Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 1745734, 13 pages
https://doi.org/10.1155/2018/1745734

Research Article

Full and Reduced-Order Unknown Input Observer Design for Linear Time-Delay Systems with Multiple Delays

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Received 31 July 2018; Revised 2 November 2018; Accepted 11 November 2018; Published 26 November 2018

Academic Editor: Francesco Conte

In this work, we consider the design problem of a full-order and also reduced-order unknown input observers for a particular class of time-delay systems. Asymptotic stability and existence conditions for the designed observers are established. The quadruple-tank process is used as a benchmark to prove the efficiency of the proposed algorithms.

1. Introduction

Time delays emerge in many industrial applications such as hydraulic systems, automotive applications, and telecommunication networks [1–7]. Commonly, delays describe propagation phenomena, energy transfer, or data transmission. Furthermore, it is recognized that delays are the key sources of instability and poor dynamics [8]. Since the sixteen's century, the research theme of time-delay systems has concerned lots of interest and has been considered as one of the essential research areas in control theory for which many significant research works have been devoted. The topical survey [9] and the recent books [10–12] as well as their related references prove the richness and maturity of this field. Time delays can appear in either the state variables, control inputs, or measurement outputs and the negligence of time delays in the controller design can lead to undesirable dynamics such as oscillations, bifurcations, chaos dynamics, and instability [8]. In this trend, stabilization of linear systems with delays has obtained a huge interest (see, for example, recent works [13–15] and related references). As reported in [16], for stabilization of continuous linear systems subject to time delays, most research works only focus in systems with input delays and solve the problem in the absence of state delays. To the best of the authors' knowledge, few research works have considered the stabilization problem with multiple delays in states and inputs where the problem is generally solved by assuming fully available state variables [1] or by resorting to output-feedback controllers [17]. The design problem of observer-based stabilization for systems with multiple delays in states and inputs is rarely solved [16, 18]. Until now, this issue of theoretical and practical importance has been remained a challenging problem for systems subject to unknown inputs, uncertainties, and input saturation.

In control theory, modeling and controlling benchmark systems is another fundamental issue. Such systems are considered as challenging control problems, yet, in spite of their uncomplicated arrangement. In this outline, several systems are commonly used in control theory for testing novel control algorithms. We can cite in this framework the three-tank process [19], mass-spring-damper system [20], bouncing ball [21], TORA system [22], hard-disk drive system [23], magnetic levitation system [24], cart-inverted pendulum [25], Furuta pendulum [26], reaction wheel pendulum [27], beam-and-ball [28], two-link flexible manipulator [29], and so on. In the same direction, the well-known quadruple-tank benchmark [30–36] has attracted many attention since it can show elementary notions in estimation and control theory, particularly performance limitation due to the

nonminimum-phase zeros and their output directions for multivariable systems. This system is composed of four coupled tanks, two pumps, and two valves [35]. Many research papers have been devoted to the problem of observer and controller design of this benchmark system without consideration of time delays [37–43] and to the best of our information, no researches have been investigated for unknown input observer design of time-delay quadruple-tank systems.

Motivated by the authors' early works for linear systems without delays [44], in this paper, we expand the problem of full-order unknown input observer (FOUIO) and *reduced-order unknown input observer* (ROUIO) design for linear systems with multiple time delays in states and inputs affected by unknown inputs. In this framework, it is important to note that a FOUIO is designed in [45–47] for linear systems with a single delay in the states and two delays in the inputs whereas another FOUIO is proposed in [48] for linear systems with a single delay in the states and a single delay in the inputs. For the last works, only the state information is used to design observers in open loops and to the best of our knowledge, there are no related works for which both the state vector and the input vectors are used to design the observer. Furthermore, in our best knowledge, there is not in the vast literature ROIU design approach that attempts to solve such a problem.

The design procedures of both reduced-order and full-order observers are considered here. The key contributions of this work can be recapitulated as follows:

(i) The designed observers' approaches do not require approximation of the infinite dimensional model of the delayed system in a lumped one as described in our previous works [45, 46].

(ii) Different from [46] where only the state vector is used to construct observers in open loop, in this paper, we construct the novel observers by using both state and input vectors. The structure of the proposed observers allows us to introduce more observer gains that can accomplish better control performance.

(iii) Different from the observer designed in [47] where the delayed system introduces a single delay in the states and a single delay in the inputs, in this paper, multiple time delays in the inputs are considered.

The rest of this paper is organized as follows: the problem statement is considered in Section 2. The full-order observer is designed in Section 3 whereas the reduced-order observer is developed in Section 4. Section 5 is devoted to the application of the proposed approaches to the quadruple-tank benchmark. Finally, conclusions are provided in Section 6.

2. Problem Statement

Consider a class of time-delay systems described by

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau_0) + \sum_{i=0}^{k} \left(B_i u(t-\tau_{i+1})\right) \\ &\quad + Dd(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathscr{R}^n$ is the state vector, $u(t) \in \mathscr{R}^m$ is the input vector, $d(t) \in \mathscr{R}^q$ is the unknown input vector, and $y(t) \in \mathscr{R}^p$ is the output vector; $A_0 \in \mathscr{R}^{n\times n}$, $A_1 \in \mathscr{R}^{n\times n}$, $B_i \in \mathscr{R}^{n\times m}$, $C \in \mathscr{R}^{p\times n}$, and $D \in \mathscr{R}^{n\times q}$ are constant matrices with appropriate dimensions, while τ_0 and τ_{i+1} for i=0, . . ., k are known and constant time delays.

The objective of this paper is to design two different approaches of unknown input observers (UIOs) for system (1):

(1) An FOUIO is described by

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Ly(t) + Jy(t-\tau_0) + \sum_{i=0}^{k} G_i u(t-\tau_{i+1}), \\ \hat{x}(t) &= \zeta(t) - Ey(t), \end{aligned} \tag{2}$$

where $\zeta(t) \in \mathscr{R}^n$ and $\hat{x}(t) \in \mathscr{R}^n$ are the observer state vector and the estimated state vector, respectively. N, L, J, E, and G_i (i=0. . .k) are matrices of appropriate dimensions to be calculated such that $\hat{x}(t)$ converges asymptotically to x(t) under the following assumptions: (A1) rank(D) = q; (A2) rank(C) = p; (A3) the pair (A_0, C) is observable (detectable).

