

Research Article

Mean-Square Stability of Split-Step Theta Milstein Methods for Stochastic Differential Equations

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The fundamental analysis of numerical methods for stochastic differential equations (SDEs) has been improved by constructing new split-step numerical methods. In this paper, we are interested in studying the mean-square (MS) stability of the new general drifting split-step theta Milstein (DSS θ M) methods for SDEs. First, we consider scalar linear SDEs. The stability function of the DSS θ M methods is investigated. Furthermore, the stability regions of the DSS θ M methods are compared with those of test equation, and it is proved that the methods with $\theta \geq 3/2$ are stochastically A-stable. Second, the nonlinear stability of DSS θ M methods is studied. Under a coupled condition on the drifting and diffusion coefficients, it is proved that the methods with $\theta > 1/2$ can preserve the MS stability of the SDEs with no restriction on the step-size. Finally, numerical examples are given to examine the accuracy of the proposed methods under the stability conditions in approximation of SDEs.

1. Introduction

Many real-world phenomena in different fields of science, such as biology, financial engineering, neural network, and wireless communications, can be simulated by the Itô stochastic differential equations (SDEs) of the form

$$\begin{aligned} dy(t) &= f(t, y(t)) dt + g(t, y(t)) dW(t), \quad t > 0, \\ y(t_0) &= y_0, \end{aligned} \quad (1)$$

where $f(t, y(t))$ is the drift coefficient and $g(t, y(t))$ is the diffusion coefficient and the Wiener process $W(t)$ is defined on a given probability space (Ω, \mathcal{F}, P) with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ which satisfied the usual conditions. In order to describe the properties of SDEs systems, numerical solutions have attracted a lot of attention. There is an extensive literature concerned with explicit and implicit numerical methods for SDEs (see [1–7]).

In order to get insight into the stability behavior of the numerical methods for SDEs (1), the scalar equation has been used as a test problem as follows:

$$\begin{aligned} dy(t) &= ay(t) dt + by(t) dW(t), \quad t > 0, \\ y(t_0) &= y_0, \end{aligned} \quad (2)$$

where $a, b \in \mathbb{R}$. Mean-square (MS) stability conditions for several numerical methods have been derived (see [8–10]). Saito and Mitsui [10] proposed the concept of the MS stability for a numerical methods solving (2). The MS stability of the classes of theta methods for scalar SDEs (2) is introduced in [11]. Higham [9] introduced allowing $\theta > 1$ in the semi-implicit Milstein scheme benefits in terms of the stability for (2).

In literature, there are several attempts to construct numerical methods based on split-step techniques to improve the fundamental analysis that contains convergence and stability for SDEs. Various implicit split-step numerical

methods have been derived based on Euler method. For example, Higham et al. [12] derived the split-step backward Euler (SSBE) method. In addition, the split-step theta (SS θ) methods which generalize the SSBE method when $\theta = 1$ were discussed in [13, 14]. Although these methods are considered stochastically A-stable (i.e., the MS stability region of the test equation is subset of or equal to that of numerical method for any step-size (see [15])), these methods have convergence order 0.5.

Recently, there are many split-step methods that have been constructed based on Milstein method, to improve the fundamental analysis that contains convergence and stability for SDEs. For example, in 2009, Wang and Liu [7] presented the drifting split-step backward Milstein (DSSBM) method. Moreover, in 2015, Voss and Abdul Khaliq [6] combined the predictor corrector method with a Milstein method to investigate split-step Adams-Moulton Milstein (SSAMM) method. The two-step Milstein schemes are defined, Adams-Bashforth Milstein (ABM), Adams-Moulton Milstein (AMM), and two-step midpoint-Simpson Milstein (BDFM) schemes by Tocino and Senosiain [15], in 2015. The necessary and sufficient conditions for their MS stability are given, and stability regions and A-stable approaches have been discussed. *Although these methods improved the convergence order to be 1.0, unfortunately, we can see that the MS stability conditions of these methods have some restriction on the parameters and step-size. Furthermore, these methods are not A-stable.*

There are a few works that discussed the nonlinear stability of numerical methods for SDEs (1). The exponential MS stability of the Euler and SSBE methods has been proved for nonlinear SDEs, under the assumption that the drift coefficient f satisfies a one-sided Lipschitz condition and the diffusion coefficient g satisfies a global Lipschitz condition (see [16]). Furthermore, without the global Lipschitz condition, the two classes of theta methods were proved to reproduce the exponential MS stability of the exact solution (see [13, 17, 18]). *Note that these numerical methods converge with order 0.5.* Recently, the exponential MS stability has been discussed for split-step Milstein type methods, which converge with order 1.0. In 2016, Jiang et al. [19] proved that the double-implicit and split two-step Milstein schemes can preserve the exponential MS stability of the original SDEs under Lipschitz condition with restriction on step-size. *Nevertheless, the MS stability analysis of the multistep methods is focused on linear SDEs, and there exist very little results on the stability analysis for nonlinear SDEs. So, the particular literature is still limited and welcomes attempts to fill this gap.*

Following the MS stability analysis of Milstein type methods for linear and nonlinear SDEs. Zong et al. [20] discussed the exponential MS stability of two classes of theta Milstein methods, the split-step theta Milstein (SSTM), and stochastic theta Milstein (STM) methods. With restriction on parameters and step-size, they proved the MS stability under $\theta \in [0, 1]$ of two numerical methods. However, they did not discuss the stability regions and A-stability approaches. Also, with restriction on step-size, the authors proved that under one-sided Lipschitz condition the two classes of theta Milstein methods can share the exponential MS stability of the exact solution for nonlinear SDEs.

Nowadays, split-step numerical methods have attracted a lot of attention from scholars for SDEs and have been proven to be a very efficient approach. In order to improve the fundamental analysis of numerical methods, the drifting split-step theta Milstein (DSS θ M) methods are considered for solving SDEs (1), in this paper. First, the MS stability results of the methods are investigated for scalar linear SDEs (2) as follows: (a) If $\theta \in [0, 1/2]$, the DSS θ M methods are MS stable with restriction on step-size. (b) If $\theta > 1/2$, the proposed methods are MS stable for all $h > 0$ under constraints on the parameters a, b , and θ . (c) The restrictions on the parameters can be avoided when $\theta \geq 3/2$ and the numerical methods are stochastically A-stable. The regions of MS stability are examined to explain the effectiveness of the methods. Second, we prove that the DSS θ M methods are MS stable for all step-size with $\theta > 1/2$ under a coupled condition on the drift and diffusion coefficients for nonlinear SDEs (1). Finally, we give a linear and nonlinear numerical examples to check the accuracy of the proposed methods in the light of the stability conditions, especially the value of parameter θ . *This work is different from [20] in that we extend the stability results for asymptomatic MS stability with $\theta > 1$. In addition, we prove for $\theta \geq 3/2$, the DSS θ M methods can preserve the asymptomatic MS stability of the exact solution with no restriction on parameters and step-size. Furthermore, we discussed the MS stability regions and A-stable approaches of DSS θ M methods for linear SDEs. Also, we proved that, under local Lipschitz condition, the numerical methods can preserve the asymptomatic MS stability of the nonlinear SDEs without restriction on step-size.*

The rest of this paper is organized as follows. In Section 2, we present some necessary notations and preliminaries, then the drifting split-step theta Milstein methods are given. The linear MS stability of the proposed methods is studied, in Section 3. In Section 4, our attention is turned to the nonlinear stability of numerical methods for SDEs. In Section 5, numerical results are given in order to demonstrate the accuracy of the proposed methods under the stability conditions and compared with existing methods. Conclusion is given in Section 6.

2. Notations and Preliminaries

Let (Ω, \mathcal{F}, P) be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, which satisfies the usual conditions (i.e., the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is right-continuous and each \mathcal{F}_t , $t \geq 0$, contains all P -null sets in \mathcal{F}). Let $(W(t))_{t \geq 0}$ be one-dimensional Brownian motion defined on this probability space and be \mathcal{F}_t -adapted and independent of \mathcal{F}_0 . $|\cdot|$ is the Euclidean norm in \mathbb{R}^d . $a \vee b$ presents $\max(a, b)$ and $a \wedge b$ presents $\min(a, b)$. \mathbb{N}_+ represents the positive integer set, namely, $\mathbb{N}_+ = \{1, 2, 3, \dots\}$. Moreover, we assume y_0 to be \mathcal{F}_0 -measurable and $E|y_0|^2 < \infty$.

