehat{x}_k$ that is given by [17]:

$$\widehat{x} = \left[\widehat{\Pi} + A^{T}\widehat{W}A\right]^{-1}\left[A^{T}\widehat{W}b + \widehat{\lambda}N_a^{T}N_b\right], \quad (3)$$

where the modified weighting matrices $\{\widehat{\Pi}, \widehat{W}\}$ are defined by

$$\widehat{\Pi} = \Pi + \widehat{\lambda} N_a^T N_a,$$
$$\widehat{W} = W + WH\left(\widehat{\lambda} I - H^T W H\right)^{\dagger} H^T W \quad (4)$$

and $\widehat{\lambda}$ is a nonnegative scalar parameter obtained by the following optimization problem:

$$\widehat{\lambda} = \arg \min_{\widehat{\lambda} \geq \|H^T W H\|} G(\lambda), \quad (5)$$

where

$$G(\lambda) \coloneqq \|x(\lambda)\|_{\Pi}^2 + \lambda \left\|N_a x(\lambda) - N_b\right\|^2 + \|A x(\lambda) - b\|_{W(\lambda)}^2. \quad (6)$$

## 3. Problem Statement

Consider the linear stochastic uncertain discrete-time system with unknown additive fault in the form:

$$x_{k+1} = (A_k + \Delta A_k) x_k + F_k^x f_k + B_k u_k + w_k,$$
$$y_k = (C_k + \Delta C_k) x_k + F_k^y f_k + v_k, \quad (7)$$

where $x_k \in \Re^n$ is the state vector, $y_k \in \Re^p$ is the observation vector, $f_k \in \Re^m$ is the unknown additive fault vector, $w_k$ and $v_k$ are uncorrelated white noise sequences of zero-mean and with covariances matrices $Q_k = E[w_k w_l^T] \geq 0$ and $R_k = E[v_k v_l^T] > 0$, respectively.

The matrices $A_k$, $B_k$, $C_k$, $F_k^x$, and $F_k^y$ are known and have appropriate dimensions.

The initial state $x_0$ is a Gaussian random variable that is uncorrelated with $\{w_k, v_k\}$ for all $k$ with $E[x_0] = \widehat{x}_0$ and $E[(x_0 - \widehat{x}_0)(x_0 - \widehat{x}_0)^T] = P_0^x$, where $E[\cdot]$ denote the expectation operator.

$\Delta A_k$ and $\Delta C_k$ are unknown matrices which represent time-varying parameter uncertainties. These uncertainties are assumed to be of the following structure:

$$\begin{bmatrix} \Delta A_k \\ \Delta C_k \end{bmatrix} = \begin{bmatrix} H_{1,k} \\ H_{2,k} \end{bmatrix} F_k E_k, \quad (8)$$

where $H_{1,k}$, $H_{2,k}$, and $E_k$ are known time-varying matrices of appropriate dimensions, while $F_k$ is an unknown time-varying matrix satisfying arbitrary contraction, $F_k F_k^T \leq I$, $\forall k \in [0 \quad N]$.

The aim of this paper is to design a new robust proportional integral filter (RPIF) to obtain a robust fault and state estimation when $0 < \operatorname{rank}[F_k^y] \leq m$ in spite of the presence of parametric uncertainties.

Initially, we seek to change the coordinate of system (7) by using the same technique developed in [26].

Let $r_k = \operatorname{rank}(F_k^y) < m$, and then the singular value decomposition of the matrix $F_k^y$ is given by

$$F_k^y = \begin{bmatrix} U_{1,k} & U_{2,k} \end{bmatrix} \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1,k}^T \\ V_{2,k}^T \end{bmatrix}, \quad (9)$$

where $\Sigma_k \in \Re^{r_k \times r_k}$, $U_{1,k} \in \Re^{p \times r_k}$, $U_{2,k} \in \Re^{p \times (p - r_k)}$, $V_{1,k} \in \Re^{m \times r_k}$, and $V_{2,k} \in \Re^{m \times (m - r_k)}$. $[U_{1,k} \quad U_{2,k}]$ and $[V_{1,k} \quad V_{2,k}]$ are unitary matrices.

Using the notations

$$f_k = V_{1,k} f_{1,k} + V_{2,k} \overline{f}_{2,k}, \quad (10)$$

$$F_{1,k}^x = F_k^x V_{1,k},$$
$$\overline{F}_{2,k}^x = F_k^x V_{2,k}, \quad (11)$$
$$F_{1,k}^y = U_{1,k} \Sigma_k,$$

where $f_{1,k} = V_{1,k}^T f_k$ and $\overline{f}_{2,k} = V_{2,k}^T f_k$, we obtain the following equivalent system of the original system (7).

$$x_{k+1} = (A_k + \Delta A_k) x_k + F_{1,k}^x f_{1,k} + \overline{F}_{2,k}^x \overline{f}_{2,k} + B_k u_k + w_k, \quad (12)$$

$$y_k = (C_k + \Delta C_k) x_k + F_{1,k}^y f_{1,k} + v_k, \quad (13)$$

where $F_{1,k}^y$ is of full-column rank due to (11).

Note that $\overline{F}_{2,k}^x$ in (12) may not have full-column rank and the unknown fault $\overline{f}_{2,k}$ may not be estimable. However it can be solved by finding a full-rank factorization of $\overline{F}_{2,k}^x$, that is, $\overline{F}_{2,k}^x = F_{2,k} \breve{F}_{2,k}$, where $F_{2,k}$ is $n \times r$ full-column rank matrix and $\breve{F}_{2,k}$ is $r(m - r_k)$ full-row rank matrix [27].

Thus by defining the notation

$$f_{2,k} = \breve{F}_{2,k}^x \overline{f}_{2,k} \quad (14)$$

(12) becomes

$$x_{k+1} = (A_k + \Delta A_k) x_k + F_{1,k}^x f_{1,k} + F_{2,k}^x f_{2,k} + B_k u_k + w_k. \quad (15)$$

## 4. Robust Proportional Filter Design

In this section, we propose to solve equivalent system (13), (15) for $\widehat{x}_{k/k}$, $\widehat{f}_{1,k/k}$ and $\widehat{f}_{2,k-1/k}$, such that $\operatorname{trace}\{E[e_k e_k^T]\}$ is minimized, where $e_k = x_k - \widehat{x}_{k/k}$ is the state estimation error.

Here, we adopt a robust least-squares estimation approach to obtain a robust estimate for the state variable and the unknown faults by following a two-step procedure.