(2) An *ROUIO* is also designed in this paper for the estimation of the state vector x(t) by obtaining the estimation of the vector: $\bar{\zeta}(t) = Fx(t), F \in \mathscr{R}^{s\times n}$, where $\bar{\zeta}(t) \in \mathscr{R}^s$ is the vector enclosing a part of the system's states or a combination of them. The dynamics of the ROUIO is

$$\begin{aligned} Z\dot{\hat{\bar{\zeta}}}(t) &= \bar{N}\hat{\bar{\zeta}}(t) + \bar{L}y(t) + \bar{J}y(t-\tau_0) \\ &\quad + \sum_{i=0}^{k} \bar{G}_i u(t-\tau_{i+1}), \\ \hat{x}(t) &= H\hat{\bar{\zeta}}(t) + \bar{E}y(t), \end{aligned} \tag{3}$$

where $\hat{\bar{\zeta}}(t) \in \mathscr{R}^s$ and $\hat{x}(t) \in \mathscr{R}^n$ are the observer state vector and estimated state vector, respectively. $\bar{N} \in \mathscr{R}^{s\times s}$, $\bar{L} \in \mathscr{R}^{s\times p}$, $\bar{J} \in \mathscr{R}^{s\times p}$, $\bar{G}_i \in \mathscr{R}^{s\times m}$ for (i = 0...k), $H \in \mathscr{R}^{n\times s}$, $\bar{E} \in \mathscr{R}^{n\times p}$, $F \in \mathscr{R}^{s\times n}$ and $Z \in \mathscr{R}^{s\times s}$ are matrices of appropriate dimensions to be calculated such that $\hat{\bar{\zeta}}(t)$ and $\hat{x}(t)$ converge asymptotically to $\bar{\zeta}(t)$ and x(t), respectively, under the assumptions as

(A4) rank(D) = q; (A5) rank(C) = p; (A6) the pair (A_0, C) is observable; (A7) $n > s > p \geq q$; (A8) $n = p + q$.

Remark 1. For the design procedure of ROUIO, a particular form for the matrix C can be chosen as $C = \begin{bmatrix} C_1 & 0_{p\times q} \end{bmatrix}$, where $C_1 \in \mathscr{R}^{p\times p}$ is a full rank (nonsingular) matrix. This is not restrictive as long as the matrix C is full row rank (conditions A2≡A5), and there will always be an orthogonal transformation which leads to the equation $y(t) = \begin{bmatrix} C_1 & 0 \end{bmatrix} x(t)$.

Either for the *FOUIO* (2) or the *ROUIO* (3), system (1) is controlled using the state-feedback controller as

$$u = K\hat{x}(t) \tag{4}$$

where $K \in \mathscr{R}^{m\times n}$ is designed using one of the appropriate approaches for system (1), for example, the approach proposed in [1].

3. Design of the Full-Order Observer

Theorem 2. *Consider system (1) with assumptions (A1-A3) presented above. The FOUIO (2) is convergent, i.e.,* $e(t) \longrightarrow 0$ *as* $t \longrightarrow \infty$ *with an arbitrary convergence rate for any x(0), d(t), and u(t), if there exists a positive-definite matrix* $R \in \mathscr{R}^{n\times n}$ *verifying the matrix inequality*

$$N^T R + RN < 0 \tag{5}$$

and if the matrix $P = (I + EC) \in \mathscr{R}^{n\times n}$ *fulfills the following conditions:*

$$\begin{aligned} NP + LC - PA_0 &= 0, \\ JC - PA_1 &= 0 \\ G_i - PB_i &= 0 \quad \forall i = 0 \cdots k, \\ PD &= 0. \end{aligned} \tag{6}$$

Proof. Define the observer reconstruction error vector as the difference between the estimated state $\hat{x}(t)$ described by (2) and the state vector related to system (1) as

$$\begin{aligned} e(t) &= \hat{x}(t) - x(t) = \zeta(t) - x(t) - Ey(t) \\ &= \zeta(t) - (I + EC)\, x(t). \end{aligned} \tag{7}$$

Using (1) and (2), the dynamics of the estimation error becomes

$$\dot{e}(t) = \dot{\zeta}(t) - (I + EC)\, \dot{x}(t), \tag{8}$$

or

$$\begin{aligned} \dot{e}(t) = {} & Ne(t) + \left[N(I + EC) + LC - (I + EC) A_0\right] x(t) \\ & + \left[JC - (I + EC) A_1\right] x(t - \tau_0) \\ & + \sum_{i=0}^{k} \left(\left[G_i - (I + EC) B_i\right] u(t - \tau_{i+1})\right) \\ & - (I + EC) Dd(t). \end{aligned} \tag{9}$$

Using $P = I + EC$, the error dynamics (9) is written in the following form:

$$\begin{aligned} \dot{e}(t) = {} & Ne(t) + \left(NP + LC - PA_0\right) x(t) \\ & + \left(JC - PA_1\right) x(t - \tau_0) \\ & + \sum_{i=0}^{k} \left(\left(G_i - PB_i\right) u(t - \tau_{i+1})\right) - PDd(t). \end{aligned} \tag{10}$$

If conditions (6) are satisfied, the error dynamics of the observer is expressed as $\dot{e}(t) = Ne(t)$. FOUIO's convergence is then achieved if the LMI (5) is verified. □

The determination of the matrices N, L, J, and P from the first, second, and fourth equations of (6) is a difficult task because we have to calculate four matrices by using only three equations. In order to use the well-known results obtained for the classical full-order observer without unknown inputs [2], by means of the notation

$$K_0 = L + NE, \tag{11}$$

the first term of (6) becomes

$$N = PA_0 - K_0 C. \tag{12}$$

Now, using (11) and (12), the dynamics (2) can be written as

$$\begin{aligned} \dot{\zeta}(t) = {} & \left(PA_0 - K_0 C\right) \zeta(t) + Ly(t) + Jy(t - \tau_0) \\ & + \sum_{i=0}^{k} G_i u(t - \tau_{i+1}), \\ \hat{x}(t) = {} & \zeta(t) - Ey(t). \end{aligned} \tag{13}$$

If the pair (PA_0, C) is not observable, the calculation of matrix K_0 is made such that the observer is asymptotically stable if and only if (PA_0, C) is detectable [2]. By extending the approach from [2], we can conclude that the necessary and sufficient conditions to design a stable observer are given by the following theorem.

Theorem 3 (see [2]). *For system (1), the full-order observer (13) exists if and only if* (1) $\operatorname{rank}(CD) = q$; (2) $\operatorname{rank}\begin{bmatrix} sI - PA_0 \\ C \end{bmatrix} = n$, $(\forall)\mathbf{s} \in Ł$, $\operatorname{Re}(\mathbf{s}) \geq 0$.

