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Research Article

Analysis of a Constant Retrial Queue with Joining Strategy and Impatient Retrial Customers

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This paper treats an M/M/1 retrial queue with constant retrial times. If the server is busy at the arrival epoch, the arriving customer decides to join the retrial orbit with probability q or balk with probability $1 - q$. Only the customer at the head of the orbit is permitted to access the server. Upon retrial, the customer immediately receives his service if the server is idle; otherwise, he may enter the orbit again or leave the system because of impatience. First, we give the performance analysis for this retrial queue and give some important performance indices. Second, based on a natural reward-cost structure, we analyze the Nash equilibrium customers' joining strategies and give some numerical examples.

1. Introduction

Retrial queueing systems have been extensively used to stochastically model many problems arising in computer networks, telecommunication, telephone systems, and daily life. About comprehensive surveys on retrial queues, readers are referred to the book of Falin and Templeton [1], the book of Artalejo and Gómez-Corral [2], the references listed in Artalejo [3, 4], and the papers of [5, 6] and references therein. In the majority of the above-mentioned, the interest of the investigators lies in the transient and/or the stationary distribution of the process of interest.

Recently, Economou and Kanta [7] considered a single-server M/M/1 retrial queue with constant retrial policy from an economic analysis viewpoint, in which customers are permitted to freely make decisions to maximize their own benefit based on some reward-cost structure. Such system can be analyzed by using the game theoretic analysis, and the fundamental consideration is to find the Nash equilibria; see Hassin and Haviv [8]. The pioneering work on the economic analysis of queueing systems can date back to Naor [9]. Naor [9] considered an M/M/1 queue with a simple linear reward-cost structure, where each customer observes the queue length before his decision. Later, Edelson and Hildebrand [10] investigated Naor's model by assuming that there is no information on the queue length for an arriving customer. Moreover, several researchers have studied the same problem for various queueing systems considering diverse characteristics, for example, retrials, breakdown and repairs, working vacations, priorities, reneging and jockeying, and schedules. The fundamental results in this area with extensive bibliographical references can be found in the comprehensive monograph by Hassin and Haviv [8]. Further studies can be referred to [9–17] and references therein.

In the present paper, we consider the Markovian single-server constant retrial queue with joining strategy and impatient retrial customers, which extends the model in Economou and Kanta [7] by considering the impatient phenomenon of the retrial customers. The customers observe the server state upon arrival and not the number of customers in the orbit and then decide whether to join the orbit or balk. We consider some performance indices, such as the stationary distribution and retrial numbers, and the (Nash) equilibrium balking strategies.

The rest of this paper is organized as follows. In Section 2, we give the model description and derive the stationary

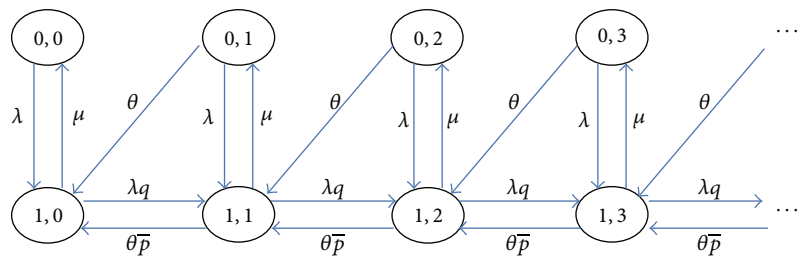

Figure 1: Transition rate diagram of $X(t) = \{(J(t), N(t)),\ t \geq 0\}$.

distribution and some performance measures and the distribution of the retrial numbers. In Section 3, we study the Nash equilibrium joining probability in unobservable case. Section 4 gives some numerical examples. Section 5 gives concluding remarks and possible related future research.

2. Model Formulation and Performance Analysis

We study here a retrial queue with constant retrial times, where customers arrive according to a Poisson process at rate λ. Upon arrival, the arriving customer will immediately receive his service and leave the system after his service completion if he finds the server idle. Otherwise, if he finds the server busy, the arriving customer may enter the retrial orbit with some probability q $(0 \leq q \leq 1)$ according to FCFS discipline or may leave the system with complementary probability $\overline{q} = 1 - q$ (called balking customer). The service times are exponentially distributed with rate μ. Only the customer at the head of the retrial orbit is allowed to retry to access the server. Upon retrial, the customer begins the service if the server is idle; otherwise, he may rejoin the orbit with probability $0 \leq p \leq 1$ or leave the system (called impatient customer) with probability $\overline{p} = 1 - p$. The retrial times are assumed to follow exponential distribution with parameter θ. We assume that all random variables are mutually independent.

By assumption of the retrial queue, the state of the system under consideration can be described by a continuous Markov process $X(t) = \{(J(t), N(t)),\ t \geq 0\}$ with state space $\mathbb{S} = \{(j,k),\ j = 0,1;\ k \geq 0\}$, where $J(t)$ is the state of the server at time t. $J(t) = 0$ or 1 according to the server is idle or busy; $N(t)$ denotes the number of customers in the orbit at time t. Transition rate diagram of the Markov chain $\{(J(t), N(t)),\ t \geq 0\}$ is depicted in Figure 1.