*4.1. Bounded Uncertainties in $C_k$ Alone.* Assume first that there are no uncertainties in $A_k$; we will incorporate the uncertainties in $A_k$ later. With bounded uncertainties in the output matrix $C_k$ alone and by using the robust least-squares estimation procedure, we can transform the original system

(13), (15) into the following augmented output equation (AOE):

$$\begin{bmatrix} y_k - (C_k + \Delta C_k)\hat{x}_{k/k-1} \\ 0 \end{bmatrix} = \begin{bmatrix} (C_k + \Delta C_k) & F^y_{1,k} & 0 \\ I & 0 & -F^x_{2,k-1} \end{bmatrix} \begin{bmatrix} x_k - \hat{x}_{k/k-1} \\ f_{1,k/k} \\ f_{2,k-1/k} \end{bmatrix} + \begin{bmatrix} v_k \\ -w_{k-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A_{k-1,c} & F^x_{1,k-1} \end{bmatrix} v_{k-1}, \tag{16}$$

where

$$\hat{x}_{k/k-1} = A_{k-1}\hat{x}_{k-1/k-1} + B_{k-1}u_{k-1} + F^x_{1,k-1}\hat{f}_{1,k-1/k-1}, \tag{17}$$

$$v_k = -\begin{bmatrix} e^T_k & \tilde{f}^T_{1,k} \end{bmatrix}^T, \tag{18}$$

where $\tilde{f}_{1,k} = f_{1,k} - \hat{f}_{1,k/k}$. Here $\hat{f}_{1,k/k}$ is the estimate of the fault and $v_k$ is zero-mean Gaussian and independent of $w_k$ and $v_{k+1}$.

The filtered estimate $\hat{x}_{k/k}$, the unknown fault $\hat{f}_{1,k/k}$, and the delayed unknown fault $\hat{f}_{2,k-1/k}$ can be obtained by solving the following robust least-squares problem:

$$\min_{x_k, f_{1,k}, f_{2,k-1}} \max_{\Delta C_k} \left\| \begin{bmatrix} y_k - (C_k + \Delta C_k)\hat{x}_{k/k-1} \\ 0 \end{bmatrix} - \begin{bmatrix} (C_k + \Delta C_k) & F^y_{1,k} & 0 \\ I & 0 & -F^x_{2,k-1} \end{bmatrix} \begin{bmatrix} x_k - \hat{x}_{k/k-1} \\ f_{1,k/k} \\ f_{2,k-1/k} \end{bmatrix} \right\|_{\mathrm{diag}(R^{-1}_k, \overline{P}^{-1}_k)}, \tag{19}$$

where

$$\begin{aligned} \overline{P}_k &= E\left[(x_k - \hat{x}_{k/k-1})(x_k - \hat{x}_{k/k-1})^T\right] \\ &= \begin{bmatrix} A_{k-1} & F^x_{1,k-1} \end{bmatrix} \begin{bmatrix} P_{k-1} & P^{xf_1}_{k-1} \\ P^{f_1x}_{k-1} & P^{f_1}_{k-1} \end{bmatrix} \begin{bmatrix} A^T_{k-1} \\ F^{xT}_{1,k-1} \end{bmatrix} + Q_{k-1}. \end{aligned} \tag{20}$$

Here $P^{xf_1}_{k-1}$ and $P^{f_1}_{k-1}$ will be defined later.

Note that problem (19) can be written more compactly in the form (1) and (2) with the identifications

$$\begin{aligned} x &\longleftarrow \mathrm{col}\,(x_k - \hat{x}_{k/k-1}, f_{1,k/k}, f_{2,k-1/k}), \\ b &\longleftarrow \begin{bmatrix} y_k - C_k\hat{x}_{k/k-1} \\ 0 \end{bmatrix}, \\ A &\longleftarrow \begin{bmatrix} C_k & F^y_{1,k} & 0 \\ I & 0 & -F^x_{2,k-1} \end{bmatrix}, \\ \delta A &\longleftarrow \begin{bmatrix} \Delta C_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \delta b &\longleftarrow \begin{bmatrix} -\Delta C_k\hat{x}_{k/k-1} \\ 0 \end{bmatrix}, \\ W &\longleftarrow \mathrm{diag}\left(R^{-1}_k, \overline{P}^{-1}_k\right), \\ H &\longleftarrow \begin{bmatrix} H_{2,k} & 0 \\ 0 & 0 \end{bmatrix}, \\ N_a &\longleftarrow \begin{bmatrix} E_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ N_b &\longleftarrow \begin{bmatrix} -E_k\hat{x}_{k/k-1} \\ 0 \end{bmatrix}, \\ \Delta &\longleftarrow F_k. \end{aligned} \tag{21}$$

From (1)–(3) the solution to (19) is given by

$$\begin{bmatrix} \hat{x}_k - \hat{x}_{k/k-1} \\ \hat{f}_{1,k/k} \\ \hat{f}_{2,k-1/k} \end{bmatrix} = \begin{bmatrix} \breve{P}^{-1}_k & C^T_k\overline{R}^{-1}_k F^y_{1,k} & -\overline{P}^{-1}_k F^x_{2,k-1} \\ F^{yT}_{1,k}\overline{R}^{-1}_k C_k & \breve{D}^{-1}_{1,k} & 0 \\ -F^{xT}_{2,k-1}\overline{P}^{-1}_k & 0 & \breve{D}^{-1}_{2,k-1} \end{bmatrix}^{-1} \cdot \begin{bmatrix} C^T_k\overline{R}^{-1}_k (y_k - C_k\hat{x}_{k/k-1}) - \hat{\lambda}_k E^T_k E_k\hat{x}_{k/k-1} \\ F^{yT}_{1,k}\overline{R}^{-1}_k (y_k - C_k\hat{x}_{k/k-1}) \end{bmatrix}, \tag{22}$$

where

$$\begin{aligned} \breve{P}^{-1}_k &= \overline{P}^{-1}_k + C^T_k\overline{R}^{-1}_k C_k, \\ \breve{D}^{-1}_{1,k} &= F^{yT}_{1,k}\overline{R}^{-1}_k F^y_{1,k}, \end{aligned}$$

$$\breve{D}_{2,k-1}^{-1} = F_{2,k-1}^{xT}\overline{P}_k^{-1}F_{2,k-1}^{x},$$
$$\overline{R}_k^{-1} = \left(R_k - \widehat{\lambda}_k^{-1}H_{2,k}H_{2,k}^{T}\right)^{-1}. \tag{23}$$

Moreover, $\widehat{\lambda}_k$ is the minimizing argument in the interval $\widehat{\lambda}_k > \|H_{2,k}^T R_k^{-1} H_{2,k}\| = \lambda_{l,k}$ of the corresponding scalar-valued function $G(\lambda)$ in (6) constructed with identification (21).

