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Research Article

Analysis of Effects of Delays and Diffusion on a Predator-Prey System

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A reaction-diffusion predator-prey system with two delays is investigated. It is found that the spatially homogeneous periodic solution will occur when the sum of two delays crosses some critical values and Hopf bifurcation takes place. For the fixed domain and diffusion, some numerical simulations are also given to illustrate the theoretical analysis. In addition, special attention is paid to effects of diffusion on the bifurcating periodic solution. It is found that the diffusion would lead to the bifurcating period solution to destabilize by calculating the relevant expression of the Floquet exponent.

1. Introduction

It is well known that the reaction-diffusion equations with delays are usually used to describe the biological system. Some results have proved that diffusion and delay take the very important role in the biological systems and can induce many spatiotemporal patterns (see monographs by Wu [1], Arino [2], and Murray [3]). Spatiotemporal patterns of predator-prey population models with diffusion and delay have been studied by many authors [4–9] in recent years. In particular, when the biological system with a single delay has the different boundary conditions such as Neumann boundary and Dirichlet boundary, the spatial homogeneous Hopf bifurcation and the spatially nonhomogeneous Hopf bifurcation near the spatially uniform equilibrium of the predator-prey system have been studied by some authors [10–13].

Faria [14] and Chen [15] have studied Lotka-Volterra type prey-predator model with two delays as the single delay varies

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= d_1\Delta u(t,x) \\ &\quad + u(t,x)\left[r_1 - a_{11}u(t,x) - a_{12}v(t-\tau_2,x)\right], \\ \frac{\partial v(t,x)}{\partial t} &= d_2\Delta v(t,x) \\ &\quad + v(t,x)\left[-r_2 + a_{21}u(t-\tau_1,x) - a_{22}v(t,x)\right],\end{aligned} \tag{1}$$

where r_1, r_2, a_{12}, a_{21}, τ_1, τ_2, a_{11}, a_{22}, d_1, d_2 are positive constants and have the following biological interprets, respectively. r_1 is the birth rate of the prey; r_2 is the death rate of the predator; a_{12} and a_{21} represent the strength of the relative effects of the interaction on the two interspecies; a_{11} and a_{22} denote the strength of the interaction of the intraspecies; τ_1 is reaction time of the prey to the predator and τ_2 is capture time of the predator.

$d_i (i = 1, 2)$ are the diffusion coefficients of prey and predator species, respectively. The variables $u(t,x)$ and $v(t,x)$ are densities of population of the prey and the predator at time t and space x, respectively. Δ denotes the Laplacian operator in $R^N (N \geq 1)$. System (1) with a single delay varying has the existence of the spatially homogeneous and nonhomogeneous Hopf bifurcation. However, in order to investigate joint effects of the two delays, we will mainly consider effects of the sum of two delays and diffusion on the species in system (1). For convenience, the new variables are introduced as follows:

$$\begin{aligned}\widetilde{u}(t,x) &= u(t-\tau_1,x),\\ \widetilde{v}(t,x) &= v(t,x),\\ \tau &= \tau_1+\tau_2.\end{aligned} \quad (2)$$

System (1) can be transformed into the following form and meanwhile dropping the tildes

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= d_1\Delta u(t,x) + u(t,x)\left[r_1 - a_{11}u(t,x) - a_{12}v(t-\tau,x)\right],\\ \frac{\partial v(t,x)}{\partial t} &= d_2\Delta v(t,x) + v(t,x)\left[-r_2 + a_{21}u(t,x) - a_{22}v(t,x)\right],\end{aligned} \quad (3)$$

which has the following initial conditions:

$$\begin{aligned}u(t,x) &= \phi(t,x)\geq 0, \quad (t,x)\in[-\tau,0]\times\overline{\Omega},\\ v(t,x) &= \varphi(t,x)\geq 0, \quad (t,x)\in[-\tau,0]\times\overline{\Omega},\end{aligned} \quad (4)$$

where Ω is a bounded open domain in $R^N(N\geq 1)$ with a smooth boundary $\partial\Omega$ and the following no flux boundary condition:

$$\frac{\partial u(t,x)}{\partial n} = \frac{\partial v(t,x)}{\partial n} = 0, \quad t\geq 0,\ x\in\partial\Omega, \quad (5)$$

where n is the outward unit normal vector on the boundary.

In what follows, we investigate effects of the delay τ and diffusion on the dynamics of (3) with initial conditions (4) and boundary conditions (5), respectively. We also assume that $\phi,\varphi\in C = C([-\tau,0],X)$ and X is defined by

$$X = \left\{u,v\in W^{2,2}(\Omega): \frac{\partial u(t,x)}{\partial t} = \frac{\partial v(t,x)}{\partial t} = 0,\ x \in\partial\Omega\right\}, \quad (6)$$

with the inner product $\langle\cdot,\cdot\rangle$. In addition, for convenience we restrict ourselves to the one-dimensional spatial domain $\Omega = (0,\pi)$ throughout this paper.

In this paper, we not only consider the bifurcation phenomenon of system (1) as the sum of the two delays varies but also investigate effects of diffusion on the bifurcating periodic solutions. We find that as the sum of two delays crosses some critical values, the bifurcating periodic solution would occur through the spatially homogeneous Hopf bifurcation. In addition, once we change the value of diffusion and fix the domain, we also find that the diffusions of the species can destabilize the bifurcating stable periodic solution under certain conditions.

The rest of the paper is organized as follows. In Section 2, the local analysis is given by taking the sum of two delays as parameter to discuss stability of the positive constant equilibrium and the existence of local Hopf bifurcation. In Section 3, by employing the theory of the center manifold and normal formal theory about the partial functional differential equation developed by Wu in [1], the bifurcation direction of Hopf bifurcation and the stability of the bifurcating periodic solutions are pointed out. In order to testify theoretical analysis results, some numerical simulation figures are also given. In Section 4, by using the Fredholm alternative theory about the periodic solution, we also investigate the effects of diffusion on the bifurcating periodic solution and obtain the conditions which determine the stability of the bifurcating periodic solution. Finally, some discussions and conclusions are drawn in Section 5.

2. Linear Stability Analysis and the Existence of Hopf Bifurcation

Obviously, if $r_1a_{21} - r_2a_{11} > 0$, system (1) or (3) has a positive equilibrium point $E^* = (u_*, v_*)$, where

$$\begin{aligned}u_* &= \frac{r_1a_{22}+r_2a_{12}}{a_{11}a_{22}+a_{12}a_{21}},\\ v_* &= \frac{r_1a_{21}-r_2a_{11}}{a_{11}a_{22}+a_{12}a_{21}}.\end{aligned} \quad (7)$$

Let $u = u_* + \overline{u}$, $v = v_* + \overline{v}$, substitute them into system (3), and meanwhile drop the bars for simplicity of notations, then system (3) can be transformed into the following vector form:

$$\frac{\partial U}{\partial t} = d\Delta u + BU + CU(t-\tau) + N, \quad (8)$$

where

$$U = \begin{pmatrix} u\\ v\end{pmatrix},\quad d = \begin{pmatrix} d_1 & 0\\ 0 & d_2\end{pmatrix},\quad B = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix},\quad C = \begin{pmatrix} c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix},\quad N = \begin{pmatrix} N_1\\ N_2\end{pmatrix}, \quad (9)$$

and

$$\begin{aligned}b_{11} &= r_1 - 2a_{11}u_* - a_{12}v_* = -a_{11}u_*,\\ b_{12} &= 0,\\ b_{21} &= a_{21}v_*,\end{aligned}$$

$$
\begin{aligned}
b_{22} &= -r_2 - 2a_{22}v_* + a_{21}u_* = -a_{22}v_*,\\
c_{11} &= 0,\\
c_{12} &= -a_{12}u_*,\\
c_{21} &= 0,\\
c_{22} &= 0,\\
N_1 &= -a_{11}u^2 - a_{12}v(t-\tau)u,\\
N_2 &= a_{21}uv - a_{22}v^2.
\end{aligned}
\tag{10}
$$