Proof. The proof of the theorem is an extension of the approach from [2]. Because system (13) is in the form of a standard observer equation, then the matrix K_0 can be calculated such that the observer (2) is asymptotically stable if and only if the pair (PA_0, C) is observable. In [2], it is proved that the above condition (2) is equivalent with the pair (PA_0, C)– observable or at least detectable (first existence condition of the full-order observer). The second constraint is the condition (1) from Theorem 3; bearing in mind the assumptions A1 and A2, as well as the dimensions of the matrices C and D, this condition is always fulfilled. Therefore, up to this point in the design of the unknown input observer, the only existence condition of FOUIO is related to the observability or at least detectability of the pair (PA_0, C). □

To calculate N, L, J, P, G_0, and G_1, we partition matrices A_0, A_1, C, N, P, L, and J as follows:

$$A_0 = \begin{bmatrix} A_{0(11)} & A_{0(12)} \\ A_{0(21)} & A_{0(22)} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} A_{1(11)} & A_{1(12)} \\ A_{1(21)} & A_{1(22)} \end{bmatrix},$$

$$C^T = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix},$$

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix},$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \quad J = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}, \tag{14}$$

where

$$\begin{aligned} A_0 &\in \mathscr{R}^{n\times n}, \\ A_{0(11)} &\in \mathscr{R}^{(n-p)\times(n-p)}, \\ A_{0(12)} &\in \mathscr{R}^{(n-p)\times p}, \\ A_{0(21)} &\in \mathscr{R}^{p\times(n-p)}, \\ A_{0(22)} &\in \mathscr{R}^{p\times p}, \\ A_1 &\in \mathscr{R}^{n\times n}, \\ A_{1(11)} &\in \mathscr{R}^{(n-p)\times(n-p)}, \\ A_{1(12)} &\in \mathscr{R}^{(n-p)\times p}, \\ A_{1(21)} &\in \mathscr{R}^{p\times(n-p)}, \\ A_{1(22)} &\in \mathscr{R}^{p\times p}, \\ N &\in \mathscr{R}^{n\times n}, \\ N_{11} &\in \mathscr{R}^{(n-p)\times(n-p)}, \\ N_{12} &\in \mathscr{R}^{(n-p)\times p}, \\ N_{21} &\in \mathscr{R}^{p\times(n-p)}, \\ N_{22} &\in \mathscr{R}^{p\times p}, \\ P &\in \mathscr{R}^{n\times n}, \\ P_{11} &\in \mathscr{R}^{(n-p)\times(n-p)}, \\ P_{12} &\in \mathscr{R}^{(n-p)\times p}, \\ P_{21} &\in \mathscr{R}^{p\times(n-p)}, \\ P_{22} &\in \mathscr{R}^{p\times p}, \\ L &\in \mathscr{R}^{n\times p}, \\ L_1 &\in \mathscr{R}^{(n-p)\times p}, \\ L_2 &\in \mathscr{R}^{p\times p}, \\ J &\in \mathscr{R}^{n\times p}, \\ J_1 &\in \mathscr{R}^{(n-p)\times p}, \\ J_2 &\in \mathscr{R}^{p\times p}, \\ C &\in \mathscr{R}^{p\times n}, \\ C_1 &\in \mathscr{R}^{p\times(n-p)}, \\ C_2 &\in \mathscr{R}^{p\times p}. \end{aligned} \tag{15}$$

The observer's design is concentrated into the following theorem.

Theorem 4. *If the rank conditions in Theorem 3 are satisfied, the UIO (2) can be designed for system (1) as follows:*

$$\begin{aligned} E &= D(CD)^+ C, \\ P &= I_n - D(CD)^+ C, \\ G_i &= PB_i \quad (\textit{for } k = 0\cdots k), \\ J &= PA_1C^+, \\ N &= \begin{bmatrix} N_{11} & \left(F_{12} - \left(F_{11} - F_{12}P_{22}^{-1}P_{21}\right)\left(C_1 - C_2P_{22}^{-1}P_{21}\right)^+ C_2\right)P_{22}^{-1} \\ N_{21} & \left(F_{22} - \left(F_{21} - F_{22}P_{22}^{-1}P_{21}\right)\left(C_1 - C_2P_{22}^{-1}P_{21}\right)^+ C_2\right)P_{22}^{-1} \end{bmatrix}, \\ L &= \begin{bmatrix} \left(F_{11} - F_{12}P_{22}^{-1}P_{21}\right)\left(C_1 - C_2P_{22}^{-1}P_{21}\right)^+ \\ \left(F_{21} - F_{22}P_{22}^{-1}P_{21}\right)\left(C_1 - C_2P_{22}^{-1}P_{21}\right)^+ \end{bmatrix}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} F_{11} &= P_{11}A_{0(11)} + P_{12}A_{0(21)} - N_{11}P_{11}, \\ F_{12} &= P_{11}A_{0(12)} + P_{12}A_{0(22)} - N_{11}P_{12}, \\ F_{21} &= P_{21}A_{0(11)} + P_{22}A_{0(21)} - N_{21}P_{11}, \\ F_{22} &= P_{21}A_{0(12)} + P_{22}A_{0(22)} - N_{21}P_{21}, \end{aligned} \tag{17}$$

while N_{11} *and* N_{21} *are arbitrarily chosen;* P_{22} *should be a nonsingular matrix.*

Proof. Since rank(CD) = q, i.e., the matrix (CD) has full column rank, we can use the pseudoinverse of (CD), given by [17]: $(CD)^+ = [(CD)^T(CD)]^{-1}(CD)^T$. By means of this pseudoinverse matrix, we can solve the fourth equation of (6); one successively gets

$$\begin{aligned} (I + EC)D = 0 \Longleftrightarrow D + ECD = 0 \Longleftrightarrow E \\ = -D(CD)^+. \end{aligned} \tag{18}$$

Hence, one gets $P = I_n + EC = I_n - D(CD)^+C$. Replacing P into the third equation of (6), we obtain $G_i = PB_i = (I_n - D(CD)^+C)B_i$. Now, substituting (14) into the first equation of (6), we obtain

$$\begin{aligned} &N_{11}P_{11} + N_{12}P_{21} + L_1C_1 - P_{11}A_{0(11)} - P_{12}A_{0(21)} \\ &\quad = 0_{(n-p)\times(n-p)}, \\ &N_{11}P_{12} + N_{12}P_{22} + L_1C_2 - P_{11}A_{0(12)} - P_{12}A_{0(22)} \\ &\quad = 0_{(n-p)\times p}, \\ &N_{21}P_{11} + N_{22}P_{21} + L_2C_1 - P_{21}A_{0(11)} - P_{22}A_{0(21)} \\ &\quad = 0_{p\times(n-p)}, \end{aligned}$$

$$N_{21}P_{12} + N_{22}P_{22} + L_2C_2 - P_{21}A_{0(12)} - P_{22}A_{0(22)} = 0_{p\times p} \tag{19}$$

