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*Research Article*

# Extension Dependent Degree Method with Mapping Transformation for Three-Parameter Interval Number Decision Making

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In view of the multiattribute decision making problem that the attribute values and weights are both three-parameter interval numbers, a new decision making approach and framework based on extension simple dependent degree are proposed. According to traditional extension simple dependent function, the new approach proposes a new extension dependent function for three-parameter interval number. Then through an interval mapping transformation method, the process for obtaining dependent degree for the interval with its optimal value not being the endpoint is transformed to the monotonous process for the interval with its optimal value being the endpoint. The method can not only perform uncertain analysis of decision results by different settings of attitude coefficients, but also take dynamic analysis and rule finding by some extension transformation. At last, an example is presented to examine the effectiveness and stability of our method.

## 1. Introduction

In multiattribute decision making (MADM), attributes information often shows some ambiguity and uncertainty due to the complexity from objective things and the finiteness from decision makers, so it is difficult to describe by some accurate numerical values. Therefore, several methods that can describe this uncertainty information, such as interval number [1], fuzzy number [2], gray number [3], and connection number [4, 5], have been widely studied and applied. Among them, interval number theory, as one of the most possible solutions, has produced many valuable research results. However, the traditional interval number is too rough to describe the uncertain information. It only focuses on the upper and lower bounds of the interval and possibly ignores its intrinsic preference information from users, which greatly limits its practicability in many actual application scenarios. In comparison, the three-parameter interval number [6] not only retains the upper and lower bounds of the interval, but also emphasizes the gravity value which has the maximum hit possibility in all values. Therefore, it is superior to the traditional interval number method in describing the uncertainty information. In recent years, the three-parameter interval number theory has become the research focus in uncertainty decision making domain. Literature [7–9] establishes the concept of three-parameter interval gray number, correlation degree, and closeness calculation method according to the gray system theory. In literature [10], the ratio of the sums of three parameters is directly used as the interval comparison results, and then the pairwise comparison matrix is built to sort the scheme. In literature [11], the traditional interval likelihood ranking method is extended to the three-parameter interval ranking, and the TOPSIS model is established for three-parameter interval number. Literature [12] also adopts TOPSIS method to sort three-parameter interval numbers by redefining an Euclidean distance. In literature [13], combining the center value, the gravity value, and the interval's length of the three-parameter interval gray number, an exact score function is defined as the basis for comparison. Literature [14] proposes to convert the three-parameter interval into traditional two-parameter interval, so that the likelihood sorting method for the traditional interval can be used directly. In some earlier literature [15, 16], some triangle fuzzy numbers are used to describe the three-parameter interval, and the scheme order was determined based on the fuzzy number operation process and the distance measure of ideal point. Literature [17] defines the concept of three-parameter interval fuzzy set, its operation rules, and distance measure formula. Literature [18] proposes a three-parameter interval gray linguistic variable decision making method based on projection model and prospect theory.

In general, although the existing research has made some great progress, there are still some things to be improved. First, most of the research is based on the fuzzy number correlation theory including score function, fuzzy distance, fuzzy similarity, and likelihood method. These fuzzy number concepts and measures are only extended to the three-parameter interval number field, so there is no new method and framework. Second, many decision making models are too deterministic and lose their uncertainty in the process. Hence, it is difficult to carry out stability checking and uncertainty analysis for the decision results in the later stage. In this regard, the set pair analysis method is proposed [19], which converts the three-parameter interval number into the connection number expression and maintains the uncertainty of the result and the calculation simplicity. Therefore, it is another idea worthy of further study. Third, the existing decision making model is basically static. Researches on dynamic decision judgment and rule discovery are very inadequate.

In order to solve the above problems, the paper attempts to propose a new multiattribute decision making method and framework based on extension dependent degree. This thinking and method are rarely seen in the existing research literature. In this method, firstly, according to the extension simple dependent degree calculation method and its mapping transformation rule, the dependent degree calculation expression for the three-parameter interval and its interval map transformation method are given. This will transform the process of calculating dependent degree of the interval with its optimal value not being the endpoint to the monotonous process of the interval with its optimal value being the endpoint. It not only makes the calculation process simple and unified, but also expresses a new three-parameter interval sorting method. Secondly, six typical coefficient setting schemes are given for the attitude coefficient in the dependent degree calculation expression, which can reflect the different preference attitudes from the decision makers for the upper, lower, and average evaluation scores. It can make the model perform some uncertainty analysis for the decision results. After that, the comparison between our method and the existing other research results is shown by numerical examples, which illustrates the effectiveness and stability of the proposed method and its ability to perform uncertainty analysis. Finally, based on the extension dependent degree calculation, the dynamic analysis and rule discovery of the decision process through extension transformation are proposed, which shows the dynamic applicability of our method.

## 2. Extension Simple Dependent Degree for Three-Parameter Interval

### 2.1. Three-Parameter Interval Number

*Definition 1.* Let $R$ be the real number set, and for any $a, b \in R$, $a \leq b$, $X = [a, b]$ is noted as interval number. Here $a$ is the lower bound of the interval, and $b$ is the upper bound. When $a = b$, $X$ degenerates into a real number.

*Definition 2* (see [10–12]). Let $R$ be a real number set, and for any $a, b, m \in R$, $a \leq m \leq b$, $\overline{X} = [a, m, b]$ is noted as a three-parameter interval number. Here $a$ and $b$, respectively, represent the lower and upper bound of the interval number. $m$ represents a special point with some statistical meaning in this interval such as mean value or maximum possibility value, which is called gravity value or ideal value.

In fact, the three-parameter interval number is the expansion of the traditional interval number but may describe uncertainty and fuzzy numbers more abundantly and accurately. For example, there is an evaluation score for the performance of a product noted as a three-parameter interval number [4, 6, 8]. The number may mean that the lower and upper bounds from user group evaluations are 4 and 8, respectively. It can also mean that the score from one user covers between 4 and 8, and 6 is the most preferred score. It can also indicate that the lowest score and highest score for single attribute are 4 and 8, and the average score is 6. By some means, the three-parameter interval number is similar to the triangular fuzzy number [20, 21] expression. Both of them consider that the left and right interval endpoints are the critical points, and the middle points are the most representative ones. However, the triangular fuzzy number focuses more on the degree of the elements belonging to some fuzzy set. The three-parameter interval number pays more attention to the probability or the statistical meaning for the interval, and so its meaning is more universal and rich. It is worthy of further research and application.

*2.2. Extension Simple Dependent Degree and Mapping Transformation.* In extenics [22], extension simple dependent function describes the relationship between a point $x$ and interval covering $X$. Its calculation is based on the given range of values and does not need to rely on subjective judgment or empirical value from decision makers, so it is convenient to quantitatively describe the nature of things. It has been used in some evaluation and forecasting applications [23, 24].

*Definition 3* (see [22]). Suppose a finite interval $X = [a, b]$, and its optimal value is $m \in X$; then

$$k(x, X) = \begin{cases} \dfrac{x - a}{m - a}, & x \leq m \\ \dfrac{b - x}{b - m}, & x \geq m. \end{cases} \tag{1}$$

Here $k(x, X)$ is called extension simple dependent degree of the point $x$ and interval $X$, as shown in Figure 1, and it satisfies the following properties:

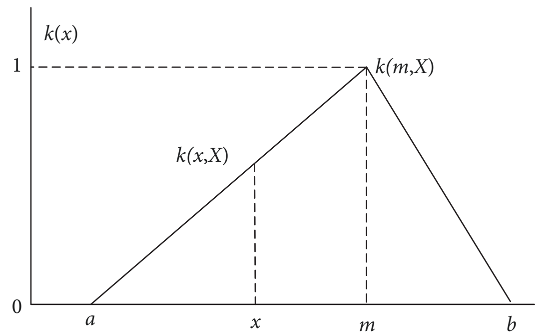



Figure 1: The extension simple dependent degree.

(1) When $x=m$, $k(x, X)$ reaches the maximum value and $k(x, X)=k(m, X)=1$.