The vector form of the corresponding linearization system of system (8) is as follows:

$$\frac{\partial U}{\partial t} = d\Delta u + BU + CU(t-\tau). \tag{11}$$

The characteristic equation of linearization system (11) of system (8) at the equilibrium point E^* has the following form:

$$
\begin{vmatrix}
\lambda + k^2 d_1 - b_{11} - c_{11}e^{-\lambda\tau} & -b_{12} - c_{12}e^{-\lambda\tau}\\
-b_{21} - c_{21}e^{-\lambda\tau} & \lambda + k^2 d_2 - b_{22} - c_{22}e^{-\lambda\tau}
\end{vmatrix}
= 0,
\tag{12}
$$

i.e.,

$$D_k(\lambda,\tau) = \lambda^2 + p(k)\lambda + r(k) + q(k)e^{-\lambda\tau} = 0, \tag{13}$$

where

$$
\begin{aligned}
p(k) &= (d_1 + d_2)k^2 + a_{11}u_* + a_{22}v_*,\\
r(k) &= d_1 d_2 k^4 + (d_1 a_{22}v_* + d_2 a_{11}u_*)k^2\\
&\quad + a_{11}a_{22}u_*v_*,\\
q(k) &= a_{12}a_{21}u_*v_*.
\end{aligned}
\tag{14}
$$

Case 1. When $\tau = 0$, the associated characteristic equation (13) becomes

$$D_k(\lambda,0) = \lambda^2 + p(k)\lambda + r(k) + q(k) = 0 \tag{15}$$

If $q(k) + r(k) > 0$, the corresponding characteristic equation (15) has two roots with negative part for any $k \in \mathbb{N}_0$.

Case 2. When $\tau \neq 0$, let us get the conditions of taking place spatially homogeneous Hopf bifurcation. It is known that when $k = 0$ the associated characteristic equation $D_0(\lambda,\tau) = 0$ becomes the following form:

$$\lambda^2 + p(0)\lambda + r(0) + q(0)e^{-\lambda\tau} = 0, \tag{16}$$

$$\text{i.e. } D_k(\lambda,\tau) = \lambda^2 + p\lambda + q + ne^{-\lambda\tau} = 0. \tag{17}$$

Let $\pm i\omega(\omega > 0)$ be roots of characteristic equation (13), then

$$
\begin{aligned}
-\omega^2 + p(0)\omega i + r(0) + q(0)\cos(\omega\tau)&\\
- iq(0)\sin(\omega\tau) = 0.&
\end{aligned}
\tag{18}
$$

Separating the real and imaginary parts, we have

$$
\begin{aligned}
\omega^2 - r(0) &= q(0)\cos(\omega\tau),\\
p(0)\omega &= q(0)\sin(\omega\tau),
\end{aligned}
\tag{19}
$$

which leads to

$$\omega^4 + \left(p^2(0) - 2r(0)\right)\omega^2 + r^2(0) - q^2(0) = 0. \tag{20}$$

By the simple analysis, we can immediately obtain the following results:

(a) If $(p^2(0)-2r(0)) > 0, r^2(0)-q^2(0) > 0$ or $\Delta = (p^2(0)-2r(0))^2 - 4(r^2(0)-q^2(0)) < 0$, then (20) has no positive real root.

(b) If $(p^2(0) - 2r(0)) < 0, r^2(0) - q^2(0) > 0$ and $\Delta > 0$, then (20) has two positive roots as follows:

$$
\begin{aligned}
&\omega_{\pm}{}^{*}\\
&= \frac{-\left(p^2(0) - 2r(0)\right) \pm \sqrt{\left(p^2(0) - 2r(0)\right)^2 - 4\left(r^2(0) - q^2(0)\right)}}{2}.
\end{aligned}
\tag{21}
$$

(c) If $r^2(0)-q^2(0) < 0$ or $(p^2(0)-2r(0)) < 0$, and $\Delta = 0$, then (20) has only one positive root $\omega_+^* = -(p^2(0) - 2r(0))/2$.

From the direct calculations and testing the above conditions, we can get that if $a_{11}a_{22}-a_{12}a_{21} < 0$ and $d_1d_2+d_1a_{22}v_*+d_2a_{11}u_*+a_{11}a_{22}u_*v_*-a_{12}a_{21}u_*v_* \geq 0$, then (20) has only one positive root ω^* defined by

$$
\begin{aligned}
&\omega^{*2}\\
&= \frac{-\left(a_{11}^2u_*^2 + a_{22}^2v_*^2\right) + \sqrt{\left(a_{11}^2u_*^2 - a_{22}^2v_*^2\right)^2 + 4a_{12}^2a_{21}^2u_*^2v_*^2}}{2},
\end{aligned}
\tag{22}
$$

and the corresponding

$$\tau_j^* = \frac{1}{\omega^*}\arccos\left(\frac{\omega^{*2} - a_{11}a_{22}u_*v_*}{a_{12}a_{21}u_*v_*}\right) + \frac{2j\pi}{\omega^*}, \tag{23}$$

where $j = 0, 1, 2, \ldots$. Then when $\tau = \tau_j^*$, the characteristic equation has a pair of purely imaginary roots $\pm i\omega^*$.

Lemma 1. *(1) If $a_{11}a_{22} - a_{12}a_{21} \geq 0$ holds, then (15) and (17) have the same number roots with positive real parts for all $\tau \geq 0$ for any $k \geq 0$.*

(2) If $a_{11}a_{22} - a_{12}a_{21} < 0$ and $d_1d_2 + d_1a_{22}v_* + d_2a_{11}u_* + a_{11}a_{22}u_*v_* - a_{12}a_{21}u_*v_* \geq 0$ hold, then (15) and (17) have the same number roots with positive real parts for $\tau \in [0, \tau_0^*)$ for any $k \geq 0$.

Lemma 2. *Suppose that*

$$
\begin{aligned}
&a_{11}a_{22} - a_{12}a_{21} < 0,\\
&d_1d_2 + d_1a_{22}v_* + d_2a_{11}u_* + a_{11}a_{22}u_*v_* - a_{12}a_{21}u_*v_*\\
&\quad \geq 0
\end{aligned}
\tag{24}
$$

hold and $\tau = \tau_j^*$, then $\operatorname{sign}\{\operatorname{Re}[d\lambda/d\tau]|_{\tau=\tau_j^*}\} > 0$.

Proof. Denote by $\lambda(\tau) = \alpha(\tau)+i\omega(\tau)$ the root of characteristic equation such that

$$\omega\left(\tau_j^*\right) = \omega^*. \tag{25}$$

Substituting $\lambda(\tau)$ into the characteristic equation and taking the derivative with respect to τ, we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{(2\lambda + p(0))e^{\lambda\tau}}{\lambda q(0)} - \frac{\tau}{\lambda}, \tag{26}$$

which leads to

$$\begin{aligned}
&\operatorname{sign}\left\{\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}\Bigg|_{\tau=\tau_j^*}\right\} \\
&\quad= \operatorname{sign}\left\{\operatorname{Re}\left[\frac{(2\lambda + p(0))e^{\lambda\tau}}{\lambda q(0)}\right]\Bigg|_{\tau=\tau_j^*}\right\} \\
&\quad= \operatorname{sign}\left\{\frac{1}{q^2(0)}\left[2\omega^{*2} - 2r(0) + p^2(0)\right]\right\} \\
&\quad= \operatorname{sign}\left\{\frac{\sqrt{\left(a_{11}^2u_*^2 - a_{22}^2v_*^2\right) + 4a_{12}^2a_{21}^2u_*^2v_*^2}}{a_{12}^2a_{21}^2u_*^2v_*^2}\right\} > 0.
\end{aligned} \tag{27}$$

This proof is completed. □

Therefore, employing Lemma 1, Lemma 2, and Hopf bifurcation theorem for partial functional differential equations in [1], we can obtain the following conclusions.