From Figure 1, we see that $X(t) = \{(J(t), N(t)),\ t \geq 0\}$ is a continuous time quasi-birth-death process, and the infinitesimal generator $\mathbb{Q}$ of $X(t)$ is given by

$$\mathbb{Q} = \begin{pmatrix} \mathbf{A}_0 & \mathbf{C} & & & \\ \mathbf{A} & \mathbf{B} & \mathbf{C} & & \\ & \mathbf{A} & \mathbf{B} & \mathbf{C} & \\ & & \mathbf{A} & \mathbf{B} & \mathbf{C} \\ & & & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1}$$

where

$$\begin{aligned} \mathbf{A}_0 &= \begin{pmatrix} -\lambda & \lambda \\ \mu & -(\lambda q + \mu) \end{pmatrix}, \\ \mathbf{C} &= \begin{pmatrix} 0 & 0 \\ 0 & \lambda q \end{pmatrix}, \\ \mathbf{A} &= \begin{pmatrix} 0 & \theta \\ 0 & \theta\overline{p} \end{pmatrix}, \\ \mathbf{B} &= \begin{pmatrix} -(\lambda + \theta) & \lambda \\ \mu & -(\lambda q + \mu + \theta\overline{p}) \end{pmatrix}. \end{aligned} \tag{2}$$

Let $\mathbf{D} = \mathbf{A} + \mathbf{B} + \mathbf{C}$; we obtain that

$$\mathbf{D} = \begin{pmatrix} -(\lambda + \theta) & \lambda + \theta \\ \mu & -\mu \end{pmatrix}, \tag{3}$$

and then $\mathbf{D}$ is obviously a generator matrix, and its associated stationary probability vector $\boldsymbol{\pi} = (\pi_0, \pi_1)$ can be derived as $(\pi_0, \pi_1) = (\mu/(\lambda + \mu + \theta), (\lambda + \theta)/(\lambda + \mu + \theta))$. Now, from the Theorem 3.1.1 in Neuts [18] which states that the necessary and sufficient condition for stability of QBD process is $\boldsymbol{\pi}\mathbf{C}\mathbf{e} < \boldsymbol{\pi}\mathbf{A}\mathbf{e}$, where $\mathbf{e}$ is a 2×1 column vector of 1s, we can get that

$$\rho = \frac{\lambda q (\lambda + \theta)}{\theta \mu_p} < 1 \tag{4}$$

is the stationary condition for our retrial system, where $\mu_p = \mu + (\lambda + \theta)\overline{p}$. Hereinafter, we assume that the inequality $\rho < 1$ holds.

2.1. Stationary Distribution. Under the condition $\rho < 1$, we aim to find the stationary distribution $P_{j,k} = \lim_{t\to\infty} P(J(t) = j,\ N(t) = k)$, $j = 0,1; k \geq 0$. By our assumption of the model, we can get the steady-state equations as follows:

$$\begin{aligned} \lambda P_{0,0} &= \mu P_{1,0}, \\ (\lambda + \theta) P_{0,k} &= \mu P_{1,k}, \quad k \geq 1 \\ (\lambda q + \mu) P_{1,0} &= \lambda P_{0,0} + \theta\overline{p} P_{1,1} + \theta P_{0,1}, \\ (\lambda q + \mu + \theta\overline{p}) P_{1,k} &= \lambda P_{0,k} + \theta\overline{p} P_{1,k+1} + \theta P_{0,k+1} \\ &\quad + \lambda q P_{1,k-1}, \quad k \geq 1. \end{aligned} \tag{5}$$

From (5), we can obtain that

$$P_{0,k} = \frac{\mu}{\lambda + \theta\left(1 - \delta_{0,k}\right)}\rho^k P_{1,0}, \quad k \geq 0,$$
$$P_{1,k} = \rho^k P_{1,0}, \quad k \geq 0, \tag{6}$$

where $\delta_{0,k} = 1$ if $k = 0$ or $\delta_{0,k} = 0$ if $k \neq 0$.

Using the normalizing condition yields

$$P_{1,0} = \frac{\lambda\mu_p}{(\lambda + \mu)\mu_p - \lambda q\mu}(1 - \rho). \tag{7}$$

Based on the above results, we discuss some performance measures in the following subsection.