(2) When $x\in X$ and $x\neq a,b$, then $k(x, X) > 0$.

(3) When $x\notin X$ and $x\neq a,b$, then $k(x, X) < 0$.

(4) When $x=a$ or $x=b$, then $k(x, X) = 0$.

A special case is when $m=a$ or $b$, and $x\in X$; then

$$k(x, X) = \begin{cases} \dfrac{b-x}{b-a}, & (m=a) \\ \dfrac{x-a}{b-a}, & (m=b). \end{cases} \tag{2}$$

Obviously, as shown in formula (2), when the optimal point $m$ is just at the endpoint of the interval, extension simple dependent degree formula becomes a monotonic increasing or monotonic decreasing function, and its calculation process is very concise and intuitive. However, as shown in formula (1), when the optimal point $m$ is not at the endpoint, the dependent degree formula must be changed according to the location of $m$, which will bring some complexity and inconveniences. Therefore, here we propose a mapping transformation method which will transform the dependent degree calculation with the optimal point not being the endpoint into the calculation with the optimal point being the endpoint, so as to keep the monotonicity and simplicity of the process.

*Definition 4.* Suppose a finite interval $X=[a,b]$ and its optimal value is $m\in X$, $m\neq a$, $b$. For $\forall x\in X$, there is a transformation $\theta(x) = x'$ that can make $x' \in [a,m)$ and $k(x',X) = k(x,X)$ hold. Then, $x'$ is called the dependent degree mapping point of $x$, and $\theta$ is called the dependent degree mapping transformation.

**Theorem 5.** *Suppose a finite interval $X = [a,b]$ and its optimal value is $m\in X$, $m\neq a$, $b$. For $\forall x\in X$, if $x'$ is the dependent degree mapping point of $x$, then $x' \in [a,m)$, and*

$$x' = \theta(x) = \begin{cases} k(x,X)(m-a)+a = \dfrac{b-x}{b-m}(m-a)+a & x\in(m,b] \\ x & x\in[a,m]. \end{cases} \tag{3}$$

*Proof.* From Definition 3, when $x \in (m,b]$, $k(x, X) \in [0,1)$, then $a \leq x' = k(x,X)(m-a)+a < m$; i.e., $x' \in [a,m)$. By Definition 3, $k(x',X) = (x'-a)/(m-a) = (k(x,X)(m-a)+a-a)/(m-a) = k(x,X)$. □

Obviously, for a finite interval $X = [a,b]$ with its optimal value $m\in X$, $m\neq a$, $b$, through the above mapping transformation, any point in the interval $X$ and its dependent degree calculation are mapped to the left half of $X$, that is, the monotonic increasing interval $[a,m)$. Here, this leads to Theorem 6.

**Theorem 6.** *Suppose a finite interval $X = [a,b]$ with its optimal value $m\in X$, $m\neq a$, $b$. $\forall x_0 \in X$, taking a mapping transformation $\theta(x_0) = x_0'$, and transforming interval $X$ as its left half interval $X' = [a,m]$. Then $k(x_0', X') = k(x_0, X)$ holds.*

The proof is very easy according to Theorem 5, so it is omitted. As we seen, this theorem successfully transforms the dependent degree calculation with the optimal point not at the endpoint of the interval to the calculation with the optimal point at the right endpoint. That makes dependent degree calculation process very simple and unified.

According to the above content, we will deduce extension dependent degree for three-parameter interval and its mapping transformation method.

### 2.3. Three-Parameter Interval Extension Dependent Degree and Mapping Transformation

*Definition 7.* Suppose a finite interval $X = [a,b]$ with its optimal value $m\in X$; there is a subinterval $X_0=[a_0, b_0]$ and $X_0 \subseteq X$; then

$$k(X_0,X) = \alpha k(a_0,X) + (1-\alpha)k(b_0,X), \quad \alpha\in[0,1]. \tag{4}$$

$k(X_0,X)$ is called the interval extension dependent degree of the subinterval $X_0$ and the interval $X$, and it satisfies the following properties,

(1) When $a_0 = b_0$, $k(X_0,X)$ degenerates into the extension simple dependent degree in Definition 3.

(2) When $a_0 = b_0= m$, $k(X_0,X)$ reaches the maximum value and $k(X_0,X) = k(m,X) = 1$.

(3) When $a_0 = a$, $b_0 = b$, $k(X_0,X)$ reaches the minimum value and $k(X_0,X) = 0$.

(4) When $X_0 \subset X$, $0 < k(X_0,X) < 1$.

Obviously, when $m=a$ or $b$, according formula (2) and (4), the interval dependent degree function is simplified to a monotonic increasing or monotonic decreasing function.

*Definition 8.* Suppose a finite interval $X = [a,b]$ with its optimal value $m\in X$; there is a three-parameter subinterval $\overline{X_0} = [a_0,m_0,b_0]$ and $\overline{X_0} \subseteq X$; then,

$$k(\overline{X_0},X) = \alpha k(a_0,X) + (1-\alpha-\beta)k(m_0,X) + \beta k(b_0,X), \quad (\alpha,\beta\in[0,1],\ \alpha+\beta\leq 1) \tag{5}$$

$k(\overline{X_0}, X)$ is called the three-parameter interval extension dependent degree of the subinterval $\overline{X_0}$ and the interval $X$, and it satisfies the following properties:

(1) When $a_0 = b_0 = m_0$, $k(\overline{X_0}, X)$ degenerates into the extension simple dependent degree.

(2) When $a_0 = b_0 = m_0 = m$, $k(\overline{X_0}, X)$ reaches the maximum value and $k(\overline{X_0}, X) = k(m_0, X) = k(m, X) = 1$.

(3) When $a_0 = a$, $b_0 = b$, $k(\overline{X_0}, X) = (1 - \alpha - \beta)k(m_0, X)$. In particular, when $m_0 = a_0$ or $m_0 = b_0$, $k(\overline{X_0}, X)$ reaches the minimum value and $k(\overline{X_0}, X) = 0$.

(4) When $\overline{X_0} \subset X$, $0 < k(\overline{X_0}, X) < 1$.

Obviously, when $m=a$ or $b$, the three-parameter interval extension dependent degree function is simplified to a monotonic increasing or monotonic decreasing function. The three-parameter interval extension dependent degree shows the relationship between the three-parameter subinterval $\overline{X_0}$ and the interval $X$. In it, $\alpha$ and $\beta$ represent the preference attitude coefficient of the decision makers, which will reflect the degree of tendency from the decision makers to the upper bound, the lower bound, and the gravity value of the interval $\overline{X_0}$. That means that for an attribute, the decision makers either pay more attention to the high evaluation or pay more attention to the low evaluation or the overall statistical value of the evaluation.

As above, when $m=a$ or $b$, the three-parameter interval extension dependent degree calculation is simple and intuitive. However, when $m \neq a, b$, the calculation and comparison of the three-parameter interval dependent degree become complicated. The calculation formula of the dependent degree will be complicatedly changed due to the difference relative position of the upper bound, the lower bound, and the optimal point $m$. The attitude coefficients also cannot directly correspond to the dependent degree of the upper and lower bounds of the interval. This paper proposes a method of interval mapping transformation. It will transform the interval dependent degree calculation with the optimal point not at the endpoint to the calculation with the optimal point at the endpoint. That keeps monotonicity and simplicity of the interval dependent degree calculation process.

*Definition 9.* Suppose a finite interval $X = [a, b]$ with its optimal value $m \in X$, $m \neq a$, $b$. For the three-parameter subinterval $\overline{X_0} = [a_0, m_0, b_0]$ and $\overline{X_0} \subseteq X$, there is a mapping transformation $\theta(\overline{X_0}) = \overline{X_0}'$ which makes $\overline{X_0}' \subseteq [a, m]$ and $k(\overline{X_0}', X) = k(\overline{X_0}, X)$ hold. Then $\overline{X_0}'$ is called the dependent degree mapping interval of $\overline{X_0}$.

