Research Article

On the Sums of Powers of Chebyshev Polynomials and Their Divisible Properties

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The main purpose of this paper is using mathematical induction and the Girard and Waring formula to study a problem involving the sums of powers of the Chebyshev polynomials and prove some divisible properties. We obtained two interesting congruence results involving Fibonacci numbers and Lucas numbers as some applications of our theorem.

1. Discussion and Conclusions

For any integer $n \geq 0$, the first-kind Chebyshev polynomials $\{T_n(x)\}$ and the second-kind Chebyshev polynomials $\{U_n(x)\}$ are defined as $T_0(x) = 1$, $T_1(x) = x$, $U_0(x) = 1$, $U_1(x) = 2x$ and $T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$, $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$ for all $n \geq 0$. If we write $\alpha = x + \sqrt{x^2 - 1}$ and $\beta = x - \sqrt{x^2 - 1}$, then we have

$$\begin{aligned} T_n(x) &= \frac{1}{2} (\alpha^n + \beta^n), \\ U_n(x) &= \frac{1}{\alpha - \beta} (\alpha^{n+1} - \beta^{n+1}) \end{aligned} \quad (1)$$

$$\forall n \geq 0.$$

Since Chebyshev polynomials occupy very important position in the theory and application of mathematics, many scholars have studied their various properties and obtained a series of important results. See [1–11]. For example, Li [1] proved some identities involving power sums of $T_n(x)$ and $U_n(x)$. She obtained some divisibility properties involving Chebyshev polynomials as some applications of these results. Ma and Lv [2] studied the computational problem of the

reciprocal sums of Chebyshev polynomials and obtained some identities. Some theoretical results related to Chebyshev polynomials can be found in Ma and Zhang [3], Cesarano [4], Lee and Wong [5], Bhrawy and others (see [6–9]), and Wang and Zhang [10]. Bircan and Pommerenke [11] also obtained many important applications of the Chebyshev polynomials.

In this paper, we will focus on the problem involving the sums of powers of Chebyshev polynomials. These contents not only are widely used in combinatorial mathematics, but also have important theoretical significance for the study of Chebyshev polynomials themselves. Here we will use mathematical induction and the Girard and Waring formula (see [12, 13]) to prove some interesting divisible properties for Chebyshev polynomials. That is, we will prove the following.

Theorem 1. Let n and h be any positive integers; then we have the congruence

$$\begin{aligned} U_0(x) U_2(x) \cdots U_{2n}(x) \sum_{m=1}^h U_{2m-1}^{2n+1}(x) \\ \equiv 0 \pmod{\left(\frac{T_{2h+1}(x) - x}{x^2 - 1} \right)^2}. \end{aligned} \quad (2)$$

Taking $x = i/2$ and $i^2 = -1$ in $U_n(x)$ and $T_n(x)$, then we have

$$\begin{aligned} U_n\left(\frac{i}{2}\right) &= \frac{i^n}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right] \\ &= i^n \cdot F_{n+1}, \\ T_n\left(\frac{i}{2}\right) &= \frac{i^n}{2} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \right] = \frac{i^n}{2} \cdot L_n, \end{aligned} \quad (3)$$

where F_n and L_n are Fibonacci numbers and Lucas numbers, respectively.

There are many very interesting and important results related to Fibonacci numbers and Lucas numbers; some of them can be found in Yi and Zhang [14], Ozeki [15], Prodinger [16], Melham [17], and Wang and Zhang [18].

We can also deduce interesting congruence properties involving Fibonacci numbers and Lucas numbers as an application of our theorem. That is, we have the following corollaries.

Corollary 2. For any positive integers n and h , we have the congruence

$$\begin{aligned} F_1 F_3 F_5 \cdots F_{2n+1} \sum_{m=1}^{2h} (-1)^m F_{2m}^{2n+1} \\ \equiv 0 \pmod{\left(\frac{L_{4h+1}-1}{5}\right)^2}. \end{aligned} \quad (4)$$

Corollary 3. For any positive integers n and h , we have the congruence

$$\begin{aligned} F_1 F_3 F_5 \cdots F_{2n+1} \sum_{m=1}^{2h-1} (-1)^m F_{2m}^{2n+1} \\ \equiv 0 \pmod{\left(\frac{L_{4h-1}+1}{5}\right)^2}. \end{aligned} \quad (5)$$

2. Several Simple Lemmas

To complete the proof of our main result, we need several simple lemmas. First we have the following.

