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Research Article

A Simultaneous Iteration Algorithm for Solving Extended Split Equality Fixed Point Problem

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We study a kind of split equality fixed point problem which is an extension of split equality problem. We propose a kind of simultaneous iterative algorithm with a way of selecting the step length which does not need any a priori information about the operator norms and prove that the sequences generated by the iterative method converge weakly to the solution of this problem. Some numerical results are shown to confirm the feasibility and efficiency of the proposed methods.

1. Introduction

Let C and Q be nonempty closed and convex subsets of the real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem is to find

$$x \in C \quad \text{such that } Ax \in Q. \tag{1}$$

It can be used in various disciplines such as image restoration and radiation therapy treatment planning [1, 2]. These applications are in finite-dimensional Hilbert spaces [3, 4]. It also can be found in an infinite-dimensional real Hilbert space [5, 6].

Recently, Moudafi [7] introduced a new split equality feasibility problem. Let H_1, H_2, and H_3 be real Hilbert spaces. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. The split equality feasibility problem is to find

$$\begin{aligned} &x \in C, \\ &y \in Q \\ &\qquad \text{such that } Ax = By, \end{aligned} \tag{2}$$

which allows asymmetric and partial relations between the variables x and y. The interest is to cover many situations, for instance, applications in decomposition methods for PDEs, in game theory and in intensity-modulated radiation therapy (for short, IMRT).

Moudafi [8] introduced the simultaneous iterative method to solve the split equality feasibility problem. Furthermore, Moudafi studied the fixed point formulation to avoid using the projection. Assume $\mathrm{Fix}(T)$ and $\mathrm{Fix}(S)$ are the sets of fixed points of T and S, respectively, where $T : H_1 \to H_1$ and $S : H_2 \to H_2$ are nonlinear operators such that $\mathrm{Fix}(T) \neq \emptyset$ and $\mathrm{Fix}(S) \neq \emptyset$. So the split equality fixed point problem is to find

$$\begin{aligned} &x \in \mathrm{Fix}(T), \\ &y \in \mathrm{Fix}(S) \\ &\qquad \text{such that } Ax = By, \end{aligned} \tag{3}$$

where T and S are firmly quasi-nonexpansive mappings. In order to find the solution of the split equality problem, Che and Li [9] proposed the following iterative algorithm:

$$\begin{aligned} u_n &= x_n - \gamma_n A^* (Ax_n - By_n), \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) T u_n, \\ v_n &= y_n + \gamma_n B^* (Ax_n - By_n), \\ y_{n+1} &= \alpha_n y_n + (1 - \alpha_n) S v_n, \end{aligned} \tag{4}$$

and the weak convergence of the scheme (4) can also be established. Furthermore, Chang et al. [10] modified the

iterative scheme (4) and provided a unified framework for solving this problem without using the projection. The framework is as follows.

$$\begin{aligned} u_n &= x_n - \gamma_n A^* (Ax_n - By_n), \\ x_{n+1} &= \alpha_n x_n \\ &\quad + (1-\alpha_n)[(1-\xi) I + \xi T((1-\eta) I + \eta T)] u_n, \\ v_n &= y_n + \gamma_n B^* (Ax_n - By_n), \\ y_{n+1} &= \alpha_n y_n \\ &\quad + (1-\alpha_n)[(1-\xi) I + \xi S((1-\eta) I + \eta S)] v_n. \end{aligned} \tag{5}$$

They got the following conclusion that the sequence (x_n, y_n) generated by the above modification converges weakly to a solution of problem (3). Furthermore, some authors [11–14] studied the problems (1)–(3) in Banach space. They proposed effective algorithms and proved their convergence under some conditions.

Recently, He and Sun [15] studied the problem of split convex feasibility and established a strongly convergent alternating algorithm. They proposed the following iterative algorithm to find

$$\begin{aligned} &x \in C, \\ &y \in Q \\ &\qquad \text{such that } Ax = By \in K, \end{aligned} \tag{6}$$

via the formula

$$\begin{aligned} \omega_n &= P_{K_n}\left(\frac{Ax_n + By_n}{2}\right), \\ K_n &= \left\{z \in H_3 : h\left(\frac{Ax_n + By_n}{2}\right)\right. \\ &\quad \left. + \left\langle \varsigma_n, z - \left(\frac{Ax_n + By_n}{2}\right)\right\rangle \le 0\right\}, \\ u_n &= P_C(x_n - \xi A^*(Ax_n - \omega_n)), \\ v_n &= P_Q(y_n - \xi B^*(Bx_n - \omega_n)), \\ C_{n+1} \times Q_{n+1} &= \left\{(x, y) \in C_n \times Q_n : \|u_n - x\|^2\right. \\ &\quad + \|v_n - y\|^2 \le \|x_n - x\|^2 + \|y_n - y\|^2, \ f(x_n) \\ &\quad \left. + \langle \mu_n, x - x_n\rangle \le 0, \ g(y_n) + \langle \eta_n, y - y_n\rangle \le 0\right\}, \\ x_{n+1} &= P_{C_{n+1}}(x_1), \\ y_{n+1} &= P_{Q_{n+1}}(y_1), \quad \forall n \in N, \end{aligned} \tag{7}$$

where $\xi \in (0, \min(1/\|A\|^2, 1/\|B\|^2))$, $\mu_n \in \partial f(x_n)$, $\eta_n \in \partial g(y_n)$, and $\varsigma_n \in \partial h((Ax_n + By_n)/2)$. In some conditions such as $\Omega = \{(x, y) \in C \times Q : Ax = By \in K\} \neq \emptyset$, they proved that the sequences $(x_n, y_n) \to (s, t)$, $(u_n, v_n) \to (s, t)$, and $\omega_n \to z^* := As = Bt \in K$, where $(s, t) \in \Omega$.