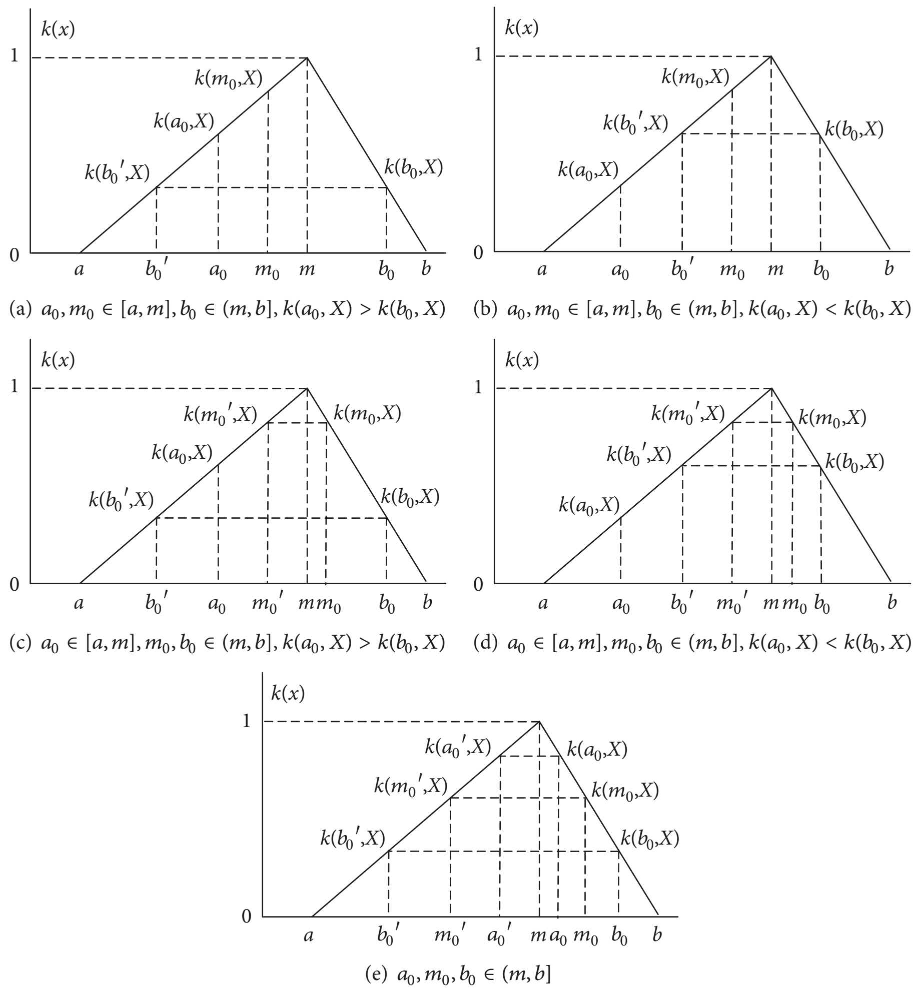
**Theorem 10.** *Suppose a finite interval $X = [a, b]$ and its optimal value $m \in X$, $m \neq a$, $b$. For the three-parameter subinterval $\overline{X_0} = [a_0, m_0, b_0]$ and $\overline{X_0} \subseteq X$, if $\overline{X_0}'$ is the dependent degree mapping interval of $\overline{X_0}$, then,*

$$\overline{X_0}' = \theta\left(\overline{X_0}\right) = \begin{cases} [a_0, m_0, b_0] = \overline{X_0} & (a_0, m_0, b_0 \in [a, m]) \\ \left[a_0, m_0, b_0'\right] = \left[a_0, m_0, \dfrac{b - b_0}{b - m}(m - a) + a\right] & (a_0, m_0 \in [a, m], b_0 \in (m, b]) \\ or\ \left[b_0', m_0, a_0\right] = \left[\dfrac{b - b_0}{b - m}(m - a) + a, m_0, a_0\right] & \\ \left[a_0, m_0', b_0'\right] = \left[a_0, \dfrac{b - m_0}{b - m}(m - a) + a, \dfrac{b - b_0}{b - m}(m - a) + a\right] & (a_0 \in [a, m], m_0, b_0 \in (m, b]) \\ or\ \left[b_0', m_0', a_0\right] = \left[\dfrac{b - b_0}{b - m}(m - a) + a, \dfrac{b - m_0}{b - m}(m - a) + a, a_0\right] & \\ \left[b_0', m_0', a_0'\right] = \left[\dfrac{b - b_0}{b - m}(m - a) + a, \dfrac{b - m_0}{b - m}(m - a) + a, \dfrac{b - a_0}{b - m}(m - a) + a\right] & (a_0, m_0, b_0 \in (m, b]). \end{cases} \quad (6)$$

*Proof.* Without loss of generality, suppose $\alpha$ is the preference coefficient of low dependent degree, $\beta$ is the preference coefficient of high dependent degree, and $1 - \alpha - \beta$ is the preference coefficient of the dependent degree of gravity value. Here, we proved the second case. When $a_0, m_0 \in [a, m]$, $b_0 \in (m, b]$, there are two cases:

(1) When $k(a_0, X) > k(b_0, X)$, as shown in Figure 2(a). Since $b_0 \in (m, b]$, according to Theorem 5, it takes a mapping transformation $\theta(b_0) = b_0' = k(b_0, X)(m - a) + a$ which makes $k(b_0', X) = k(b_0, X)$ and $b_0' \in [a, m)$ hold. Then, since the dependent degree increases monotonically in the interval $[a, m)$, and $k(b_0', X) < k(a_0, X)$, there is $b_0' < a_0$. Thus, $\overline{X_0}' = [b_0', m_0, a_0]$; then,

$$\begin{aligned} k\left(\overline{X_0'}, X\right) &= \alpha k\left(b_0', X\right) + (1 - \alpha - \beta) k\left(m_0, X\right) \\ &\quad + \beta k\left(a_0, X\right) \\ &= \alpha k\left(\frac{b - b_0}{b - m}(m - a) + a, X\right) \\ &\quad + (1 - \alpha - \beta) k\left(m_0, X\right) + \beta k\left(a_0, X\right) \end{aligned}$$



(a) $a_0, m_0 \in [a, m], b_0 \in (m, b], k(a_0, X) > k(b_0, X)$ (b) $a_0, m_0 \in [a, m], b_0 \in (m, b], k(a_0, X) < k(b_0, X)$

(c) $a_0 \in [a, m], m_0, b_0 \in (m, b], k(a_0, X) > k(b_0, X)$ (d) $a_0 \in [a, m], m_0, b_0 \in (m, b], k(a_0, X) < k(b_0, X)$

(e) $a_0, m_0, b_0 \in (m, b]$

Figure 2: The interval dependent degree mapping transformation.