Theorem 3. *(1) If $a_{11}a_{22} - a_{12}a_{21} \geq 0$ holds, then the coexistence positive equilibrium E^* of system (8) is locally asymptotically stable for all $\tau \geq 0$.*

(2) If $a_{11}a_{22} - a_{12}a_{21} < 0$ and $d_1d_2 + d_1a_{22}v_* + d_2a_{11}u_* + a_{11}a_{22}u_*v_* - a_{12}a_{21}u_*v_* \geq 0$ hold, then the coexistence positive equilibrium E^* of system (8) is locally asymptotically stable for $\tau \in [0, \tau_0^*)$ and unstable when $\tau > \tau_0^*$.

(3) If $a_{11}a_{22} - a_{12}a_{21} < 0$ and $d_1d_2 + d_1a_{22}v_* + d_2a_{11}u_* + a_{11}a_{22}u_*v_* - a_{12}a_{21}u_*v_* \geq 0$ hold, then system (8) undergoes the Hopf bifurcation at the coexistence positive equilibrium E^* for $\tau = \tau_j^*$.

Remark 4. For $k_0 = 0$, this bifurcation is called spatially homogeneous Hopf bifurcation. For $k_0 \neq 0$, this bifurcation is called spatially nonhomogeneous Hopf bifurcation. This spatially homogeneous Hopf bifurcation is the same as the nondispersal equation in [16] if we impose the restrictive diffusion condition as follows:

$$\begin{aligned}
&d_1d_2 + d_1a_{22}v_* + d_2a_{11}u_* + a_{11}a_{22}u_*v_* - a_{12}a_{21}u_*v_* \\
&\quad\geq 0.
\end{aligned} \tag{28}$$

3. Direction and Stability of the Bifurcating Periodic Solution

In this section, we mainly devote our interests to the properties of the bifurcating periodic solution, including the direction of bifurcation and stability of the bifurcating periodic solution.

Let $t = \tau\tilde{t}$, and $\alpha = \tau - \tau^*$, substitute them into system (8), and meanwhile drop the tildes above t, then system (8) can be denoted in the form of abstract partial functional differential equations

$$\begin{aligned}
\dot{U}(t) &= (\alpha + \tau^*)d\Delta U + (\alpha + \tau^*)L(U_t) \\
&\quad + (\alpha + \tau^*)N(U_t) \\
&= \tau^*d\Delta U + \tau^*L(U_t) \\
&\quad + \left[\alpha d\Delta U + \alpha L(U_t) + (\alpha + \tau^*)N(U_t)\right],
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
L(U_t) &= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \\
&\quad + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} u(t-1) \\ v(t-1) \end{pmatrix}, \\
N(U_t) &= \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} = \begin{pmatrix} -a_{11}u^2 - a_{12}v(t-1)u \\ a_{21}uv - a_{22}v^2 \end{pmatrix}.
\end{aligned} \tag{30}$$

For convenience, we rewrite system (29) in the following form:

$$\dot{U}(t) = \tau^*d\Delta U + L_*(U_t) + N_*(U_t) \tag{31}$$

where

$$\begin{aligned}
L_*(U_t) &= \tau^*L(U_t), \\
N_*(U_t) &= \left[\alpha d\Delta U + \alpha L(U_t) + (\alpha + \tau^*)N(U_t)\right].
\end{aligned} \tag{32}$$

And the corresponding linearization system of system (29) becomes

$$\dot{U}(t) = \tau^*d\Delta U + L_*(U_t). \tag{33}$$

From the discussion of Section 2, we know that $(0, 0)$ is an equilibrium point of system (29), and the characteristic equation of (33) has a pair of simple purely imaginary eigenvalues $\pm i\omega^*\tau^*$ when $\tau = \tau^*$, $k = 0$.

Consider the ordinary functional differential equation:

$$\dot{u}(t) = -\tau^*k^2du(t) + L_*(u_t), \tag{34}$$

and by the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta, \tau^*)$ $(-1 \leq \theta \leq 0)$, whose elements are of bounded variation such that

$$\begin{aligned}
-\tau^*k^2d\phi(0) + L_*(\phi) &= \int_{-1}^{0} d[\eta(\theta)]\phi(\theta) \\
&\qquad \text{for } \phi \in C\left([-1, 0], \mathbb{R}^2\right).
\end{aligned} \tag{35}$$

In fact, we can choose

$$\eta(\theta,\tau^*) = \begin{cases} \tau^* \begin{pmatrix} -d_1k^2+b_{11} & b_{12} \\ b_{21} & -d_2k^2+b_{22} \end{pmatrix}, & \theta = 0, \\ 0, & \theta \in (-1,0), \\ -\tau^* \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, & \theta = -1. \end{cases} \tag{36}$$

Let $A(\tau^*)$ denote the infinitesimal generators of the semigroup induced by the solutions of (34) and $A^*(\tau^*)$ be the formal adjoint operator of $A(\tau^*)$ under the following bilinear functional:

$$\begin{aligned} (\varphi(\theta),\psi(\theta)) &= \sum_{i=1}^{2}\overline{\psi_i(0)}\varphi_i(0) - \int_{-1}^{0}\int_{0}^{\theta}\overline{\psi(\xi+1)}d\eta(\theta)\varphi(\xi)\,d\xi \\ &= \sum_{i=1}^{2}\overline{\psi_i(0)}\varphi_i(0) + \tau^*\int_{-1}^{0}\overline{\psi(\xi+1)}C\varphi(\xi)\,d\xi \end{aligned} \tag{37}$$

for $\varphi \in C([-1,0],\mathbb{R}^2)$, $\psi \in C([0,1],\mathbb{R}^2)$. From the discussion of Section 2, we know that $A(\tau^*)$ and $A^*(\tau^*)$ have a pair of simple pure imaginary roots $\pm i\omega^*\tau^*$. Let $p_1(\theta)$ and $p_2(\theta)$ be eigenvector of $A(\tau^*)$ about eigenvalues $\pm i\omega^*\tau^*$, respectively, and $q_1(\theta)$ and $q_2(\theta)$ be eigenvector of $A^*(\tau^*)$ about eigenvalues $\pm i\omega^*\tau^*$, respectively.

Lemma 5.

$$\text{Let } \xi = \frac{a_{21}v^*}{i\omega^*\tau^* + a_{22}v^*}, \qquad \eta = \frac{a_{11}u^* - i\omega^*\tau^*}{a_{21}v^*}, \tag{38}$$

then

$$p_1(\theta) = e^{i\omega\tau^*\theta}\begin{pmatrix}1\\ \xi\end{pmatrix}, \qquad p_2(\theta) = \overline{p_1(\theta)}, \qquad -1 \le \theta \le 0, \tag{39}$$

is a basis of P with Λ_0 *and*

$$q_1(\theta) = \frac{1}{D}q_1^*, \quad q_1^* = e^{i\omega\tau^*\theta}(1,\eta), \qquad q_2(\theta) = \overline{q_1(\theta)}, \qquad 0 \le \theta \le 1, \tag{40}$$

is a basis of Q with Λ_0, *where* $D = (p_1(\theta), q_1^*(s))$, *P and Q are the center subspace. That is to say, P and Q are the generalized eigenspace of operators* $A(\tau^*)$ *and* $A^*(\tau^*)$ *associated with* Λ_0, *respectively,* $\dim P = \dim Q = 2$.