Let $f, g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be Borel measurable functions. Let us consider the d -dimensional SDEs (1) with initial data $y(t_0) = y_0 \in \mathbb{R}^d$. Assume that f and g satisfy the following assumption.

Assumption 1 (local Lipschitz condition). There exists a positive constant k_j for all $j > 0$ such that for all $t \in \mathbb{R}$ and $(t, x), (t, y) \in [t_0, \infty) \times \mathbb{R}^d$ with $|x| \vee |y| \leq j$,

$$|f(t, x) - f(t, y)| \vee |g(t, x) - g(t, y)| \leq k_j |x - y|. \quad (3)$$

Assumption 2. Assume that there exists a constant σ such that for any $(t, x), (t, y) \in [t_0, \infty) \times \mathbb{R}^d$, the function g in (1) also satisfies

$$\left| \frac{\partial g(t, x)}{\partial y} g(t, x) - \frac{\partial g(t, y)}{\partial y} g(t, y) \right|^2 \leq \sigma |x - y|^2. \quad (4)$$

From [21], we note that if SDEs satisfy Assumption 1, then the linear growth condition or a class of monotone conditions will ensure a global solution for the SDEs (1).

In order to improve the fundamental analysis that contains convergence and stability of numerical solutions for SDEs (1), there are many drifting split-step methods that have been constructed over the past several years [6, 7, 12–14, 20, 22]. In this paper, we consider general split-step numerical methods; the drifting split-step theta Milstein (DSS θ M) methods are

$$y_n^* = y_n + \theta h f(t_n, y_n^*), \quad (5)$$

$$y_{n+1} = y_n^* + (1 - \theta) h f(t_n, y_n^*) + g(t_n, y_n^*) \Delta W_n + \frac{1}{2} \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) [(\Delta W_n)^2 - h], \quad (6)$$

where y_n is approximation to $y(t_n)$ at the discrete points in time t_n , $t_n = nh$ with step-size $h = T/N$, N is a given positive integer, θ is a free parameter, with increments $\Delta W_n := W(t_{n+1}) - W(t_n)$ being independent $N(0, h)$ -distributed Gaussian random variables and $y(0) = y_0$. Moreover, y_n is $\{\mathcal{F}_{t_n}\}$ -measurable at the mesh point t_n . The DSS θ M ((5)-(6)) with parameter $\theta = 1$ reduce to the DSSBM [7], while with parameter $\theta = 0$, the DSS θ M become the classical Milstein [23].

The methods ((5)-(6)) are discussed in [20], where it is called the split-step theta Milstein (SSTM) method. The strong convergence order 1.0 of the DSS θ M methods ((5)-(6)) for SDEs (1) has been given in [20, 22]. Here, we only list the definition of the numerical solutions and omit the counterparts for the continuous solutions to the SDEs. We will further explain the relevance of these concepts to numerical theory and practice in Section 4.

Definition 3. The numerical method is said to be MS stable, if there exists $h^* > 0$, such that any application of the method to SDEs (1) generates numerical approximation y_n , which satisfy

$$\lim_{n \rightarrow \infty} E |y_n|^2 = 0, \quad (7)$$

for all step-size $h \in (0, h^*)$.

3. Linear Stability

Recently, split-step numerical methods have attracted a lot of attention from scholars SDEs and have been proven to

be a very efficient approach. It is well known that there are many split-step numerical methods with convergence order 0.5 that are A-stable for scalar SDEs (2), for example, SSBE and SS θ methods. In addition, split-step Milstein schemes with convergence order 1.0 have been introduced for SDEs in literature review such as SSAMM, DSSBM, and (ABM, AMM, BDFM) methods [6, 7, 15], respectively. Unfortunately, when comparing the stability regions of these Milstein type schemes with that of test equation (2), we can see that these methods are not A-stable, and the MS stability conditions of these methods have some restriction on parameters and step-size. In this section, we discuss the MS stability of new general split-step numerical methods, the DSS θ M methods ((5)-(6)) for scalar linear SDEs (2). First, we recall some notations of stability.

Definition 4 (see [10]). Suppose that the equilibrium position of SDEs is asymptotically MS stable. Then a numerical method that produces the iterations y_n to approximate the solution $y(t)$ of SDEs (1) is said to be asymptotically MS stable if

$$\lim_{n \rightarrow \infty} |y_n| = 0. \quad (8)$$

We can apply one-step method to the test equation (2) which is represented by

$$y_{n+1} = R(a, b, h, J) y_n, \quad (9)$$

where J is the standard Gaussian random variable $J = \Delta W_n / \sqrt{h} \sim N(0, 1)$. The following condition characterizes the MS stability of (2)

Lemma 5 (see [10]). *If the constants a, b in (2) satisfy*

$$a + \frac{1}{2} b^2 < 0, \quad (10)$$

then the solution of (2) is asymptotically stable in the MS sense; that is,

$$\lim_{t \rightarrow \infty} E |y(t)|^2 = 0. \quad (11)$$

Definition 6 (see, [10]). The numerical method is said to be MS stable for a, b , and h if

$$\bar{R}(a, b, h) = E(R^2(a, b, h, J)) < 1, \quad (12)$$

where $\bar{R}(a, b, h)$ is called MS stability function of the numerical method.

Applying the DSS θ M methods ((5)-(6)) to the scalar linear SDEs (2), we obtained the following form:

$$y_n^* = y_n + \theta h a y_n^*,$$

$$y_{n+1} = y_n^* + (1 - \theta) h a y_n^* + b y_n^* \Delta W_n + \frac{1}{2} b^2 y_n^* [(\Delta W_n)^2 - h]. \quad (13)$$

During the rest of this section, following Higham [9], we extend the stability results in [20] for asymptotic MS stability with $\theta > 1$. In addition, we prove that (a) for $\theta \geq 3/2$, the DSS θ M methods can preserve the asymptotic MS stability of the exact solution with no restriction on parameters and step-size. Furthermore, the methods are stochastically A-stable. (b) The stability regions of the DSS θ M methods are compared with those of considerable numerical methods to explain the effectiveness of the proposed methods.

Now, we can state the main results of the linear stability of the DSS θ M methods ((5)-(6)) for SDEs (2) in the following theorem.

Theorem 7. *Supposing that MS stability condition (10) holds, one has the following:*

- (1) *If $\theta \in [0, 1/2]$, then the DSS θ M methods are MS stable for all $h \in (0, h^*(a, b, \theta))$,*

$$h^* < \frac{-(2a + b^2)}{(1 - 2\theta)a^2 + (1/2)b^4}. \tag{14}$$

- (2) *If $\theta > 1/2$ and $a^2 < b^4/2(2\theta - 1)$, then the DSS θ M methods are MS stable for all $h \in (0, h^*(a, b, \theta))$.*

- (3) *If $\theta > 1/2$ and $a^2 \geq b^4/2(2\theta - 1)$, then the DSS θ M methods are MS stable for all $h > 0$.*

Proof. The proof of result number (1), when $\theta \in [0, 1/2]$, has been given in ([20], Theorem 5.3). So, we omit the proof here. In the following, we prove results numbers (2) and (3), when $\theta > 1/2$.

Note that y_n is \mathcal{F}_{t_n} -measurable at the mesh point t_n ; we easily know from ((5)-(6)) that y_n^* is also \mathcal{F}_{t_n} -measurable at related mesh-points, and ΔW_n is independent of \mathcal{F}_{t_n} . From $E[\Delta W_n] = 0$, $E[|\Delta W_n|^2] = h$, $E[\Delta W_n^3] = 0$, and $E[|\Delta W_n|^4] = 3h^2$, it is easy to see from (13) that

$$E(y_{n+1})^2 = P(a, b, \theta, h) E(y_n)^2, \tag{15}$$

where

$$P(a, b, \theta, h) = \frac{1}{(1 - \theta ha)^2} \left((1 + (1 - \theta)ha)^2 + b^2h + \frac{1}{2}b^4h^2 \right). \tag{16}$$

Hence $E(y_{n+1})^2 \rightarrow 0$, (when $n \rightarrow \infty$) if and only if

$$\left((1 - 2\theta)a^2 + \frac{1}{2}b^4 \right)h + (2a + b^2) < 0. \tag{17}$$

With respect to condition (10), we know that $2a + b^2 < 0$ and the following results can be obtained.