Using [28, Prop. 2.8.7], we find that the inverse in (22) can be written as

$$\begin{bmatrix} \breve{P}_k^{-1} & C_k^T\overline{R}_k^{-1}F_{1,k}^{y} & -\overline{P}_k^{-1}F_{2,k-1}^{x} \\ F_{1,k}^{yT}\overline{R}_k^{-1}C_k & \breve{D}_{1,k}^{-1} & 0 \\ -F_{2,k-1}^{xT}\overline{P}_k^{-1} & 0 & \breve{D}_{2,k-1}^{-1} \end{bmatrix}^{-1} \tag{24}$$
$$= \begin{bmatrix} P_k & P_k^{xf_1} & P_k^{xf_2} \\ P_k^{f_1x} & P_k^{f_1} & P_k^{f_1f_2} \\ P_k^{f_2x} & P_k^{f_2f_1} & P_{k-1}^{f_2} \end{bmatrix},$$

$$P_k^{xf_1} = \left(P_k^{f_1x}\right)^T = -P_kC_k^T\overline{R}_k^{-1}F_{1,k}^{y}\breve{D}_{1,k}^{-1}, \tag{25}$$

$$P_k^{xf_2} = \left(P_k^{f_2x}\right)^T = P_k\overline{P}_k^{-1}F_{2,k-1}^{x}\breve{D}_{2,k-1}^{-1}, \tag{26}$$

$$P_k^{f_1f_2} = \left(P_k^{f_2f_1}\right)^T = -\breve{D}_{1,k}F_{1,k}^{yT}\overline{R}_k^{-1}C_kP_k\overline{P}_k^{-1}F_{2,k-1}^{x}\breve{D}_{2,k-1}^{-1}. \tag{27}$$

Note that $P_k$, $P_k^{f_1}$, and $P_{k-1}^{f_2}$ can be identified as the covariance matrices of $\widehat{x}_{k/k}$, $\widehat{f}_{1,k}$, and $\widehat{f}_{2,k-1}$; that is,

$$P_k = E\left[\left(x_k - \widehat{x}_{k/k}\right)\left(x_k - \widehat{x}_{k/k}\right)^T\right],$$
$$P_k^{f_1} = E\left[\left(f_{1,k/k} - \widehat{f}_{1,k/k}\right)\left(f_{1,k/k} - \widehat{f}_{1,k/k}\right)^T\right], \tag{28}$$
$$P_{k-1}^{f_2} = E\left[\left(f_{2,k-1/k} - \widehat{f}_{2,k-1/k}\right)\left(f_{2,k-1/k} - \widehat{f}_{2,k-1/k}\right)^T\right],$$

where the inverse of $P_k$, $P_k^{f_1}$, and $P_{k-1}^{f_2}$ are given by

$$P_k^{-1} = \breve{P}_{k,1}^{-1} - C_k^T\overline{R}_k^{-1}F_{1,k}^{y}\breve{D}_{1,k}F_{1,k}^{yT}\overline{R}_k^{-1}C_k,$$
$$\left(P_k^{f_1}\right)^{-1} = \breve{D}_{1,k}^{-1} - F_{1,k}^{yT}\overline{R}_k^{-1}C_k\breve{P}_{k,1}C_k^T\overline{R}_k^{-1}F_{1,k}^{y},$$
$$\left(P_{k-1}^{f_2}\right)^{-1} = \breve{D}_{2,k-1}^{-1} - F_{2,k-1}^{xT}\overline{P}_k^{-1}\widehat{P}_{k,1}\overline{P}_k^{-1}F_{2,k-1}^{x},$$
$$\breve{P}_{k,1}^{-1} = \breve{P}_k^{-1} - \overline{P}_k^{-1}F_{2,k-1}^{x}\breve{D}_{2,k-1}F_{2,k-1}^{xT}\overline{P}_k^{-1},$$
$$\widehat{P}_{k,1}^{-1} = \breve{P}_k^{-1} - C_k^T\overline{R}_k^{-1}F_{1,k}^{y}\breve{D}_{1,k}F_{1,k}^{yT}\overline{R}_k^{-1}C_k. \tag{29}$$

Finally substituting (24) in (22) yields

$$\widehat{x}_{k/k} = \left(I - \widehat{\lambda}_kP_kE_k^TE_k\right)\widehat{x}_{k/k-1} + P_kC_k^T\overline{R}_k^{-1}\left(y_k - C_k\widehat{x}_{k/k-1} - F_{1,k}^{y}\widehat{f}_{1,k/k}\right), \tag{30}$$
$$\widehat{f}_{1,k/k} = M_k^{f_1}\left(y_k - \overline{C}_k\widehat{x}_{k/k-1}\right), \tag{31}$$
$$\widehat{f}_{2,k-1/k} = M_k^{f_2}\left(y_k - \overline{C}_k\widehat{x}_{k/k-1}\right), \tag{32}$$

where

$$M_k^{f_1} = \left(F_{1,k}^{yT}\widetilde{R}_{1,k}^{-1}F_{1,k}^{y}\right)^{-1}F_{1,k}^{yT}\widetilde{R}_{1,k}^{-1}, \tag{33}$$
$$M_k^{f_2} = \left(F_{2,k-1}^{xT}\widetilde{R}_{2,k}^{-1}F_{2,k-1}^{x}\right)^{-1}F_{2,k-1}^{xT}\widetilde{R}_{2,k}^{-1}, \tag{34}$$
$$\overline{C}_k = C_k - \widehat{\lambda}_kC_k\overline{P}_kE_k^TE_k, \tag{35}$$
$$\widetilde{R}_{1,k} = C_k\widetilde{P}_kC_k^T + \overline{R}_k, \tag{36}$$
$$\widetilde{R}_{2,k}^{-1} = \overline{P}_k^{-1} - \overline{P}_k^{-1}\widehat{P}_{k,1}\overline{P}_k^{-1}, \tag{37}$$
$$\widetilde{P}_k^{-1} = \overline{P}_k^{-1} - \overline{P}_k^{-1}F_{2,k-1}^{x}\breve{D}_{2,k-1}F_{2,k-1}^{xT}\overline{P}_k^{-1}. \tag{38}$$

Furthermore, we find that $P_k^{f_1}$ and $P_{k-1}^{f_2}$ are given, respectively, by

$$P_k^{f_1} = \left(F_{1,k}^{yT}\widetilde{R}_k^{-1}F_{1,k}^{y}\right)^{-1},$$
$$P_{k-1}^{f_2} = \left(F_{2,k-1}^{xT}\widetilde{R}_{2,k}^{-1}F_{2,k-1}^{x}\right)^{-1}. \tag{39}$$

Finally, it follows from (17) and (30) that

$$\widehat{x}_{k+1/k} = A_{p,k}\widehat{x}_{k/k-1} + F_{1,k}^{x}\widehat{f}_{1,k/k} + B_ku_k + K_{p,k}\left(y_k - C_k\widehat{x}_{k/k-1} - F_{1,k}^{y}\widehat{f}_{1,k/k}\right), \tag{40}$$

where $A_{p,k}$ and $K_{p,k}$ are defined in terms of the parameter $P_k$ as

$$A_{p,k} = A_k\left(I - \widehat{\lambda}_kP_kE_k^TE_k\right),$$
$$K_{p,k} = A_kP_kC_k^T\overline{R}_k^{-1}. \tag{41}$$

Note that these expressions for $A_{p,k}$ and $K_{p,k}$ are defined linearly in terms of $P_k$ and have been determined by assuming uncertainties in $C_k$ alone. We will incorporate the uncertainties in $A_k$ in the next section.