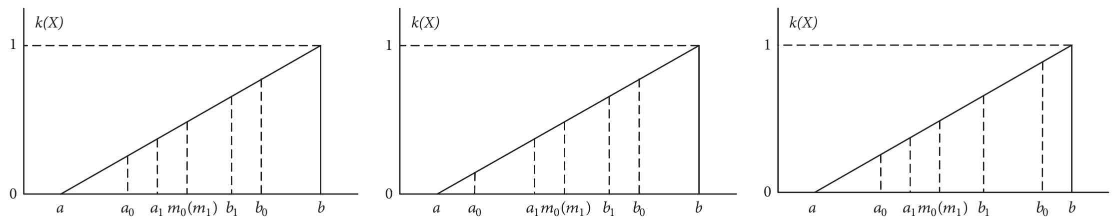
$$\begin{aligned}
&= \alpha k\left(k\left(b_0, X\right)(m-a)+a, X\right) \\
&\quad + \left(1-\alpha-\beta\right) k\left(m_0, X\right) + \beta k\left(a_0, X\right) \\
&= \alpha\left(\frac{k\left(b_0, X\right)(m-a)+a}{m-a}\right) \\
&\quad + \left(1-\alpha-\beta\right) k\left(m_0, X\right) + \beta k\left(a_0, X\right) \\
&= \alpha k\left(b_0, X\right) + \left(1-\alpha-\beta\right) k\left(m_0, X\right) \\
&\quad + \beta k\left(a_0, X\right) = k\left(\overline{X_0}, X\right).
\end{aligned} \tag{7}$$

(2) When $k(a_0, X) < k(b_0, X)$, as shown in Figure 2(b). Similarly, it is easy to get $\theta(b_0) = b_0' = k(b_0)(m-a)+a$, $b_0' \in [a, m)$ and $b_0' > a_0$. Therefore, $\overline{X_0}' = [a_0, m_0, b_0']$; then,

$$\begin{aligned}
k\left(\overline{X_0}', X\right) &= \alpha k\left(a_0, X\right) + \left(1-\alpha-\beta\right) k\left(m_0, X\right) \\
&\quad + \beta k\left(b_0', X\right) \\
&= \alpha k\left(a_0, X\right) + \left(1-\alpha-\beta\right) k\left(m_0, X\right) \\
&\quad + \beta k\left(\frac{b-b_0}{b-m}(m-a)+a, X\right) \\
&= \alpha k\left(a_0, X\right) + \left(1-\alpha-\beta\right) k\left(m_0, X\right) \\
&\quad + \beta k\left(k\left(b_0, X\right)(m-a)+a, X\right) \\
&= \alpha k\left(a_0, X\right) + \left(1-\alpha-\beta\right) k\left(m_0, X\right) \\
&\quad + \beta\left(\frac{k\left(b_0, X\right)(m-a)+a-a}{m-a}\right) \\
&= \alpha k\left(a_0, X\right) + \left(1-\alpha-\beta\right) k\left(m_0, X\right) \\
&\quad + \beta k\left(b_0, X\right) = k\left(\overline{X_0}, X\right).
\end{aligned} \tag{8}$$

□

Therefore, when $k(a_0, X) > k(b_0, X)$, there is $\overline{X_0}' = [b_0', m_0, a_0]$. When $k(a_0, X) < k(b_0, X)$, there is $\overline{X_0}' = [a_0, m_0, b_0']$. The other cases are shown in Figures 2(c)–2(e) and may be proved at the same way. Here we will no longer repeat them.




(a) $a_0 - a_1 = b_1 - b_0$ (b) $a_0 - a_1 < b_1 - b_0$ (c) $a_0 - a_1 > b_1 - b_0$

Figure 3: The interval dependent degree of benefit attribute interval.

Obviously, for a finite interval $X = [a, b]$ and its optimal value $m \in X$, $m \neq a$, $b$, by performing the interval mapping transformation defined above, all of the subintervals and their dependent degree calculations are mapped to the left half of the $X$, i.e., $[a, m]$. It leads to Theorem 11.

**Theorem 11.** *Suppose a finite interval $X = [a, b]$ with its optimal value $m \in X$, $m \neq a$, $b$. For the three-parameter subinterval $\overline{X_0} = [a_0, m_0, b_0]$ and $\overline{X_0} \subseteq X$, by taking interval mapping transformation $\theta(\overline{X_0}) = \overline{X_0}'$ for $X_0$, and transforming interval $X$ to its left half interval $X' = [a, m]$, then $k(\overline{X_0}', X') = k(\overline{X_0}, X)$ holds.*

*The proof is very easy according to Theorem 10, so it is omitted.*

*As we see, this theorem successfully transforms the three-parameter interval extension dependent degree calculation with the optimal point not at the endpoint to the calculation with the optimal point at the endpoint. That makes the three-parameter interval dependent degree calculation process very simple and uniform.*

*Different from the traditional interval theories such as gray correlation analysis, interval closeness, and the other measurement methods, the three-parameter interval extension dependent degree not only measures the relationship between two intervals, but also measures the relationship between the gravity points of them. Therefore, the measure is more comprehensive and reasonable. Compared to some existing methods in recent years such as three-parameter interval gray correlation degree, three-parameter interval closeness, three-parameter set pair connection number, and three-parameter interval projection sorting [25], the extension dependent degree method is not only simpler in calculation process but also capable of performing dynamic and uncertainty analysis.*

### *2.4. Dependent Degree Sorting Methods for Different Attribute Types*

**Theorem 12.** *Suppose a benefit attribute interval $X = [a, b]$; there are three-parameter subintervals $\overline{X_0} = [a_0, m_0, b_0]$ and $\overline{X_1} = [a_1, m_1, b_1]$, $\overline{X_0}, \overline{X_1} \subseteq X$. Then,*

*(1) when $\alpha a_0 + (1-\alpha-\beta)m_0 + \beta b_0 = \alpha a_1 + (1-\alpha-\beta)m_1 + \beta b_1$, there is $k(\overline{X_0}, X) = k(\overline{X_1}, X)$;*

*(2) when $\alpha a_0 + (1-\alpha-\beta)m_0 + \beta b_0 < \alpha a_1 + (1-\alpha-\beta)m_1 + \beta b_1$, there is $k(\overline{X_0}, X) < k(\overline{X_1}, X)$;*

*(3) when $\alpha a_0 + (1-\alpha-\beta)m_0 + \beta b_0 > \alpha a_1 + (1-\alpha-\beta)m_1 + \beta b_1$, there is $k(\overline{X_0}, X) > k(\overline{X_1}, X)$.*

*Proof.* Since $X = [a, b]$ is the benefit attribute interval, $b$ is the optimal value. Then,

$$
\begin{aligned}
&k(\overline{X_0}, X) - k(\overline{X_1}, X) \\
&= \alpha k(a_0, X) + (1-\alpha-\beta) k(m_0, X) + \beta k(b_0, X) \\
&\quad - (\alpha k(a_1, X) + (1-\alpha-\beta) k(m_1, X) + \beta k(b_1, X)) \\
&= \alpha\left(\frac{a_0 - a}{b - a}\right) + (1-\alpha-\beta)\left(\frac{m_0 - a}{b - a}\right) + \beta\left(\frac{b_0 - a}{b - a}\right) \\
&\quad - \left(\alpha\left(\frac{a_1 - a}{b - a}\right) + (1-\alpha-\beta)\left(\frac{m_1 - a}{b - a}\right) + \beta\left(\frac{b_1 - a}{b - a}\right)\right) \\
&= \frac{\alpha a_0 + (1-\alpha-\beta) m_0 + \beta b_0 - (\alpha a_1 + (1-\alpha-\beta) m_1 + \beta b_1)}{b - a}.
\end{aligned} \tag{9}
$$

□

**Corollary 13.** *Suppose a benefit attribute interval $X = [a, b]$; there are three-parameter subintervals $\overline{X_0} = [a_0, m_0, b_0]$ and $\overline{X_1} = [a_1, m_1, b_1]$, $\overline{X_0}, \overline{X_1} \subseteq X$. When $m_0 = m_1$, then,*

*(1) when $\alpha a_0 + \beta b_0 = \alpha a_1 + \beta b_1$, there is $k(\overline{X_0}, X) = k(\overline{X_1}, X)$;*

*(2) when $\alpha a_0 + \beta b_0 < \alpha a_1 + \beta b_1$, there is $k(\overline{X_0}, X) < k(\overline{X_1}, X)$;*

*(3) when $\alpha a_0 + \beta b_0 > \alpha a_1 + \beta b_1$, there is $k(\overline{X_0}, X) > k(\overline{X_1}, X)$.*

*The following part analyzes the practical significance of Corollary 13 in the decision making for benefit attributes. For no loss of generality and more intuitive process, it is assumed that $\alpha = \beta$. At this point, $a_0 + b_0$ and $a_1 + b_1$ determine the values of the dependent degree, respectively.*