In this article, we study the following problem of extended split equality fixed point, which is to find

$$\begin{aligned} &x \in \mathrm{Fix}(G), \\ &y \in \mathrm{Fix}(M) \\ &\qquad \text{such that } Ax = By \in K. \end{aligned} \tag{8}$$

When $K = H_3$, problem (8) is problem (3). Therefore, problem (8) is the extension of the split equality fixed point problem. We propose the simultaneous iterative algorithm for solving this problem, which avoids using the projection and the step length sequences do not depend on the operator norms $\|A\|$ and $\|B\|$. Furthermore, we prove the sequences generated by the algorithm weakly converge to a solution of the extended split equality fixed point problem. Numerical examples show the feasibility and efficiency of this algorithm.

2. Preliminaries

In this paper, we recall some concepts, definitions, and conclusions, which are prepared for proving our main results. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. We denote by H_1 a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$.

A mapping $T : C \to C$ is called

(i) quasi-nonexpansive, if $\mathrm{Fix}(T) \neq \emptyset$,

$$\|Tx - x^*\| \le \|x - x^*\|, \quad \forall x \in C, \ x^* \in \mathrm{Fix}(T), \tag{9}$$

(ii) quasi-pseudo-contractive, if $\mathrm{Fix}(T) \neq \emptyset$,

$$\begin{aligned} &\|Tx - x^*\| \le \|x - x^*\| + \|Tx - x\|, \\ &\qquad \forall x \in C, \ x^* \in \mathrm{Fix}(T). \end{aligned} \tag{10}$$

A mapping P_C is said to be metric projection of H_1 onto C if, for every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{11}$$

It is well known that P_C is a nonexpansive mapping and is characterized by the following properties:

$$\begin{aligned} &\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y\rangle, \quad \forall x, y \in H_1, \\ &\langle x - P_C x, y - P_C x\rangle \le 0, \quad \forall x \in H_1, \ y \in C, \\ &\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \\ &\qquad \forall x \in H_1, \ y \in C, \\ &\|(x - y) - (P_C x - P_C y)\|^2 \\ &\quad \ge \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H_1. \end{aligned} \tag{12}$$

In the proof of our results, we need the following lemmas.

Lemma 1 (see [16]). *Let H be a real Hilbert space; then the following conclusions hold.*

$$\begin{aligned} &\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \quad \forall x, y \in H, \\ &\|\alpha x + (1-\alpha) y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|, \quad \forall x, y \in H,\ \alpha \in [0,1]. \end{aligned} \tag{13}$$

Lemma 2 (see [16]). *Let H be a real Hilbert space and $T : H \to H$ be a L-Lipschitzian mapping with $L \geq 1$. Denote*

$$G := (1-\xi) I + \xi T((1-\eta) I + \eta T). \tag{14}$$

If $0 < \xi < \eta < 1/(1+\sqrt{1+L^2})$, then the following conclusions hold.

(i) $\mathrm{Fix}(T) = \mathrm{Fix}(T((1-\eta)I + \eta T)) = \mathrm{Fix}(G)$.

(ii) *If T is demiclosed at 0, then G is also demiclosed at 0.*

(iii) *In addition, if $T : H \to \mathrm{H}$ is quasi-pseudo-contractive, then the mapping G is quasi-nonexpansive; that is,*

$$\|Gx - u^*\| \leq \|x - u^*\|, \quad \forall x \in H,\ u^* \in \mathrm{Fix}(T) = \mathrm{Fix}(G). \tag{15}$$

3. Main Results

In this section, we assume that

(i) $G : H_1 \to H_1$ and $M : H_2 \to H_2$ are two L-Lipschitzian and quasi-pseudo-contractive mappings with $L \geq 1$, $\mathrm{Fix}(G) \neq \emptyset$, and $\mathrm{Fix}(M) \neq \emptyset$.

(ii) H_1, H_2, and H_3 be real Hilbert spaces. $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators; A^* and B^* are their adjoint operators, respectively. Let $K \subset H_3$ be nonempty closed convex set. We consider the extended split equality fixed point problem (8).

Theorem 3. *Let H_1, H_2, and H_3 be real Hilbert spaces and K be a closed convex level set*

$$K = \{x \in H_3 : h(x) \leq 0\}, \tag{16}$$

where $h : H_3 \to R$ is convex function which is subdifferentiable on K and its subdifferentials are bounded on bounded sets. $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators with their adjoint operators A^* and B^*, respectively, $\sigma \in (0,1)$ is a parameter controlling step length, $0 < \gamma < \eta < 1/(1+\sqrt{1+L^2})$, and $\{\alpha_n\} \subset (0,1)$. Let $\{x_n\}$, $\{y_n\}$, $\{\omega_n\}$, $\{u_n\}$, and $\{v_n\}$ be sequences generated by

$$\begin{aligned} \omega_n &= P_{K_n}\left(\frac{Ax_n + By_n}{2}\right), \\ K_n &= \left\{z \in H_3 : h\left(\frac{Ax_n + By_n}{2}\right) + \left\langle \varsigma_n, z - \frac{Ax_n + By_n}{2}\right\rangle \leq 0\right\}, \\ u_n &= x_n - \xi_n A^*(Ax_n - \omega_n), \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n) T u_n, \\ v_n &= y_n - \xi_n B^*(By_n - \omega_n), \\ y_{n+1} &= \alpha_n y_n + (1-\alpha_n) S v_n, \end{aligned} \tag{17}$$

where

$$\begin{aligned} &\varsigma_n \in \partial h\left(\frac{Ax_n + By_n}{2}\right), \\ &\xi_n \in \left(0, (1-\sigma) \cdot \frac{\|Ax_n - \omega_n\|^2 + \|By_n - \omega_n\|^2}{\|A^*(Ax_n - \omega_n)\|^2 + \|B^*(By_n - \omega_n)\|^2}\right), \\ &T := (1-\gamma) I + \gamma G((1-\eta) I + \eta G), \\ &S := (1-\gamma) I + \gamma M((1-\eta) I + \eta M), \end{aligned} \tag{18}$$

where G and M are demiclosed at 0. Assume $\Omega = \{(x,y) \in \mathrm{Fix}(G) \times \mathrm{Fix}(M) : Ax = By \in K\} \neq \emptyset$. Then $(x_n, y_n) \rightharpoonup (t,s)$, $(u_n, v_n) \rightharpoonup (t,s)$, and $\omega_n \rightharpoonup z^* := At = Bs$, where $(t,s) \in \Omega$.