*4.2. Bounded Uncertainties in $A_k$ and $C_k$.* We now incorporate uncertainties into $A_k$. That is, we consider norm bounded uncertainties in $A_k$ and $C_k$ as in (13) and (15). We will move to select the parameter $P_k$ by assuming uncertainties in $A_k$ alone and one that meets robustness criterion (19) when there are uncertainties in $C_k$.

Denoting $\widetilde{x}_k = x_k - \widehat{x}_{k/k-1}$, we define the extended weight vector $\zeta_k = \left[\begin{smallmatrix}\widetilde{x}_k \\ \widehat{x}_{k/k-1}\end{smallmatrix}\right]$. Then in the absence of uncertainties in $C_k$, we find that $\zeta_k$ satisfies

$$\zeta_{k+1} = \left(\overline{A}_{k,c} + \overline{H}_{k,c}F_k\overline{E}_{k,c}\right)\zeta_k + \overline{F}_{k,c}\overline{f}_{k,c} + \overline{G}_{k,c}\overline{w}_k, \tag{42}$$

where

$$\overline{A}_{k,c} = \begin{pmatrix} A_k - K_{p,k}C_k & A_k - A_{p,k} \\ K_{p,k}C_k & A_{p,k}\end{pmatrix},$$
$$\overline{F}_{k,c} = \begin{bmatrix} F_k^x & -K_{p,k}F_{1,k}^y & 0 \\ 0 & K_{p,k}F_{1,k}^y & F_{1,k}^x\end{bmatrix}, \tag{43}$$
$$\overline{G}_{k,c} = \begin{bmatrix} I & -K_{p,k} & F_{2,k}^x \\ 0 & K_{p,k} & 0\end{bmatrix},$$

$$\overline{H}_{k,c} = \begin{pmatrix} H_{1,k} & 0 \\ 0 & 0\end{pmatrix},$$
$$\overline{E}_{k,c} = \begin{pmatrix} E_k & E_k \\ 0 & 0\end{pmatrix},$$
$$\overline{f}_{k,c} = \begin{pmatrix} \widetilde{f}_k \\ \widetilde{f}_{1,k} \\ \widehat{f}_{1,k}\end{pmatrix}, \tag{44}$$
$$\overline{w}_k = \begin{pmatrix} w_k \\ v_k \\ \widehat{f}_{2,k}\end{pmatrix}.$$

The covariance matrix of $\zeta_k$ satisfies

$$\begin{aligned}\Sigma_{k+1} &= E\left\{\left(\overline{A}_{k,c} + \overline{H}_{k,c}F_k\overline{E}_{k,c}\right)\Sigma_k\left(\overline{A}_{k,c} + \overline{H}_{k,c}F_k\overline{E}_{k,c}\right)^T\right\} \\ &+ E\left\{\left(\overline{A}_{k,c} + \overline{H}_{k,c}F_k\overline{E}_{k,c}\right)\Gamma_k\overline{F}_{k,c}^T\right\} \\ &+ E\left\{\overline{F}_{k,c}\Gamma_k^T\left(\overline{A}_k + \overline{H}_kF_k\overline{E}_{k,c}\right)^T\right\} + \overline{F}_{k,c}\Upsilon_k\overline{F}_{k,c}^T \\ &+ \overline{G}_{k,c}S_k\overline{G}_{k,c}^T,\end{aligned} \tag{45}$$

where

$$S_k = \begin{pmatrix} Q_k & 0 & 0 \\ 0 & R_k & 0 \\ 0 & 0 & P_k^{f_2}\end{pmatrix},$$
$$\Gamma_k = \begin{pmatrix} P_k^{xf} & P_k^{xf_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}, \tag{46}$$
$$\Upsilon_k = \begin{pmatrix} P_k^{f} & P_k^{ff_1} & 0 \\ P_k^{f_1f} & P_k^{f_1} & 0 \\ 0 & 0 & P_k^{f_1}\end{pmatrix},$$

where

$$\begin{aligned} P_k^f &= \left[V_{1,k}\ \ V_{2,k}\right]\begin{bmatrix} P_k^{f_1} & P_k^{f_1f_2} \\ P_k^{f_2f_1} & P_k^{f_2}\end{bmatrix}\begin{bmatrix} V_{1,k}^T \\ V_{2,k}^T\end{bmatrix}, \\ P_k^{xf} &= V_{1,k}P_k^{xf_1} + V_{2,k}P_k^{xf_2}, \\ P_k^{ff_1} &= \left(P_k^{f_1f}\right)^T = V_1P_k^{f_1} + V_2\breve{F}_{2,k}^{-1}P_k^{f_2f_1}.\end{aligned} \tag{47}$$

Now observe that the expressions for $\{A_{p,k}, K_{p,k}\}$ are parameterized linearly in terms of the parameter $P_k$. We shall choose $P_k$ by minimizing an upper bound of $\Sigma_{k+1}$ in the absence of uncertainties in $C_k$.

Let $\alpha_k$ be a scalar such that $\alpha_k I - \overline{E}_{c,k}\Sigma_k\overline{E}_{c,k}^T > 0$ and let $\beta_k$ be a scalar such that $\beta_k I - \overline{E}_{c,k}\Upsilon_k\overline{E}_{c,k}^T > 0$. Bearing in mind (45), we can see that $\Sigma_{k+1}$ is bounded by

$$\begin{aligned}\Sigma_{k+1} \geq\ & \overline{A}_{k,c}\Sigma_k\overline{A}_{k,c}^T + \overline{A}_{k,c}\Gamma_k\overline{F}_{k,c}^T + \overline{F}_{k,c}\Gamma_k^T\overline{A}_{k,c}^T \\ &+ \overline{F}_{k,c}\Upsilon_k\overline{F}_{k,c}^T + \overline{G}_{k,c}S_k\overline{G}_{k,c}^T \\ &+ \left(\alpha_k + \beta_k\right)\overline{H}_{k,c}\overline{H}_{k,c}^T.\end{aligned} \tag{48}$$

We choose $P_k > 0$, by solving

$$\min_{P_k>0}\ \ \operatorname{tr}\left(\Sigma_{k+1}\right), \tag{49}$$

$$\begin{aligned}\text{subject to}\quad \Sigma_{k+1} &\geq \overline{A}_{k,c}\Sigma_k\overline{A}_{k,c}^T + \overline{A}_{k,c}\Gamma_k\overline{F}_{k,c}^T + \overline{F}_{k,c}\Gamma_k^T\overline{A}_{k,c}^T \\ &+ \overline{F}_{k,c}\Upsilon_k\overline{F}_{k,c}^T + \overline{G}_{k,c}S_k\overline{G}_{k,c}^T \\ &+ \left(\alpha_k + \beta_k\right)\overline{H}_{k,c}\overline{H}_{k,c}^T.\end{aligned} \tag{50}$$