Lemma 4. For any positive integer h , we have

$$\begin{aligned} (x^2 - 1, T_{2h+1}(x) - x) &= x^2 - 1, \\ (x^2 - 1, \frac{T_{2h+1}(x) - x}{x^2 - 1}) &= 1. \end{aligned} \quad (6)$$

Proof. If $h = 1$, then $T_{2h+1}(x) = T_3(x) = 4x^3 - 3x$ and $T_3(x) - x = 4x(x^2 - 1)$. It is clear that $(x^2 - 1, T_3(x) - x) = x^2 - 1$ and

$$(x^2 - 1, \frac{T_3(x) - x}{x^2 - 1}) = (x^2 - 1, 4x) = 1. \quad (7)$$

So without loss of generality we can assume that $h > 1$. Then from the definition of $T_n(x)$ and binomial theorem we have the identity

$$\begin{aligned} T_{2h+1}(x) &= \frac{1}{2} \sum_{k=0}^{2h+1} \binom{2h+1}{k} x^k (x^2 - 1)^{(2h+1-k)/2} \\ &\quad + \frac{1}{2} \sum_{k=0}^{2h+1} \binom{2h+1}{k} x^k (-1)^{2h+1-k} (x^2 - 1)^{(2h+1-k)/2} \\ &= \sum_{k=0}^h \binom{2h+1}{2k+1} x^{2k+1} (x^2 - 1)^{h-k}. \end{aligned} \quad (8)$$

From (8), we have the polynomial congruence

$$\begin{aligned} T_{2h+1}(x) &= \sum_{k=0}^h \binom{2h+1}{2k+1} x^{2k+1} (x^2 - 1)^{h-k} \equiv x^{2h+1} \\ &\equiv x(x^2 - 1 + 1)^h \equiv x \pmod{(x^2 - 1)} \end{aligned} \quad (9)$$

or

$$T_{2h+1}(x) - x \equiv 0 \pmod{(x^2 - 1)}, \quad (10)$$

which implies that

$$(x^2 - 1, T_{2h+1}(x) - x) = x^2 - 1. \quad (11)$$

On the other hand, from (8) we also have

$$\begin{aligned} T_{2h+1}(x) &= \sum_{k=0}^h \binom{2h+1}{2k+1} x^{2k+1} (x^2 - 1)^{h-k} \\ &\equiv x^{2h+1} + h(2h+1)x^{2h-1}(x^2 - 1) \\ &\equiv x(x^2 - 1 + 1)^h \\ &\quad + h(2h+1)x(x^2 - 1 + 1)^{h-1}(x^2 - 1) \\ &\equiv x + hx(x^2 - 1) \\ &\quad + h(2h+1)x(x^2 - 1) \pmod{(x^2 - 1)^2}, \end{aligned} \quad (12)$$

which implies that

$$(x^2 - 1, \frac{T_{2h+1}(x) - x}{x^2 - 1}) = (x^2 - 1, 2h(h+1)x) = 1. \quad (13)$$

It is clear that Lemma 4 follows from (11) and (13). $\square$

Lemma 5. Let n and h be any positive integers; then we have the polynomial congruence

$$\begin{aligned} 4xT_{(2h+1)(2n+1)}(x) + U_{2n-2}(x) - U_{2n+2}(x) \\ \equiv 0 \pmod{(T_{2h+1}(x) - x)}. \end{aligned} \quad (14)$$

Proof. For any fixed positive integer h , we prove this polynomial congruence by complete induction for positive integer n . Note that $U_0(x) = 1$, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$, $U_4(x) = (4x^2 - 1)^2 - 4x^2 = 16x^4 - 12x^2 + 1$, and $4T_{2h+1}^3(x) = T_{3(2h+1)}(x) + 3T_{2h+1}(x)$, so if $n = 1$, then we have

$$\begin{aligned}
 & 4xT_{(2h+1)(2n+1)}(x) + U_{2n-2}(x) - U_{2n+2}(x) \\
 &= 4xT_{3(2h+1)}(x) + U_0(x) - U_4(x) \\
 &= 4xT_{3(2h+1)}(x) + 1 - 16x^4 + 12x^2 - 1 \\
 &= 4x(4T_{2h+1}^3(x) - 3T_{2h+1}(x)) - 16x^2 + 12x^2 \\
 &\equiv 4x(4x^3 - 3x) - 16x^2 + 12x^2 \\
 &\equiv 0 \pmod{(T_{2h+1}(x) - x)}.
 \end{aligned} \tag{15}$$

That is to say, Lemma 5 is true for $n = 1$.