Proof. According to the definition of operators $A(\tau^*)$ and $A^*(\tau^*)$, we have the following equations:

$$\begin{aligned} i\omega^*\tau^*\begin{pmatrix}1\\ \xi\end{pmatrix} &= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix}1\\ \xi\end{pmatrix} + e^{-i\omega^*\tau^*}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}\begin{pmatrix}1\\ \xi\end{pmatrix}, \\ -i\omega^*\tau^*\begin{pmatrix}1\\ \eta\end{pmatrix}^T &= \begin{pmatrix}1\\ \eta\end{pmatrix}^T\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix} + e^{i\omega^*\tau^*}\begin{pmatrix}1\\ \eta\end{pmatrix}^T\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}. \end{aligned} \tag{41}$$

We can obtain the values of ξ and η,

$$\xi = \frac{a_{21}v^*}{i\omega^*\tau^* + a_{22}v^*}, \qquad \eta = \frac{a_{11}u^* - i\omega^*\tau^*}{a_{21}v^*}. \tag{42}$$

Let

$$\Phi = (p_1(\theta),\ p_2(\theta)), \qquad \Psi = \begin{pmatrix} q_1(\theta) \\ q_2(\theta)\end{pmatrix}, \tag{43}$$

such that

$$(\Psi,\ \Phi) = \begin{pmatrix} (p_1(\theta)\ \ q_1(\theta)) & (p_1(\theta)\ \ q_2(\theta)) \\ (p_2(\theta)\ \ q_1(\theta)) & (p_2(\theta)\ \ q_2(\theta)) \end{pmatrix} = I_{2\times 2}. \tag{44}$$

The proof is completed. □

In addition, $f_k = \binom{\beta_k^1}{\beta_k^2}$ and $c \cdot f_k = c_1\beta_k^1 + c_2\beta_k^2$, for any $c = \binom{c_1}{c_2} \in \mathbb{C}^2$. The center subspace of linear equation (33) is given by $P_{CN}\mathfrak{F}$, where

$$P_{CN}\mathfrak{F}(\phi) = \Phi(\Psi,\langle\phi, f_k\rangle)\cdot f_k, \quad \phi \in \mathfrak{F}, \tag{45}$$

and $\mathfrak{F} = P_{CN}\mathfrak{F} \oplus P_S\mathfrak{F}$, where $P_S\mathfrak{F}$ denotes the complement subspace of $P_{CN}\mathfrak{F}$ in $\mathfrak{F}$.

Let A_{τ^*} be the infinitesimal generator induced by the solution of (33). Then (29) can be rewritten as the abstract form

$$\dot{u}_t = A_{\tau^*}u_t + R(\alpha, u_t), \tag{46}$$

where $R(\alpha,u_t) = \begin{cases} 0, & \theta \in [-1,0), \\ N(u_t,\alpha), & \theta = 0. \end{cases}$

Using the decomposition $\mathfrak{I} = P_{CN}\mathfrak{I} \oplus P_S\mathfrak{I}$ and (45), the solution of (31) can be written as

$$u_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k + h(x_1, x_2, \alpha), \tag{47}$$

where $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = (\Psi, \langle u_t, f_k \rangle)$ and $h(x_1, x_2, \alpha) \in P_S\mathfrak{I}$, $h(0,0,0) = 0, Dh(0,0,0) = 0$.

In particular, the solution of (31) on the center manifold is given by

$$u_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k + h(x_1, x_2, 0). \tag{48}$$

Let $z = x_1 - ix_2$, $\Psi(0) = \begin{pmatrix} \Psi_1(0) \\ \Psi_2(0) \end{pmatrix}$, and notice that $p_1 = \Phi_1 + i\Phi_2$. Then

$$\begin{aligned} \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k &= (\Phi_1, \Phi_2) \begin{pmatrix} \dfrac{z+\overline{z}}{2} \\ \dfrac{i(z-\overline{z})}{2} \end{pmatrix} \cdot f_k \\ &= \frac{1}{2}(p_1 z + \overline{p}_1 \overline{z}) \cdot f_k. \end{aligned} \tag{49}$$

Then system (48) can be transformed into

$$u_t = \frac{1}{2}(p_1 z + \overline{p}_1 \overline{z}) \cdot f_k + w(z, \overline{z}), \tag{50}$$

where

$$\begin{aligned} w(z, \overline{z}) &= h\left(\frac{z+\overline{z}}{2}, \frac{i(z-\overline{z})}{2}, 0\right) \\ &= w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta) z\overline{z} + w_{02}(\theta)\frac{\overline{z}^2}{2} \\ &\quad + w_{30}(\theta)\frac{z^3}{6} + \cdots, \\ &\qquad w_{ij}(\theta) \in C([-1,0], Q). \end{aligned} \tag{51}$$

Moreover by [1], z satisfies

$$\dot{z} = i\omega\tau^* z + g(z, \overline{z}), \tag{52}$$

where

$$\begin{aligned} g(z, \overline{z}) &= \frac{1}{D}(1, \eta)\langle N(u_t, 0), f_k \rangle \\ &= g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \end{aligned} \tag{53}$$

From (31) and (48), it follows that

$$\begin{aligned} U_t &= \Phi \begin{pmatrix} z(t) \\ \overline{z}(t) \end{pmatrix} \cdot f_k + W(z, \overline{z}) \\ &= \frac{1}{2}(p_1 z + p_2 \overline{z}) \cdot f_k + W_{20}(\theta)\frac{z^2}{2} + W_{20}(\theta) z\overline{z} \\ &\quad + W_{02}(\theta)\frac{\overline{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots \\ &= \frac{1}{2}\begin{pmatrix} e^{i\omega\tau^*\theta} z + e^{-i\omega\tau^*\theta}\overline{z} \\ \xi e^{i\omega\tau^*\theta} z + \overline{\xi} e^{-i\omega\tau^*\theta}\overline{z} \end{pmatrix} \cdot f_k + W_{20}(\theta)\frac{z^2}{2} \\ &\quad + W_{20}(\theta) z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots \\ &= \frac{1}{2}\left(e^{i\omega\tau^*\theta} z + e^{-i\omega\tau^*\theta}\overline{z}\right)\beta_k^1 \\ &\quad + \frac{1}{2}\left(\xi e^{i\omega\tau^*\theta} z + \overline{\xi} e^{-i\omega\tau^*\theta}\overline{z}\right)\beta_k^2 + W_{20}(\theta)\frac{z^2}{2} \\ &\quad + W_{20}(\theta) z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots \\ &= \frac{1}{2}\left(e^{i\omega\tau^*\theta} z + e^{-i\omega\tau^*\theta}\overline{z}\right)\begin{pmatrix} \gamma_k \\ 0 \end{pmatrix} \\ &\quad + \frac{1}{2}\left(\xi e^{i\omega\tau^*\theta} z + \overline{\xi} e^{-i\omega\tau^*\theta}\overline{z}\right)\begin{pmatrix} 0 \\ \gamma_k \end{pmatrix} \\ &\quad + \begin{pmatrix} W_{20}^{(1)}(\theta) \\ W_{20}^{(2)}(\theta) \end{pmatrix}\frac{z^2}{2} + \begin{pmatrix} W_{11}^{(1)}(\theta) \\ W_{11}^{(2)}(\theta) \end{pmatrix} z\overline{z} \\ &\quad + \begin{pmatrix} W_{02}^{(1)}(\theta) \\ W_{02}^{(2)}(\theta) \end{pmatrix}\frac{\overline{z}^2}{2} + \cdots \end{aligned} \tag{54}$$