If $\theta > 1/2$ and $(1 - 2\theta)a^2 < (1/2)b^4$, that is,

$$a^2 < \frac{b^4}{2(2\theta - 1)}, \tag{18}$$

then the DSS θ M methods are MS stable for $h \in (0, h^*(a, b, \theta))$.

If $\theta > 1/2$ and $(1 - 2\theta)a^2 \geq (1/2)b^4$, that is,

$$a^2 \geq \frac{b^4}{2(2\theta - 1)}, \tag{19}$$

then (17) holds for all $h > 0$. Namely, the DSS θ M methods are MS stable for all $h > 0$. \square

The stability regions of the proposed methods with $\theta \in [0, 1]$ are strictly contained in that for the problem. Theorem 7 shows that this behavior extends to DSS θ M methods (13) with $\theta \geq 1/2$ when the diffusion term dominates. However, Theorem 7 also shows that if the drift term dominates, then the unconditional stability holds.

The following corollary shows that, for $\theta \geq 3/2$, the DSS θ M methods are MS stable with no restrict on the parameters.

Corollary 8. *If $\theta \geq 3/2$, then the DSS θ M methods (13) are MS stable for all $h > 0$.*

Proof. Condition (10) and $\theta \geq 3/2$ imply

$$a^2 > \frac{b^4}{4} \geq \frac{b^4}{2(2\theta - 1)}, \tag{20}$$

which yields (17). Hence, the DSS θ M methods are MS stable for all $h > 0$. \square

Remark 9. Zong et al. ([20], Theorem 5.3) discussed the exponential MS stability of the DSS θ M (13) with $\theta \in [0, 1]$. The authors proved that, for $\theta \in [0, 1/2]$, the methods can share the exponential MS stability of the exact solution with restriction on step-size. For $\theta \in (1/2, 1]$, if the diffusion term plays a crucial role, then the restriction on the step-size still holds with restriction on parameters, and if the drift term plays a crucial role, then the numerical methods are MS stable for all step-size with restriction on parameters. In this paper, Theorem 7 extended the same results with free parameter θ , which can be greater than 1. In addition, Corollary 8 proved that for $\theta \geq 3/2$, the DSS θ M methods can preserve the asymptomatic MS stability of the exact solution for SDEs (2) with no restriction on parameters and step-size.

By substituting $y_n = y_n^* - \theta hf(t_n, y_n^*)$ into (6), the approximation $\{y_n\}_{n \geq 0}$ in the DSS θ M ((5)-(6)) methods is in fact the stochastic theta Milstein (STM) approximation, which is provided for linear SDEs (2)

$$y_{n+1} = y_n + \theta hay_{n+1} + (1 - \theta)hay_n + by_n \Delta W_n + \frac{1}{2}b^2 y_n [(\Delta W_n)^2 - h]. \tag{21}$$

The exponential MS stability of STM (21) methods was investigated for linear SDEs with restriction on step-size in [20]. Theorem 7 and Corollary 8 illustrated the asymptotic MS stability results of the STM (21) for linear SDEs (2).

The MS stability regions are the areas under the plotted curves and symmetric about the $x - y$ plane with $x = ah$ and $y = b^2h$. In this case, the stability conditions (MS stability)

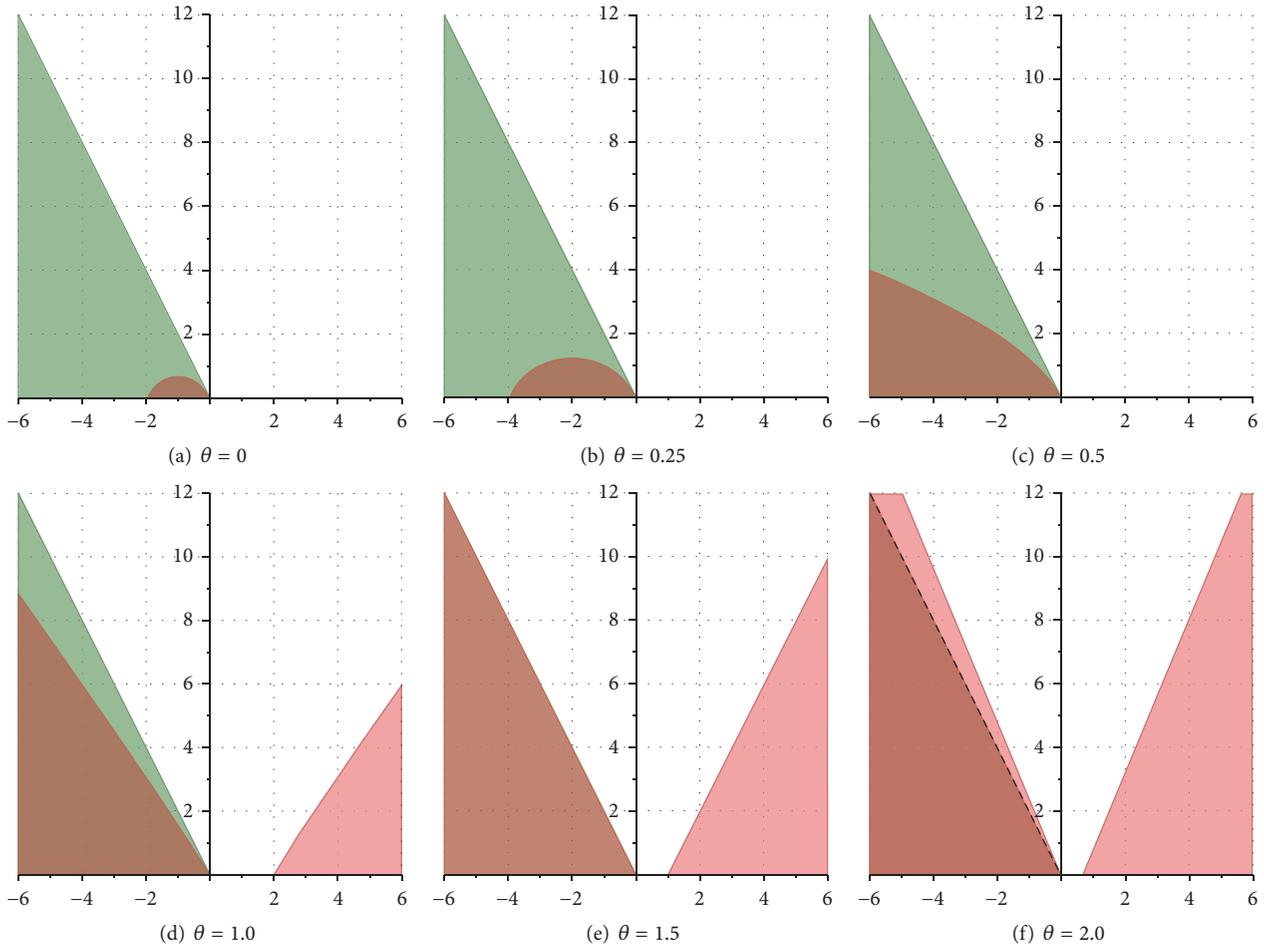


FIGURE 1: Real MS stability regions for test problem (green) and DSSθM methods (red).

(10) and (17) for the test problem and DSSθM methods, respectively, become

$$x + \frac{1}{2}y < 0 \quad (\text{Problem}), \quad (22)$$

$$(1 - 2\theta)x^2 + \frac{1}{2}y^2 + 2x + y < 0 \quad (\text{DSS}\theta\text{M}). \quad (23)$$

We plot the stability regions of the DSSθM and test problem as follows. Let $R_{\text{SDE}} = \{x, y \in \mathbb{R} : y \geq 0, x + (1/2)y < 0\}$ denote the MS stability region of the SDEs (2), and let $R_{\text{DSS}\theta\text{M}}(\theta) = \{x, y \in \mathbb{R} : y \geq 0, (23) \text{ hold}\}$ be the MS stability regions of DSSθM methods. Figure 1 illustrates how the $R_{\text{DSS}\theta\text{M}}(\theta)$ varies with θ . The green shading marks the regions R_{SDE} and the red shading superimposes $R_{\text{DSS}\theta\text{M}}(\theta)$ for various values of θ . Figure 1 confirms our results.