By using (17), system (19) is divided into the following systems:

$$\begin{aligned} N_{12}P_{21} + L_1C_1 &= F_{11}, \\ N_{12}P_{22} + L_1C_2 &= F_{12}, \end{aligned} \tag{20}$$

$$\begin{aligned} N_{22}P_{22} + L_2C_2 &= F_{22}, \\ N_{22}P_{21} + L_2C_1 &= F_{21}. \end{aligned} \tag{21}$$

Working in the hypothesis: the matrix $(C_1 - C_2P_{22}^{-1}P_{21}) \in \mathscr{R}^{p\times(n-p)}$ is full column rank. By solving systems (20) and (21), we obtain

$$\begin{aligned} L_1 &= \left(F_{11} - F_{12}P_{22}^{-1}P_{21}\right)\left(C_1 - C_2P_{22}^{-1}P_{21}\right)^+, \\ L_2 &= \left(F_{21} - F_{22}P_{22}^{-1}P_{21}\right)\left(C_1 - C_2P_{22}^{-1}P_{21}\right)^+, \\ N_{12} &= \Big(F_{12} \\ &\quad - \left(F_{11} - F_{12}P_{22}^{-1}P_{21}\right)\left(C_1 - C_2P_{22}^{-1}P_{21}\right)^+ C_2\Big)P_{22}^{-1}, \\ N_{22} &= \Big(F_{22} \\ &\quad - \left(F_{21} - F_{22}P_{22}^{-1}P_{21}\right)\left(C_1 - C_2P_{22}^{-1}P_{21}\right)^+ C_2\Big)P_{22}^{-1}. \end{aligned} \tag{22}$$

Now, solving the second equation of (6), we get

$$J = PA_1C^+. \tag{23}$$

The unknown matrices of the FOUIO (2), i.e., N, L, J, P, and G_i, have been successfully determined with respect to the arbitrarily chosen matrices N_{11} and N_{21}. According to Theorem 2, the obtained matrix N should satisfy two constraints: (1) there exists a symmetric and positive-definite matrix $R \in \mathscr{R}^{n\times n}$ which verifies the LMI (5), i.e., the matrix N is Hurwitz; (2) the pair (PA_0, C) is observable. If these two conditions are satisfied, the matrix N has been obtained properly, the dynamics of the observer error has an homogeneous form, and $e(t) \longrightarrow 0$ as $t \longrightarrow \infty$ with an arbitrary convergence rate for any x(0), d(t), and u(t). Otherwise, other matrices N_{11}, N_{21} are chosen, the matrices from (17) are again calculated, and systems (20) and (21) are solved until the above presented constraints are fulfilled. In fact, no condition regarding the observability or detectability of the pairs (PA_0, C) is required in our design approach. Therefore, the unique existence condition of the observer is rank(CD) = q. The theorem's proof is now complete. □

The algorithm associated to the delay-dependent observer is summarized below.

Algorithm 5. Design of FOUIO

Step 1. Given system (1), verify the related assumptions A1-A3.

Step 2. If the existence conditions given in Theorem 3 hold, one calculates the generalized inverse matrix $(CD)^+$ related to the matrix CD. By using it, one computes the matrix E by means of the first equation of (16); then, one computes the matrices P, G_0, and G_1 with the second, third, and fourth equations of (16), respectively.

Step 3. One partitions the matrices A_0, A_1, C, N, P, L, and J according to (14).

Step 4. Matrices N_{11} and N_{21} are arbitrarily chosen; the matrices from (17) are determined.

Step 5. One solves systems (20) and (21); after that, the matrices N, L, and J are built.

Step 6. Using the obtained matrix N, if the required constraints are satisfied, then (1) the matrix N is Hurwitz, i.e., there exists a symmetric and positive-definite matrix $R \in \mathscr{R}^{n\times n}$ verifying the LMI (5); (2) the pair (PA_0, C) is observable or at least detectable; the matrix N has been obtained properly; otherwise, Steps 4–6 are repeated (in a "while" loop) until these conditions are fulfilled. Bearing in mind that there is no risk for infinite "while" loop, there can be concluded that the FOUIO's design algorithm has no existence conditions, being characterized by lack of a priori restrictions on the class of systems to be considered.

Step 7. The observer described by (2) is completely designed and the time history of the system's estimated states can be obtained.

4. Design of the Reduced-Order Observer

Theorem 6. *Consider the LTI multivariable system (1) with multiple delays under the five assumptions (A4-A8) presented above; the ROUIO (3) is asymptotically stable, if and only if there exist symmetric and positive-definite matrix* $\overline{R} \in \mathscr{R}^{s\times s}$ *verifying the LMI:*

$$\widetilde{N}^T\overline{R} + \overline{R}\widetilde{N} < 0, \tag{24}$$

with $\widetilde{\boldsymbol{N}} = \boldsymbol{Z}^{-1}\overline{\boldsymbol{N}}$, *and if there exists a vector* $\overline{\zeta}(t) = Fx(t)$ *such that the next conditions are satisfied:*

$$\begin{aligned} \overline{N}F + \overline{L}C - ZFA_0 &= 0, \\ \overline{J}C - ZFA_1 &= 0, \\ \overline{G}_i - ZFB_i &= 0 \quad \forall i = 0\cdots k, \\ ZFD &= 0, \\ \overline{E}C + HF &= I_n; \end{aligned} \tag{25}$$

Proof. Define the observer error vector defined as the difference between the estimated vector $\hat{\overline{\zeta}}(t)$ described by (3) and the state vector related to system (1) as

$$\overline{e}(t) = \hat{\overline{\zeta}}(t) - \overline{\zeta}(t) \tag{26}$$

Using (1), (3), and (4), the dynamics of the estimation error becomes

$$\begin{aligned} Z\dot{\bar{e}}(t) &= \overline{N}\bar{e}(t) + \left[\overline{N}F - ZFA_0 + \overline{L}C\right] x(t) \\ &\quad + \left[\bar{J}C - ZFA_1\right] x(t-\tau_0) \\ &\quad + \sum_{i=0}^{k} \left(\left[\overline{G}_i - ZFB_i\right] u(t-\tau_{i+1})\right) - ZFDd(t). \end{aligned} \tag{27}$$

If $\overline{N}F + \overline{L}C - ZFA_0 = 0, \bar{J}C - ZFA_1 = 0, \overline{G}_i - ZFB_i = 0\ \forall i = 0 \cdots k$, $ZFD = 0$ the error dynamics (27) can be written as

$$\dot{\bar{e}}(t) = \widetilde{N}\bar{e}(t), \tag{28}$$

where $\widetilde{\boldsymbol{N}} = \boldsymbol{Z}^{-1}\overline{\boldsymbol{N}}$.