We can obtain the following results:

$$\begin{aligned} u_t(0) &= \frac{1}{2}\gamma_k z + \frac{1}{2}\gamma_k \overline{z} + w_{20}^{(1)}(0)\frac{z^2}{2} + w_{11}^{(1)}(0) z\overline{z} \\ &\quad + \cdots, \\ v_t(0) &= \frac{1}{2}\gamma_k \xi z + \frac{1}{2}\gamma_k \overline{\xi}\overline{z} + w_{20}^{(2)}(0)\frac{z^2}{2} + w_{11}^{(2)}(0) z\overline{z} \\ &\quad + \cdots, \\ v_t(-1) &= \frac{1}{2}e^{-i\omega\tau^*}\gamma_k \xi z + \frac{1}{2}e^{i\omega\tau^*}\gamma_k \overline{\xi}\overline{z} + w_{20}^{(2)}(-1)\frac{z^2}{2} \\ &\quad + w_{11}^{(2)}(-1) z\overline{z} + \cdots, \\ u_t^2(0) &= \left(\frac{1}{4}\gamma_k^2\right) z^2 + \left(\frac{1}{2}\gamma_k^2\right) z\overline{z} + \left(\frac{1}{4}\gamma_k^2\right)\overline{z}^2 \\ &\quad + \left(w_{11}^{(1)}(0)\gamma_k + \frac{1}{2}w_{20}^{(1)}(0)\gamma_k\right) z^2\overline{z} + \cdots, \end{aligned}$$

$$\begin{aligned}
u_t(0)\,u_{t2}(-1) ={}& \left(\frac{1}{4}e^{-i\omega\tau^*}\xi\gamma_k^2\right)z^2 + \left(\frac{1}{4}e^{-i\omega\tau^*}\xi\gamma_k^2\right.\\
&\left.+\frac{1}{4}e^{i\omega\tau^*}\overline{\xi}\gamma_k^2\right)z\overline{z} + \left(\frac{1}{4}e^{i\omega\tau^*}\overline{\xi}\gamma_k^2\right)\overline{z}^2\\
&+\left(\frac{1}{2}w_{11}^{(2)}(-1)\gamma_k + \frac{1}{2}w_{11}^{(1)}(0)e^{-i\omega\tau^*}\xi\gamma_k\right.\\
&\left.+\frac{1}{4}w_{20}^{(2)}(-1)\gamma_k + \frac{1}{4}w_{20}^{(1)}(0)e^{i\omega\tau^*}\overline{\xi}\gamma_k\right)z^2\overline{z} + \cdots,\\
u_t(0)\,v_t(0) ={}& \left(\frac{1}{4}\xi\gamma_k^2\right)z^2 + \left(\frac{1}{4}\xi\gamma_k^2 + \frac{1}{4}\overline{\xi}\gamma_k^2\right)z\overline{z}\\
&+\left(\frac{1}{4}\overline{\xi}\gamma_k^2\right)\overline{z}^2 + \left(\frac{1}{2}\gamma_k w_{11}^{(2)}(0) + \frac{1}{4}\gamma_k w_{20}^{(2)}(0)\right.\\
&\left.+\frac{1}{2}\xi\gamma_k w_{11}^{(1)}(0) + \frac{1}{4}\overline{\xi}\gamma_k w_{20}^{(1)}(0)\right)z^2\overline{z} + \cdots\\
v_t^2(0) ={}& \left(\frac{1}{4}\xi^2\gamma_k^2\right)z^2 + \left(\frac{1}{2}\xi\overline{\xi}\gamma_k^2\right)z\overline{z} + \left(\frac{1}{4}\overline{\xi}^2\gamma_k^2\right)\overline{z}^2\\
&+\left(\xi\gamma_k w_{11}^{(2)}(0) + \frac{1}{2}\overline{\xi}\gamma_k w_{20}^{(2)}(0)\right)z^2\overline{z} + \cdots,\\
N_1(U_t) ={}& -a_{11}u_t^2(0) - a_{12}u_t(0)\,v_t(-1),\\
N_2(U_t) ={}& a_{21}u_t(0)\,v_t(0) - a_{22}v_t^2(0),\\
N(U_t) ={}& \begin{pmatrix} N_1(U_t)\\ N_2(U_t)\end{pmatrix}\\
={}& \begin{pmatrix} -a_{11}u_t^2(0) - a_{12}u_t(0)\,v_t(-1)\\ a_{21}u_t(0)\,v_t(0) - a_{22}v_t^2(0)\end{pmatrix},\\
\langle N(U_t,0), f_k\rangle ={}& \begin{pmatrix}\langle N(U_t,0),\beta_k^1\rangle\\ \langle N(U_t,0),\beta_k^2\rangle\end{pmatrix} = \frac{1}{\pi}\\
&\cdot\begin{pmatrix}\int_0^{\pi} N_1(U_t,0)\,\gamma_k dx\\ \int_0^{\pi} N_2(U_t,0)\,\gamma_k dx\end{pmatrix}.
\end{aligned} \tag{55}$$

According to the following equality,

$$\begin{aligned}
g(z,\overline{z}) &= q_1(0)\,\langle N_*(U_t,0), f_k\rangle\\
&= q_1(0)\,\langle N(U_t,0), f_k\rangle,
\end{aligned} \tag{56}$$

we can obtain the following results:

$$\begin{aligned}
g_{20} &= \frac{1}{\pi}term_{z^2}\left(\int_0^{\pi} N_1(U_t,0)\,\gamma_k dx\right.\\
&\quad\left.+\eta\int_0^{\pi} N_2(U_t,0)\,\gamma_k dx\right),\\
g_{11} &= \frac{1}{\pi}term_{z\overline{z}}\left(\int_0^{\pi} N_1(U_t,0)\,\gamma_k dx\right.\\
&\quad\left.+\eta\int_0^{\pi} N_2(U_t,0)\,\gamma_k dx\right),\\
g_{02} &= \frac{1}{\pi}term_{\overline{z}^2}\left(\int_0^{\pi} N_1(U_t,0)\,\gamma_k dx\right.\\
&\quad\left.+\eta\int_0^{\pi} N_2(U_t,0)\,\gamma_k dx\right),\\
g_{21} &= \frac{1}{\pi}term_{z^2\overline{z}}\left(\int_0^{\pi} N_1(U_t,0)\,\gamma_k dx\right.\\
&\quad\left.+\eta\int_0^{\pi} N_2(U_t,0)\,\gamma_k dx\right),
\end{aligned} \tag{57}$$

where $term_{z^2}$, $term_{z\overline{z}}$ $term_{\overline{z}^2}$ $term_{z^2\overline{z}}$ denote the coefficients of the z^2, $z\overline{z}$, $\overline{z}^2$, $z^2\overline{z}$ in the polynomial $(1/\overline{D})\int_0^{\pi} N_1(U_t,0)\gamma_k dx + (\eta/\overline{D})\int_0^{\pi} N_2(U_t,0)\gamma_k dx$ about z and $\overline{z}$, respectively.