Remark 10. By comparing the MS stability regions of DSSθM methods with various values of parameter θ and that of scalar SDEs (2) (see Figure 1), we can see that the DSSθM methods are stochastically A-stable with $\theta \geq 3/2$.

In Figure 2, we compare the MS stability regions of the proposed DSSθM methods (23) (on the figure, DSSTM) with

various value of parameter θ with those of test equation (22), Milstein, drifting split-step backward Milstein (DSSBM), and split-step Adams-Moulton Milstein (SSAMM) methods (note that Voss and Abdul Khaliq [6] presented that the SSAMM methods with $\theta = -1/2 - 1/\sqrt{2}$ give the large MS stability regions). The MS stability functions (MS stability conditions) of these numerical methods are shown in Table 1. We examine the MS stability regions of the DSSθM methods with $\theta = 1.0, 1.5, 2.0$, respectively. From Figure 2, we can see that when $\theta = 1.0$, the MS stability regions of the DSSθM methods and those of the DSSBM method match exactly (see Figure 2(a)). With $\theta = 1.5$, the MS stability regions of the DSSθM methods are identical exactly to the test problem stability regions. Also, the MS stability regions of DSSθM methods are better than those of the others. Furthermore, when $\theta = 2.0$, we note that the test problem stability regions are subset of those of the proposed methods; that is, the DSSθM methods are stochastically A-stable under $\theta \geq 3/2$ with stability regions being better than those of existing Milstein type methods.

Remark 11. If we look closely to the stability conditions of the split-step methods which are based on Milstein method to approximate the diffusion part in SDEs (2), we can find that the term $(1/2)y^2$ plays an important role in the stability

TABLE 1: The split-step numerical methods.

N. methods	One step meth. (1)	One step meth. (2)	Stability condition
Milstein	Milstein meth.	—	$x^2 + 2x + y + \frac{1}{2}y^2 < 0$
DSSBM	Backward Euler meth.	Milstein meth.	$\frac{1 + y + (1/2)y^2}{(1 - x)^2} < 1$
SSAMM	Predictor corrector meth.	Milstein meth.	$\frac{(1 + y + (1/2)y^2)(1 + 2\theta x)^2}{(1 - ((1/2) - \theta)x)^4} < 1$
DSS θ M	Theta method.	Milstein meth.	$(1 - 2\theta)x^2 + 2x + \frac{1}{2}y^2 + y < 0$

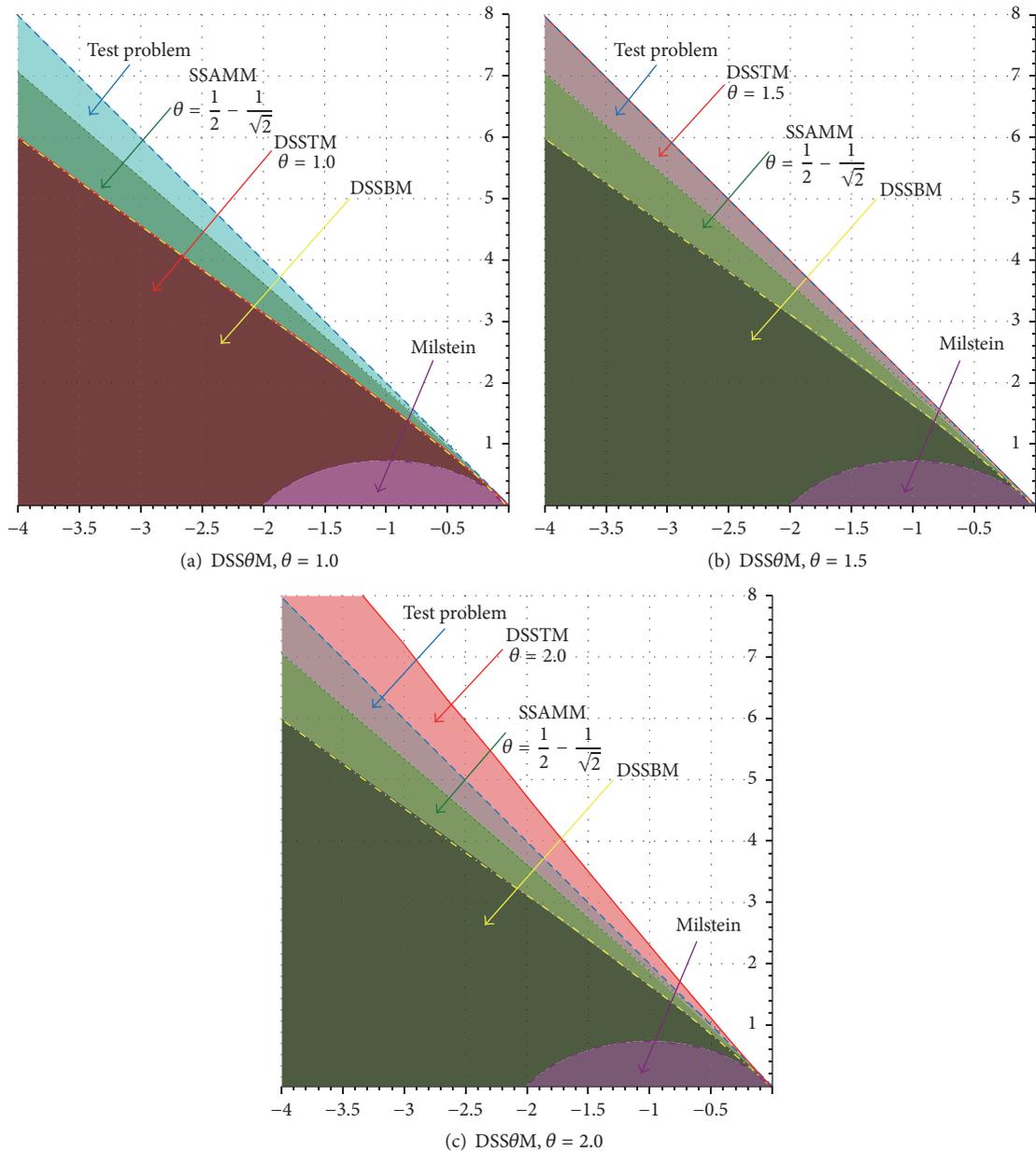


FIGURE 2: Real MS stability regions.

functions (see Table 1). In addition, this term should be handled to improve the stability properties of the split-step method by the one-step method which is used for the drift part in SDEs (2). We can see from Table 1 that the backward Euler method and the predictor corrector method could not handle this term in DSSBM and SSAMM methods, respectively. So, the MS stability properties of these methods are not A-stable and have restriction on the parameters and step-size. In the case of the proposed DSS θ M methods, we find that the free parameter θ plays the main role to handle that term $(1/2)y^2$. So, the MS stability properties of the DSS θ M methods are A-stable with $\theta \geq (3/2)$ (see Figure 1) and have no restriction on the parameters and step-size (see Corollary 8).

4. Nonlinear Stability

In this section, we discuss the nonlinear stability of the DSS θ M methods for SDEs (1). It is impossible to find a sufficient and necessary condition for analytical stability for nonlinear SDEs. For the purpose of stability, assume that $f(0, 0) = g(0, 0) = 0$. This shows that SDEs (1) admit a trivial solution. Then inequality (4) reduces to

$$\left| \frac{\partial g(t, x)}{\partial y} g(t, x) \right|^2 \leq \sigma |x|^2. \quad (24)$$

We assume that f and g satisfy Assumption 1 (local Lipschitz condition), which is classical for the nonlinear SDEs. To investigate stability of numerical approximation, let us firstly give the stability criterion of SDEs (1).