Or equivalently

$$\begin{bmatrix} -\Sigma_{k+1}+\overline{A}_{k,c}\Gamma_k\overline{F}_{k,c}^T+\overline{F}_{k,c}\Gamma_k^T\overline{A}_{k,c}^T+(\alpha_k+\beta_k)\overline{H}_{k,c}\overline{H}_{k,c}^T & \overline{A}_{k,c}\Sigma_k & \overline{G}_{k,c}S_k^{1/2} & \overline{F}_{k,c}\Upsilon_k \\ \Sigma_k\overline{A}_{k,c}^T & -\Sigma_k & 0 & 0 \\ S_k\overline{G}_{k,c}^{T/2} & 0 & -I & 0 \\ \Upsilon_k\overline{F}_{k,c}^T & 0 & 0 & -\Upsilon_k \end{bmatrix} \le 0. \quad (51)$$

Since inequality (51) is affine in $A_k$, $P_k$ thus found will ensure minimum error covariance $\Sigma_k$ over all possible $A_k$ in the bounded domain. Therefore the desired robust proportional filter is given by (40)–(41) and (31)–(38). So initializing $\Sigma_0 = \text{diag}\{\varepsilon I, P_0\}$ for $P_0 > 0$ and a scalar $\varepsilon > 0$, the resulting filter is listed in the following section.

*4.3. Summary of the RPF.* In this section we summarize the filter equations, we assume that $\hat{x}_0$ is the estimate of the initial state a zero-mean and has known variance $P_0$.

*Initial Condition.* $\hat{x}_0 = 0$, $\Sigma_0 = \text{diag}\{\varepsilon I, P_0\}$, where $P_0 > 0$ and $\varepsilon > 0$.

*Step 1.* If $H_{2,k} = 0$, then set $\hat{\lambda}_k = 0$. Otherwise, construct $G(\lambda)$ of (6) with identification (21) and determine $\hat{\lambda}_k$ by minimizing $G(\lambda)$ over the interval

$$\hat{\lambda}_k > \lambda_{l,k} = \left\| H_{2,k}^T R_k^{-1} H_{2,k} \right\|. \quad (52)$$

*Step 2.* Using $\Sigma_k$, compute $\{P_k, \Sigma_{k+1}\}$ by solving (49) subject to inequality (51), where $\{\overline{A}_{k,c}, \overline{F}_{k,c}, \overline{G}_{k,c}, \overline{H}_{k,c}\}$ and $\{S_k, \Gamma_k, \Upsilon_k\}$ are given, respectively, by (43) and (46).

*Step 3.* Robust simultaneous fault and state estimation are as follows:

$$\begin{aligned} \hat{f}_{1,k/k} &= M_k^{f_1}\left(y_k - \overline{C}_k\hat{x}_{k/k-1}\right), \\ \hat{f}_{2,k-1/k} &= M_k^{f_2}\left(y_k - \overline{C}_k\hat{x}_{k/k-1}\right), \\ \hat{f}_{2,k} &= \breve{F}_{2,k}^x\hat{\hat{f}}_{2,k}, \\ \hat{f}_k &= V_{1,k}\hat{f}_{1,k} + V_{2,k}\hat{\hat{f}}_{2,k}, \end{aligned} \quad (53)$$

where $M_k^{f_1}$, $M_k^{f_2}$, and $\overline{C}_k$ are given by (33)–(37).

Update $\hat{x}_{k/k-1}$ to $\hat{x}_{k+1/k}$ as

$$\begin{aligned} \hat{x}_{k+1/k} = {} & A_{p,k}\hat{x}_{k/k-1} + F_{1,k}^x\hat{f}_{1,k/k} + B_k u_k \\ & + K_{p,k}\left(y_k - C_k\hat{x}_{k/k-1} - F_{k,1}^y\hat{f}_{1,k/k}\right), \end{aligned} \quad (54)$$

where $A_{p,k}$ and $K_{p,k}$ are given by (41).

*Remark 1.* Note that the robust filter (RPF) developed in the previous section gives a better estimation of the state and the fault; however, when the unknown fault is time-varying, the performances of the robust filter can deteriorate. So we will extend the RPF to further propose a new robust proportional integral filter (RPIF) structure, in which the integral action is believed to improve robust estimation of the unknown time-varying faults and to improve robustness against uncertainties.

## 5. Robust Proportional Integral Filter Design

In this section, we propose to design a new robust proportional integral filter (RPIF) for stochastic linear uncertain system (13) and (15) to improve the estimation of unknown time-varying faults and to improve robustness against uncertainties such as disturbances and modeling errors.

The proposed filter has the following structure:

$$\begin{aligned} \hat{x}_{k+1/k} = {} & A_{p,k}\hat{x}_{k/k-1} + F_{1,k}^x\hat{f}_{1,k/k} + B_k u_k \\ & + K_{p,k}\left(y_k - C_k\hat{x}_{k/k-1} - F_{1,k}^y\hat{f}_{1,k/k}\right) \\ & + K_{z,k}z_k, \\ z_{k+1} = {} & \alpha_0 z_k + H_k\left(y_k - C_k\hat{x}_{k/k-1} - F_{1,k}^y\hat{f}_{1,k/k}\right), \\ \hat{f}_{1,k/k} = {} & M_k^{f_1}\left(y_k - \overline{C}_k\hat{x}_{k/k-1}\right), \\ \hat{f}_{2,k-1/k} = {} & M_k^{f_2}\left(y_k - \overline{C}_k\hat{x}_{k/k-1}\right), \\ \hat{f}_{k/k} = {} & V_{1,k}\hat{f}_{1,k/k} + V_{2,k}\hat{\hat{f}}_{2,k/k}, \\ \hat{f}_{2,k} = {} & \breve{F}_{2,k}^x\hat{\hat{f}}_{2,k}, \end{aligned} \quad (55)$$

where the matrices $K_{p,k}$ and $K_{z,k}$ represent a proportional gain and an integral gain, respectively. The variable $z_k$ is related to the weighted integral of the output estimation error. The constant value $\alpha_0$ stands for a fading effect coefficient that regulates the transient response. The matrix $H_k$ is an integral effect coefficient. The two design parameters are assumed to be preselected by designers. Theses expressions for $A_{p,k}$, $K_{p,k}$, $K_{z,k}$, $M_k^{f_1}$, and $M_k^{f_2}$ have been determined by assuming uncertainties in $C_k$ alone. We now move on to select the parameter $P_k$ by assuming uncertainties in $A_k$ alone. By doing so, we will arrive at a filter that minimizes a bound on the state error covariance matrix when there are uncertainties in $A_k$ alone.

Proceeding in the same manner as in the previous section, we know that the expression for $\{A_{p,k}, K_{p,k}, M_k^{f_1}, M_k^{f_2}\}$ can be parameterized linearly in terms of the parameter $P_k$.