Next, in order to get the value of g_{21}, we need to compute the expression of $W_{20}(\theta), W_{11}(\theta)$. It follows from (50) that

$$\begin{aligned}
\dot{w}(z,\overline{z}) &= \dot{w}_z\dot{z} + \dot{w}_{\overline{z}}\dot{\overline{z}}\\
&= w_{20}z\dot{z} + w_{11}\dot{z}\overline{z} + w_{11}z\dot{\overline{z}} + w_{02}\overline{z}\dot{\overline{z}} + \cdots,
\end{aligned} \tag{58}$$

$$\begin{aligned}
A_{\tau^*}w &= A_{\tau^*}w_{20}(\theta)\frac{z^2}{2} + A_{\tau^*}w_{11}(\theta)\,z\overline{z}\\
&\quad+ A_{\tau^*}w_{02}(\theta)\frac{\overline{z}^2}{2} + A_{\tau^*}w_{30}(\theta)\frac{z^3}{6} + \cdots.
\end{aligned} \tag{59}$$

In addition, by [1], $w(z,\overline{z})$ satisfies

$$\begin{aligned}
\dot{w}(t) &= A_{\tau^*}w + H(z,\overline{z},\theta)\\
&= A_{\tau^*}w + X_0N_*(U_t,0)\\
&\quad- \Phi(\Psi, \langle X_0N_*(U_t,0), f_k\rangle)\\
&\quad\cdot f_k,
\end{aligned} \tag{60}$$

$$\begin{aligned}
\text{where } H(z,\overline{z},\theta) &= H_{20}\frac{z^2}{2} + H_{11}(\theta)\,z\overline{z} + H_{02}\frac{\overline{z}^2}{2}\\
&\quad+\cdots.
\end{aligned} \tag{61}$$

Thus from (50), (52), (58), (59), (60), and (61) we can obtain the following results:

$$\begin{aligned}
(2i\omega^*\tau^* - A_{\tau^*})\,W_{20}(\theta) &= H_{20}(\theta),\\
-A_{\tau^*}W_{11}(\theta) &= H_{11}(\theta),\\
(-2i\omega^*\tau^* - A_{\tau^*})\,W_{02}(\theta) &= H_{02}(\theta).
\end{aligned} \tag{62}$$

From (60)–(62), when $-1 \le \theta < 0$, we have

$$\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}}{\omega^*\tau^*}q(\theta)\cdot f_k + \frac{i\overline{g}_{02}}{3\omega^*\tau^*}\overline{q(\theta)}\cdot f_k\\
&\quad+ E_1e^{2i\omega^*\tau^*\theta},
\end{aligned} \tag{63}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega^*\tau^*}q(\theta)\cdot f_k + \frac{i\overline{g}_{11}}{\omega^*\tau^*}\overline{q(\theta)}\cdot f_k + E_2.$$

Meanwhile when $\theta = 0$, employing the definition of operator A_{τ^*} and (60), we have

$$\begin{aligned} \left(2i\omega^*\tau^* - A_{\tau^*}\right) W_{20}(\theta) &= H_{20}(\theta), \\ -A_{\tau^*} W_{11}(\theta) &= H_{11}(\theta), \\ \left(-2i\omega^*\tau^* - A_{\tau^*}\right) W_{02}(\theta) &= H_{02}(\theta). \end{aligned} \quad (64)$$

Combining (63) and (64), we can get the following results:

$$\begin{aligned} &2i\omega^*\tau^* \left[\frac{ig_{20}}{\omega^*\tau^*} q(0) \cdot f_k + \frac{i\overline{g}_{02}}{3\omega^*\tau^*} \overline{q(0)} \cdot f_k + E_1 \right] \\ &- \Delta \left[\frac{ig_{20}}{\omega^*\tau^*} q(0) \cdot f_k + \frac{i\overline{g}_{02}}{3\omega^*\tau^*} \overline{q(0)} \cdot f_k + E_1 \right] \\ &- L_* \left[\frac{ig_{20}}{\omega^*\tau^*} q(\theta) \cdot f_k + \frac{i\overline{g}_{02}}{3\omega^*\tau^*} \overline{q(\theta)} \cdot f_k \right. \\ &\left. + E_1 e^{2i\omega^*\tau^*\theta} \right] = -\left[g_{20} q(0) + \overline{g_{02}} \overline{q(0)} \right] \cdot f_k \\ &+ \begin{pmatrix} N_1(z^2) \\ N_2(z^2) \end{pmatrix}, \end{aligned} \quad (65)$$

and

$$\begin{aligned} &- \Delta \left[-\frac{ig_{11}}{\omega^*\tau^*} q(0) \cdot f_k + \frac{i\overline{g}_{11}}{\omega^*\tau^*} \overline{q(0)} \cdot f_k + E_2 \right] \\ &- L_* \left[-\frac{ig_{11}}{\omega^*\tau^*} q(\theta) \cdot f_k + \frac{i\overline{g}_{11}}{\omega^*\tau^*} \overline{q(\theta)} \cdot f_k + E_2 \right] \\ &= -\left[g_{11} q(0) + \overline{g_{11}} \overline{q(0)} \right] \cdot f_k + \begin{pmatrix} N_1(z\overline{z}) \\ N_2(z\overline{z}) \end{pmatrix}. \end{aligned} \quad (66)$$

Meanwhile note the following equalities:

$$\begin{aligned} &\tau^* d\Delta \left[p_1(0) \cdot f_k \right] + L(\tau^*) \left[p_1(0) \cdot f_k \right] \\ &\quad = i\omega\tau^* p_1(0) \cdot f_k, \\ &\tau^* d\Delta \left[p_2(0) \cdot f_k \right] + L(\tau^*) \left[p_2(0) \cdot f_k \right] \\ &\quad = -i\omega\tau^* p_2(0) \cdot f_k. \end{aligned} \quad (67)$$

Associating with (65)–(67), we can obtain the following two equalities:

$$-\tau^* d\Delta E_2 - L_*(E_2) = \begin{pmatrix} N_1(z\overline{z}) \\ N_2(z\overline{z}) \end{pmatrix}, \quad (68)$$

$$2i\omega\tau^* E_1 - \tau^* d\Delta E_1 - L_*\left(E_1 e^{2i\omega\tau^*\theta}\right) = \begin{pmatrix} N_1(z^2) \\ N_2(z^2) \end{pmatrix}. \quad (69)$$

Thus from (68) and (69), we can get the values of E_1 and E_2,

$$\begin{aligned} E_1 &= \begin{pmatrix} 2i\omega\tau^* + d_1k^2 + a_{11}u_1^* & a_{12}u_1^* e^{-2\omega^*\tau^*} \\ -a_{21}u_2^* & 2i\omega\tau^* + d_2k^2 + a_{22}u_2^* \end{pmatrix}^{-1} \\ &\quad \cdot \begin{pmatrix} N_1(z^2) \\ N_2(z^2) \end{pmatrix}, \\ E_2 &= \begin{pmatrix} d_1k^2 + a_{11}u_1^* & a_{12}u_1^* \\ -a_{21}u_2^* & d_2k^2 + a_{22}u_2^* \end{pmatrix}^{-1} \begin{pmatrix} N_1(z\overline{z}) \\ N_2(z\overline{z}) \end{pmatrix}. \end{aligned} \quad (70)$$

Summarizing the above analysis, we can obtain the expression of $W_{20}(\theta)$ and $W_{11}(\theta)$ and get the value of g_{21}. According to the normal form theory developed by Wu [1], we can get the normal form of system (8).

Theorem 6 ([1]).

$$\dot{\xi} = \lambda(\mu)\xi + C(\mu)\xi|\xi|^2 + o\left(|\xi|\,|(\xi,\mu)|^3\right), \quad (71)$$

where

$$C(0) = \frac{i}{2\omega^*\tau^*}\left(g_{20}g_{11} - 2|g_{11}| - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}. \quad (72)$$

Further, the following several terms which determine the properties of bifurcating periodic solution are given as follows:

$\mu_2 = -\mathrm{Re}\, C(0)/\tau^* \mathrm{Re}(\lambda'(\tau^*))$ *which determines the direction of bifurcation.*

$\beta_2 = 2\mathrm{Re}\, C(0)$ *which determines the stability of the bifurcating periodic solution.*

$T_2 = (-\mathrm{Im}(C(0)) + \mu_2 \mathrm{Im}(\lambda'(\tau^*)))/\omega^*\tau^*$ *which determines the variation of period of the bifurcating periodic solution.*

For system (8), employing the above discussion, we can obtain the following further results.