If $n = 2$, then note that the identities

$$\begin{aligned}
 U_6(x) - U_2(x) &= 64x^4 - 80x^2 + 20x^2, \\
 4T_{2h+1}^3(x) &= T_{3(2h+1)}(x) + 3T_{2h+1}(x), \\
 16T_{2h+1}^5(x) &= T_{3(2h+1)}(x) + 5T_{3(2h+1)}(x) \\
 &\quad + 10T_{2h+1}(x);
 \end{aligned} \tag{16}$$

we have the polynomial congruence

$$\begin{aligned}
 & 4xT_{(2h+1)(2n+1)}(x) + U_{2n-2}(x) - U_{2n+2}(x) \\
 &= 4xT_{5(2h+1)}(x) + U_2(x) - U_6(x) \\
 &= 4x(16T_{2h+1}^5(x) - 5T_{3(2h+1)}(x) - 10T_{2h+1}(x)) \\
 &\quad - (64x^4 - 80x^2 + 20x^2) \equiv 4x[16x^5 \\
 &\quad - 5(4T_{2h+1}^3(x) - 3T_{2h+1}(x)) - 10T_{2h+1}(x)] \\
 &\quad - (64x^4 - 80x^2 + 20x^2) = 4x[16x^5 \\
 &\quad - 5(4x^3 - 3x) - 10x] - (64x^4 - 80x^2 + 20x^2) \\
 &\equiv 0 \pmod{(T_{2h+1}(x) - x)}.
 \end{aligned} \tag{17}$$

So Lemma 5 is true for $n = 2$.

Suppose Lemma 5 is true for all integers $n = 1, 2, \dots, k$. That is,

$$\begin{aligned}
 & 4xT_{(2h+1)(2n+1)}(x) + U_{2n-2}(x) - U_{2n+2}(x) \\
 &\equiv 0 \pmod{(T_{2h+1}(x) - x)}
 \end{aligned} \tag{18}$$

for all $1 \leq n \leq k$.

Then, for $n = k + 1 \geq 3$, note that the identities

$$\begin{aligned}
 & 2T_{2(2h+1)}(x)T_{(2h+1)(2k+1)}(x) \\
 &= T_{(2h+1)(2k+3)}(x) + T_{(2h+1)(2k-1)}(x), \\
 & U_{2k+4}(x) = (4x^2 - 1)U_{2k+2}(x) - 2xU_{2k+1}(x) \\
 &= (4x^2 - 1)U_{2k+2}(x) - U_{2k+2}(x) - U_{2k}(x), \\
 & T_{2(2h+1)}(x) = 2T_{2h+1}^2(x) - 1 \\
 &\equiv 2x^2 - 1 \pmod{(T_{2h+1}(x) - x)};
 \end{aligned} \tag{19}$$

from inductive assumption (18), we have

$$\begin{aligned}
 & 4xT_{(2h+1)(2n+1)}(x) + U_{2n-2}(x) - U_{2n+2}(x) \\
 &= 4xT_{(2h+1)(2k+3)}(x) + U_{2k}(x) - U_{2k+4}(x) \\
 &= 8xT_{2(2h+1)}(x)T_{(2h+1)(2k+1)}(x) - 4xT_{(2h+1)(2k-1)}(x) \\
 &\quad + U_{2k}(x) - U_{2k+4}(x) \\
 &\equiv 8x(2x^2 - 1)T_{(2h+1)(2k+1)}(x) - 4xT_{(2h+1)(2k-1)}(x) \\
 &\quad + (4x^2 - 1)U_{2k-2}(x) - U_{2k-2}(x) - U_{2k-4}(x) \\
 &\quad - (4x^2 - 1)U_{2k+2}(x) + U_{2k+2}(x) + U_{2k}(x) \\
 &= (4x^2 - 1)(4xT_{(2h+1)(2k+1)}(x) + U_{2k-2}(x) - U_{2k+2}(x)) \\
 &\quad - (4xT_{(2h+1)(2k+1)}(x) + U_{2k-2}(x) - U_{2k+2}(x)) \\
 &\quad - (4xT_{(2h+1)(2k-1)}(x) + U_{2k-4}(x) - U_{2k}(x)) \\
 &\equiv 0 \pmod{(T_{2h+1}(x) - x)}.
 \end{aligned} \tag{20}$$