As shown in Figure 3(a), for a benefit attribute interval $X=[a,b]$, there are two evaluation intervals $\overline{X_0} = [a_0, m_0, b_0]$ and $\overline{X_1} = [a_1, m_1, b_1]$ with the same gravity value $m_0$ $(m_1)$, which indicates that their overall evaluation is consistent. Although $\overline{X_0}$ has a higher upper bound and a lower bound, it expands out to the same extent on the upper and lower bounds for $\overline{X_1}$; i.e., $a_0 - a_1 = b_1 - b_0$. This only illustrates that the evaluation of $\overline{X_0}$ is more relatively controversial but cannot distinguish which is better. Therefore, $k(\overline{X_0}, X) = k(\overline{X_1}, X)$, which is in line with people's habit of making decision. As shown in Figure 3(b), the overall evaluation of $\overline{X_0}$ and $\overline{X_1}$ is consistent, but relative to $\overline{X_1}$, the expanding segment of the lower bound of $\overline{X_0}$ is larger than that of the upper bound; i.e., $a_0 - a_1 < b_1 - b_0$. This indicates that $\overline{X_0}$

Table 1: An example of the interval dependent degree calculation for benefit attribute interval.

| $\alpha=\beta=1/3$ | $\overline{X_1}=[2,4,8]$ | $\overline{X_0}=[3,4,7]$ | $\overline{X_0}=[1,4,7]$ | $\overline{X_0}=[2,4,9]$ |
|---|---|---|---|---|
| $X=[0,10]$ | $k(\overline{X_1},X)=0.467$ | $k(\overline{X_0},X)=0.467$ | $k(\overline{X_0},X)=0.4$ | $k(\overline{X_0},X)=0.5$ |
| Sorting result | | $k(\overline{X_0},X)=k(\overline{X_1},X)$ | $k(\overline{X_0},X)<k(\overline{X_1},X)$ | $k(\overline{X_0},X)>k(\overline{X_1},X)$ |

Table 2: An example of the interval dependent degree calculation for fixed attribute interval.

| $\alpha=\beta=1/3$ | $\overline{X_1}=[2,4,8]$ | $\overline{X_0}=[3,4,7]$ | $\overline{X_0}=[1,4,7]$ | $\overline{X_0}=[2,4,9]$ |
|---|---|---|---|---|
| $X=[0,10]$, $m=4$ | $\theta(\overline{X_1})=\overline{X_1'}=[4/3,4,2]$ | $\theta(\overline{X_0})=\overline{X_0'}=[2,4,3]$ | $\theta(\overline{X_0})=\overline{X_0'}=[1,4,2]$ | $\theta(\overline{X_0})=\overline{X_0'}=[2/3,4,2]$ |
| $X'=[0,4]$ | $k(\overline{X_1},X)=k(\overline{X_1}',X')=0.611$ | $k(\overline{X_0},X)=k(\overline{X_0}',X')=0.75$ | $k(\overline{X_0},X)=k(\overline{X_0}',X')=0.583$ | $k(\overline{X_0},X)=k(\overline{X_0}',X')=0.556$ |
| Sorting result | | $k(\overline{X_0},X)>k(\overline{X_1},X)$ | $k(\overline{X_0},X)<k(\overline{X_1},X)$ | $k(\overline{X_0},X)<k(\overline{X_1},X)$ |

contains a lower evaluation value, so the overall evaluation of $\overline{X_0}$ should be slightly lower than $\overline{X_1}$; i.e., $k(\overline{X_0},X)<k(\overline{X_1},X)$. This is also in line with people's thinking habits of making decision. On the contrary, as shown in Figure 3(c), when $a_0-a_1>b_1-b_0$, the expanding segment of the upper bound of $\overline{X_0}$ is larger than that of the lower bound, which means $\overline{X_0}$ contains a higher evaluation value. Therefore, the overall evaluation of $\overline{X_0}$ should be slightly higher than $\overline{X_1}$. Table 1 shows a simple example of the above content.

**Corollary 14.** *Suppose a benefit attribute interval* $X=[a,b]$; *there are three-parameter subintervals* $\overline{X_0}=[a_0,m_0,b_0]$ *and* $\overline{X_1}=[a_1,m_1,b_1]$, $\overline{X_0},\overline{X_1}\subseteq X$. *When* $a_0=a_1$, $b_0=b_1$, *then,*

*(1) when* $m_0=m_1$, *there is* $k(\overline{X_0},X)=k(\overline{X_1},X)$;

*(2) when* $m_0<m_1$, *there is* $k(\overline{X_0},X)<k(\overline{X_1},X)$;

*(3) when* $m_0>m_1$, *there is* $k(\overline{X_0},X)>k(\overline{X_1},X)$.

*The practical significance of Corollary 14 is that when the upper and lower bounds of two evaluation intervals both are the same, the one with the larger gravity value is better.*

**Theorem 15.** *Suppose a cost attribute interval* $X=[a,b]$; *there are three-parameter subintervals* $\overline{X_0}=[a_0,m_0,b_0]$ *and* $\overline{X_1}=[a_1,m_1,b_1]$, $\overline{X_0},\overline{X_1}\subseteq X$. *Then,*

*(1) when* $\alpha a_0+(1-\alpha-\beta)m_0+\beta b_0=\alpha a_1+(1-\alpha-\beta)m_1+\beta b_1$, *there is* $k(\overline{X_0},X)=k(\overline{X_1},X)$;

*(2) when* $\alpha a_0+(1-\alpha-\beta)m_0+\beta b_0>\alpha a_1+(1-\alpha-\beta)m_1+\beta b_1$, *there is* $k(\overline{X_0},X)<k(\overline{X_1},X)$;

*(3) when* $\alpha a_0+(1-\alpha-\beta)m_0+\beta b_0<\alpha a_1+(1-\alpha-\beta)m_1+\beta b_1$, *there is* $k(\overline{X_0},X)>k(\overline{X_1},X)$.

*The proof is the same as that of Theorem 12.*

**Corollary 16.** *Suppose a cost attribute interval* $X=[a,b]$; *there are three-parameter subintervals* $\overline{X_0}=[a_0,m_0,b_0]$ *and* $\overline{X_1}=[a_1,m_1,b_1]$, $\overline{X_0},\overline{X_1}\subseteq X$. *When* $m_0=m_1$, *then,*

*(1) when* $\alpha a_0+\beta b_0=\alpha a_1+\beta b_1$, *there is* $k(\overline{X_0},X)=k(\overline{X_1},X)$;

*(2) when* $\alpha a_0+\beta b_0>\alpha a_1+\beta b_1$, *there is* $k(\overline{X_0},X)<k(\overline{X_1},X)$;

*(3) when* $\alpha a_0+\beta b_0<\alpha a_1+\beta b_1$, *there is* $k(\overline{X_0},X)>k(\overline{X_1},X)$.

*The practical significance of Corollary 16 is the same as that of Corollary 13.*

**Corollary 17.** *Suppose a cost attribute interval* $X=[a,b]$; *there are three-parameter subintervals* $\overline{X_0}=[a_0,m_0,b_0]$ *and* $\overline{X_1}=[a_1,m_1,b_1]$, $\overline{X_0},\overline{X_1}\subseteq X$. *When* $a_0=a_1$, $b_0=b_1$, *then,*

*(1) when* $m_0=m_1$, *there is* $k(\overline{X_0},X)=k(\overline{X_1},X)$;

*(2) when* $m_0<m_1$, *there is* $k(\overline{X_0},X)>k(\overline{X_1},X)$;

*(3) when* $m_0>m_1$, *there is* $k(\overline{X_0},X)<k(\overline{X_1},X)$.