2.2. Performance Measures

2.2.1. Queue Length and the Server State Probabilities. (1) Let P_0 and P_1 be the probabilities that the server is idle and busy, respectively; then

$$P_0 = \sum_{k=0}^{\infty} P_{0,k} = \left(\frac{\mu}{\lambda} + \frac{\mu}{\lambda + \theta}\frac{\rho}{1 - \rho}\right)P_{1,0}$$
$$= \frac{\mu\left(\mu_p - \lambda q\right)}{(\lambda + \mu)\mu_p - \lambda q\mu}, \tag{8}$$
$$P_1 = \sum_{k=0}^{\infty} P_{1,k} = \frac{1}{1 - \rho}P_{1,0} = \frac{\lambda\mu_p}{(\lambda + \mu)\mu_p - \lambda q\mu}.$$

(2) Let N_O and N_s be the mean numbers of customers in the retrial orbit and in the system, respectively; then

$$N_O = \sum_{k=0}^{\infty} k\left(P_{0,k} + P_{1,k}\right) = \frac{\lambda + \theta + \mu}{\lambda + \theta}\frac{\rho}{(1 - \rho)^2}P_{1,0}, \tag{9}$$

$$N_S = N_O + P_1. \tag{10}$$

2.2.2. Waiting Time in the Orbit. In this subsection, we focus on the analysis of the mean waiting time in the orbit of a tagged customer, denoted by W_O. Let $\tau_{j,k}$ be the waiting time in the orbit of the tagged customer in the k-th position given that the server is currently at state j, and $T(j,k) = E(\tau_{j,k})$. Then we have the following theorem.

Theorem 1. *W_O and $T(j,k)$ satisfy the following equations:*

$$W_O = q\frac{\lambda + \theta + \mu}{\theta\mu_p}\frac{1}{(1 - \rho)^2}P_{1,0}, \tag{11}$$

$$T(1,k) = \frac{\lambda + \theta + \mu}{\theta\mu_p}k, \quad k = 1, 2, 3, \ldots, \tag{12}$$

$$T(0,k) = \frac{\lambda + \theta + \mu}{\theta\mu_p}k - \frac{p}{\mu_p}, \quad k = 1, 2, 3, \ldots. \tag{13}$$

Proof. For $T(1,k)$, we remark that the arriving customer who finds the server busy after the tagged customer has no effect on $T(1,k)$, but the service time and the impatience of the customer at the head of the orbit do affect the value of $T(1,k)$. However, $T(0,k)$ depends on arrival rate, because the new arrival who finds the server idle immediately receives his service. We obtain that

$$T(1,1) = \frac{1}{\mu + \theta} + \frac{\mu}{\mu + \theta}T(0,1) + \frac{\theta p}{\mu + \theta}T(1,1), \tag{14}$$

$$T(1,k) = \frac{1}{\mu + \theta} + \frac{\mu}{\mu + \theta}T(0,k) + \frac{\theta p}{\mu + \theta}T(1,k)$$
$$+ \frac{\theta\overline{p}}{\mu + \theta}T(1,k-1), \quad k = 2, 3, \ldots, \tag{15}$$

$$T(0,1) = \frac{1}{\lambda + \theta} + \frac{\lambda}{\lambda + \theta}T(1,1), \tag{16}$$

$$T(0,k) = \frac{1}{\lambda + \theta} + \frac{\lambda}{\lambda + \theta}T(1,k) + \frac{\theta}{\lambda + \theta}T(1,k-1),$$
$$k = 2, 3, \ldots. \tag{17}$$

It follows from (14) and (16) that (12) and (13) hold for $k = 1$.

Equation (15) indicates that

$$T(1,k) = \frac{1}{\mu + \theta\overline{p}} + \frac{\mu}{\mu + \theta\overline{p}}T(0,k)$$
$$+ \frac{\theta\overline{p}}{\mu + \theta\overline{p}}T(1,k-1), \quad k = 2, 3, \ldots. \tag{18}$$

Inserting (17) into (18) leads to

$$T(1,k) - T(1,k-1) = \frac{\lambda + \theta + \mu}{\theta\mu_p}, \quad k = 2, 3, \ldots. \tag{19}$$

Then we obtain (12) that holds for $k \geq 2$. Substituting (12) into (17), we obtain that

$$T(0,k) = \frac{\lambda + \theta + \mu}{\theta\mu_p}k - \frac{p}{\mu_p}, \quad k = 1, 2, 3, \ldots. \tag{20}$$

Then by PASTA property, we have

$$W_O = 0 \cdot \left(P_0 + \overline{q}P_1\right) + q\sum_{k=0}^{\infty} P_{1,k}T(1,k+1)$$
$$= q\frac{\lambda + \theta + \mu}{\theta\mu_p}P_{1,0}\sum_{k=0}^{\infty}\rho^k(k+1) \tag{21}$$
$$= q\frac{\lambda + \theta + \mu}{\theta\mu_p}\frac{1}{(1 - \rho)^2}P_{1,0}.$$

This ends the proof of Theorem 1. □

Using (4) and comparing (11) with (9), we find that

$$W_O = \frac{\lambda + \theta + \mu}{\lambda + \theta}\frac{\rho}{(1 - \rho)^2}P_{1,0} \times \frac{q(\lambda + \theta)}{\theta\mu_p\rho} = \frac{N_O}{\lambda}, \tag{22}$$

which shows that Little's law holds for our retrial queue with impatient customers.