Now Lemma 5 follows from complete induction. $\square$

Lemma 6. For any integers $h \geq 1$ and $n \geq 0$, we have the congruence

$$\begin{aligned}
 & U_0(x)U_2(x) \cdots U_{2n}(x) \\
 &\cdot \sum_{m=1}^h [U_{2m(2n+1)-1}(x) - (2n+1)U_{2m-1}(x)] \\
 &\equiv 0 \pmod{\left(\frac{T_{2h+1}(x) - x}{x^2 - 1}\right)^2}.
 \end{aligned} \tag{21}$$

Proof. For integer $n \geq 0$, note that the identities $\alpha\beta = 1$, $T_n(x) = (1/2)[\alpha^n + \beta^n]$,

$$\begin{aligned}
 & \sum_{m=1}^h U_{2m(2n+1)-1}(x) = \frac{1}{2\sqrt{x^2 - 1}} \\
 &\cdot \sum_{m=1}^h [\alpha^{2m(2n+1)} - \beta^{2m(2n+1)}] \\
 &= \frac{1}{2\sqrt{x^2 - 1}} \left[\frac{\alpha^{2(2n+1)}(\alpha^{2h(2n+1)} - 1)}{\alpha^{2(2n+1)} - 1} \right]
 \end{aligned}$$

$$\begin{aligned}
& - \frac{\beta^{2(2n+1)} (\beta^{2h(2n+1)} - 1)}{\beta^{2(2n+1)} - 1} \Big] \\
& = \frac{1}{2\sqrt{x^2 - 1}} \left[\frac{\alpha^{(2h+1)(2n+1)} - \alpha^{2n+1}}{\alpha^{2n+1} - \beta^{2n+1}} \right. \\
& \quad \left. + \frac{\beta^{(2h+1)(2n+1)} - \beta^{2n+1}}{\alpha^{2n+1} - \beta^{2n+1}} \right] \\
& = \frac{1}{2(x^2 - 1)U_{2n}(x)} [T_{(2h+1)(2n+1)}(x) - T_{2n+1}(x)], \\
& \sum_{m=1}^h U_{2m-1}(x) = \frac{1}{2(x^2 - 1)} [T_{(2h+1)}(x) - x].
\end{aligned} \tag{22}$$

It is clear that from (22) we know that, to prove Lemma 6, we need only to prove the polynomial congruence

$$\begin{aligned}
& T_{(2h+1)(2n+1)}(x) - T_{2n+1}(x) \\
& \quad - (2n+1)U_{2n}(x)(T_{2h+1}(x) - x) \\
& \equiv 0 \pmod{(T_{2h+1}(x) - x)^2}.
\end{aligned} \tag{23}$$

Now we use complete induction to prove (23). It is clear that polynomial congruence (23) is true for $n = 0$. If $n = 1$, then note that $U_2(x) = 4x^2 - 1$, $T_3(x) = 4x^3 - 3x$, and $T_{3(2h+1)}(x) = 4T_{2h+1}^3(x) - 3T_{2h+1}(x)$; we have

$$\begin{aligned}
& T_{(2h+1)(2n+1)}(x) - T_{2n+1}(x) \\
& \quad - (2n+1)U_{2n}(x)(T_{2h+1}(x) - x) = T_{3(2h+1)}(x) \\
& \quad - T_3(x) - 3(4x^2 - 1)(T_{2h+1}(x) - x) \\
& = 4T_{2h+1}^3(x) - 3T_{2h+1}(x) - (4x^3 - 3x) \\
& \quad - 3(4x^2 - 1)(T_{2h+1}(x) - x) \\
& = 4(T_{2h+1}(x) - x)^2(T_{2h+1}(x) + 2x) \\
& \equiv 0 \pmod{(T_{2h+1}(x) - x)^2}.
\end{aligned} \tag{24}$$

So congruence (23) is true for $n = 1$. Suppose (23) is true for all integers $n = 0, 1, 2, \dots, k$. Namely,

$$\begin{aligned}
& T_{(2h+1)(2n+1)}(x) - T_{2n+1}(x) \\
& \quad - (2n+1)U_{2n}(x)(T_{2h+1}(x) - x) \\
& \equiv 0 \pmod{(T_{2h+1}(x) - x)^2}
\end{aligned} \tag{25}$$

for all $n = 0, 1, \dots, k$.