*The practical significance of Corollary 17 is that when the upper and lower bounds of two evaluation intervals both are the same, the one with the smaller gravity value is better.*

**Theorem 18.** *Suppose a fixed attribute interval* $X=[a,b]$ *with its optimal value* $m\in X$, $m\neq a,b$. *For the three-parameter subintervals* $\overline{X_0}=[a_0,m_0,b_0]$ *and* $\overline{X_1}=[a_1,m_1,b_1]$, $X_0$, $X_1\subseteq X$, *by taking the interval mapping transformation* $\theta(\overline{X_0})=\overline{X_0}'$ *and* $\theta(\overline{X_1})=\overline{X_1}'$ *and transforming interval* $X$ *to the benefit interval* $X'=[a,m]$, *then,* $k(\overline{X_0}',X')=k(\overline{X_0},X)$, $k(\overline{X_1}',X')=k(\overline{X_1},X)$ *hold.*

*The proof is easy by Definition 9 and Theorem 11, so it is omitted.*

*Theorem 18 successfully transforms the three-parameter interval dependent degree calculation for the fixed attribute interval into the calculation for the benefit attribute interval with monotonic increasing feature. It not only makes the calculation process in multiattribute decision making application more simple and uniform, but also facilitates performing uncertainty and dynamic analysis. Table 2 shows an example of the interval dependent degree calculation for fixed attribute interval.*

*2.5. Preference Attitude Coefficient.* In the three-parameter interval extension dependent degree formula of Definition 8, $\alpha$ and $\beta$ represent the preference attitude coefficient of the decision maker. Different coefficients reflect the tendency extent from decision makers to the dependent degree of the upper bound, the lower bounds, and gravity value. Table 3 shows various typical coefficient settings.

Among them, type I represents decision makers having no preference. Type II represents that decision makers focusing more on the dependent degree of gravity values. Type III represents that decision makers preferring the dependent degree of the upper and the lower bounds. Type IV represents that decision makers preferring the dependent degree of gravity values and considering the evaluation values within

Table 3: Setting of preference attitude coefficient.

| | I | II | III | IV | V | VI |
|---|---|---|---|---|---|---|
| $\alpha$ | 0.333 | 0.25 | 0.4 | 0.1065 | 0.222 | 0.444 |
| $\beta$ | 0.333 | 0.25 | 0.4 | 0.1065 | 0.444 | 0.222 |
| $1-\alpha-\beta$ | 0.333 | 0.5 | 0.2 | 0.787 | 0.333 | 0.333 |

the interval presenting a standard normal distribution. Type V and IV, respectively, represent decision makers tending to the dependent degree of the upper bound and the lower bound.

In type IV, assuming the evaluation value $x$ obeying the standard normal distribution with $\mu = 0, \sigma = 1$, the probability density function is $\varphi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$. According to the $2\sigma$ principle, the probability of gravity value is $\varphi(\mu) = 0.399$, the probabilities of the upper and lower bounds are $\varphi(\mu+2\sigma) = \varphi(\mu-2\sigma) = 0.054$. Then, the preference attitude coefficient of gravity value is $(1-\alpha-\beta) = 0.399/(0.399+2\times 0.054) = 0.787$, and the coefficients of the dependent degree of the upper and lower bounds are $\alpha = \beta = 0.1065$.

## 3. The Process of Decision Making

Suppose, for multiattribute decision making, there are solution set $S = \{s_1, s_2, \ldots, s_m\}$ and attribute set $U = \{u_1, u_2, \ldots, u_n\}$. The evaluation value of attribute $u_j$ of solution $s_i$ is the three-parameter interval number $\overline{X_{ij}} = [a_{ij}, m_{ij}, b_{ij}]$. Then, the evaluation value matrix is $(\overline{X_{ij}})_{m\times n}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$.

*Step 1.* According to Theorems 10 and 11, make interval mapping transformation towards the evaluation matrix $(\overline{X_{ij}})_{m\times n}$, and transform the fixed attribute intervals and their three-parameter interval evaluations value into the appropriate benefit attribute intervals and the appropriate evaluations value. A new evaluation matrix $(\overline{X_{ij}}')_{m\times n}$ is obtained.

*Step 2.* Determine the coefficient setting of $\alpha$ and $\beta$ in Table 3 as needed.

*Step 3.* Get the value range of each attribute $u_j$. If the value range of the attribute is determined in advance, this range is taken as the value interval of the attribute. If not, take $X_j' = [\min(\overline{X_{ij}}'), \max(\overline{X_{ij}}')]$ $(i = 1, 2, \ldots, m)$ as the value interval of the attribute according to the max-min principle. Then calculate the three-parameter interval dependent degree of each evaluation value $k(\overline{X_{ij}}', X_j')$ according to Definition 8.

*Step 4.* If the attribute weight set has been given $W = \{w_1, w_2, \ldots, w_n\}$, then get the comprehensive three-parameter interval dependent degree towards each solution $s_i$, which is described as $K(s_i) = \sum_{j=1}^{n} w_j k(\overline{X_{ij}}', X_j')$ and sorted. Turn to Step 6.

*Step 5.* If the attribute weights have been given in the form of three-parameter interval number set $W = \{\overline{W_1}, \overline{W_2}, \ldots, \overline{W_n}\}$ and each weight value is $\overline{W_j} = [w_j^a, w_j^m, w_j^b] (j = 1, 2, \ldots, n)$, $w_j^a \le w_j^b$, then calculate the comprehensive interval dependent degree under three weights of each solution $s_i$, which are described as $K(s_i)^a = \sum_{j=1}^{n} w_j^a k(\overline{X_{ij}}', X_j')$, $K(s_i)^m = \sum_{j=1}^{n} w_j^m k(\overline{X_{ij}}', X_j')$, and $K(s_i)^b = \sum_{j=1}^{n} w_j^b k(\overline{X_{ij}}', X_j')$. Then, perform the stability test and comprehensive sorting according to the three dependent degrees. Turn to Step 6.

*Step 6.* Perform uncertainty analysis for decision result through different settings of $\alpha$ and $\beta$.

*Step 7.* Perform dynamics analysis and rule discovery on decision result through the interval extension transformation.

## 4. Example Analysis

*4.1. Multiattribute Decision Making Process and Its Uncertainty Analysis.* For convenience of comparison and illustration, the example uses the data from [15, 26]. One organization decides to give an annual evaluation for 5 candidates ($s_1$, $s_2$, $s_3$, $s_4$, $s_5$) according to 6 evaluation attributes ($u_1$, $u_2$, $u_3$, $u_4$, $u_5$, $u_6$) which include moral quality, working attitude, working style, educational level, leadership, and development ability. Each attribute of each candidate was scored, respectively, from the group decisions and then the evaluation of each attribute of each one is obtained by some basic statistical processing. The score range was determined in advance. Obviously, the evaluation for the same candidate varies from person to person; as a result, the attribute value is given in the form of three-parameter interval number. The first five are benefit attributes which are the bigger the better, ranging from 0.80 to 1.00. The last one is a fixed attribute, ranging from 0.80 to 1.20. For the fixed attribute, 1.00 is the optimal score, and the score beyond it means too radical and vice versa (too conservative). The attribute values are shown in Table 4. Here, the optimal candidate is determined according to the above information.

(1) The evaluation matrix is obtained according to Table 4.

(2) According to Step 1, do the interval mapping transformation for the evaluation matrix. The evaluation values in the benefit attribute intervals keep unchanged, and those belonging to the fixed attribute intervals are transformed to those in the benefit attribute, as shown in Table 5.

(3) Determine the setting scheme of coefficient $\alpha$ and $\beta$ according to Step 2. For comparing consistency, the setting of type I in Table 3 is used ($\alpha = \beta = 1/3$).