Theorem 12 (see [21]). *Let Assumption 1 hold. If there exists a positive constant γ such that for all $(t, x) \in [t_0, \infty) \times \mathbb{R}^d$,*

$$2xf(t, x) + |g(t, x)|^2 \leq -\gamma |x|^2, \quad (25)$$

then the solution of (1) is MS stable.

Following Higham [9], we can prove that the DSS θ M methods share the asymptotic MS stability of the exact solution when $\theta > 1/2$ with no restriction on step-size. Before giving the main theorem of the MS stability, we establish the key role in the following lemma.

Lemma 13. *Let conditions ((24), (25)) hold. Moreover, the functions f and g satisfy the local Lipschitz condition (3) for any $(t, x), (t, y) \in [t_0, \infty) \times \mathbb{R}^d$, with a positive constant k_l for all $l > 0$. Then there exists a constant \bar{K}_l for all $l > 0$ and $n > 0$, such that the DSS θ M methods ((5)-(6)) have the properties*

$$\begin{aligned} E |y_n|^2 &\leq \bar{K}_l, \\ E |y_n^*|^2 &\leq \bar{K}_l. \end{aligned} \quad (26)$$

Proof. For sufficiently large $l > 0$, we define the stopping time

$$\lambda_l := \inf \{i > 0 : |y_i^*| > l \text{ or } |y_i| > l\}. \quad (27)$$

It is observed that, for $n \in [0, \lambda_l]$,

$$\begin{aligned} |y_{n-1}^*| &\leq l, \\ |y_{n-1}| &\leq l. \end{aligned} \quad (28)$$

Then we can complete the proof of our theorem in the same idea and similar process, which were used in the first part of the proof of Theorem 5.2 in [20]. So, we omit the detailed proof here. \square

Theorem 14. *Let conditions ((24), (25)) hold. Then the DSS θ M methods ((5)-(6)) have the following stability results:*

(1) *If $\theta > 1/2$ and for all $(t, x) \in [t_0, \infty) \times \mathbb{R}^d$,*

$$(1 - 2\theta) |f(t, x)|^2 + \frac{1}{2} \left| \frac{\partial g}{\partial x}(t, x) g(t, x) \right|^2 \leq 0, \quad (29)$$

then the DSS θ M methods are MS stable for all $h > 0$.

(2) *If $\theta \in [0, 1/2]$ and if there exists a positive constant K such that, for any $(t, x) \in [t_0, \infty) \times \mathbb{R}^d$,*

$$|f(t, x)|^2 \leq K |x|^2, \quad (30)$$

then the DSS θ M methods are MS stable for all $h \in (0, h^*)$,

$$h^* \leq \frac{\gamma}{(1 - 2\theta)K + (1/2)\sigma}. \quad (31)$$

Proof. By (5), we get

$$|y_n^*|^2 = |y_n|^2 + 2\theta h y_n^* f(t_n, y_n^*) - \theta^2 h^2 |f(t_n, y_n^*)|^2. \quad (32)$$

Squaring both sides of (6) and substituting (32), we have

$$\begin{aligned} |y_{n+1}|^2 &= |y_n|^2 + 2h y_n^* f(t_n, y_n^*) + h |g(t_n, y_n^*)|^2 \\ &\quad + (1 - 2\theta) h^2 |f(t_n, y_n^*)|^2 \\ &\quad + \frac{1}{2} h^2 \left| \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) \right|^2 + M_n, \end{aligned} \quad (33)$$

where

$$\begin{aligned} M_n &= |g(t_n, y_n^*)|^2 [|\Delta W_n|^2 - h] \\ &\quad + \frac{1}{4} \left| \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) \right|^2 [|\Delta W_n|^4 - 3h^2] \\ &\quad + 2y_n^* g(t_n, y_n^*) \Delta W_n + y_n^* \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) \\ &\quad \cdot [|\Delta W_n|^2 - h] + 2(1 - \theta) h f(t_n, y_n^*) g(t_n, y_n^*) \\ &\quad \cdot \Delta W_n + (1 - \theta) \\ &\quad \cdot h f(t_n, y_n^*) \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) [|\Delta W_n|^2 - h] \\ &\quad + g(t_n, y_n^*) \Delta W_n \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) \\ &\quad \cdot [|\Delta W_n|^2 - h]. \end{aligned} \quad (34)$$

Note that y_n is \mathcal{F}_{t_n} -measurable at the mesh point t_n , and we easily know from ((5)-(6)) that y_n^* is also \mathcal{F}_{t_n} -measurable at related mesh point, and ΔW_n is independent of \mathcal{F}_{t_n} . So, $E[\Delta W_n] = 0$, $E[|\Delta W_n|^2] = h$, $E[|\Delta W_n|^4] = 3h^2$, and $E[|M_n|] = 0$.

Case I. If $\theta > 1/2$, then $(1 - 2\theta)|f(t_n, y_n^*)|^2 < 0$. Hence, we need to discuss the sign of the following term:

$$(1 - 2\theta) |f(t_n, y_n^*)|^2 + \frac{1}{2} \left| \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) \right|^2. \quad (35)$$

First. If $(1-2\theta)|f(t_n, y_n^*)|^2 + (1/2)|(\partial g/\partial y)(t_n, y_n^*)g(t_n, y_n^*)|^2 \leq 0$, we obtain from (33), with respect to condition (25) and Lemma 13, that the DSS θ M methods are MS stable for all $h > 0$.

Second. If $(1 - 2\theta)|f(t_n, y_n^*)|^2 + (1/2)|(\partial g/\partial y)(t_n, y_n^*)g(t_n, y_n^*)|^2 > 0$, using condition (24), we obtain from (33)

$$|y_{n+1}|^2 \leq |y_n|^2 + h \left[\frac{1}{2}\sigma h - \gamma \right] |y_n^*|^2 + M_n. \quad (36)$$

Taking expectation and summation on both sides of (36) over n from $n = 0$ with respect to Lemma 13 gives

$$E[|y_{n+1}|^2] + h \left[\gamma - \frac{1}{2}\sigma h \right] \sum_{i=0}^n E[|y_i^*|^2] \leq E[|y_0|^2]. \quad (37)$$

Let

$$h_1 \leq \frac{2\gamma}{\sigma}. \quad (38)$$

Then for any $h \in (0, h_1)$, the DSS θ M methods are MS stable.

Case II. If $\theta \in [0, 1/2]$, using conditions ((24), (30)), we get from (33)

$$|y_{n+1}|^2 \leq |y_n|^2 + h \left[\left((1 - 2\theta)K + \frac{1}{2}\sigma \right) h - \gamma \right] |y_n^*|^2 + M_n. \quad (39)$$

Let

$$h^* \leq \frac{\gamma}{(1 - 2\theta)K + (1/2)\sigma}, \quad (40)$$

with respect to Lemma 13. We get that the DSS θ M methods are MS stable for any $h \in (0, h^*)$. \square

Remark 15. Zong et al. ([20], Theorem 5.2) proved that, under one-sided Lipschitz condition, when $\theta \in [0, 1]$ the DSS θ M methods can share the exponential MS stability of the exact solution with restriction on step-size. In this paper, Theorem 14 proved that, under local Lipschitz condition when $\theta \in [0, 1/2]$, the DSS θ M methods can preserve the asymptotic MS stability of the exact solution with restriction on step-size. In addition, with suitable condition on the drift and diffusion functions (29), when $\theta > 1/2$ the numerical methods can preserve the asymptotic MS stability of the exact solution with no restriction on step-size.

From the extension of MS stability for the semi-implicit theta Milstein methods [9] and the results of the DSS θ M methods ((5)-(6)) for linear SDEs in Section 3, we know that the stability for the methods with $\theta \in [0, 1]$ is strictly contained in that for the problem. Theorem 14 shows that these behaviors extend to the numerical methods with $\theta > 1/2$ when the diffusion term dominates. Also, if the drift term dominates, then the unconditional stability holds. The following remark shows that, for $\theta \geq 3/2$, the numerical methods are MS stable.