Proof. It is obvious that $K \subset K_n$ for any $n \in N$. Let $(x^*, y^*) \in \Omega$; namely, $x^* \in \mathrm{Fix}(G)$, $y^* \in \mathrm{Fix}(M)$, and $\omega^* = Ax^* = By^* \in K$. By Lemma 1, we have

$$\begin{aligned} \|\omega_n - \omega^*\| &= \left\|P_{K_n}\left(\frac{Ax_n + By_n}{2}\right) - P_{K_n}(\omega^*)\right\|^2 \\ &\leq \left\|\frac{1}{2}(Ax_n - \omega^*) + \frac{1}{2}(By_n - \omega^*)\right\|^2 \\ &= \frac{1}{2}\|Ax_n - \omega^*\|^2 + \frac{1}{2}\|By_n - \omega^*\|^2 - \frac{1}{4}\|Ax_n - By_n\|^2. \end{aligned} \tag{19}$$

$$\begin{aligned} \|u_n - x^*\|^2 &= \|x_n - x^* - \xi_n A^*(Ax_n - \omega_n)\|^2 \\ &= \|x_n - x^*\|^2 + \xi_n^2 \|A^*(Ax_n - \omega_n)\|^2 - 2\langle x_n - x^*, \xi_n A^*(Ax_n - \omega_n)\rangle. \end{aligned} \tag{20}$$

By (20), we have

$$\begin{aligned}\|u_n - x^*\|^2 = {} & \|x_n - x^*\|^2 + \xi_n^2 \|A^*(Ax_n - \omega_n)\|^2 \\ & - \xi_n \|Ax_n - \omega^*\|^2 - \xi_n \|Ax_n - \omega_n\|^2 \\ & + \xi_n \|\omega_n - \omega^*\|^2 .\end{aligned} \tag{21}$$

Similarly,

$$\begin{aligned}\|v_n - y^*\|^2 = {} & \|y_n - y^*\|^2 + \xi_n^2 \|B^*(By_n - \omega_n)\|^2 \\ & - \xi_n \|By_n - \omega^*\|^2 - \xi_n \|By_n - \omega_n\|^2 \\ & + \xi_n \|\omega_n - \omega^*\|^2 .\end{aligned} \tag{22}$$

By (21), (22), (17), and (19), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \\
&\quad = \|\alpha_n x_n + (1-\alpha_n) Tu_n - x^*\|^2 \\
&\quad + \|\alpha_n y_n + (1-\alpha_n) Sv_n - y^*\|^2 \\
&\quad = \|\alpha_n x_n + (1-\alpha_n) Tu_n - (1-\alpha_n) x^* - \alpha_n x^*\|^2 \\
&\quad + \|\alpha_n y_n + (1-\alpha_n) Sv_n - (1-\alpha_n) y^* - \alpha_n y^*\|^2 \\
&\quad = \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n) \|Tu_n - x^*\|^2 \\
&\quad - \alpha_n (1-\alpha_n) \|x_n - Tu_n\|^2 + \alpha_n \|y_n - y^*\|^2 \\
&\quad + (1-\alpha_n) \|Sv_n - y^*\|^2 - \alpha_n (1-\alpha_n) \|y_n - Sv_n\|^2 \\
&\quad \le \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n) \|u_n - x^*\|^2 \\
&\quad - \alpha_n (1-\alpha_n)(1-\alpha_n) \|x_n - Tu_n\|^2 + \alpha_n \|y_n - y^*\|^2 \\
&\quad + (1-\alpha_n) \|v_n - y^*\|^2 - \alpha_n (1-\alpha_n) \|y_n - Sv_n\|^2 \\
&\quad = \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n) \|x_n - x^*\|^2 + (1-\alpha_n) \\
&\quad \cdot \xi_n^2 \|A^*(Ax_n - \omega_n)\|^2 - \xi_n (1-\alpha_n) \|Ax_n - \omega^*\|^2 \\
&\quad - \xi_n (1-\alpha_n) \|Ax_n - \omega_n\|^2 + 2\xi_n (1-\alpha_n) \\
&\quad \cdot \|\omega_n - \omega^*\|^2 - \alpha_n (1-\alpha_n) \|x_n - Tu_n\|^2 \\
&\quad + \alpha_n \|y_n - y^*\|^2 + (1-\alpha_n) \|y_n - y^*\|^2 + (1-\alpha_n) \\
&\quad \cdot \xi_n^2 \|B^*(By_n - \omega_n)\|^2 - \xi_n (1-\alpha_n) \|By_n - \omega^*\|^2 \\
&\quad - \xi_n (1-\alpha_n) \|By_n - \omega_n\|^2 - \alpha_n (1-\alpha_n) \|y_n - Sv_n\|^2 \\
&\quad \le \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n) \|x_n - x^*\|^2 + (1-\alpha_n) \\
&\quad \cdot \xi_n^2 \|A^*(Ax_n - \omega_n)\|^2 - \xi_n (1-\alpha_n) \|Ax_n - \omega^*\|^2 \\
&\quad - \xi_n (1-\alpha_n) \|Ax_n - \omega_n\|^2 + \xi_n (1-\alpha_n) \\
&\quad \cdot \|Ax_n - \omega^*\|^2 + \xi_n (1-\alpha_n) \|By_n - \omega^*\|^2 - \frac{1}{2} \\
&\quad \cdot \xi_n (1-\alpha_n) \|Ax_n - By_n\|^2 - \alpha_n (1-\alpha_n) \\
&\quad \cdot \|x_n - Tu_n\|^2 + \alpha_n \|y_n - y^*\|^2 + (1-\alpha_n) \\
&\quad \cdot \|y_n - y^*\|^2 + (1-\alpha_n) \xi_n^2 \|B^*(By_n - \omega_n)\|^2 \\
&\quad - \xi_n (1-\alpha_n) \|By_n - \omega^*\|^2 - \xi_n (1-\alpha_n) \\
&\quad \cdot \|By_n - \omega_n\|^2 - \alpha_n (1-\alpha_n) \|y_n - Sv_n\|^2 \\
&\quad = \|x_n - x^*\|^2 + \|y_n - y^*\|^2 + (1-\alpha_n) \\
&\quad \cdot \xi_n^2 \left(\|A^*(Ax_n - \omega_n)\|^2 + \|B^*(By_n - \omega_n)\|^2\right) \\
&\quad - (1-\alpha_n) \xi_n \left(\|Ax_n - \omega_n\|^2 + \|By_n - \omega_n\|^2\right) \\
&\quad - \alpha_n (1-\alpha_n) \left(\|x_n - Tu_n\|^2 + \|y_n - Sv_n\|^2\right) \\
&\quad - \frac{1}{2} (1-\alpha_n) \xi_n \|Ax_n - By_n\|^2 .
\end{aligned} \tag{23}$$