(4) Determine the value range of each attribute according to Step 3. In this example, the first five have the same

Table 4: Evaluation value table.

| candidate | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ |
|---|---|---|---|---|---|---|
| $s_1$ | [0.80 0.85 0.90] | [0.90 0.92 0.95] | [0.91 0.94 0.95] | [0.93 0.96 0.99] | [0.90 0.91 0.92] | [0.95 0.97 1.01] |
| $s_2$ | [0.90 0.95 1.00] | [0.89 0.90 0.93] | [0.90 0.92 0.95] | [0.90 0.92 0.95] | [0.94 0.97 0.98] | [0.90 0.93 1.05] |
| $s_3$ | [0.88 0.91 0.95] | [0.84 0.86 0.90] | [0.91 0.94 0.97] | [0.91 0.94 0.96] | [0.86 0.89 0.92] | [1.06 1.08 1.09] |
| $s_4$ | [0.85 0.87 0.90] | [0.91 0.93 0.95] | [0.85 0.88 0.90] | [0.86 0.89 0.93] | [0.87 0.90 0.94] | [1.04 1.07 1.08] |
| $s_5$ | [0.86 0.89 0.95] | [0.90 0.92 0.95] | [0.90 0.95 0.97] | [0.91 0.93 0.95] | [0.90 0.92 0.96] | [1.10 1.13 1.15] |

Table 5: The evaluation table after the interval mapping transformation.

| candidate | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ |
|---|---|---|---|---|---|---|
| $s_1$ | [0.80 0.85 0.90] | [0.90 0.92 0.95] | [0.91 0.94 0.95] | [0.93 0.96 0.99] | [0.90 0.91 0.92] | [0.95 0.97 0.99] |
| $s_2$ | [0.90 0.95 1.00] | [0.89 0.90 0.93] | [0.90 0.92 0.95] | [0.90 0.92 0.95] | [0.94 0.97 0.98] | [0.90 0.93 0.95] |
| $s_3$ | [0.88 0.91 0.95] | [0.84 0.86 0.90] | [0.91 0.94 0.97] | [0.91 0.94 0.96] | [0.86 0.89 0.92] | [0.91 0.92 0.94] |
| $s_4$ | [0.85 0.87 0.90] | [0.91 0.93 0.95] | [0.85 0.88 0.90] | [0.86 0.89 0.93] | [0.87 0.90 0.94] | [0.92 0.93 0.96] |
| $s_5$ | [0.86 0.89 0.95] | [0.90 0.92 0.95] | [0.90 0.95 0.97] | [0.91 0.93 0.95] | [0.90 0.92 0.96] | [0.85 0.87 0.90] |

Table 6: Dependent degree of the three-parameter interval attribute values.

| | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ |
|---|---|---|---|---|---|---|
| $k_{1j}$ | 0.2500 | 0.6167 | 0.6667 | 0.8000 | 0.5500 | 0.8500 |
| $k_{2j}$ | 0.7500 | 0.5333 | 0.6167 | 0.6167 | 0.8167 | 0.6333 |
| $k_{3j}$ | 0.5667 | 0.3333 | 0.7000 | 0.6833 | 0.4500 | 0.6167 |
| $k_{4j}$ | 0.3667 | 0.6500 | 0.3833 | 0.4667 | 0.5167 | 0.6833 |
| $k_{5j}$ | 0.5000 | 0.6167 | 0.7000 | 0.6500 | 0.6333 | 0.3667 |

Table 7: The three-parameter interval weight values.

| $w_j$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ |
|---|---|---|---|---|---|---|
| $[w_j{}^a, w_j{}^m, w_j{}^b]$ | $[0.10, 0.15, 0.20]$ | $[0.05, 0.10, 0.15]$ | $[0.20, 0.25, 0.30]$ | $[0.05, 0.10, 0.15]$ | $[0.15, 0.20, 0.25]$ | $[0.10, 0.15, 0.20]$ |

Table 8: Dependent degree of the three-parameter interval weight values.

| $K(s_i)$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|
| $[K(s_i)^a, K(s_i)^m, K(s_i)^b]$ | $[0.397, 0.583, 0.770]$ | $[0.442, 0.640, 0.838]$ | $[0.377, 0.544, 0.712]$ | $[0.315, 0.468, 0.622]$ | $[0.385, 0.558, 0.732]$ |

Table 9: The scheme sorting results and their stability test.

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | sorting result |
|---|---|---|---|---|---|---|
| $K(s_i)^a$ | 0.3967 | 0.4417 | 0.3767 | 0.3150 | 0.3850 | $s_2 \succ s_1 \succ s_5 \succ s_3 \succ s_4$ |
| $K(s_i)^m$ | 0.5833 | 0.6400 | 0.5442 | 0.4683 | 0.5583 | $s_2 \succ s_1 \succ s_5 \succ s_3 \succ s_4$ |
| $K(s_i)^b$ | 0.7700 | 0.8383 | 0.7117 | 0.6217 | 0.7317 | $s_2 \succ s_1 \succ s_5 \succ s_3 \succ s_4$ |

determined range $[0.80, 1.00]$. The range of the last one is also changed to $[0.80, 1.00]$ after the interval mapping transformation. Then, the three-parameter interval dependent degree of each attribute evaluation value is calculated, as shown in Table 6.

(5) In order to maintain comparison consistency and validity, the weights of the attributes are set the same as those in [15, 26], as shown in the form of three-parameter interval number in Table 7. According to Step 5, the comprehensive dependent degree of each scheme under three weight parameters is calculated, respectively, as shown in Table 8. Then, the scheme sorting results and their stability test are shown in Table 9. By sorting under the upper bound, lower bound, and the gravity value of weight parameter, respectively, the sorting results are all the same, which means higher stability of the sorting results. The results are also the same as those in [26].

(6) According to the different settings of $\alpha$ and $\beta$ in Table 3, the uncertainty analysis is performed. $\alpha$ and $\beta$ show the different tendency from decision makers for the dependent degree of the upper bound, the lower bound, and the gravity value. Table 10 shows the sorting results under six different settings of $\alpha$ and $\beta$. The results illustrate that the sorting results are still the same under different attitude coefficient settings, which means the current sorting result has lower level of uncertainty and it does not change when

Table 10: the sorting results under different attitude coefficient settings.

| coefficient settings | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | sorting result |
|---|---|---|---|---|---|---|
| I | 0.5833 | 0.6400 | 0.5442 | 0.4683 | 0.5583 | $s_2 \succ s_1 \succ s_5 \succ s_3 \succ s_4$ |
| II | 0.5850 | 0.6400 | 0.5425 | 0.4662 | 0.5569 | $s_2 \succ s_1 \succ s_5 \succ s_3 \succ s_4$ |
| III | 0.5820 | 0.6400 | 0.5455 | 0.4700 | 0.5595 | $s_2 \succ s_1 \succ s_5 \succ s_3 \succ s_4$ |
| IV | 0.5879 | 0.6400 | 0.5396 | 0.4627 | 0.5544 | $s_2 \succ s_1 \succ s_5 \succ s_3 \succ s_4$ |
| V | 0.5578 | 0.6111 | 0.5147 | 0.4400 | 0.5253 | $s_2 \succ s_1 \succ s_5 \succ s_3 \succ s_4$ |
| VI | 0.6089 | 0.6689 | 0.5736 | 0.4966 | 0.5913 | $s_2 \succ s_1 \succ s_5 \succ s_3 \succ s_4$ |

Table 11: Sorting reversion caused by the move transformation of evaluation of attribute $s_1$.

| move transformation $T(\overline{X_{1j}})$ | $K(s_1)$ | $K(s_2)$ | sorting reversion |
|---|---|---|---|
| $\overline{X_{11}}[0.80, 0.85, 0.90] \longrightarrow [0.88, 0.93, 0.98]$ | 0.6433 | 0.6400 | $s_1 \succ s_2 \succ s_5 \succ s_3 \succ s_4$ |
| $\overline{X_{13}}[0.91, 0.94, 0.95] \longrightarrow [0.96, 0.99, 1.00]$ | 0.6458 | 0.6400 | $s_1 \succ s_2 \succ s_5 \succ s_3 \succ s_4$ |
| $\overline{X_{15}}[0.90, 0.91, 0.92] \longrightarrow [0.96, 0.97, 0.98]$ | 0.6433 | 0.6400 | $s_1 \succ s_2 \succ s_5 \succ s_3 \succ s_4$ |

the preference tendency from decision makers for the upper or lower bound of attribute evaluations changes.