Let the state estimation error be defined as $e_x(t) = \hat{x}(t) - x(t)$. From (1) and (3), one obtains $e_x(t) = (HF - I_n + \overline{E}C)x(t)$. If $HF - I_n + \overline{E}C = 0$, $e_x(t) = 0$ for any $x(t)$ results.

Furthermore, if condition (24) is satisfied then the observer's error dynamics (28) is asymptotically stable. Now, the proof of the Theorem 6 is complete. □

To compute the matrices $F, Z, \overline{E}, \overline{N}, \overline{L}, \bar{J}, H, \overline{G}_0$, and $\overline{G}_1$, let us partition the matrices A_0, A_1, C, $\overline{N}$, H, $\overline{L}$, $\bar{J}$, F, Z, and D as follows:

$$A_0 = \begin{bmatrix} A_{0(11)} & A_{0(12)} \\ A_{0(21)} & A_{0(22)} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} A_{1(11)} & A_{1(12)} \\ A_{1(21)} & A_{1(22)} \end{bmatrix},$$

$$C = \begin{bmatrix} C_1 & 0_{p\times q} \end{bmatrix} = \begin{bmatrix} \widetilde{C}_1 & \widetilde{C}_2 \end{bmatrix},$$

$$D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

$$\overline{N} = \begin{bmatrix} \overline{N}_{11} & \overline{N}_{11} \\ \overline{N}_{21} & \overline{N}_{22} \end{bmatrix} = \begin{bmatrix} \overline{N}_1 & \overline{N}_2 \end{bmatrix},$$

$$H = \begin{bmatrix} H_{11} & H_{11} \\ H_{21} & H_{22} \end{bmatrix},$$

$$\overline{L} = \begin{bmatrix} \overline{L}_1 \\ \overline{L}_2 \end{bmatrix},$$

$$\bar{J} = \begin{bmatrix} \bar{J}_1 \\ \bar{J}_2 \end{bmatrix},$$

$$\overline{E} = \begin{bmatrix} \overline{E}_1 \\ \overline{E}_2 \end{bmatrix},$$

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} 0_{p\times q} & I_{p\times p} \\ 0_{(s-p)\times q} & 0_{(s-p)\times p} \end{bmatrix},$$

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_1 & Z_2 \end{bmatrix},$$

$$Z_1 = \begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix},$$

$$Z_2 = \begin{bmatrix} Z_{12} \\ Z_{22} \end{bmatrix}, \tag{29}$$

where

$$\begin{aligned} A_0 &\in \mathscr{R}^{n\times n}, \\ A_{0(11)} &\in \mathscr{R}^{q\times q}, \\ A_{0(12)} &\in \mathscr{R}^{q\times p}, \\ A_{0(21)} &\in \mathscr{R}^{p\times q}, \\ A_{0(22)} &\in \mathscr{R}^{p\times p}, \\ A_1 &\in \mathscr{R}^{n\times n}, \\ A_{1(11)} &\in \mathscr{R}^{q\times q}, \\ A_{1(12)} &\in \mathscr{R}^{q\times p}, \\ A_{1(21)} &\in \mathscr{R}^{p\times q}, \\ A_{1(22)} &\in \mathscr{R}^{p\times p}, \\ \overline{N} &\in \mathscr{R}^{s\times s}, \\ \overline{N}_{11} &\in \mathscr{R}^{q\times p}, \\ \overline{N}_{12} &\in \mathscr{R}^{q\times(s-p)}, \\ \overline{N}_{21} &\in \mathscr{R}^{(s-q)\times p}, \\ \overline{N}_{22} &\in \mathscr{R}^{(s-q)\times(s-p)}, \\ H &\in \mathscr{R}^{n\times s}, \\ H_{11} &\in \mathscr{R}^{q\times p}, \\ H_{12} &\in \mathscr{R}^{q\times(s-p)}, \\ H_{21} &\in \mathscr{R}^{p\times p}, \\ H_{22} &\in \mathscr{R}^{p\times(s-p)}, \\ \overline{L} &\in \mathscr{R}^{s\times p}, \\ \overline{L}_1 &\in \mathscr{R}^{q\times p}, \\ \overline{L}_2 &\in \mathscr{R}^{(s-q)\times p}, \\ \bar{J} &\in \mathscr{R}^{s\times p}, \\ \bar{J}_1 &\in \mathscr{R}^{q\times p}, \\ \bar{J}_2 &\in \mathscr{R}^{(s-q)\times p}, \\ \overline{E} &\in \mathscr{R}^{n\times p}, \end{aligned}$$

$$\begin{aligned}
\overline{E}_1 &\in \mathscr{R}^{q\times p},\\
\overline{E}_2 &\in \mathscr{R}^{p\times p},\\
D &\in \mathscr{R}^{n\times q},\\
D_1 &\in \mathscr{R}^{q\times q},\\
D_2 &\in \mathscr{R}^{p\times q},\\
Z &\in \mathscr{R}^{s\times s},\\
Z_{11} &\in \mathscr{R}^{q\times p},\\
Z_{12} &\in \mathscr{R}^{q\times (s-p)},\\
Z_{21} &\in \mathscr{R}^{(s-q)\times p},\\
Z_{22} &\in \mathscr{R}^{(s-q)\times (s-p)},\\
Z_1 &\in \mathscr{R}^{s\times p},\\
Z_2 &\in \mathscr{R}^{s\times (s-p)},\\
\overline{N}_1 &\in \mathscr{R}^{s\times p},\\
\overline{N}_2 &\in \mathscr{R}^{s\times (s-p)},\\
C &\in \mathscr{R}^{p\times n},\\
C_1 &\in \mathscr{R}^{p\times p},\\
\widetilde{C}_1 &\in \mathscr{R}^{p\times q},\\
\widetilde{C}_2 &\in \mathscr{R}^{p\times p}.
\end{aligned} \tag{30}$$

The ROUIO's design is concentrated into the following theorem.