Theorem 7. *(1) If* $\mathrm{Re}\, C(0) > 0$, *then the bifurcating periodic solution exists in the side of* $\tau < {\tau_0}^*$ *and is unstable.*

(2) If $\mathrm{Re}\, C(0) < 0$, *then the bifurcating periodic solution exists in the side of* $\tau > {\tau_0}^*$ *and is stable.*

Next we give the corresponding numerical results and set the parameters as follows:

$$\begin{aligned} d_1 &= d_2 = 1, \\ r_1 &= 0.6, \\ r_2 &= 0.5, \\ a_{11} &= 0.5, \\ a_{12} &= 0.5, \\ a_{21} &= 0.8, \\ a_{22} &= 0.2. \end{aligned} \quad (73)$$

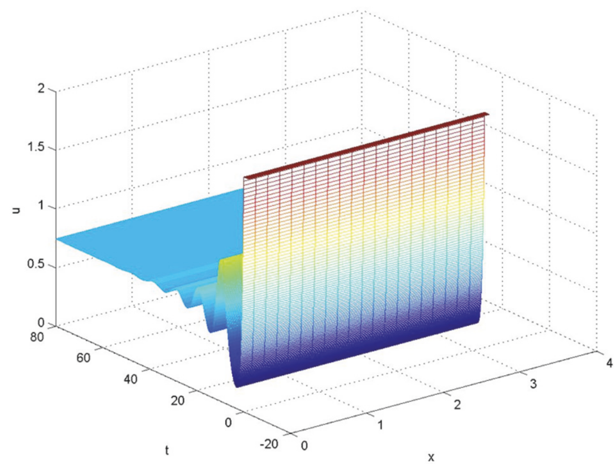

Figure 1: The trajectory of prey densities versus time t and position x with the initial condition $u = 2, v = 1$ when $\tau = 2 < \tau_0^*$.

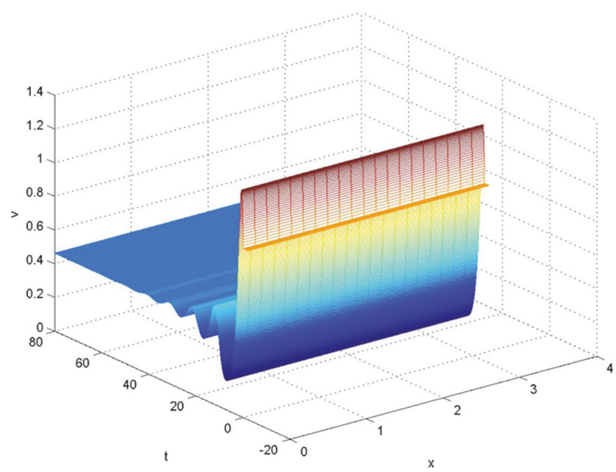

Figure 2: The trajectory of predator densities versus time t and position x with the initial condition $u = 2, v = 1$ when $\tau = 2 < \tau_0^*$.

We can get the equilibrium $E^* = (0.74, 0.46)$ and $\tau_0^* = 2.725$. Figures 1 and 2 present that the equilibrium of system (3) is locally asymptotically stable when the sum of two delays is less than τ_0^*. But when the sum of two delays is greater than τ_0^*, the equilibrium point of system (3) will be unstable and lead to the spatially homogeneous periodic solution to occur; see Figures 3 and 4.

4. Effects of Diffusion on the Bifurcating Periodic Solution

In the previous section, we mainly discuss the spatially homogeneous Hopf bifurcation and the relevant properties of the bifurcating periodic solution for the fixed domain and diffusion. However, in this present section, we will employ the method based on Fredholm alternative to investigate the effects of diffusions on the bifurcating periodic solution. For simplicity, we take the same notations of [1].

From the previous section, we know that $p_1(\theta)(-1 \le \theta \le 0)$ is the eigenfunction of $N_\phi(0) = A(0)$ corresponding to the simple eigenvalue $i\omega^*\tau^*$, that is, $A(0)p_1 = i\omega^*\tau^* p_1$; and $q_1(\theta)(0 \le \theta \le 1)$ is the eigenfunction of $A^*(0)$ corresponding to the simple eigenvalue $-i\omega^*\tau^*$, that is, $A^*(0)q_1 = -i\omega^*\tau^* q_1$, where

$$p_1 = p_1(0) e^{i\omega^*\tau^*\theta} = \begin{pmatrix} 1 \\ \xi \end{pmatrix} e^{i\omega^*\tau^*\theta} \quad (-1 \le \theta \le 0), \tag{74}$$

$$q_1 = q_1(0) e^{i\omega^*\tau^*\theta} = \frac{1}{\overline{D}} \begin{pmatrix} 1 \\ \eta \end{pmatrix}^T e^{i\omega^*\tau^*\theta} \quad (0 \le \theta \le 1). \tag{75}$$

Now we define $\zeta, \widehat{\zeta} \in C([-1,0];\mathbb{R}^2)$ by

$$\begin{aligned} (2i\omega^*\tau^* - A(0))\zeta &= \frac{1}{2} N_{\phi\phi}(0)(p_1, p_1), \\ -A(0)\widehat{\zeta} &= N_{\phi\phi}(0)(p_1, \overline{p}_1), \end{aligned} \tag{76}$$

where ζ has the form

$$\zeta(\theta) = \zeta(0) e^{2i\omega^*\tau^*\theta} \quad (-1 \le \theta \le 0), \tag{77}$$

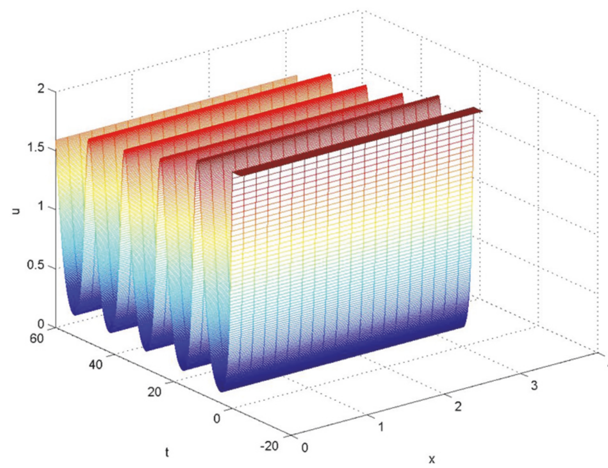

FIGURE 3: The trajectory of prey densities versus time t and position x with the initial condition $u = 2, v = 1$ when $\tau = 3 > \tau_0^*$.