Remark 16. In Theorem 14, if $\theta \geq 3/2$, then the DSS θ M methods ((5)-(6)) are MS stable for all $h > 0$, under a coupled condition on the drift and diffusion coefficients (29). In case of $\theta = 3/2$, this condition reduces to

$$-2|f(t_n, y_n^*)|^2 + \frac{1}{2} \left| \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) \right|^2 \leq 0. \quad (41)$$

From Remark 16, we get that the approximation solution can reproduce the MS stability of trivial solution when $\theta \geq 3/2$. To illustrate these results, we consider the following nonlinear SDEs:

$$dy(t) = (-y - y^3)dt + b \sin(y) dW(t), \quad t \geq 0. \quad (42)$$

Clearly, the coefficients of (42) satisfy conditions ((3), (4)). Now, we explain that the stability condition (41) of the numerical methods satisfies the exact solution stability condition (25).

The Exact Solution Stability Condition

$$\begin{aligned} 2y(-y - y^3) + |b \sin(y)|^2 &\leq 0, \\ (-2 + b^2)y^2 - 2y^4 &\leq 0, \\ 4 &\geq b^4. \end{aligned} \quad (43)$$

The Stability Condition for DSS θ M Methods

$$-2(-y - y^3)^2 + \frac{1}{2} |b^2 \sin(y) \cos(y)|^2 \leq 0. \quad (44)$$

We get

$$\begin{aligned} \left(-2 + \frac{1}{2}b^4 \right) y^2 - 4y^4 - 2y^6 &\leq 0 \\ 4 &\geq b^4. \end{aligned} \quad (45)$$

Remark 17. By substituting $y_n = y_n^* - \theta hf(t_n, y_n^*)$ into (6), the approximation $\{y_n\}_{n \geq 0}$ in the DSS θ M methods ((5)-(6)) is in fact the stochastic theta Milstein (STM) approximation

$$\begin{aligned} y_{n+1} &= y_n + \theta hf(t_n, y_{n+1}) + (1 - \theta) hf(t_n, y_n) \\ &\quad + g(t_n, y_n) \Delta W_n \\ &\quad + \frac{1}{2} \frac{\partial g}{\partial y}(t_n, y_n) g(t_n, y_n) [(\Delta W_n)^2 - h]. \end{aligned} \quad (46)$$

TABLE 2: Endpoint mean of absolute errors for (48) with $a = -1/2$ and $b = 1/2$.

Step-size	SS θ	SSAMM	DSS θ M
	$\theta = 0.1$	$\theta = -1/2 - 1/\sqrt{2}$	$\theta = 0.9$
2^{-5}	4.27×10^{-4}	7.12×10^{-4}	7.11×10^{-6}
2^{-6}	2.23×10^{-4}	3.80×10^{-4}	1.78×10^{-6}
2^{-7}	1.20×10^{-4}	1.90×10^{-4}	4.80×10^{-7}
2^{-8}	5.63×10^{-5}	9.52×10^{-5}	1.21×10^{-7}
2^{-9}	2.64×10^{-5}	4.8×10^{-5}	2.98×10^{-8}

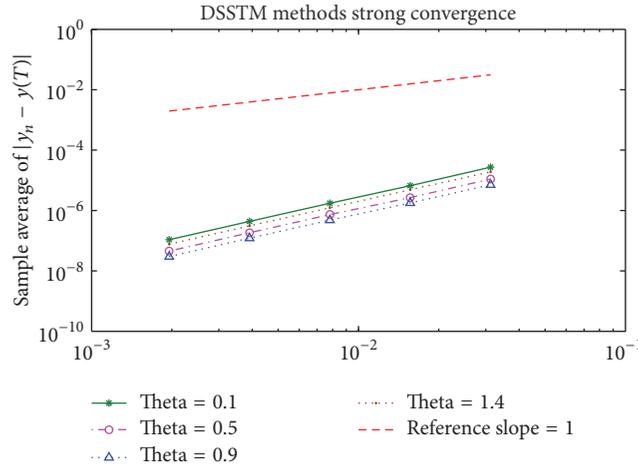


FIGURE 3: Strong convergence of DSS θ M methods with $a = -1/2$ and $b = 1/2$.

The exponential MS stability of STM (46) is investigated for nonlinear SDEs with restriction on step-size in [20]. Theorem 14 and Remark 16 illustrate the asymptotic MS stability results of the STM (46) without restriction on step-size for SDEs (1).

5. Numerical Experiments

In order to examine the accuracy of the proposed methods under the stability conditions, we consider several illustrative numerical examples for showing the strong convergence order and MS stability of DSS θ M methods for SDEs. The MS errors at time T versus the step-size h are analyzed under different values of the parameter θ in a log-log diagram. The MS errors of the numerical approximations are defined by [14] as follows:

$$\varepsilon = \frac{1}{N} \sum_{i=1}^N |y_n^{(i)} - y^{(i)}(T)|^2, \quad (47)$$

where $y_n^{(i)}$ is a numerical approximation to $y^{(i)}(T)$ and $y^{(i)}(T)$ is the value of the exact solution of SDEs at time T . The superscript i means the i th sample path, $i = 1, 2, \dots, N$.

Example 18 (scalar linear SDEs). We apply the DSS θ M methods to the scalar linear SDEs

$$\begin{aligned} dy(t) &= ay(t) dt + by(t) dW(t), \quad t \in [0, T], \\ y(0) &= 1, \end{aligned} \quad (48)$$

whose exact solution is

$$y(t) = y(0) \exp\left(\left(a - \frac{1}{2}b^2\right)t + bW(t)\right). \quad (49)$$

We use the parameters $a = -1/2$ and $b = 1/2$ the same as in [6, 14] and demonstrate the strong convergence rate of the DSS θ M methods at the terminal time $T = 1$. We compute 4000 different discretized Brownian paths over $[0, 1]$ with step-size $dt = 2^{-9}$. For each path, the DSS θ M methods are applied with five different step-sizes: $\Delta t = 2^{p-1}dt$, $1 \leq p \leq 5$. Table 2 compares the mean of absolute errors over the sample paths for SS θ methods [14] and SSAMM methods [6], with the proposed DSS θ M methods (note that the value of parameter θ for SS θ and SSAMM has been chosen to give the best absolute errors of those methods according to [6, 14]). We find that the proposed DSS θ M methods are more efficient than SS θ and SSAMM methods. *Figure 3* shows the results of the MS errors at time T versus the step-size h under different values of the parameter θ in a log-log diagram. This explains that there is a balance between the value of parameter θ and the accuracy.

Example 19 (scalar nonlinear SDEs). We consider the nonlinear SDEs

$$\begin{aligned} dy(t) &= -(\alpha + \beta^2 y)(1 - y^2) dt \\ &\quad + \beta(1 - y^2) dW(t), \quad t \in [0, T], \end{aligned} \quad (50)$$

TABLE 3: Errors for (50) with $\alpha = 1$ and $\beta = 1$.

Step-size	SS θ	SSAMM	DSS θ M
	$\theta = 0.1$	$\theta = -1/2 - 1/\sqrt{2}$	$\theta = 0.9$
2^{-5}	7.03×10^{-3}	1.40×10^{-3}	3.50×10^{-4}
2^{-6}	3.11×10^{-3}	6.60×10^{-3}	8.02×10^{-5}
2^{-7}	1.55×10^{-3}	3.20×10^{-3}	1.83×10^{-5}
2^{-8}	7.40×10^{-4}	1.60×10^{-3}	4.58×10^{-6}
2^{-9}	3.64×10^{-4}	8.00×10^{-4}	1.23×10^{-6}

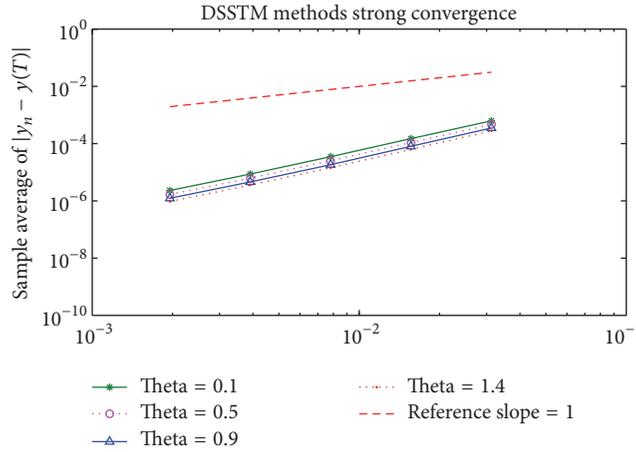


FIGURE 4: Strong convergence rate of DSS θ M methods with $\alpha = 1$ and $\beta = 1$.