Theorem 7. *Consider the LTI multivariable (1) under the assumptions A4-A8; the ROUIO (3) is asymptotically stable if and only if the following conditions are satisfied:* (1) *there exists the matrix* $\overline{R} > 0$ *verifying the LMI (24);* (2) *the matrices* $\overline{G}_0, \overline{G}_1, \overline{N}, \overline{L}, \overline{J}, \overline{E}$, *and* H *are given by*

$$\begin{aligned}
Z_1 &= I_{s\times p} - \overline{D}_2 \left(C_1 D_2\right)^+ C_1,\\
\overline{G}_i &= \left[0_{s\times q} \quad Z_1\right] B_i \quad \forall i = 1\cdots k,\\
\overline{N} &= \left[Z_1 \left(A_{0(14)} - A_{0(13)}\widetilde{C}_1^+\widetilde{C}_2\right) \quad \overline{N}_2\right],\\
\overline{L} &= Z_1 A_{0(21)}\widetilde{C}_1^+,\\
\overline{J} &= \left(ZFA_1\right) C^+,\\
\overline{E} &= \begin{bmatrix} \widetilde{C}_1^+ \\ I_p - \widetilde{C}_1 \left(C_1\widetilde{C}_1\right)^+ C_1 \end{bmatrix},\\
H &= \begin{bmatrix} -\widetilde{C}_1^+\widetilde{C}_2 & 0_{q\times(s-p)} \\ I_p - \widetilde{C}_2 + \widetilde{C}_1 \left(C_1\widetilde{C}_1\right)^+ C_1\widetilde{C}_2 & 0_{p\times(s-p)} \end{bmatrix},
\end{aligned} \tag{31}$$

where the matrices $Z_2 \in \mathscr{R}^{s\times(s-p)}$ *and* $\overline{N}_2 \in \mathscr{R}^{s\times(s-p)}$ *are arbitrarily chosen,* $\widetilde{C}_1 \in \mathscr{R}^{p\times q}$ *and* $\widetilde{C}_2 \in \mathscr{R}^{p\times p}$ *are submatrices of the matrix* C, *and* $\overline{D}_2 = I_{s\times p}D_2$.

Proof. Using (29) and (30), the fourth equation of (25) becomes

$$Z_1 D_2 = 0_{s\times q}. \tag{32}$$

To compute the matrix Z_1 from this equation, one can choose

$$Z_1 = I_{s\times p} + T_1 C_1, \tag{33}$$

with $T_1 \in \mathscr{R}^{s\times p}$ an unknown matrix to be computed. With the notation $\overline{D}_2 = I_{s\times p}D_2$, one gets $T_1 = -\overline{D}_2(C_1D_2)^+$, where $(C_1D_2)^+$ is the generalized pseudoinverse of (C_1D_2), given by $(C_1D_2)^+ = [(C_1D_2)^T(C_1D_2)]^{-1}(C_1D_2)^T$. According to the assumption A5 and to the fact that the matrix C_1 is full rank, we can conclude that $\operatorname{rank}(C_1D_2) = q$, and the generalized pseudoinverse of (C_1D_2) can be always obtained.

Equation (33) can be then written as

$$Z_1 = I_{s\times p} - \overline{D}_2 \left(C_1 D_2\right)^+ C_1, \tag{34}$$

and the matrix $Z_2 \in \mathscr{R}^{s\times(s-p)}$ is chosen arbitrarily.

Now, using the expression of the matrix Z, we obtain the matrices $\overline{G}_i$ in the third equation of (25). We arbitrarily choose the matrices N_{12} and N_{22}; as a consequence, the matrix $\overline{N}_2$ is arbitrarily chosen. Using this as well as the form of matrix C given in (29) and (30), i.e., $C = \left[C_1 \quad 0_{p\times q}\right] = \left[\widetilde{C}_1 \quad \widetilde{C}_2\right]$, the first equation (25) can be written in a form of matrix system as follows:

$$\begin{aligned}
-Z_{11}A_{0(21)} + \overline{L}_1\widetilde{C}_1 &= 0_{q\times q},\\
\overline{N}_{11} - Z_{11}A_{0(22)} + \overline{L}_1\widetilde{C}_2 &= 0_{q\times p},\\
-Z_{21}A_{0(21)} + \overline{L}_2\widetilde{C}_1 &= 0_{(s-q)\times q},\\
\overline{N}_{21} - Z_{21}A_{0(22)} + \overline{L}_2\widetilde{C}_2 &= 0_{(s-q)\times p};
\end{aligned} \tag{35}$$

by solving system (35), we obtain

$$\begin{aligned}
\overline{N}_{11} &= Z_{11}\left(A_{0(22)} - A_{0(21)}\widetilde{C}_1^+\widetilde{C}_2\right),\\
\overline{L}_1 &= Z_{11}A_{0(21)}\widetilde{C}_1^+,\\
\overline{N}_{21} &= Z_{21}\left(A_{0(22)} - A_{0(21)}\widetilde{C}_1^+\widetilde{C}_2\right),\\
\overline{L}_2 &= Z_{21}A_{0(21)}\widetilde{C}_1^+;
\end{aligned} \tag{36}$$

bearing in mind the fact that $q < p$, it results that $\widetilde{C}_1$ is a submatrix of the nonsingular matrix C_1; from this, one deduces that $\operatorname{rank}(\widetilde{C}_1) = q$, i.e., $\widetilde{C}_1$ is a full column rank matrix.

Now, solving the second equation of (25), we get

$$\overline{J} = \left(ZFA_1\right) C^+. \tag{37}$$

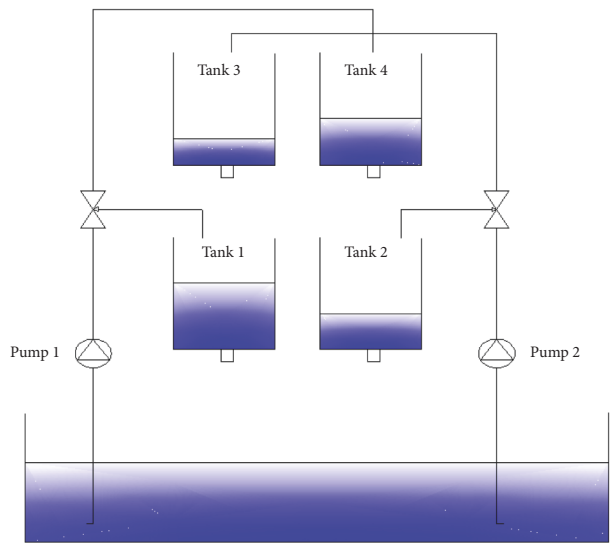

Figure 1: The quadruple-tank process.