Remark 2. To interpret (14)–(17), we take (15) as an example. Let B be the length of the service time and Θ be length of the retrial time. For $T(1,k), k \geq 2$, by comparing the length of the service time of the customer being served and the retrial time of the customer at the head of the orbit and using memoryless property of exponential distribution, we have

$$\begin{aligned} T(1,k) &= P(B \leq \Theta) E\left[B + \tau_{0,k} \mid B \leq \Theta\right] \\ &\quad + pP(B > \Theta) E\left[\Theta + \tau_{1,k} \mid B > \Theta\right] \\ &\quad + \overline{p}P(B > \Theta) E\left[\Theta + \tau_{1,k-1} \mid B > \Theta\right] \\ &= P(B \leq \Theta) E[B \mid B \leq \Theta] \\ &\quad + P(B > \Theta) E[\Theta \mid B > \Theta] \\ &\quad + P(B \leq \Theta) E\left[\tau_{0,k}\right] + pP(B > \Theta) E\left[\tau_{1,k}\right] \\ &\quad + \overline{p}P(B > \Theta) E\left[\tau_{1,k-1}\right] \qquad (23) \\ &= E[\min\{B, \Theta\}] + P(B \leq \Theta) T(0,k) \\ &\quad + pP(B > \Theta) T(1,k) \\ &\quad + \overline{p}P(B > \Theta) T(1,k-1) \\ &= \frac{1}{\mu+\theta} + \frac{\mu}{\mu+\theta} T(0,k) + \frac{\theta p}{\mu+\theta} T(1,k) \\ &\quad + \frac{\theta \overline{p}}{\mu+\theta} T(1,k-1). \end{aligned}$$

2.2.3. Number of Retrials. Define R as the number of repeated attempts made by a tagged customer before he departs the system, either with his service completion or without getting his service because of his impatience. Obviously, when, upon arrival, the tagged customer finds the server idle or the server busy but the customer chooses to leave the system, then $R = 0$. So we mainly focus on the case that the server is busy when the tagged customer arrives and enters the orbit. In this case R is obviously independent of the customer's position in the orbit. Without loss of generality, suppose that the tagged customer is in the 1st position in the orbit. We define the following:

(i) $R_s(j,k)$: the conditional probability that the tagged customer makes a total of k retrials, where the last one finds the server idle, and the rest $k-1$ unsuccessful retrials are made and the customer enters the orbit, given that the server is at state j currently.

(ii) $R_v(j,k)$: the conditional probability that the tagged customer makes a total of k retrials, where the last one finds the server busy and chooses to leave the system, and the rest $k-1$ vain retrials are made and the customer enters the orbit, given that the server is at state j currently.

(iii) P_s: the probability that the tagged customer in the orbit successfully accepts his service.

(iv) P_v: the probability that the tagged customer in the orbit leaves the system without getting his service because of his impatience.

Then $P_s = \sum_{k=1}^{\infty} R_s(1,k)$ and $P_v = \sum_{k=1}^{\infty} R_v(1,k)$. To find the expressions of $E(R)$, P_s, and P_v, we need to give the expressions of $R_s(j,k)$ and $R_v(j,k)$, $k \geq 1$. Following the similar analysis as Section 2.2.2, we have

$$\begin{aligned} R_s(1,1) &= \frac{\mu}{\theta+\mu} R_s(0,1), \\ R_s(1,k) &= \frac{\mu}{\theta+\mu} R_s(0,k) + \frac{\theta p}{\theta+\mu} R_s(1,k-1), \\ &\qquad k = 2,3,\ldots, \\ R_s(0,1) &= \frac{\lambda}{\lambda+\theta} R_s(1,1) + \frac{\theta}{\lambda+\theta}, \\ R_s(0,k) &= \frac{\lambda}{\lambda+\theta} R_s(1,k), \quad k = 2,3,\ldots, \qquad (24) \\ R_v(1,1) &= \frac{\mu}{\theta+\mu} R_v(0,1) + \frac{\theta \overline{p}}{\theta+\mu}, \\ R_v(1,k) &= \frac{\mu}{\theta+\mu} R_v(0,k) + \frac{\theta p}{\theta+\mu} R_v(1,k-1), \\ &\qquad k = 2,3,\ldots, \\ R_v(0,k) &= \frac{\lambda}{\lambda+\theta} R_v(1,k), \quad k = 1,2,3,\ldots. \end{aligned}$$

Solving (24) yields

$$\begin{aligned} R_s(1,k) &= \left(\frac{(\lambda+\theta)p}{\lambda+\theta+\mu}\right)^{k-1} \frac{\mu}{\lambda+\theta+\mu}, \\ &\qquad k = 1,2,3,\ldots, \\ R_v(1,k) &= \left(\frac{(\lambda+\theta)p}{\lambda+\theta+\mu}\right)^{k-1} \frac{(\lambda+\theta)\overline{p}}{\lambda+\theta+\mu}, \\ &\qquad k = 1,2,3,\ldots, \end{aligned} \qquad (25)$$

and so, we obtain the distribution of R as follows:

$$\begin{aligned} P(R=0) &= P_0 + \overline{q}P_1, \\ P(R=k) &= 0 \cdot \left(P_0 + \overline{q}P_1\right) + qP_1\left(R_s(1,k) + R_v(1,k)\right) \\ &= qP_1 \frac{\mu_p}{\lambda+\theta+\mu} \left(\frac{(\lambda+\theta)p}{\lambda+\theta+\mu}\right)^{k-1}, \qquad (26) \\ &\qquad k = 1,2,3,\ldots, \end{aligned}$$

which leads to

$$E(R) = \sum_{k=1}^{\infty} kP(R=k) = qP_1 \frac{\lambda+\theta+\mu}{\mu_p} = \frac{\lambda q(\lambda+\theta+\mu)}{(\lambda+\mu)\mu_p - \lambda q\mu}. \tag{27}$$