Defining the extended weight vector $\eta_k = \begin{pmatrix} \tilde{x}_k \\ z_k \end{pmatrix}$. Ignoring the uncertainties in $C_k$, we find that $\eta_k$ satisfies

$$\eta_{k+1} = \left(\overline{A}_k + \overline{H}_k F_k \overline{E}_{1,k}\right)\eta_k + \left(\overline{F}_k + \overline{H}_k F_k \overline{E}_{2,k}\right)\overline{f}_k + \overline{G}_k \overline{w}_k, \tag{56}$$

where

$$\begin{gathered}
\overline{A}_k = \begin{pmatrix} A_k - K_{p,k}C_k & -K_{z,k} \\ H_k C_k & \alpha I \end{pmatrix}, \\
\overline{F}_k = \begin{bmatrix} A_k - A_{p,k} & F_k^x & -K_{p,k}F_{1,k}^y \\ 0 & 0 & H_k F_{1,k}^y \end{bmatrix}, \\
\overline{G}_k = \begin{bmatrix} I & -K_{p,k} & F_{2,k}^x \\ 0 & H_k & 0 \end{bmatrix}, \\
\overline{H}_k = \begin{pmatrix} H_{1,k} & 0 \\ 0 & 0 \end{pmatrix}, \\
\overline{E}_{1,k} = \begin{pmatrix} E_k & 0 \\ 0 & 0 \end{pmatrix}, \\
\overline{E}_{2,k} = \begin{bmatrix} E_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\overline{f}_k = \begin{pmatrix} \hat{x}_{k/k-1} \\ \tilde{f}_k \\ \tilde{f}_{k,1} \end{pmatrix}, \\
\overline{w}_k = \begin{pmatrix} w_k \\ v_k \\ \hat{f}_{2,k} \end{pmatrix}.
\end{gathered} \tag{57, 58}$$

Suppose that $\overline{\Sigma}_{k+1}$ satisfies

$$\begin{aligned}
\overline{\Sigma}_{k+1} &= E\left\{\left(\overline{A}_k + \overline{H}_k F_k \overline{E}_{1,k}\right)\overline{\Sigma}_k\left(\overline{A}_k + \overline{H}_k F_k \overline{E}_{1,k}\right)^T\right\} \\
&+ E\left\{\left(\overline{A}_k + \overline{H}_k F_k \overline{E}_{1,k}\right)\overline{\Gamma}_k\left(\overline{F}_k + \overline{H}_k F_k \overline{E}_{2,k}\right)^T\right\} \\
&+ E\left\{\left(\overline{F}_k + \overline{H}_k F_k \overline{E}_{2,k}\right)\overline{\Gamma}_k^T\left(\overline{A}_k + \overline{H}_k F_k \overline{E}_{1,k}\right)^T\right\} \\
&+ E\left\{\left(\overline{F}_k + \overline{H}_k F_k \overline{E}_{2,k}\right)\overline{\Upsilon}_k\left(\overline{F}_k + \overline{H}_k F_k \overline{E}_{2,k}\right)^T\right\} \\
&+ \overline{G}_k S_k \overline{G}_k^T,
\end{aligned} \tag{59}$$

where

$$S_k = \begin{pmatrix} Q_k & 0 & 0 \\ 0 & R_k & 0 \\ 0 & 0 & P_k^{f_2} \end{pmatrix}, \tag{60}$$

$$\overline{\Gamma}_k = \begin{pmatrix} 0 & P_k^{xf} & P_k^{xf_1} \\ 0 & P_k^{zf} & P_k^{zf_1} \\ 0 & 0 & 0 \end{pmatrix}, \tag{61}$$

$$\overline{\Upsilon}_k = \begin{pmatrix} P_k^x - P_k & 0 & 0 \\ 0 & P_k^f & P_k^{ff_1} \\ 0 & P_k^{f_1 f} & P_k^{f_1} \end{pmatrix}, \tag{62}$$

$$P_k^{zf} = H_k\left(C_k P_k^{xf} + F_{1,k}^y P_k^{f_1} V_1^T + F_{1,k}^y P_k^{f_1 f_2} \check{F}_{2,k}^{-T} V_2^T\right), \tag{63}$$

$$P_k^{zf_1} = H_k\left(C_k P_k^{xf_1} + F_{1,k}^y P_k^{f_1}\right). \tag{64}$$

Let $\overline{\alpha}_k$ be a scalar such that $\overline{\alpha}_k I - \overline{E}_{1,k}\overline{\Sigma}_k\overline{E}_{1,k}^T > 0$ and let $\overline{\beta}_k$ be a scalar such that $\overline{\beta}_k I - \overline{E}_{2,k}\overline{\Upsilon}_k\overline{E}_{2,k}^T > 0$. Bearing in mind (59), we can see that $\overline{\Sigma}_{k+1}$ is bounded by

$$\begin{aligned}
\overline{\Sigma}_{k+1} \geq{}& \overline{A}_k\overline{\Sigma}_k\overline{A}_k^T + \overline{A}_k\overline{\Gamma}_k\overline{F}_k^T + \overline{F}_k\overline{\Gamma}_k^T\overline{A}_k^T + \overline{F}_k\overline{\Upsilon}_k\overline{F}_k^T \\
&+ \overline{G}_k S_k \overline{G}_k^T + \left(\overline{\alpha}_k + \overline{\beta}_k\right)\overline{H}_k\overline{H}_k^T.
\end{aligned} \tag{65}$$

We choose $P_k > 0$, by solving

$$\min_{P_k > 0} \quad \operatorname{tr}\left(\overline{\Sigma}_{k+1}\right), \tag{66}$$

$$\begin{aligned}
\text{subject to} \quad \overline{\Sigma}_{k+1} &\geq \overline{A}_k\overline{\Sigma}_k\overline{A}_k^T + \overline{A}_k\overline{\Gamma}_k\overline{F}_k^T + \overline{F}_k\overline{\Gamma}_k^T\overline{A}_k^T \\
&+ \overline{F}_k\overline{\Upsilon}_k\overline{F}_k^T + \overline{G}_k S_k \overline{G}_k^T \\
&+ \left(\overline{\alpha}_k + \overline{\beta}_k\right)\overline{H}_k\overline{H}_k^T.
\end{aligned} \tag{67}$$

Or equivalently

$$\begin{bmatrix} -\overline{\Sigma}_{k+1} + \overline{A}_k\overline{\Gamma}_k\overline{F}_k^T + \overline{F}_k\overline{\Gamma}_k^T\overline{A}_k^T + \left(\overline{\alpha}_k + \overline{\beta}_k\right)\overline{H}_k\overline{H}_k^T & \overline{A}_k\Sigma_k & \overline{G}_kS_k^{1/2} & \overline{F}_k\overline{\Upsilon} \\ \overline{\Sigma}_k\overline{A}_k^T & -\overline{\Sigma}_k & 0 & 0 \\ S_k\overline{G}_k^{T/2} & 0 & -I & 0 \\ \overline{\Upsilon}_k\overline{F}_k^T & 0 & 0 & -\overline{\Upsilon}_k \end{bmatrix} \le 0. \tag{68}$$

Therefore the desired robust proportional integral filter (RPIF) is given by (55) where $P_k$ is the positive-definite solution of (66) subject to (67) with the initial condition $\overline{\Sigma}_0 = \mathrm{diag}\{P_0, \varepsilon I\}$ for $P_0 > 0$. Note that there always exists a solution to (66)-(67). The resulting filter is listed in following section.