where $y(0) = y_0$ with α and β being real constants. The exact solution is given by [24]

$$y(t) = \frac{(1 + y_0) \exp(-2\alpha t + 2\beta W(t)) + y_0 - 1}{(1 + y_0) \exp(-2\alpha t + 2\beta W(t)) - y_0 + 1}. \quad (51)$$

Errors of SS θ methods [14] and SSAMM methods [6] with our DSS θ M methods are displayed in Table 3 at the terminal time $T = 1$. The numerical results provide a comparison of these methods for fixed parameters $\alpha = 1$ and $\beta = 1$. We compute 4000 different discretized Brownian paths over $[0, 1]$ with step-size $dt = 2^{-9}$. For each path, the methods are applied with five different step-sizes: $\Delta t = 2^{p-1}dt$, $1 \leq p \leq 5$. We can see that the proposed methods return the most accurate solution. Figure 4 shows the results of the mean of absolute errors using a log-log plot. This explains that the absolute error with $\theta = 1.4$ is the best one of the DSS θ M methods.

Finally, we can see that the numerical examples show a balance between the stability conditions which depends on values of parameter θ and the accuracy of the proposed methods in approximation of SDEs. Furthermore, numerical examples show that the numerical methods are still effective with $\theta > 1$.

In the following, we illustrate the stability properties of the DSS θ M methods by simulating SDEs (2). The set of coefficients satisfy condition (10). So, the trivial solution of the test equation is MS stable. The data used in Figures 5, 6,

and 7 is obtained by the MS of data from 4000 trajectories, that is, $(1/4000) \sum_{i=1}^{4000} E|y_n(w_i)|^2$.

We test the MS stability for the DSS θ M methods with $a = -15$, $b = 1$ and the initial value $y_0 = 0.5$. For $\theta = 0.1$ and 0.3 , we obtain $h^*(-15, 1, 0.1) = 0.1607$ and $h^*(-15, 1, 0.3) = 0.3204$. First, we fix the parameter $\theta = 0.1$ and 0.3 and change the step-size h (see Figures 5 and 6, respectively). It is shown that the DSS θ M methods are MS stable for any $\theta \in [0, 1/2]$ if $h \in (0, h^*(a, b, \theta))$. Figure 7 explains that, for any $\theta > 1/2$, the DSS θ M methods are MS stable for any $h > 0$.

6. Conclusion

We are interested in mean-square (MS) stability of the drifting split-step theta Milstein (DSS θ M) methods for stochastic differential equations (SDEs). Zong et al. [20] proved that, under one-sided Lipschitz condition, the numerical methods with $\theta \in [0, 1]$ can share the exponential MS stability of the exact solution for nonlinear SDEs with restriction on step-size. Also, the authors could prove that, for $\theta \in (1/2, 1]$, the numerical methods are exponential MS stable for linear SDEs for all step-size with restriction on parameters. In this work, for linear SDEs, we extended the stability results in [20] for asymptotic MS stability with $\theta > 1$. In addition, we proved that, for $\theta \geq 3/2$, the DSS θ M methods can preserve the MS stability of the exact solution with no restriction on parameters and step-size. Furthermore, by comparing the stability regions of the numerical methods with those of the test equation, we proved that the DSS θ M methods are

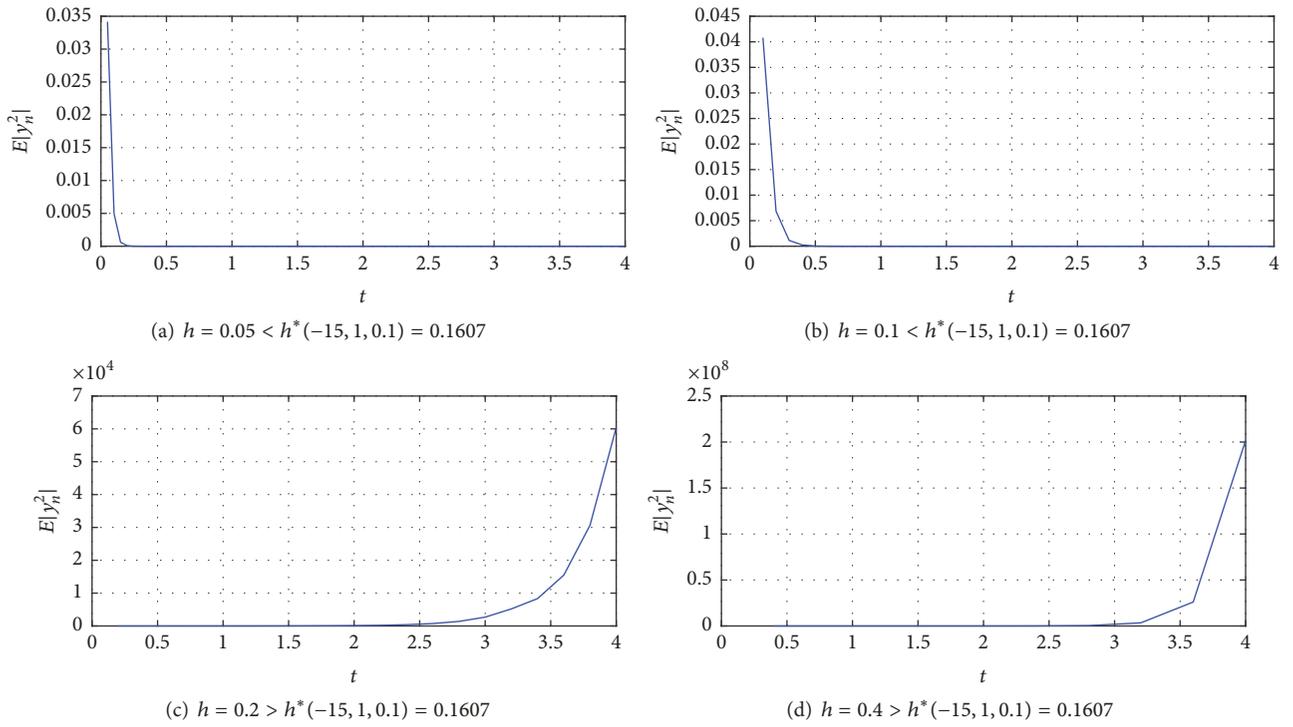


FIGURE 5: Simulating of DSSθM with fixed parameters $\theta = 0.1$, $a = -15$, and $b = 1$ for (2).