From (25), P_s and P_v can be expressed as

$$P_s = \sum_{k=1}^{\infty} R_s(1,k) = \frac{\mu}{\mu_p}, \qquad P_v = \sum_{k=1}^{\infty} R_v(1,k) = \frac{(\lambda+\theta)\overline{p}}{\mu_p}. \tag{28}$$

Remark 3. (1) Define W_S as the mean sojourn time in the system of a tagged customer and then by the PASTA property, (10) and (12), we have

$$W_S = \frac{1}{\mu}P_0 + 0\cdot\overline{q}P_1 + q\sum_{k=0}^{\infty} P_{1,k}\left(T(1,k+1) + \frac{1}{\mu}P_s\right) = W_O + \frac{1}{\mu}(P_0 + qP_1P_s) = W_O + \frac{P_1}{\lambda} = \frac{N_O + P_1}{\lambda}, \tag{29}$$

which indicates that Little's law holds.

(2) Let $E[W_S \mid J=1]$ be the conditional sojourn time in the system of a tagged customer given that he finds the server busy upon his arrival and decided to enter the retrial orbit. Then

$$E[W_S \mid J=1] = \frac{E[W_S,\ J=1]}{P(J=1)} = \frac{\sum_{k=0}^{\infty} P_{1,k}(T(1,k+1) + (1/\mu)P_s)}{P(J=1)} = \frac{\lambda+\theta+\mu}{\theta\mu_p - \lambda q(\lambda+\theta)} + \frac{1}{\mu_p}. \tag{30}$$

3. Analysis of Nash Equilibrium Joining Strategy-Unobservable Case

In this section, we aim to analyze the Nash equilibrium joining strategy under a given reward-cost structure, which is given in the following, because customers have the right to decide whether to enter the orbit or not depending on the server's state, not on the number of customers in the orbit, which is called unobservable case.

The reward-cost structure considered in this paper is as follows:

(i) $\mathscr{R}_s$: each customer receiving a reward of $\mathscr{R}_s$ units for completing service.

(ii) $\mathscr{R}_v$: each customer receiving a reward of $\mathscr{R}_v$ units in case that he is forced to leave the system due to a retrial failure.

(iii) C: a waiting cost of C units per time unit where a customer remains in the system.

Assume that all customers follow a given joining strategy q and $\lambda(\lambda+\theta)/\theta\mu_p < 1$, which implies that the retrial queueing system in this paper is also stable because of $\rho < 1$ holding for the joining strategy q. Customers are assumed to be risk neutral and maximize their expected net benefit and

$$\frac{\mathscr{R}_s}{C} > \frac{1}{\mu} \tag{31}$$

which ensures that the customer who finds the server idle will join the system because of the positive benefit $\mathscr{R}_s - C/\mu$.

First we introduce some notations about Nash equilibrium (see [8, 11] for details).

Assume that all customers are indistinguishable; we can consider the situation as a symmetric game among them. Denote the common set of strategies and the payoff function of a symmetric game by S and F, respectively. More concretely, let $F(s_{\text{tagged}}, s_{\text{others}})$ be the payoff of a tagged customer who adopts strategy s_{tagged} when all other customers select s_{others}. Strategy s_e is a (symmetric) Nash equilibrium, if and only if $F(s_e, s_e) \geq F(s, s_e)$, $\forall s \in S$. The intuitive interpretation of a Nash equilibrium is that an equilibrium strategy is the best response against itself, so that if all customers agree to follow it, no one can benefit by deviating from it. Strategy s_1 is said to dominate strategy s_2 if $F(s_1, s) \geq F(s_2, s)$, for every $s \in S$ and for at least one s the inequality is strict. Strategy s_* is said to be dominant if it dominates all other strategies in S. We should remark that the notion of a dominant strategy is stronger than the notion of a Nash equilibrium.

For a tagged customer who finds the server busy and decides to join the orbit, under the given reward-cost structure, he will receive a reward $\mathscr{R}_s$ with probability P_s and a reward $\mathscr{R}_s$ with probability P_v, and he will be charged total cost $CE[W_S \mid J=1]$. Then from (28) and (30), his net benefit is given by

$$S_{un}(1,q) = P_s\mathscr{R}_s + P_v\mathscr{R}_v - CE[W_S \mid J=1] = \frac{\mu\mathscr{R}_s + (\lambda+\theta)\overline{p}\mathscr{R}_v}{\mu_p} - C\left(\frac{\lambda+\theta+\mu}{\theta\mu_p - \lambda q(\lambda+\theta)} + \frac{1}{\mu_p}\right). \tag{32}$$

Then the expected net benefit of a tagged customer that enters the orbit with probability q' given that the system is found busy, when everyone else follows a strategy q, is given by

$$S_{un}(q', q) = q' S_{un}(1, q). \tag{33}$$

In the following theorem, we aim to present the Nash equilibrium joining strategy.