Using (29) and (30), the sixth equation (25) can be transformed into a matrix system as

$$\begin{aligned} \overline{E}_1\widetilde{C}_1 &= I_q, \\ \overline{E}_1\widetilde{C}_2 + H_{11} &= 0_{q\times p}, \\ \overline{E}_2\widetilde{C}_1 &= 0_{p\times q}, \\ \overline{E}_2\widetilde{C}_2 + H_{21} &= I_p; \end{aligned} \tag{38}$$

from the first and second equations of (38), we obtain $\overline{E}_1 = \widetilde{C}_1^+$ and $H_{11} = -\overline{E}_1\widetilde{C}_2 = -\widetilde{C}_1^+\widetilde{C}_2$. On the other hand, the matrix $\overline{E}_2$ is calculated from the third equation of (35) as follows:

$$\overline{E}_2 = I_p + T_2C_1, \tag{39}$$

where $T_2 \in \mathscr{R}^{p\times p}$ is an unknown matrix to be computed. Because $C_1 \in \mathscr{R}^{p\times p}$ is a nonsingular matrix and $\text{rank}(\widetilde{C}_1) = q$, one gets $\text{rank}(C_1\widetilde{C}_1) = q$, i.e., $(C_1\widetilde{C}_1)$ is a full column rank matrix. Taking this into account, one successively gets $T_2 = -\widetilde{C}_1(C_1\widetilde{C}_1)^+$ and $\overline{E}_2 = I_p - \widetilde{C}_1(C_1\widetilde{C}_1)^+C_1$. From the fourth equation of system (38), we obtain $H_{12} = I_p - \widetilde{C}_2 + \widetilde{C}_1(C_1\widetilde{C}_1)^+C_1\widetilde{C}_2$; by concatenation of the matrices $\overline{E}_1, \overline{E}_2, H_{11}, H_{21}, H_{12} = 0_{q\times(s-p)}$ and $H_{22} = 0_{p\times(s-p)}$, the last two equations (31) result. Now, the proof of the Theorem 7 is complete. □

Algorithm 8. Design of the ROUIO.

Step 1. Given system (1), verify the five related assumptions A4-A8.

Step 2. The matrix F of from (29) is chosen; the matrices $A_0, A_1, C, D, \overline{N}, H, \overline{L}, \overline{J}, \overline{E}$, and Z are partitioned with (29) and (30). The matrices Z_{12} and Z_{22} and thus the matrix Z_2 are arbitrarily chosen; the matrices Z_1, $\overline{G}_i$ $\forall i = 1\cdots k$, are obtained via (31).

Step 3. The matrices $\overline{N}_{12}, \overline{N}_{22}$ and thus the matrix $\overline{N}_2$ are arbitrarily chosen; by solving system (35) with respect to the matrices $\overline{N}_{11}, \overline{N}_{21}, \overline{L}_1, \overline{L}_2$ or, directly, by means of (36), these four matrices are obtained; the matrices $\overline{N}, \overline{J}$ and $\overline{L}$ are then calculated with (31). Matrices $\overline{E}$ and H are computed by solving system (38).

Step 4. Using the above calculated matrix $\overline{N}$, one checks if there exists a symmetric and positive-definite matrix $\overline{R} \in \mathscr{R}^{n\times n}$ verifying the LMI (24); if so, the matrix $\overline{N}$ has been properly obtained; otherwise, Steps 2–4 are repeated (in a "while" loop) until this condition is fulfilled.

Step 5. The observer described by (3) is completely designed and the time history of the system's estimated states can be obtained.

5. Numerical Simulation Results

5.1. The Quadruple-Tank Process. The quadruple-tank process is shown in Figure 1. It is composed of four connected tanks and two pumps that split water into two tanks [30, 31, 49]. The inlet flow of each tank is measured by an electromagnetic flow-meter and regulated by a pneumatic valve whereas the level of each tank h_i (I =1...4) is measured by means of a pressure sensor. The process inputs are the input voltages of the pumps (ϑ_1 and ϑ_2) and the output variables are the tank levels y_1 and y_2.

5.2. The Quadruple-Tank Process Model. For practical considerations, we assume that the operating regime of the process is well known that the quadruple-tank process is

affected by delays and that the transport delays are perfectly symmetric. In such a case, the model of the process can be written as case study of model (1) and can be described by [30]

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau_1) + B_0 u(t-\tau_2) \\ &\quad + B_1 u(t-\tau_3) + Dd(t), \\ y(t) &= Cx(t),\end{aligned} \tag{40}$$

with $x = [h_1 \;\; h_2 \;\; h_3 \;\; h_4]^T$ being the state vector, $u = [\vartheta_1 \;\; \vartheta_2]^T$ being the input vector, and $y = [y_1 \;\; y_2]^T$ being the output vector, respectively, and with

$$\begin{aligned}
A_0 &= \begin{bmatrix} -\frac{a_1}{A_1}\sqrt{\frac{g}{2h_{10}}} & 0 & 0 & 0 \\ 0 & -\frac{a_2}{A_2}\sqrt{\frac{g}{2h_{20}}} & 0 & 0 \\ 0 & 0 & -\frac{a_3}{A_3}\sqrt{\frac{g}{2h_{30}}} & 0 \\ 0 & 0 & 0 & -\frac{a_4}{A_4}\sqrt{\frac{g}{2h_{40}}} \end{bmatrix}, \\
B_0 &= \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
A_1 &= \begin{bmatrix} 0 & 0 & \frac{a_3}{A_1}\sqrt{\frac{g}{2h_{30}}} & 0 \\ 0 & 0 & 0 & \frac{a_4}{A_2}\sqrt{\frac{g}{2h_{40}}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix}.
\end{aligned} \tag{41}$$

Using the following numerical values

$$\begin{aligned}
D_1 &= D_2 = D_3 = D_4 = 9.2\,\text{cm}, \\
A_1 &= A_2 = A_3 = A_4 = \pi\left(\frac{D_1}{2}\right)^2, \\
h_{max} &= 50\,\text{cm}, \\
h_{i0} &= \frac{h_{max}}{2} = 25\,\text{cm}, \\
g &= 981\,\text{cm}\cdot\text{s}^{-2}, \\
d_1 &= 0.2\,\text{cm}, \\
a_1 &= a_2 = \pi\left(\frac{d_1}{2}\right), \\
d_3 &= 0.9, \\
a_3 &= a_4 = \pi\left(\frac{d_3}{2}\right)^2, \\
k_1 &= 7.39, \\
k_2 &= 6.92,
\end{aligned} \tag{42}$$

the matrices from (1) are given by

$$\begin{aligned}
A_0 &= \begin{bmatrix} -0.0021 & 0 & 0 & 0 \\ 0 & -0.0021 & 0 & 0 \\ 0 & 0 & -0.0424 & 0 \\ 0 & 0 & 0 & -0.0424 \end{bmatrix}, \\
B_0 &= \begin{bmatrix} 0.1113\times\gamma_1 & 0 \\ 0 & 0.1042\times\gamma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
A_1 &= \begin{bmatrix} 0 & 0 & 0.0424 & 0 \\ 0 & 0 & 0 & 0.0424 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & (1-\gamma_2)\times 0.1042 \\ (1-\gamma_1)\times 0.1042 & 0 \end{bmatrix}.
\end{aligned} \tag{43}$$