Notice that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \le {} & \|x_n - x^*\|^2 + \|y_n \\ & - y^*\|^2 + \xi_n (1-\alpha_n) \\ & \cdot \left[\xi_n \left(\|A^*(Ax_n - \omega_n)\|^2 + \|B^*(By_n - \omega_n)\|^2\right)\right. \\ & \left. - \left(\|Ax_n - \omega_n\|^2 + \|By_n - \omega_n\|^2\right)\right] .\end{aligned} \tag{24}$$

Thanks to

$$\begin{aligned}\xi_n \in \Bigg(& 0, (1-\sigma) \\ & \cdot \frac{\|Ax_n - \omega_n\|^2 + \|By_n - \omega_n\|^2}{\|A^*(Ax_n - \omega_n)\|^2 + \|B^*(By_n - \omega_n)\|^2} \Bigg),\end{aligned} \tag{25}$$

we obtain

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \\ &\quad \le \|x_n - x^*\|^2 + \|y_n - y^*\|^2 .\end{aligned} \tag{26}$$

Let

$$X_n(x^*, y^*) = \|x_n - x^*\|^2 + \|y_n - y^*\|^2 . \tag{27}$$

We obtain

$$X_{n+1}(x^*, y^*) \le X_n(x^*, y^*) . \tag{28}$$

This implies that $X_n(x^*, y^*)$ is a nonincreasing sequence; hence $\lim_{n\to\infty} X_n(x^*, y^*)$ exists. As a result, $\{x_n\}$ and $\{y_n\}$ are

bounded sequences. Rewrite (23) as

$$\begin{aligned}&(1-\alpha_n)\xi_n\left[\left(\|Ax_n-\omega_n\|^2+\|By_n-\omega_n\|^2\right)\right.\\&\quad\left.-\xi_n\left(\|A^*(Ax_n-\omega_n)\|^2+\|B^*(By_n-\omega_n)\|^2\right)\right]\\&\quad+\alpha_n(1-\alpha_n)\left(\|x_n-Tu_n\|^2+\|y_n-Sv_n\|^2\right)+\frac{1}{2}(1\\&\quad-\alpha_n)\xi_n\|Ax_n-By_n\|^2\le\|x_n-x^*\|^2+\|y_n-y^*\|^2\\&\quad-\left(\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\right)=X_n(x^*,y^*)\\&\quad-X_{n+1}(x^*,y^*).\end{aligned}\tag{29}$$

Letting $n\to\infty$ and taking the limit in (29), we have

$$\begin{aligned}\|Ax_n-By_n\|&\longrightarrow 0,\\\|Tu_n-x_n\|&\longrightarrow 0,\\\|Sv_n-y_n\|&\longrightarrow 0,\end{aligned}\tag{30}$$

$$\begin{aligned}\|Ax_n-\omega_n\|&\longrightarrow 0,\\\|By_n-\omega_n\|&\longrightarrow 0.\end{aligned}\tag{31}$$

Then,

$$\begin{aligned}\lim_{n\to\infty}\|u_n-x_n\|&=0,\\\lim_{n\to\infty}\|v_n-y_n\|&=0,\\\lim_{n\to\infty}\|x_{n+1}-x_n\|&=\lim_{n\to\infty}\|Tu_n-x_n\|=0,\\\lim_{n\to\infty}\|y_{n+1}-y_n\|&=\lim_{n\to\infty}\|Sv_n-y_n\|=0,\end{aligned}\tag{32}$$

which imply that $\{x_n\}$ and $\{y_n\}$ are asymptotically regular. Furthermore, we get

$$\begin{aligned}\lim_{n\to\infty}\|Tu_n-u_n\|&=0,\\\lim_{n\to\infty}\|Sv_n-v_n\|&=0.\end{aligned}\tag{33}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded sequences, there exist weakly convergent subsequences, say $\{x_{n_i}\}\subset\{x_n\}$ such that $x_{n_i}\rightharpoonup t$; also $\{y_{n_i}\}\subset\{y_n\}$ such that $y_{n_i}\rightharpoonup s$. The Opial property guarantees that the weakly subsequential limit of $\{(x_n,y_n)\}$ is unique. So we have $x_n\rightharpoonup t$, $y_n\rightharpoonup s$. Therefore $u_n\rightharpoonup t$, $v_n\rightharpoonup s$. Since G and M are demiclosed at 0, and from Lemma 2, by (33), we have $Tt=t$, $Ss=s$, which imply that $t\in\mathrm{Fix}(G)$, $s\in\mathrm{Fix}(M)$. Hence,