Theorem 4. *In the unobservable retrial queue with constant retrial times and impatient, assume that $\lambda(\lambda+\theta)/\theta\mu_p < 1$ and $\mathscr{R}_s/C > 1/\mu$ hold; a unique mixed Nash equilibrium joining strategy exists: enter the retrial orbit with probability q_e whenever finding the server busy. q_e is given by*

$$q_e = \begin{cases} 0, & \dfrac{1}{\mu} + \dfrac{(\lambda+\theta)\,\overline{p}}{\mu}\dfrac{\mathscr{R}_v}{C} < \dfrac{\mathscr{R}_s}{C} + \dfrac{(\lambda+\theta)\,\overline{p}}{\mu}\dfrac{\mathscr{R}_v}{C} \le \dfrac{\lambda+\theta+\mu}{\theta\mu} + \dfrac{1}{\mu}, \\ q_e^*, & \dfrac{\lambda+\theta+\mu}{\theta\mu} + \dfrac{1}{\mu} < \dfrac{\mathscr{R}_s}{C} + \dfrac{(\lambda+\theta)\,\overline{p}}{\mu}\dfrac{\mathscr{R}_v}{C} < \dfrac{\mu_p(\lambda+\theta+\mu)}{\mu\left(\theta\mu_p - \lambda(\lambda+\theta)\right)} + \dfrac{1}{\mu}, \\ 1, & \dfrac{\mathscr{R}_s}{C} + \dfrac{(\lambda+\theta)\,\overline{p}}{\mu}\dfrac{\mathscr{R}_v}{C} \ge \dfrac{\mu_p(\lambda+\theta+\mu)}{\mu\left(\theta\mu_p - \lambda(\lambda+\theta)\right)} + \dfrac{1}{\mu}, \end{cases} \tag{34}$$

where

$$q_e^* = \frac{\theta\mu_p}{\lambda(\lambda+\theta)} - \left(\frac{\mu\mathscr{R}_s + (\lambda+\theta)\,\overline{p}\mathscr{R}_v}{\mu_p C} - \frac{1}{\mu_p}\right)^{-1}\frac{\lambda+\theta+\mu}{\lambda(\lambda+\theta)}. \tag{35}$$

Proof. From (32), we can see that the function $S_{un}(1,q)$ is strictly decreasing on the interval $(0,1)$. Then the unique maximum of $S_{un}(1,q)$ is

$$S_{un}(1,0) = \frac{\mu\mathscr{R}_s + (\lambda+\theta)\,\overline{p}\mathscr{R}_v}{\mu_p} - C\left(\frac{\lambda+\theta+\mu}{\theta\mu_p} + \frac{1}{\mu_p}\right), \tag{36}$$

and the unique minimum is

$$S_{un}(1,1) = \frac{\mu\mathscr{R}_s + (\lambda+\theta)\,\overline{p}\mathscr{R}_v}{\mu_p} - C\left(\frac{\lambda+\theta+\mu}{\theta\mu_p - \lambda(\lambda+\theta)} + \frac{1}{\mu_p}\right). \tag{37}$$

Therefore, we consider the following three cases.

(i) If $1/\mu + ((\lambda+\theta)\overline{p}/\mu)(\mathscr{R}_v/C) < \mathscr{R}_s/C + ((\lambda+\theta)\overline{p}/\mu)(\mathscr{R}_v/C) \le (\lambda+\theta+\mu)/\theta\mu + 1/\mu$, that is, $S_{un}(1,0) \le 0$, which leads to $S_{un}(1,q) \le 0$ for any $q \in [0,1]$, the best response of the tagged customer who finds the server busy is balking, and the unique equilibrium joining probability is $q_e = 0$.

(ii) If $(\lambda+\theta+\mu)/\theta\mu + 1/\mu < \mathscr{R}_s/C + ((\lambda+\theta)\overline{p}/\mu)(\mathscr{R}_v/C) < \mu_p(\lambda+\theta+\mu)/\mu(\theta\mu_p - \lambda(\lambda+\theta)) + 1/\mu$, then $S_{un}(1,q) = 0$ has a unique solution in $(0,1)$, which is given by q_e^*, which leads to $S_{un}(1,q) > 0$ for $q < q_e^*$ and $S_{un}(1,q_e^*) = 0$ and $S_{un}(1,q) < 0$ for $q > q_e^*$. So the best response is q_e^*. As a result, the joining strategy with probability q_e^* is the unique equilibrium strategy.