*4.2. Extension Transformation Analysis.* At above, the sorting method and uncertainty analysis of multiattribute decision making under static environment have been described precisely though building extension interval dependent degree function. In addition, according to Step 7, the dependent degree will change by performing extension transformation of the dependent degree formula, which can precisely describe the changes and rule discoveries of decision results under dynamic environment. In this case, a most common extension transformation is element transformation. For example, the evaluation value of attribute $u_1$ of candidate $s_1$ changes from [0.80 0.85 0.90] to [0.85 0.90 0.95] after element transformation. Will it influence the decision results? Or what the extent of the influence that it makes? Literature [22] puts forward five basic interval extension transformations such as movement, expansion and contraction, replacement, addition and deletion, and decomposition. Here, movement transformation, which is the most common one, is used to analyze decision results under some dynamic environment.

*Definition 19.* Suppose a three-parameter interval $\overline{X} = [a, m, b]$ and a constant $l$. There is transformation $T$ towards them, and it makes $T(\overline{X}) = [a+l, m+l, b+l]$ hold. Then, $T$ is called the move transformation of $\overline{X}$. While $l \geq 0$, $T$ is called positive direction move transformation.

**Theorem 20.** *Suppose a benefit attribute interval $X = [a, b]$ and a three-parameter subinterval $\overline{X_0} = [a_0, m_0, b_0]$, $\overline{X_0} \subseteq X$. If a move transformation $T(\overline{X_0}) = [a_0 + l, m_0 + l, b_0 + l]$ $(0 \leq l \leq b - b_0)$ exits, then $k(T(\overline{X_0}), X) - k(\overline{X_0}, X) = l/(b-a)$ holds.*

*Proof.* Since $X = [a, b]$ is a benefit attribute interval, $b$ is the optimal value, so

$$\begin{aligned} k\left(T\left(\overline{X_0}\right), X\right) - k\left(\overline{X_0}, X\right) &= \alpha k\left(a_0 + l\right) + (1 - \alpha \\ &- \beta)\, k\left(m_0 + l\right) + \beta k\left(b_0 + l\right) - \left(\alpha k\left(a_0\right)\right. \\ &+ \left.(1 - \alpha - \beta)\, k\left(m_0\right) + \beta k\left(b_0\right)\right) = \alpha\left(\frac{a_0 + l - a}{b - a}\right) \\ &+ (1 - \alpha - \beta)\left(\frac{m_0 + l - a}{b - a}\right) + \beta\left(\frac{b_0 + l - a}{b - a}\right) \\ &- \left(\alpha\left(\frac{a_0 - a}{b - a}\right) + (1 - \alpha - \beta)\left(\frac{m_0 - a}{b - a}\right)\right. \\ &+ \left.\beta\left(\frac{b_0 - a}{b - a}\right)\right) = \frac{\alpha l + (1 - \alpha - \beta)\, l + \beta l}{b - a} = \frac{l}{b - a}. \end{aligned} \tag{10}$$

□

For multiattribute decision making, each evaluation value may be performed by the movement transformation. As a result, the combination result of all movement transformations tends to be quite complex. Here, we only discuss the impact from movement transformation of one attribute. For example, for a positive direction move transformation of one attribute of candidate $s_1$, the transformation is established as

$$\begin{aligned} T\left(\overline{X_{1j}}\right) &= \left[a_{1j} + l_j, m_{1j} + l_j, b_{1j} + l_j\right] \\ &\qquad \left(j = 1, 2, \ldots n,\ 0 \leq l_j \leq 1 - m_{1j}\right). \end{aligned} \tag{11}$$

According to Theorem 20, by iteration calculation with step size of 0.01, the transformation rule and the threshold value that is able to bring some sorting reversion are obtained, as shown in Table 11. It illustrates that, if the evaluation of scheme $s_1$ wants to overtake that of $s_2$, the evaluation of any of the attributes $u_1$, $u_3$, $u_5$ should be chosen to be promoted. Among them, attribute $u_3$ needs a minimum level of promotion. However, except for $u_1$, $u_3$, $u_5$, promoting the evaluation of any of the other attributes will not change the results.

Here is another example, by taking positive direction move transformation of one attribute of candidate $s_3$, transformation analysis will be performed. By iteration calculation with step size of 0.01, the transformation rule and the threshold value that is able to bring some sorting reversion

Table 12: Sorting reversion caused by the move transformation of evaluation of attribute $s_3$.

| move transformation $T(\overline{X_{3j}})$ | $K(s_3)$ | $K(s_5)$ | sorting reversion |
|---|---|---|---|
| $\overline{X_{31}}[0.88, 0.91, 0.95] \longrightarrow [0.90, 0.93, 0.97]$ | 0.5592 | 0.5583 | $s_2 \succ s_1 \succ s_3 \succ s_5 \succ s_4$ |
| $\overline{X_{32}}[0.84, 0.86, 0.90] \longrightarrow [0.87, 0.89, 0.93]$ | 0.5592 | 0.5583 | $s_2 \succ s_1 \succ s_3 \succ s_5 \succ s_4$ |
| $\overline{X_{33}}[0.91, 0.94, 0.97] \longrightarrow [0.93, 0.96, 0.99]$ | 0.5692 | 0.5583 | $s_2 \succ s_1 \succ s_3 \succ s_5 \succ s_4$ |
| $\overline{X_{34}}[0.91, 0.94, 0.96] \longrightarrow [0.94, 0.97, 0.99]$ | 0.5592 | 0.5583 | $s_2 \succ s_1 \succ s_3 \succ s_5 \succ s_4$ |
| $\overline{X_{35}}[0.86, 0.89, 0.92] \longrightarrow [0.88, 0.91, 0.94]$ | 0.5642 | 0.5583 | $s_2 \succ s_1 \succ s_3 \succ s_5 \succ s_4$ |
| $\overline{X_{36}}'[0.91, 0.92, 0.94] \longrightarrow [0.93, 0.94, 0.96]$ $\Longrightarrow \overline{X_{36}} \Longrightarrow [1.06, 1.08, 1.09] \longrightarrow [1.04, 1.06, 1.07]$ | 0.5592 | 0.5583 | $s_2 \succ s_1 \succ s_3 \succ s_5 \succ s_4$ |

are obtained, as shown in Table 12. It illustrates that, if the evaluation of $s_3$ is going to overtake that of $s_5$, the evaluation of any of the attributes can be chosen to be promoted. Among them, attributes $u_1$, $u_3$, $u_5$, $u_6$ need relatively low levels of promotion. It must be noted that the interval of $u_6$ has been processed by mapping transformation and, as a result, and needs to map back after movement transformation.

## 5. Conclusion

By adopting the extension dependent degree function, the paper researches the multiattribute decision making with attribute information being three-parameter interval number. It is a new thinking in the related research field. The main contributions are listed as follows: (1) Based on extension dependent function, a new decision making method and framework towards the multiattribute decision with attribute information being three-parameter interval number are put forward. (2) A formula of extension three-parameter interval dependent degree function is given, which reflects the different tendency from decision makers towards the lower bound, the upper bound, and the gravity point of the attribute evaluation by setting preference attitude coefficients. (3) Through defining the interval dependent degree mapping method, the calculation of the interval dependent degree with the optimal point not at the endpoint is transformed to the calculation with the optimal point at the endpoint, which has monotonic and simple process. (4) Six typical settings of attitude coefficients are given and the uncertainty analysis of decision results is made accordingly. (5) Based on the framework of the extension dependent degree calculation, dynamic analysis and rule discovery on decision results are performed through some extension element transformation.

The research work is quite abundant in the future. As a new thinking and framework, the decision method based on extension dependent function still needs further development and promotion. Next, the model will consider the combination of psychological behavior from decision makers such as prospect theory or regret theory, which can better reflect decision makers' risk preferences. By improving the extension dependent function, the model can describe some more complex decision processes. The model may be revised to adapt to some more complex extension transformations. Furthermore, the model can also need to be expanded to many other decision environments including mixed type data, incomplete information, and fuzzy hesitant set.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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