$$(t,s)\in\mathrm{Fix}(G)\times\mathrm{Fix}(M).\tag{34}$$

Furthermore, since $Ax_n-By_n\rightharpoonup At-Bs$, by using the weakly lower semicontinuity of squared norm, we have

$$\begin{aligned}\|At-Bs\|&\le\liminf_{n\to\infty}\|Ax_n-By_n\|=\lim_{n\to\infty}\|Ax_n-By_n\|\\&=0;\end{aligned}\tag{35}$$

that is,

$$At=Bs.\tag{36}$$

By (31), we have $\omega_n\rightharpoonup z^*:=At=Bs$. Now, we prove $z^*\in K$. We know that $\{\varsigma_n\}$ is bounded. There exists $\overline{M}>0$, such that $\|\varsigma_n\|\le\overline{M}$, where $\overline{M}$ is a constant. Note that $\omega_n=P_{K_n}((Ax_n+By_n)/2)\in K_n$, and we have

$$h\left(\frac{Ax_n+By_n}{2}\right)+\left\langle\varsigma_n,\omega_n-\frac{Ax_n+By_n}{2}\right\rangle\le 0.\tag{37}$$

Hence,

$$\begin{aligned}h\left(\frac{Ax_n+By_n}{2}\right)&\le-\left\langle\varsigma_n,\omega_n-\frac{Ax_n+By_n}{2}\right\rangle\\&\le\overline{M}\left\|\omega_n-\frac{Ax_n+By_n}{2}\right\|.\end{aligned}\tag{38}$$

By the lower semicontinuity of h, (38), and (31), we obtain

$$\begin{aligned}h(z^*)&\le\liminf_{n\to\infty}h\left(\frac{Ax_n+By_n}{2}\right)\\&\le\overline{M}\liminf_{n\to\infty}\left\|\omega_n-\frac{Ax_n+By_n}{2}\right\|=0.\end{aligned}\tag{39}$$

Thus $z^*\in K$. The proof is completed. □

4. Consequent Results

In this section, we give some corollaries, which are easily obtained from Theorem 3.

If $G=M$, we have the following corollary.

Corollary 4. *Let H_1, H_2, and H_3 be real Hilbert spaces and K be a closed convex level set*

$$K=\{x\in H_3:h(x)\le 0\},\tag{40}$$

where $h:H_3\to R$ is convex function which is subdifferentiable on K and its subdifferentials are bounded on bounded sets. $A:H_1\to H_3$ and $B:H_2\to H_3$ are two bounded linear operators with their adjoint operators A^* and B^*, respectively, $\sigma\in(0,1)$ is a parameter controlling step length, $0<\gamma<\eta<1/(1+\sqrt{1+L^2})$, and $\{\alpha_n\}\subset(0,1)$. Let $\{x_n\}$, $\{y_n\}$, $\{\omega_n\}$, $\{u_n\}$, and $\{v_n\}$ be sequences generated by

$$\begin{aligned}\omega_n&=P_{K_n}\left(\frac{Ax_n+By_n}{2}\right),\\K_n&=\left\{z\in H_3:h\left(\frac{Ax_n+By_n}{2}\right)\right.\\&\quad\left.+\left\langle\varsigma_n,z-\frac{Ax_n+By_n}{2}\right\rangle\le 0\right\},\\u_n&=x_n-\xi_nA^*(Ax_n-\omega_n),\\x_{n+1}&=\alpha_nx_n+(1-\alpha_n)Tu_n,\\v_n&=y_n-\xi_nB^*(By_n-\omega_n),\\y_{n+1}&=\alpha_ny_n+(1-\alpha_n)Tv_n,\end{aligned}\tag{41}$$

where

$$\begin{aligned} &\varsigma_n \in \partial h\left(\frac{Ax_n + By_n}{2}\right), \\ &\xi_n \in \left(0, (1-\sigma) \cdot \frac{\|Ax_n - \omega_n\|^2 + \|By_n - \omega_n\|^2}{\|A^*(Ax_n - \omega_n)\|^2 + \|B^*(By_n - \omega_n)\|^2}\right), \\ &T := (1-\gamma) I + \gamma G((1-\eta) I + \eta G), \end{aligned} \tag{42}$$

where G is demiclosed at 0. Assume $\Omega = \{(x, y) \in \mathrm{Fix}(G) \times \mathrm{Fix}(G) : Ax = By \in K\} \neq \emptyset$. Then $(x_n, y_n) \rightharpoonup (t, s)$, $(u_n, v_n) \rightharpoonup (t, s)$ and $\omega_n \rightharpoonup z^* := At = Bs$, where $(t, s) \in \Omega$.

If $A = B$, we have the following corollary.