(iii) If $\mathscr{R}_s/C + ((\lambda+\theta)\overline{p}/\mu)(\mathscr{R}_v/C) \ge \mu_p(\lambda+\theta+\mu)/\mu(\theta\mu_p - \lambda(\lambda+\theta)) + 1/\mu$, that is, $S_{un}(1,1) \ge 0$, then the net benefit given by (32) is always positive for $\mathscr{R}_s/C + ((\lambda+\theta)\overline{p}/\mu)(\mathscr{R}_v/C) > \mu_p(\lambda+\theta+\mu)/\mu(\theta\mu_p - \lambda(\lambda+\theta)) + 1/\mu$; that is, the best response for the tagged customer who finds the server busy is joining the retrial orbit; that is, $q_e = 1$.

□

4. Numerical Illustration

To illustrate the effect of some parameters on the equilibrium joining probability q_e, we give some numerical examples.

In Figures 2 and 3, we present the influence of $\mathscr{R}_v$ and $\mathscr{R}_s$ on q_e for model with $(\lambda,\theta,\mu,p,\mathscr{R}_s,C) = (2,3,4,0.5,2,3)$ and $(\lambda,\theta,\mu,p,\mathscr{R}_v,C) = (2.5,3,3.5,0.5,2.25,2.5)$, respectively. Figures 2 and 3 show that the equilibrium joining probability q_e is increasing as rewards $\mathscr{R}_v$ and $\mathscr{R}_s$ increase. The reason is that the higher the reward that the customers receive, the greater the willingness that customers take to enter the orbit.

In Figures 4 and 5, we present the curves of q_e versus μ and θ for models with $(\lambda,\theta,p,\mathscr{R}_s,\mathscr{R}_v,C) = (2.5,3.5,0.5,4,2.25,2.5)$ and $(\lambda,\mu,p,\mathscr{R}_s,\mathscr{R}_v,C) = (2,0.25,0.25,3,1,2.5)$, respectively. Customers can get more profit as the service rates μ and θ increase, because the mean sojourn time decreases with μ and θ increasing and then customers prefer to enter the orbit, so q_e is increasing as μ and θ increase.

In Figures 6 and 7, we examine the effect of λ and p on q_e for models with $(\theta,\mu,p,\mathscr{R}_s,\mathscr{R}_v,C) = (2,2,0.5,4,2.25,2.5)$ and $(\lambda,\mu,\mathscr{R}_s,\mathscr{R}_v,C) = (0.5,2,3,3,0.75,2.5)$, respectively. Figures 6 and 7 indicate that q_e increases with the values of λ and p increasing. The reason is that the larger the values of λ and p are, the longer the customers sojourn in the system, and then the less profit the customers can get, which leads to customers being less willing to enter the orbit.

5. Conclusions

This paper has first presented the performance analysis of an M/M/1 retrial queue with impatient customers. Second, adopting a linear reward-cost structure, we have provided customer Nash equilibrium strategies from an economic viewpoint. However, we considered two different types of reward: one is the reward received by the customer who leaves the system after his service completion; the other is the reward received by the customer who is forced to leave the system due to a retrial failure. As an extension of this study, one can generalize the queue to the case that the server may take Bernoulli vacation. For this generalized model, one could study the observable and unobservable cases.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

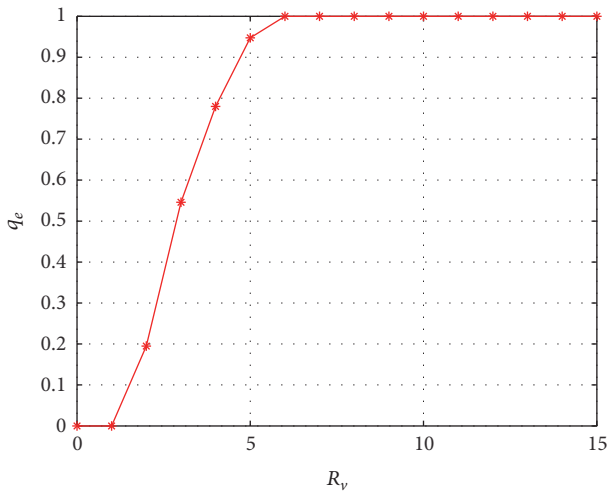

FIGURE 2: q_e versus R_v for $\lambda = 2$, $\theta = 3$, $\mu = 4$, $p = 0.5$, $R_s = 2$, and $C = 3$.

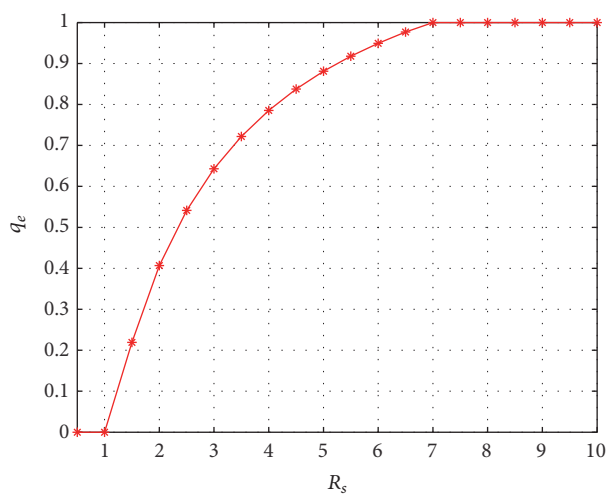

FIGURE 3: q_e versus R_s for $\lambda = 2.5$, $\theta = 3.5$, $\mu = 3.5$, $p = 0.5$, $R_v = 2.25$, and $C = 2.5$.