*Summary of the RPIF.* In this section we summarize the filter equations; we assume that $\hat{x}_0$ is the estimate of the initial state a zero-mean and has known variance $P_0$.

The initialization step of the filter is then given as follows.

*Initial Condition.* $\hat{x}_0 = 0$, $\overline{\Sigma}_0 = \mathrm{diag}\{P_0, \varepsilon I\}$, where $P_0 > 0$ and $\varepsilon > 0$.

*Step 1.* If $H_{2,k} = 0$, then set $\hat{\lambda}_k = 0$. Otherwise, construct $G(\lambda)$ of (6) with the identification (21) and determine $\hat{\lambda}_k$ by minimizing $G(\lambda)$ over the interval

$$\hat{\lambda}_k > \lambda_{l,k} = \left\| H_{2,k}^T R_k^{-1} H_{2,k} \right\|. \tag{69}$$

*Step 2.* Using $\overline{\Sigma}_k$, compute $\{P_k, K_{z,k}, \overline{\Sigma}_{k+1}\}$ by solving (66) subject to inequality (67), where $\{\overline{A}_k, \overline{F}_k, \overline{G}_k, \overline{H}_k\}$ and $\{S_k, \overline{\Gamma}_k, \overline{\Upsilon}_k\}$ are given by (57) and (60)–(62).

*Step 3.* Robust simultaneous fault and state estimation are as follows:

$$\begin{aligned} \hat{f}_{1,k/k} &= M_k^{f_1}\left(y_k - \overline{C}_k\hat{x}_{k/k-1}\right), \\ \hat{f}_{2,k-1/k} &= M_k^{f_2}\left(y_k - \overline{C}_k\hat{x}_{k/k-1}\right), \\ \hat{f}_{2,k} &= \breve{F}_{2,k}^{x}\hat{\hat{f}}_{2,k}, \\ \hat{f}_k &= V_{1,k}\hat{f}_{1,k} + V_{2,k}\hat{\hat{f}}_{2,k}, \end{aligned} \tag{70}$$

where $M_k^{f_1}$, $M_k^{f_2}$, and $\overline{C}_k$ are given by (33)–(35).

Update $\hat{x}_{k/k-1}$ to $\hat{x}_{k+1/k}$ as

$$\begin{aligned} \hat{x}_{k+1/k} = {} & A_{p,k}\hat{x}_{k/k-1} + F_{1,k}^x\hat{f}_{1,k/k} + B_k u_k \\ & + K_{p,k}\left(y_k - C_k\hat{x}_{k/k-1} - F_{k,1}^y\hat{f}_{1,k/k}\right) \\ & + K_{z,k}z_z, \end{aligned} \tag{71}$$

$$z_{k+1} = \alpha_0 z_k + H_k\left(y_k - C_k\hat{x}_{k/k-1} - F_{k,1}^y\hat{f}_{1,k/k}\right), \tag{72}$$

where $A_{p,k}$ and $K_{p,k}$ are given by (41).

## 6. Illustrative Example

Robust estimation of simultaneous actuator and sensor faults is as follows.

In this section, we propose the use of the resulting filters RPF and RPIF to solve the robust estimation of simultaneous actuator and sensor faults problem.

We consider the same numerical example used in (Chen and Patton [5, 6]). The linearized model of a simplified longitudinal flight control system is as follows:

$$\begin{aligned} x_{k+1} &= \left(A_k + \Delta A_k\right)x_k + B_k u_k + F_k^a f_k^a + w_k, \\ y_k &= \left(C_k + \Delta C_k\right)x_k + F_k^s f_k^s + v_k, \end{aligned} \tag{73}$$

where the state variables are pitch angle $\delta_z$, pitch rate $w_z$, and normal velocity $\eta_y$, the control input $u_k$ is the elevator control signal, and $F_k^a$ and $F_k^s$ are the matrices distribution of the actuator fault $f_k^a$ and sensor fault $f_k^s$.

The presented system equations (73) can be rewritten as follows:

$$\begin{aligned} x_{k+1} &= \left(A_k + \Delta A_k\right)x_k + B_k u_k + F_k^x f_k + w_k, \\ y_k &= \left(C_k + \Delta C_k\right)x_k + F_k^y f_k + v_k, \end{aligned} \tag{74}$$

where $F_k^x$ and $F_k^y$ are the matrices injection of the faults vector in the state and the measurement equations.

$$\begin{aligned} F_k^x &= \left[F_k^a \ \ 0\right], \\ F_k^y &= \left[0 \ \ F_k^s\right]. \end{aligned} \tag{75}$$

The system parameter matrices are

$$
\begin{aligned}
A_k &= \begin{bmatrix} 0.9944 & -0.1203 & -0.4302 \\ 0.0017+\delta_{21}(k) & 0.9902+\delta_{22}(k) & -0.0747 \\ 0 & 0.8187 & 0 \end{bmatrix}, \\
B_k &= \begin{bmatrix} 0.4252 \\ -0.0082 \\ 0.1813 \end{bmatrix}, \\
C_k &= I_{3\times 3}, \\
x_k &= \begin{bmatrix} \eta_y & w_z & \delta_z \end{bmatrix}^T, \\
Q_k &= \operatorname{diag}\left\{0.1^2, 0.1^2, 0.01^2\right\}, \\
R_k &= 0.1^2 I_{3\times 3},
\end{aligned}
\tag{76}
$$

where $\delta_{21}(k)$ and $\delta_{22}(k)$ represent parametric uncertainties in the state matrix satisfying

$$
\begin{aligned}
\left|\delta_{21}(k)\right| &\le 0.0008, \\
\left|\delta_{22}(k)\right| &\le 0.05.
\end{aligned}
\tag{77}
$$

We inject simultaneously two faults in the system:

$$
\begin{bmatrix} f_k^a \\ f_k^s \end{bmatrix} = \begin{bmatrix} 4u_s(k-40) - 4u_s(k-70) \\ -4u_s(k-60) + 4u_s(k-80) \end{bmatrix}, \tag{78}
$$

where $u_s(k)$ is the unit-step function. The first fault $f_k^a$ occurs in the actuator and the second fault $f_k^s$ occurs in the sensor for $\delta_z$.

The matrices injection of the fault and the unknown disturbances is taken as follows:

$$
\begin{aligned}
F_k^a &= \begin{bmatrix} 0.4252 \\ -0.0082 \\ 0.1813 \end{bmatrix}, \\
F_k^s &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\end{aligned}
\tag{79}
$$

In the simulation we set $u_k = 10$, $x_0 = 0$, and $P_0 = 0.01^2 I_{3\times 3}$.