Then, for $n = k + 1 \geq 2$, note that the identities

$$\begin{aligned}
& 2T_{2(2h+1)}(x)T_{(2h+1)(2k+1)}(x) \\
& \quad = T_{(2h+1)(2k+3)}(x) + T_{(2h+1)(2k-1)}(x), \\
& T_{2(2h+1)}(x) = 2T_{2h+1}^2(x) - 1 \\
& \quad = 2(T_{2h+1}(x) - x + x)^2 - 1 \\
& \quad \equiv 4xT_{2h+1}(x) - 2x^2 - 1 \pmod{(T_{2h+1}(x) - x)^2};
\end{aligned} \tag{26}$$

from inductive assumption (25) and Lemma 5, we have

$$\begin{aligned}
& T_{(2h+1)(2n+1)}(x) - T_{2n+1}(x) - (2n+1)U_{2n}(x) \\
& \quad \cdot (T_{2h+1}(x) - x) = T_{(2h+1)(2k+3)}(x) - T_{2k+3}(x) \\
& \quad - (2k+3)U_{2k+2}(x)(T_{2h+1}(x) - x) = 2T_{2(2h+1)}(x) \\
& \quad \cdot T_{(2h+1)(2k+1)}(x) - T_{(2h+1)(2k-1)}(x) - T_{2k+3}(x) \\
& \quad - (2k+3)U_{2k+2}(x)(T_{2h+1}(x) - x) \equiv (8xT_{2h+1}(x) \\
& \quad - 4x^2 - 2)T_{(2h+1)(2k+1)}(x) - T_{(2h+1)(2k-1)}(x) \\
& \quad - (4x^2 - 1)T_{2k+1}(x) + T_{2k+1}(x) + T_{2k-1}(x) \\
& \quad - 2U_{2k+2}(x)(T_{2h+1}(x) - x) - (2k+1) \\
& \quad \cdot ((4x^2 - 1)U_{2k}(x) - U_{2k}(x) - U_{2k-2}(x)) \\
& \quad \cdot (T_{2h+1}(x) - x) = 2(T_{2h+1}(x) - x) \\
& \quad \cdot (4xT_{(2h+1)(2k+1)}(x) + U_{2k-2}(x) - U_{2k+2}(x)) \\
& \quad + (4x^2 - 2)[T_{(2h+1)(2k+1)}(x) - T_{2k+1}(x) \\
& \quad - (2k+1)U_{2k}(x)(T_{2h+1}(x) - x)] \\
& \quad - [T_{(2h+1)(2k-1)}(x) - T_{2k-1}(x) \\
& \quad - (2k-1)U_{2k-2}(x)(T_{2h+1}(x) - x)] \\
& \equiv 0 \pmod{(T_{2h+1}(x) - x)^2}.
\end{aligned} \tag{27}$$

This proves Lemma 6 by complete induction. $\square$

Lemma 7. For all nonnegative integers u and real numbers X and Y , we have the identity

$$\begin{aligned}
& X^u + Y^u \\
& \quad = \sum_{k=0}^{\lfloor u/2 \rfloor} (-1)^k \frac{u}{u-k} \binom{u-k}{k} (X+Y)^{u-2k} (XY)^k,
\end{aligned} \tag{28}$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

Proof. This formula is given by Girard and Waring, which can be found in [12] or [13]. $\square$

3. Proof of the Theorem

In this section, we will prove our theorem by mathematical induction. Taking $X = \alpha^{2m}$, $Y = -\beta^{2m}$, and $u = 2n + 1$ in Lemma 7, note that $XY = -1$; from the definition of $U_n(x)$ and binomial theorem, we have

$$\begin{aligned} U_{2m(2n+1)-1}(x) &= \frac{1}{2\sqrt{x^2-1}} (\alpha^{2n(2n+1)} - \beta^{2m(2n+1)}) \\ &= \sum_{k=0}^n (-1)^k \frac{2n+1}{2n+1-k} \\ &\quad \cdot \binom{2n+1-k}{k} 4^{n-k} (x^2-1)^{n-k} U_{2m-1}^{2n+1-2k}(x) (-1)^k \quad (29) \\ &= \sum_{k=0}^n \frac{2n+1}{2n+1-k} \binom{2n+1-k}{k} 4^{n-k} (x^2-1)^{n-k} \\ &\quad \cdot U_{2m-1}^{2n+1-2k}(x). \end{aligned}$$

For any integer $h \geq 1$, from (29) we have the identity

$$\begin{aligned} \sum_{m=1}^h [U_{2m(2n+1)-1}(x) - (2n+1)U_{2m-1}(x)] \\ = \sum_{k=0}^{n-1} \frac{2n+1}{2n+1-k} \binom{2n+1-k}{k} 4^{n-k} (x^2-1)^{n-k} \quad (30) \\ \cdot \sum_{m=1}^h U_{2m-1}^{2n+1-2k}(x). \end{aligned}$$