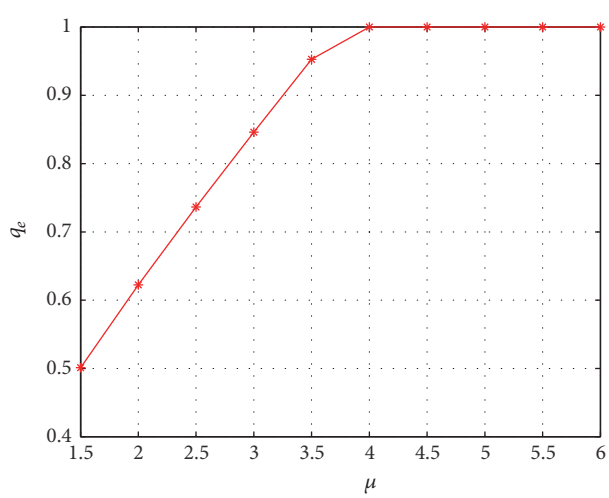

FIGURE 4: q_e versus μ for $\lambda = 2.5$, $\theta = 3.5$, $p = 0.5$, $R_s = 4$, $R_v = 2.25$, and $C = 2.5$.

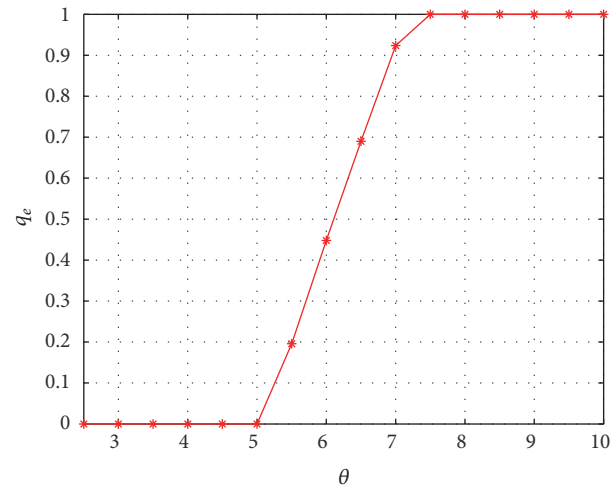

FIGURE 5: q_e versus θ for $\lambda = 2$, $\mu = 0.25$, $p = 0.25$, $R_s = 3$, $R_v = 1$, and $C = 2.5$.

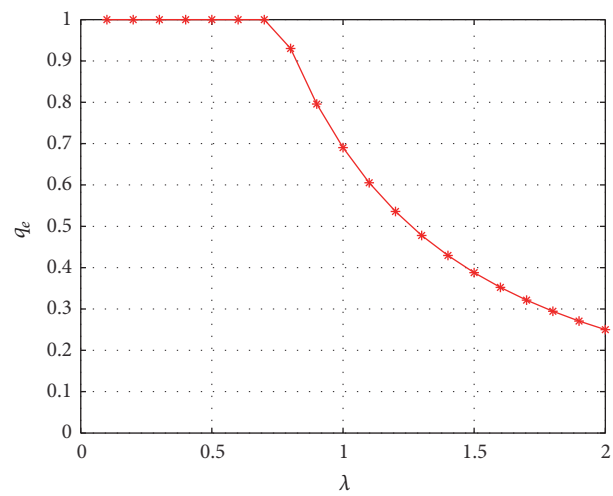

FIGURE 6: q_e versus λ for $\theta = 2$, $\mu = 2$, $p = 0.5$, $R_s = 4$, $R_v = 2.25$, and $C = 2.5$.

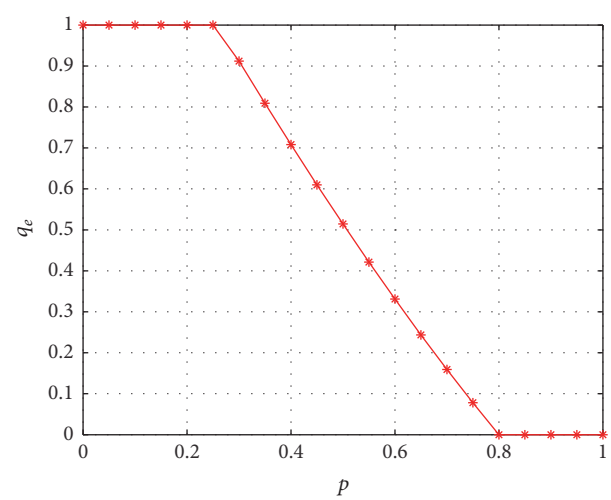

FIGURE 7: q_e versus p for $\lambda = 0.5$, $\theta = 2$, $\mu = 3$, $R_s = 3$, $R_v = 0.75$, and $C = 2.5$.

Acknowledgments

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