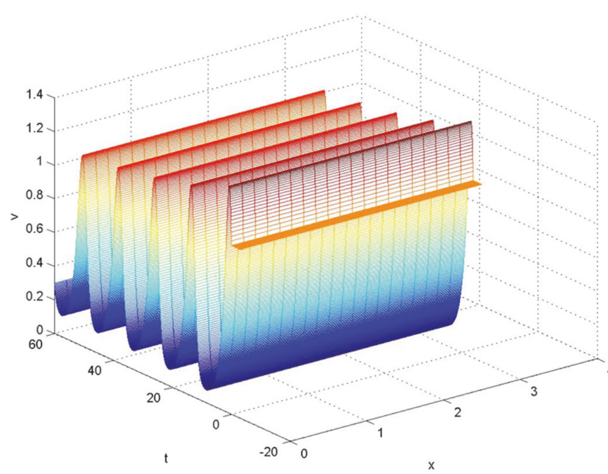

FIGURE 4: The trajectory of prey densities versus time t and position x with the initial condition $u = 2, v = 1$ when $\tau = 3 > \tau_0^*$.

and $\zeta(0)$ is determined by

$$\begin{aligned}&\left(2i\omega^*\tau^* I - L(0)\left(e^{2i\omega^*\tau^* \cdot}\right)\right)\zeta(0)\\&\quad = \frac{1}{2}N_{\phi\phi}(0)(p_1, p_1),\end{aligned} \tag{78}$$

and $\widehat{\zeta}$ is a constant 2-vector satisfying

$$-L(0)\widehat{\zeta} = N_{\phi\phi}(0)(p_1, \overline{p}_1). \tag{79}$$

With the above preparations, we can now discuss the stability of $p(t,\varepsilon)$ as a solution of (29). First of all, we note that the linear variation equation of (29) about $p(t,\varepsilon)$ is given by

$$\frac{dz(t)}{dt} = d\Delta z(t) + N_\phi\left(\mu(\varepsilon), p_t(.,\varepsilon)\right)z_t. \tag{80}$$

Adopting the new variables

$$\begin{aligned}s &= \omega(\varepsilon)t,\\ \omega(s) &= z\left(\frac{s}{\omega(\varepsilon)}\right),\end{aligned} \tag{81}$$

we can transform (80) into

$$\begin{aligned}\omega(\varepsilon)\frac{d}{ds}\omega(s) &= d\Delta\omega(s)\\&\quad + N_\phi\left(\mu(\varepsilon), y_{s,\omega(\varepsilon)}(.,\varepsilon)\right)\omega_{s,\omega(s)},\end{aligned} \tag{82}$$

where $y(s,\varepsilon) = p(s/\omega(\varepsilon);\varepsilon)$, $\omega_{s,\omega(\varepsilon)}(\theta) = \omega(s+\omega(\varepsilon)\theta)$, $-1 \le \theta \le 0$.

Let λ_m be the mth eigenvalue of the vector $-\Delta$ and let φ_m be the eigenfunction corresponding to λ_m, i.e.,

$$\begin{aligned}&\Delta\varphi_m = -\lambda_m\varphi_m,\\&\qquad m = 1, 2, \ldots, \quad (0 = \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots).\end{aligned} \tag{83}$$

Clearly, γ is a Floquet exponent of the linear 2π-periodic system (80) if and only if there exists a function $q(s) \neq 0$ and a positive integer m such that

$$[\omega(s)](x) = e^{\gamma(s)/\omega(\varepsilon)}q(s)\varphi_m(x) \tag{84}$$

satisfies (80). So calculating the Floquet exponent γ is reduced to finding a solution $y(s)$ to (80) in the form (84). Substituting

(84) into (80) and comparing the coefficients of φ_m on both sides of (80), we obtain

$$\begin{aligned}\omega(s)\frac{d}{ds}q(s) &= -\left(\gamma+\lambda_m d\right)q(s) \\ &\quad + N_\phi\left(\mu(\varepsilon), y_{s,\omega(\varepsilon)}(\cdot,\varepsilon)\right)\left(q_{s,\omega(\varepsilon)}(\cdot)e^{\gamma\cdot}\right).\end{aligned} \tag{85}$$

Equation (85) is independent of the spatial variable x.

When $m = 1$, (85) coincides with the one induced from the linearized equation of the diffusion-free equation, that is,

$$\begin{aligned}\omega(s)\frac{d}{ds}q(s) &= -\gamma q(s) + N_\phi\left(\mu(\varepsilon), y_{s,\omega(\varepsilon)}(\cdot,\varepsilon)\right)\left(q_{s,\omega(\varepsilon)}(\cdot)e^{\gamma\cdot}\right).\end{aligned} \tag{86}$$

From the above equations, we know (86) has Floquet exponents 0 and $\beta < 0$.

Next we mainly consider the case $m \neq 1$ in (85). Take any positive integer m, greater than 1, and fix it. Let us parameterize the deformation of a domain and diffusion coefficients in terms of the same parameter ε as above. In what follows, we assume the parameterization is taken in such a way that

$$\lambda_m d = \varepsilon^2 E, \tag{87}$$

for some matrix E. Then (85) can be written as

$$\begin{aligned}\omega(s)\frac{d}{ds}q(s) &= -\left(\gamma+\varepsilon^2 E\right)q(s) \\ &\quad + N_\phi\left(\mu(\varepsilon), y_{s,\omega(\varepsilon)}(\cdot;\varepsilon)\right)\left(q_{s,\omega(\varepsilon)}(\cdot)e^{\gamma\cdot}\right).\end{aligned} \tag{88}$$

Now in (88), we will find a real value function $q(s)$ and a real number γ in the following form:

$$\begin{aligned}\gamma &= \gamma_2\varepsilon^2 + \widehat{\gamma}(\varepsilon)\varepsilon^2, \quad \widehat{\gamma}(0) = 0, \\ q(s;\varepsilon) &= q_0(s;\varepsilon) + q_1(s;\varepsilon)\varepsilon + \widehat{q}(s;\varepsilon)\varepsilon,\end{aligned} \tag{89}$$

where γ is the Floquet exponent of the linear periodic system (80) whose sign determines the stability of the periodic solution $p(t;\varepsilon)$. We regard (89) as a perturbation of (86) with perturbation term (87).

According to the theory in [1] developed by Wu, after a slightly lengthy but simple calculation, we can obtain the following form expression about γ_2:

$$\begin{aligned}&\gamma_2^2 + 2\,\mathrm{Re}\left(\left(Ep_1(0), q_1(0)\right) - B_1\right)\gamma_2 \\ &\quad + \left|\left(Ep_1(0), q_1(0)\right)\right|^2 \\ &\quad - 2\,\mathrm{Re}\left(\overline{B}_1\left(Ep_1(0), q_1(0)\right)\right) = 0,\end{aligned} \tag{90}$$

which determines the stability of the bifurcating periodic solution as the diffusion coefficients vary, where

$$\begin{aligned}B_1 &= \left(N_{\phi\phi}(0)\left(p_1(\theta), \widehat{\zeta}\right), q_1(0)\right) \\ &\quad + \left(N_{\phi\phi}(0)\left(\overline{p}_1(\theta), \zeta\right), q_1(0)\right) \\ &\quad + \frac{1}{2}\left(N_{\phi\phi\phi}(0)\left(p_1(\theta), p_1(\theta), \overline{p}_1(\theta)\right), q_1(0)\right).\end{aligned} \tag{91}$$

Combining the above formulas (74), (75), (77), (78), (79), (87), we can obtain the expression of (90) whose coefficients are determined by the parameters from the original system (29).

5. Discussions

In this paper, we mainly investigate effects of the sum of two delays and diffusion on the dynamical behaviors of the Lotka-Volterra type predator-prey model with two delays. By employing theories of Hopf bifurcation for some fixed diffusions and spatial domain, we find that delays can cause the spatially homogeneous equilibrium point to destabilize and lead to the spatially homogeneous periodic solution to occur. On the other hand, we also investigate effects of diffusions on the bifurcation periodic solution and find that once the value of diffusion varies the bifurcating stable periodic solution may be unstable. As Morita [17] had pointed out, diffusion would lead to other spatiotemporal patterns to occur such as the periodic traveling wave, even spatiotemporal chaos, which will be studied in future work. So it is very important for us to consider the interaction of delay and diffusion for biological system (1) and explore the mechanism of all kinds of spatiotemporal patterns.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have read and approved the final manuscript.

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