Corollary 5. *Let H_1 and H_3 be real Hilbert spaces and K be a closed convex level set*

$$K = \{x \in H_3 : h(x) \leq 0\}, \tag{43}$$

where $h : H_3 \to R$ is convex function which is subdifferentiable on K and its subdifferentials are bounded on bounded sets. $A : H_1 \to H_3$ is a bounded linear operator with its adjoint operator A^*, $\sigma \in (0, 1)$ is a parameter controlling step length, $0 < \gamma < \eta < 1/(1 + \sqrt{1 + L^2})$, and $\{\alpha_n\} \subset (0, 1)$. Let $\{x_n\}$, $\{y_n\}$, $\{\omega_n\}$, $\{u_n\}$, and $\{v_n\}$ be sequences generated by

$$\begin{aligned} &\omega_n = P_{K_n}\left(\frac{Ax_n + Ay_n}{2}\right), \\ &K_n = \left\{z \in H_3 : h\left(\frac{Ax_n + Ay_n}{2}\right) + \left\langle \varsigma_n, z - \frac{Ax_n + Ay_n}{2}\right\rangle \leq 0\right\}, \\ &u_n = x_n - \xi_n A^*(Ax_n - \omega_n), \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n) T u_n, \\ &v_n = y_n - \xi_n A^*(Ay_n - \omega_n), \\ &y_{n+1} = \alpha_n y_n + (1-\alpha_n) S v_n, \end{aligned} \tag{44}$$

where

$$\begin{aligned} &\varsigma_n \in \partial h\left(\frac{Ax_n + By_n}{2}\right), \\ &\xi_n \in \left(0, (1-\sigma) \cdot \frac{\|Ax_n - \omega_n\|^2 + \|By_n - \omega_n\|^2}{\|A^*(Ax_n - \omega_n)\|^2 + \|B^*(By_n - \omega_n)\|^2}\right), \\ &T := (1-\gamma) I + \gamma G((1-\eta) I + \eta G), \\ &S := (1-\gamma) I + \gamma M((1-\eta) I + \eta M), \end{aligned} \tag{45}$$

where G and M are demiclosed at 0. Assume $\Omega = \{(x, y) \in \mathrm{Fix}(G) \times \mathrm{Fix}(M) : Ax = Ay \in K\} \neq \emptyset$. Then $(x_n, y_n) \rightharpoonup (s, t)$, $(u_n, v_n) \rightharpoonup (s, t)$, and $\omega_n \rightharpoonup z^* := As = At$, where $(s, t) \in \Omega$.

If $G = M$ and $A = B$, we have the following corollary.

Corollary 6. *Let H_1 and H_3 be real Hilbert spaces and K be a closed convex level set*

$$K = \{x \in H_3 : h(x) \leq 0\}, \tag{46}$$

where $h : H_3 \to R$ is convex function which is subdifferentiable on K and its subdifferentials are bounded on bounded sets. $A : H_1 \to H_3$ is a bounded linear operator with its adjoint operator A^*, and $\sigma \in (0, 1)$ is a parameter controlling step length, $0 < \gamma < \eta < 1/(1 + \sqrt{1 + L^2})$, and $\{\alpha_n\} \subset (0, 1)$. Let $\{x_n\}$, $\{y_n\}$, $\{\omega_n\}$, $\{u_n\}$, and $\{v_n\}$ be sequences generated by

$$\begin{aligned} &\omega_n = P_{K_n}\left(\frac{Ax_n + Ay_n}{2}\right), \\ &K_n = \left\{z \in H_3 : h\left(\frac{Ax_n + Ay_n}{2}\right) + \left\langle \varsigma_n, z - \frac{Ax_n + Ay_n}{2}\right\rangle \leq 0\right\}, \\ &u_n = x_n - \xi_n A^*(Ax_n - \omega_n), \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n) T u_n, \\ &v_n = y_n - \xi_n A^*(Ay_n - \omega_n), \\ &y_{n+1} = \alpha_n y_n + (1-\alpha_n) T v_n, \end{aligned} \tag{47}$$

where

$$\begin{aligned} &\varsigma_n \in \partial h\left(\frac{Ax_n + By_n}{2}\right), \\ &\xi_n \in \left(0, (1-\sigma) \cdot \frac{\|Ax_n - \omega_n\|^2 + \|By_n - \omega_n\|^2}{\|A^*(Ax_n - \omega_n)\|^2 + \|B^*(By_n - \omega_n)\|^2}\right), \\ &T := (1-\gamma) I + \gamma G((1-\eta) I + \eta G), \end{aligned} \tag{48}$$

where G is demiclosed at 0. Assume $\Omega = \{(x, y) \in \mathrm{Fix}(G) \times \mathrm{Fix}(G) : Ax = Ay \in K\} \neq \emptyset$. Then $(x_n, y_n) \rightharpoonup (t, s)$, $(u_n, v_n) \rightharpoonup (t, s)$, and $\omega_n \rightharpoonup z^* := At = As$, where $(t, s) \in \Omega$.

If $G = M = A = B$ is an identity operator, we have the following corollary.

Corollary 7. *Let H_1 and H_3 be real Hilbert spaces and K be a closed convex level set*

$$K = \{x \in H_3 : h(x) \leq 0\}, \tag{49}$$

where $h : H_3 \to R$ is convex function which is subdifferentiable on K and its subdifferentials are bounded on bounded sets.