5.3. Software Implementation of the Full-Order Observer. For simulation results, the following numerical values are chosen: $x(0) = [-4 \;\; 4 \;\; 6 \;\; -5]$, $d(t) = 0.3\sin(2\pi ft)$, $D^T = \begin{bmatrix} 10 & 20 & -12 & 15 \\ -23 & 28 & 30 & -36 \end{bmatrix}$, $\gamma_1 = 0.333$, $\gamma_2 = 0.307$, $\alpha = 0.4420$, and $\tau_1 = 5\text{s}$, $\tau_2 = 3\text{s}$, $\tau_3 = 4\text{s}$. The control gain matrix is computed using the approach proposed in [1]

$$K = \begin{bmatrix} -0.1603 & -0.1765 & -0.0795 & -0.2073 \\ -0.1977 & -0.1579 & -0.2288 & -0.0772 \end{bmatrix} \tag{44}$$

For the FOUIO software implementation, one obtains

$$E = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1.264 & -0.032 \\ -1.54 & 0.02 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.26 & -0.032 & 1 & 0 \\ -1.54 & -0.020 & 0 & 1 \end{bmatrix},$$

$$G_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.046 & -0.01 \\ -0.057 & 0.06 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0.072 \\ 0.074 & 0 \end{bmatrix},$$

$$N = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & -0.0424 & 0 \\ 1 & 1 & 0 & -0.0424 \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.05 & -0.013 \\ -0.06 & 0.08 \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.043 & 0.095 \\ -0.025 & 0.017 \end{bmatrix}, \tag{45}$$

5.4. Software Implementation of the Reduced-Order Observer. During the first step of the Algorithm 8, we find the matrices and vectors from (3). In this paper, the order is chosen as equal n=4, m=2, p=2, q=2, and s=3 and the constant time delays are chosen as $\tau_1 = 5\text{s}, \tau_2 = 2\text{s}$, and $\tau_3 = 4\text{s}$. We obtain

$$\overline{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\overline{G}_0 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0 & 0.07 \end{bmatrix},$$

$$\overline{G}_1 = \begin{bmatrix} 0 & 0.072 \\ 0.074 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\overline{N} = 10^{13} \begin{bmatrix} 0 & 0 & 2.945 \\ 0 & 0 & 0.46 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\overline{L} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.05 & -0.013 \\ -0.06 & 0.08 \end{bmatrix},$$

$$\overline{J} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.043 & 0.095 \\ -0.025 & 0.017 \end{bmatrix},$$

$$Z = 10^{-14} \begin{bmatrix} 0.2220 & -0.0439 & 0.5255 \\ 0.1377 & 0.0111 & 0.8992 \\ 0 & 0 & 0.9452 \end{bmatrix} \tag{46}$$

Figure 2 shows the effectiveness of two designed observers for the quadruple-process and their performances for which the variables x_1 and x_2 (variation of water levels in tanks 1 and 2) are measured and only the variables x_3 and x_4 (variation of water levels in tanks 3 and 4) are estimated. The effectiveness of two observers is proved by the superpose of the three curves.

6. Conclusion

This study considers the design of two approaches of UIOs for systems with multiple time-delays: a full-order delay-dependent UIO and a reduced-order delay-dependent UIO. The quadruple-tank process was used as a benchmark to prove the effectiveness of the new observer algorithms for the case study of multivariable nonminimum phase systems with multiple delays.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

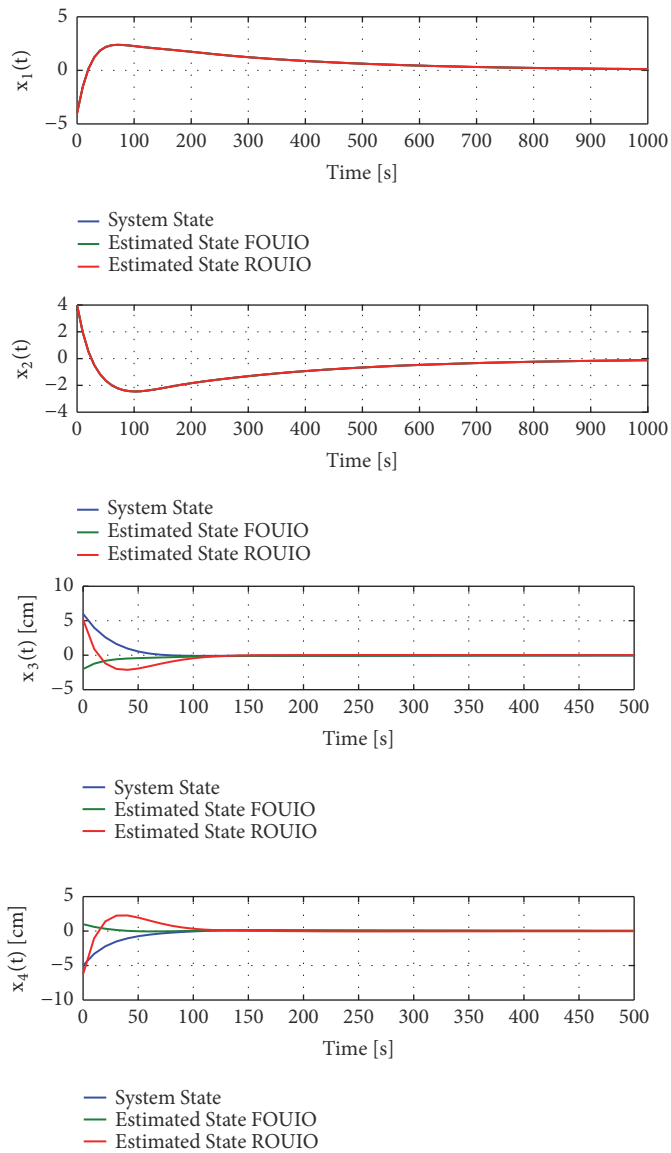

Figure 2: FOUIO and ROUIO of the quadruple-tank process.

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