In Figure 1, we have plotted the actual and the estimated value of the first element and the second element of the fault vector $f_k = [f_k^a \;\; f_k^s]^T$, respectively, using the RPF and the RPIF. Figure 1 presents the simulation results for the worst case ($\delta_{21}(k) = 0.0008$ and $\delta_{22}(k) = 0.05$). In Figure 2, we have plotted the actual and the estimated value of the state vector $x_k = [\eta_y \;\; w_z \;\; \delta_z]^T$ for the two worst cases ($\delta_{21}(k) = 0.0008$ and $\delta_{22}(k) = 0.05$).

According to the simulation results, it can be seen that both the proposed filters RPF and RPIF give a better estimation of the state and the faults. Mainly, we focus on the simultaneous estimation of the actuator and the sensor faults in spite of norm bounded uncertainties in the state and the observation matrix.

Figures 3 and 4 present, respectively, the actual and the estimated value of the first element of the fault vector $f_k^a$ using the RPF and the RPIF. We can conclude that in the case of time-varying faults the RPIF gives a better estimation results than the RPF. So the integral action is believed to improve robustness against uncertainties and to improve the estimation of time-varying faults.

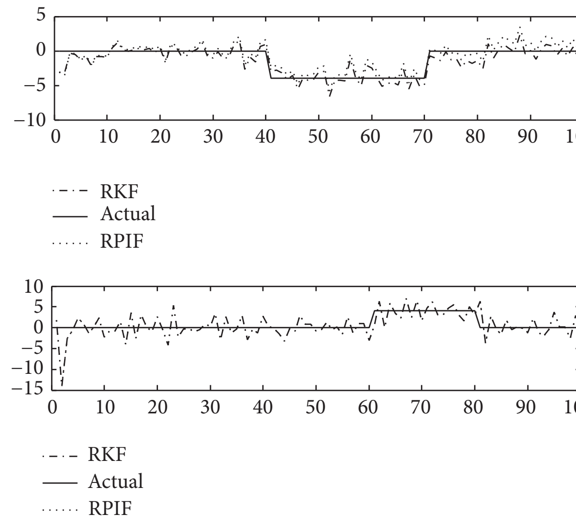


FIGURE 1: Actual fault $f_k$ and estimated fault $\widehat{f}_k$.

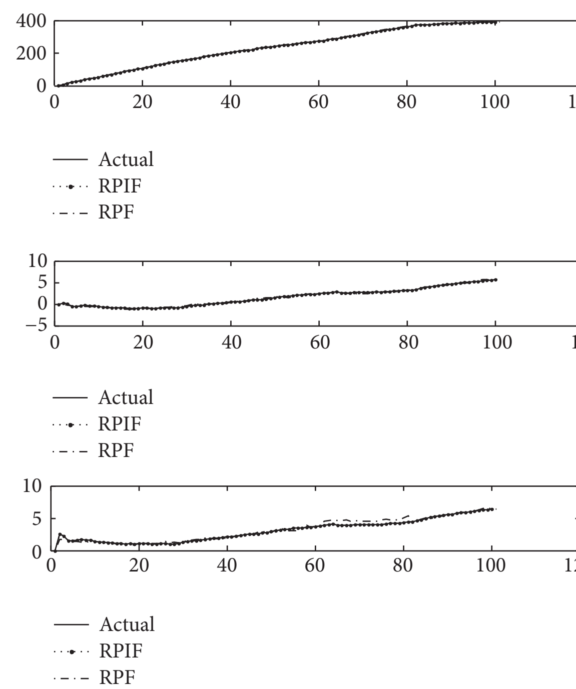


FIGURE 2: Actual state $x_k$ and estimated state $\widehat{x}_k$.

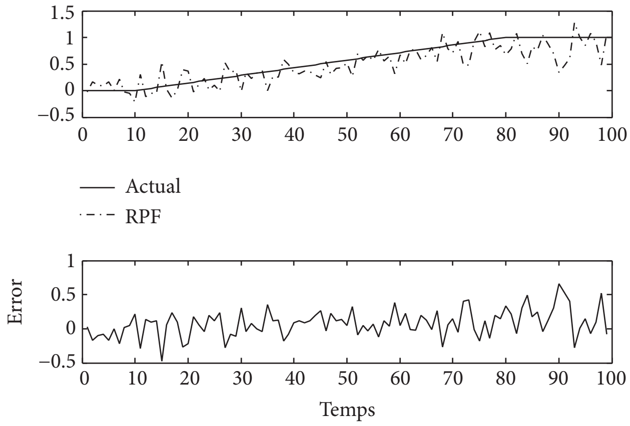


Figure 3: Actual fault $f_k$ and estimated fault $\widehat{f}_k$ (using RPF).

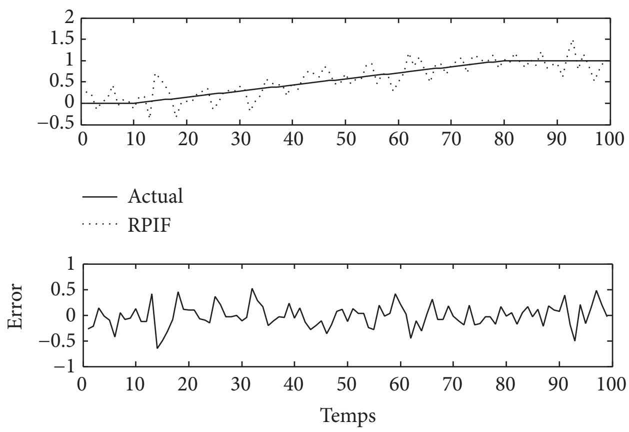


Figure 4: Actual fault $f_k$ and estimated fault $\widehat{f}_k$ (using RPIF).

## 7. Conclusion

In this paper, we have derived a new robust filter for linear time-varying discrete-time systems with unknown faults that affect both the system and the output. By including the unknown fault vector as a part of the augmented system state and using transformation of the original system, a new robust proportional filter (RPF) is developed to solve the robust estimation problem, further, and show how to enforce certain minimum error variance property. We design robust filters that bound the state error covariance matrix for all admissible uncertainties. The robustness criterion used is based on robust square estimation approach. We have extended the developed filter to further propose a new robust proportional integral filter (RPIF) to improve robust estimation of unknown faults and to improve robustness against uncertainties. The design procedure is through the solution of a robust weighted recursive least square problem and it enforces a minimum state error variance property. The advantages of this filter are especially important in the case when the direct feedthrough matrix of the fault has an arbitrary rank and when we do not have any prior information about the fault. An application of the proposed filter has been shown by an illustrative example. The proposed robust filter is able to obtain a robust state and fault estimation in spite of the presence of norm bounded uncertainties. In the next step, we can consider nonlinear systems and the extended Kalman filter can be used to solve the problem of simultaneous fault and state estimation.

## Competing Interests

The authors declare that they have no competing interests.

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