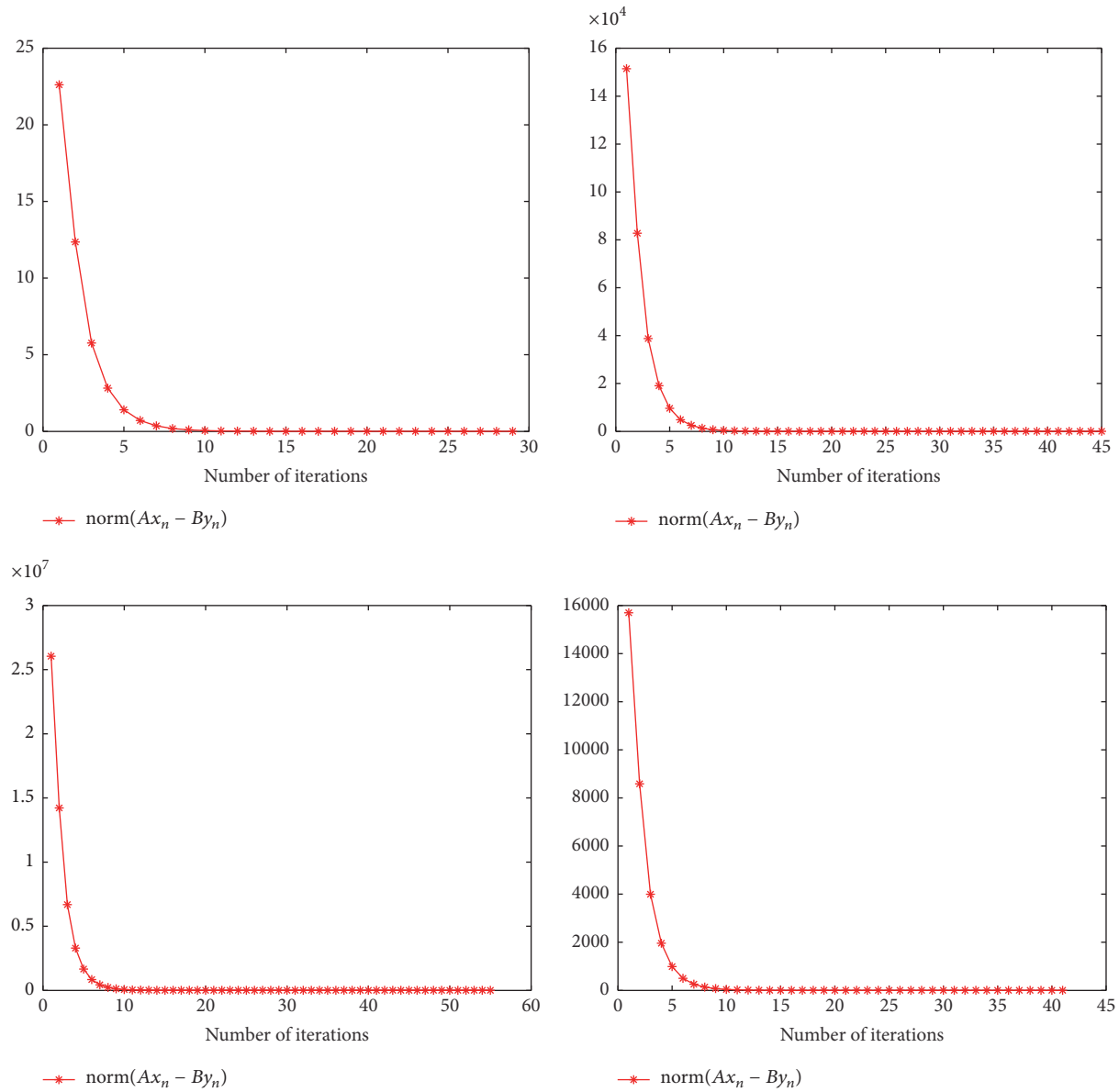

FIGURE 1: The behaviors of $\|Ax_n - By_n\|$ for different initial points.

$\sigma \in (0,1)$ is a parameter controlling step length, $0 < \gamma < \eta < 1/(1+\sqrt{1+L^2})$, and $\{\alpha_n\} \subset (0,1)$. Let $\{x_n\}$, $\{y_n\}$, $\{\omega_n\}$, $\{u_n\}$, and $\{v_n\}$ be sequences generated by

$$\begin{aligned}
\omega_n &= P_{K_n}\left(\frac{x_n + y_n}{2}\right),\\
K_n &= \left\{z \in H_3 : h\left(\frac{x_n + y_n}{2}\right) + \left\langle \varsigma_n, z - \frac{x_n + y_n}{2}\right\rangle \right.\\
&\quad \left. \le 0\right\},\\
u_n &= (1-\xi)\, x_n + \xi\omega_n,\\
x_{n+1} &= \alpha_n x_n + (1-\alpha_n)\, u_n,\\
v_n &= (1-\xi)\, y_n + \xi\omega_n,\\
y_{n+1} &= \alpha_n y_n + (1-\alpha_n)\, v_n,
\end{aligned} \tag{50}$$

where $\varsigma_n \in \partial h((x_n + y_n)/2)$, $\xi \in (0, 1-\sigma)$. Assume $\Omega = \{(x,y) \in H_1 \times H_1 : x = y \in K\} \neq \emptyset$. Then $(x_n, y_n) \rightharpoonup (t,s)$, $(u_n, v_n) \rightharpoonup (t,s)$, and $\omega_n \rightharpoonup z^* := t = s$, where $(t,s) \in \Omega$.

5. Numerical Examples

In this section, we give an example to show some insight into the behavior of the algorithm presented in this paper. The whole codes are written in Matlab 7.0. All the numerical results are carried out on a personal Lenovo ThinkPad computer with Intel(R) Core(TM) i7-6500U CPU 2.50 GHz and RAM 8.00 GB.

Let $H_1 = R^2$, $H_2 = R^3$. $A \in R^{3\times 2}$ and $B \in R^{3\times 3}$ are as follows.

$$A = \begin{pmatrix} 0.8147 & 0.9134 \\ 0.9058 & 0.6324 \\ 0.1270 & 0.0975 \end{pmatrix},$$

Table 1: Number of iterations and the CPU time for different initial points.