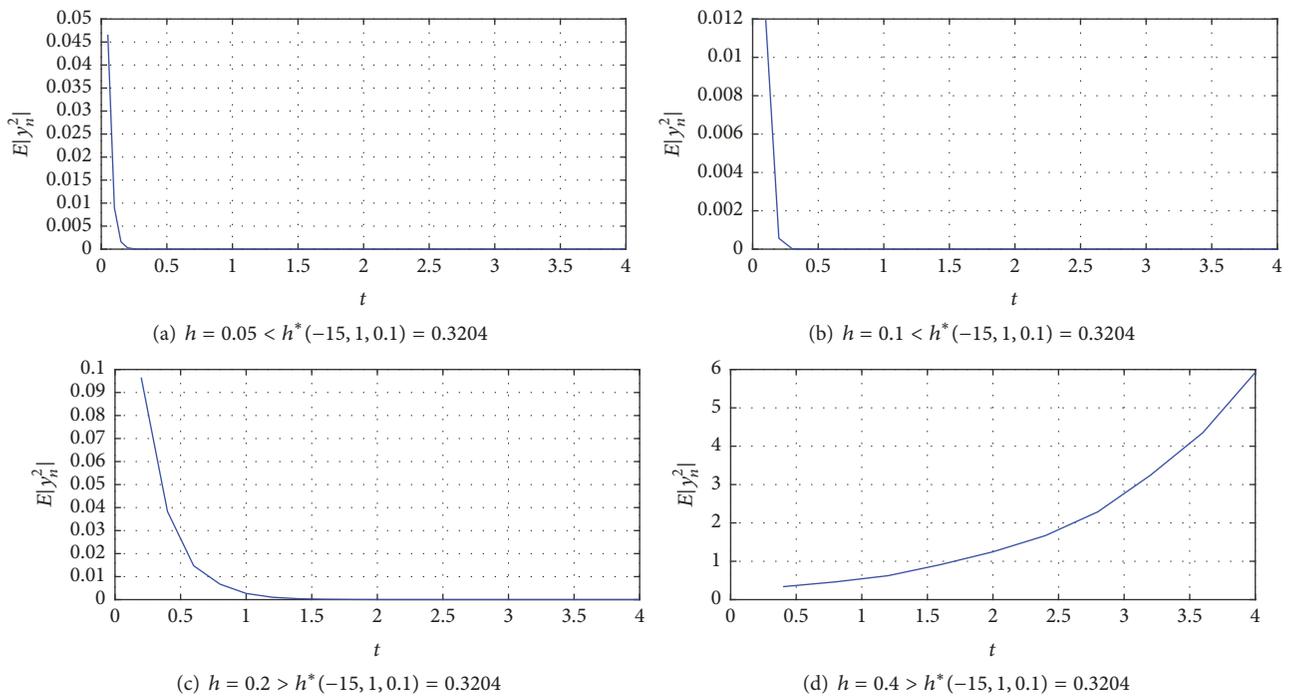


FIGURE 6: Simulating of DSSθM with fixed parameters $\theta = 0.3$, $a = -15$, and $b = 1$ for (2).

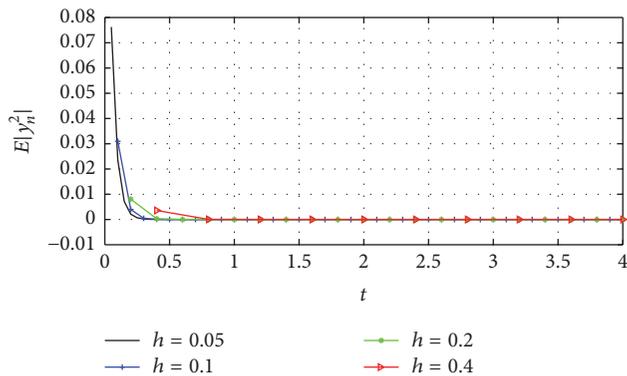


FIGURE 7: Simulating of DSS θ M with $\theta = 0.8$, $a = -15$, and $b = 1$ for (2).

stochastically A-stable with $\theta \geq 3/2$. The stability regions of various Milstein type schemes were compared to show the effectiveness of the proposed methods. For nonlinear SDEs, under local Lipschitz condition, we proved that the DSS θ M methods with $\theta > 1/2$ are asymptotic MS stable with no restriction on step-size under suitable condition on the drift and diffusion functions. In addition, we gave an example to explain that, for $\theta \geq 3/2$, the suitable condition (stability condition) of the numerical methods satisfies that of the exact solution. Finally, numerical examples are given to show that there is a balance between the stability conditions which related to values of parameter θ and the accuracy of the proposed methods in approximation of SDEs. Furthermore, numerical examples show that the numerical methods are still effective with $\theta > 1$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

- [1] K. Burrage and T. Tian, "The composite Euler method for stiff stochastic differential equations," *Journal of Computational and Applied Mathematics*, vol. 131, no. 1-2, pp. 407–426, 2001.
- [2] M. A. Eissa and B. Tian, "Lobatto-milstein numerical method in application of uncertainty investment of solar power projects," *Energies*, vol. 10, no. 1, article no. 43, 2017.
- [3] G. N. Milstein, E. Platen, and H. Schurz, "Balanced implicit methods for stiff stochastic systems," *SIAM Journal on Numerical Analysis*, vol. 35, no. 3, pp. 1010–1019, 1998.
- [4] M. Ramadan, T. S. ElDanaf, and M. A. Eissa, "System of ordinary differential equations solving using cellular neural networks," *Journal of Advanced Mathematics and Applications*, vol. 3, no. 2, pp. 182–194, 2014.
- [5] T. Tian and K. Burrage, "Implicit Taylor methods for stiff stochastic differential equations," *Applied Numerical Mathematics*, vol. 38, no. 1-2, pp. 167–185, 2001.
- [6] D. A. Voss and Q. Abdul Khaliq, "Split-step Adams-Moulton MILstein methods for systems of stiff stochastic differential equations," *International Journal of Computer Mathematics*, vol. 92, no. 5, pp. 995–1011, 2015.
- [7] P. Wang and Z. Liu, "Split-step backward balanced Milstein methods for stiff stochastic systems," *Applied Numerical Mathematics*, vol. 59, no. 6, pp. 1198–1213, 2009.
- [8] E. Buckwar, R. Horváth-Bokor, and R. Winkler, "Asymptotic mean-square stability of two-step methods for stochastic ordinary differential equations," *BIT Numerical Mathematics*, vol. 46, no. 2, pp. 261–282, 2006.
- [9] D. J. Higham, "A-stability and stochastic mean-square stability," *BIT Numerical Mathematics*, vol. 40, no. 2, pp. 404–409, 2000.
- [10] Y. Saito and T. Mitsui, "Stability analysis of numerical schemes for stochastic differential equations," *SIAM Journal on Numerical Analysis*, vol. 33, no. 6, pp. 2254–2267, 1996.
- [11] D. J. Higham, "Mean-square and asymptotic stability of the stochastic theta method," *SIAM Journal on Numerical Analysis*, vol. 38, no. 3, pp. 753–769, 2000.
- [12] D. J. Higham, X. Mao, and A. M. Stuart, "Strong convergence of Euler-type methods for nonlinear stochastic differential equations," *SIAM Journal on Numerical Analysis*, vol. 40, no. 3, pp. 1041–1063, 2002.
- [13] C. Huang, "Exponential mean square stability of numerical methods for systems of stochastic differential equations," *Journal of Computational and Applied Mathematics*, vol. 236, no. 16, pp. 4016–4026, 2012.
- [14] X.-H. Ding, Q. Ma, and L. Zhang, "Convergence and stability of the split-step θ -method for stochastic differential equations," *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1310–1321, 2010.
- [15] A. Tocino and M. J. Senosiain, "Two-step Milstein schemes for stochastic differential equations," *Numerical Algorithms*, vol. 69, no. 3, pp. 643–665, 2015.
- [16] D. J. Higham, X. Mao, and A. M. Stuart, "Exponential mean-square stability of numerical solutions to stochastic differential equations," *LMS Journal of Computation and Mathematics*, vol. 6, pp. 297–313, 2003.
- [17] X. Zong and F. Wu, "Choice of θ and mean-square exponential stability in the stochastic theta method of stochastic differential equations," *Journal of Computational and Applied Mathematics*, vol. 255, pp. 837–847, 2014.
- [18] X. Zong, F. Wu, and C. Huang, "Preserving exponential mean square stability and decay rates in two classes of theta approximations of stochastic differential equations," *Journal of Difference Equations and Applications*, vol. 20, no. 7, pp. 1091–1111, 2014.
- [19] F. Jiang, X. Zong, C. Yue, and C. Huang, "Double-implicit and split two-step Milstein schemes for stochastic differential equations," *International Journal of Computer Mathematics*, vol. 93, no. 12, pp. 1987–2011, 2016.
- [20] X. Zong, F. Wu, and G. Xu, "Convergence and stability of two classes of theta-Milstein schemes for stochastic differential equations," <https://arxiv.org/abs/1501.03695>.

- [21] X. Mao, *Stochastic Differential Equations and Applications*, Elsevier, 2007.
- [22] B. Tian, M. A. Eissa, and Z. Shijie, "Two families of theta milstein methods in a real options framework," in *Proceedings of the 5th Annual International Conference on Computational Mathematics, Computational Geometry and Statistics (CMCGS, '6, 1819*, Singapore, 2016.
- [23] S. Singh, "Split-step forward Milstein method for stochastic differential equations," *International Journal of Numerical Analysis & Modeling*, vol. 9, no. 4, pp. 970–981, 2012.
- [24] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, vol. 23, Springer-Verlag, Berlin, Germany, 1992.