Taking $n = 1$ in (30), we have the identity

$$\begin{aligned} \sum_{m=1}^h [U_{6m-1}(x) - 3U_{2m-1}(x)] \\ = 4(x^2-1) \sum_{m=1}^h U_{2m-1}^3(x). \quad (31) \end{aligned}$$

From Lemma 4, we know that $(x^2-1, (T_{2h+1}(x) - x)/(x^2-1)) = 1$, so applying Lemma 6 and (31) we can deduce that

$$\begin{aligned} U_0(x)U_2(x) \sum_{m=1}^h U_{2m-1}^3(x) \\ \equiv 0 \pmod{\left(\frac{T_{2h+1}(x) - x}{x^2-1}\right)^2}. \quad (32) \end{aligned}$$

That is to say, the theorem is true for $n = 1$.

Suppose that the theorem is true for all integers $n = 1, 2, \dots, s$. That is,

$$\begin{aligned} U_0(x)U_2(x) \cdots U_{2n}(x) \sum_{m=1}^h U_{2m-1}^{2n+1}(x) \\ \equiv 0 \pmod{\left(\frac{T_{2h+1}(x) - x}{x^2-1}\right)^2} \quad (33) \end{aligned}$$

for all integers $1 \leq n \leq s$.

Then, for $n = s + 1$, from (29) we have

$$\begin{aligned} \sum_{m=1}^h (U_{2m(2s+3)-1}(x) - (2s+3)U_{2m-1}(x)) \\ = \sum_{k=0}^s \frac{2s+3}{2s+3-k} \binom{2s+3-k}{k} 4^{s+1-k} (x^2-1)^{s+1-k} \\ \cdot \sum_{m=1}^h U_{2m-1}^{2s+3-2k}(x) = \sum_{k=1}^s \frac{2s+3}{2s+3-k} \\ \cdot \binom{2s+3-k}{k} 4^{s+1-k} (x^2-1)^{s+1-k} \\ \cdot \sum_{m=1}^h U_{2m-1}^{2s+3-2k}(x) + 4^{s+1} (x^2-1)^{s+1} \sum_{m=1}^h U_{2m-1}^{2s+3}(x). \quad (34) \end{aligned}$$

From Lemma 6, we have

$$\begin{aligned} U_0(x)U_2(x) \cdots U_{2s+2}(x) \\ \cdot \sum_{m=1}^h [U_{2m(2s+3)-1}(x) - (2s+3)U_{2m-1}(x)] \\ \equiv 0 \pmod{\left(\frac{T_{2h+1}(x) - x}{x^2-1}\right)^2}. \quad (35) \end{aligned}$$

Applying inductive assumption (33), we have

$$\begin{aligned} U_0(x)U_2(x) \cdots U_{2s}(x) \sum_{k=1}^s \frac{2s+3}{2s+3-k} \binom{2s+3-k}{k} \\ \times 4^{s+1-k} (x^2-1)^{s+1-k} \sum_{m=1}^h U_{2m-1}^{2s+3-2k}(x) \\ \equiv 0 \pmod{\left(\frac{T_{2h+1}(x) - x}{x^2-1}\right)^2}. \quad (36) \end{aligned}$$

Combining (34), (35), (36), and Lemma 6, we can deduce the polynomial congruence

$$\begin{aligned} U_0(x)U_2(x) \cdots U_{2s+2}(x) 4^{s+1} (x^2-1)^{s+1} \sum_{m=1}^h U_{2m-1}^{2s+3}(x) \\ \equiv 0 \pmod{\left(\frac{T_{2h+1}(x) - x}{x^2-1}\right)^2}. \quad (37) \end{aligned}$$

From Lemma 4, we know that $(x^2-1, (T_{2h+1}(x) - x)/(x^2-1)) = 1$, so from (37) we may immediately deduce the polynomial congruence

$$\begin{aligned} U_0(x)U_2(x) \cdots U_{2s+2}(x) \sum_{m=1}^h U_{2m-1}^{2s+3}(x) \\ \equiv 0 \pmod{\left(\frac{T_{2h+1}(x) - x}{x^2-1}\right)^2}. \quad (38) \end{aligned}$$

This completes the proof of our theorem by mathematical induction.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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