Initial point		Iter.	Sec.
$x = (1, 1)^T$,	$y = (1, 1, 1)^T$	27	0.01
$x = (5.2378, 2.6487)^T$,	$y = (6.8357, 43.6327, 17.3853)^T$	42	0.016
$x = (0.5221, 19.0936)^T$,	$y = (86.1193, 192.3197, 152.4829)^T$	50	0.017
$x = (0.0588, 5.4403)^T$,	$y = (14.1190, 12.9026, 11.0462)^T$	38	0.012

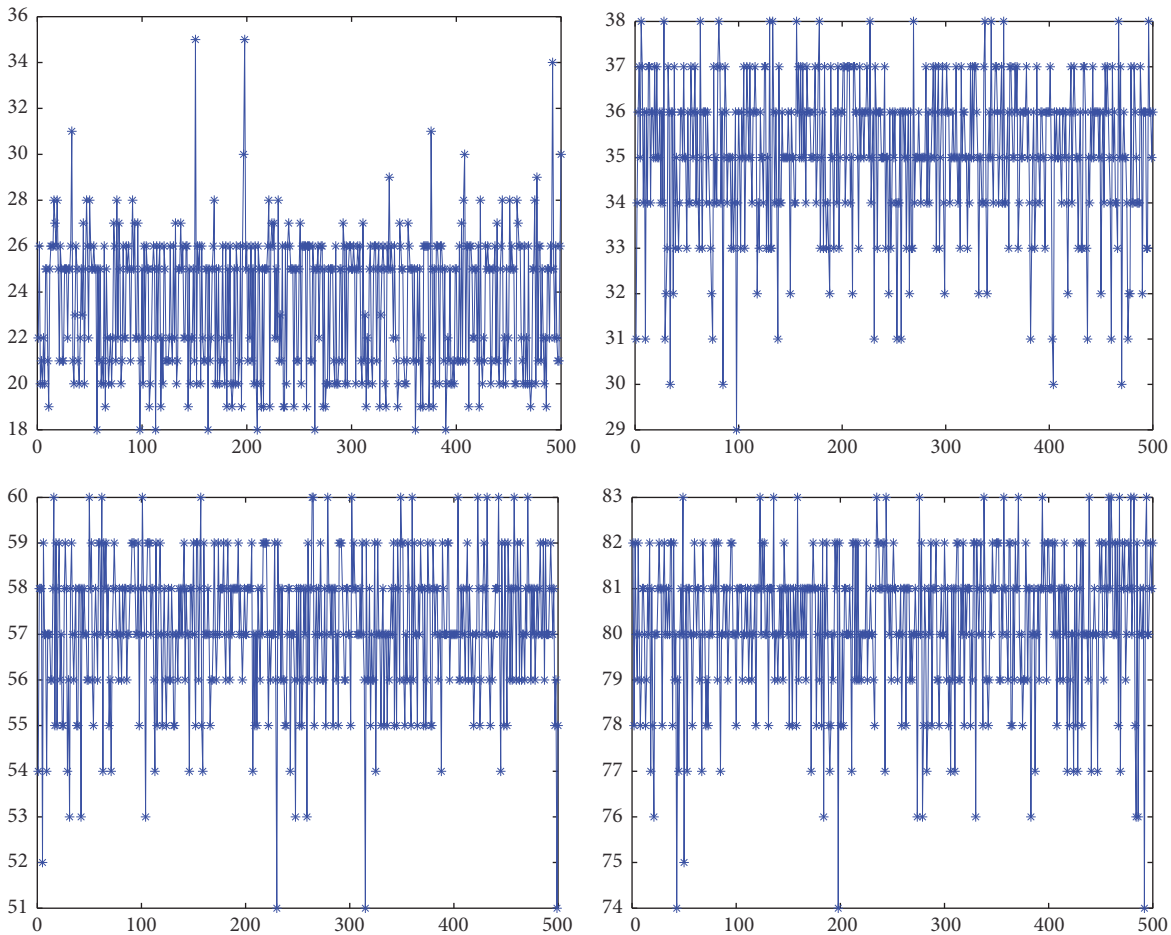

Figure 2: The behaviors of iterative numbers for different initial points.

$$B = \begin{pmatrix} 0.2785 & 0.9649 & 0.9572 \\ 0.5469 & 0.1576 & 0.4854 \\ 0.9575 & 0.9706 & 0.8003 \end{pmatrix}. \tag{51}$$

Let $K = \{x \in R^n \mid h(x) = x_1^3 + x_2^2 + x_3 - 2 \leq 0\}$, where $x = (x_1, x_2, x_3)^T$. For convenience, we take $\sigma = 0.2$, $\alpha_n = 1/3 + 1/2^n$, and $G = M = (1/2)I$. We choose $\|Ax_n - By_n\| \leq 10^{-6}$ as the stopping criterion.

Figure 1 presents the behaviors of $\|Ax_n - By_n\|$ in different initial points such as

$$\begin{aligned} x &= (1, 1)^T, \\ y &= (1, 1, 1)^T, \\ x &= (5.2378, 2.6487)^T, \\ y &= (6.8357, 43.6327, 17.3853)^T, \\ x &= (0.5221, 19.0936)^T, \\ y &= (86.1193, 192.3197, 152.4829)^T, \\ x &= (0.0588, 5.4403)^T, \\ y &= (14.1190, 12.9026, 11.0462)^T. \end{aligned} \tag{52}$$

It is easy to see that the presentation reveals that $Ax = By$. Table 1 shows the number of iterations and the CPU time

for the above four initial points. We denote Iter. and Sec. as the number of iterations and the CPU time in seconds, respectively.

Furthermore, for testing the stationary property of iterative numbers, we carry out 500 experiments for different initial points which are presented randomly, such as

$$\begin{aligned} x &= \text{rand}(2,1), \\ y &= \text{rand}(3,1), \\ x &= 10 * \text{rand}(2,1), \\ y &= 10 * \text{rand}(3,1), \\ x &= 1000 * \text{rand}(2,1), \\ y &= 600 * \text{rand}(3,1), \\ x &= 100000 * \text{rand}(2,1), \\ y &= 60000 * \text{rand}(3,1), \end{aligned} \tag{53}$$

separately. In these cases, we take $\sigma = 0.2$, $\alpha_n = 0.4$ and choose $\|Ax_n - w_n\| + \|By_n - w_n\| \leq 10^{-6}$ as the stopping criterion. Figure 2 illustrates the behaviors of iterative numbers for different initial points, which reveals the stationary property of iterative numbers of the algorithm.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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