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Research Article

Stability and Hopf Bifurcation Analysis of a Fractional-Order Epidemic Model with Time Delay

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A fractional-order epidemic model with time delay is considered. Firstly, stability of the disease-free equilibrium point and endemic equilibrium point is studied. Then, by choosing the time delay as a bifurcation parameter, the existence of Hopf bifurcation is studied. Finally, numerical simulations are given to illustrate the effectiveness and feasibility of theoretical results.

1. Introduction

Mathematical model plays an important role in describing the dynamics of biological system [1–3]. The dynamics of the epidemic models have received much attention during the recent years, and to explain the disease spreading and control strategies a series of epidemic models [4–7] was proposed. A stochastic SIRS epidemic model was formulated in [8]; it investigated the effect of stochastic environmental variability on interpandemic transmission dynamics of influenza A. In [9], an age-structured SEIR epidemic model was considered. The authors investigated an SEIR model with varying population size and vaccination strategy in [10], and different threshold parameters were obtained to govern the disease eradication. Many models in biological mathematics involve some time delays. In biological dynamics, time delay was widely applied to reflect some biological facts, such as immunity period [11] and latent period of the disease [12]. An epidemic model with time delay was proposed in [13], and the model is shown as follows:

$$\begin{aligned}\frac{dS(t)}{dt} &= rS(t)\left(1-\frac{S(t)}{k}\right)-\beta S(t-\tau)I(t-\tau)\\ &\quad +\mu S(t),\\ \frac{dI(t)}{dt} &= \beta S(t)I(t)-dI(t),\end{aligned} \tag{1}$$

where $S(t)$, $I(t)$ represent the number of susceptible and infected population. r represents the intrinsic birth rate constant, k represents carrying capacity of susceptible population, β represents the force of infection or the rate of transmission, μ represents immigration coefficient, d represents death coefficient of $I(t)$, and τ is the latent period of the disease.

Fractional calculus is a generalization of classical differentiation and integration to arbitrary (noninteger) order [14]. In the past decades, fractional-order calculus garnered considerable attention and it was applied to various fields [15–21]. Recently, many investigators started to study the fractional-order biological models [22–24]. The main reason is that fractional-order models are naturally related to systems with memory which exists in most biological systems [25, 26]. In [27], the authors introduced a fractional-order prey-predator model and deal with the biological behaviors of the model. A fractional-order SIS model with variable population size is considered in [28], and the stability of equilibrium points is studied. A fractional-order model of two-species facultative mutualism with harvesting was presented in [29], and stability of the model was analyzed. In [30], the authors introduced a fractional-order epidemic model with vaccination; it shows that the stability region of the model is related to threshold-value R_0 and value of the fractional-order α. A delayed fractional-order differential model of HIV infection of $CD4^+$

was investigated in [31]. In [32], a fractional-order prey-predator model with time delay and Monod-Haldane function was studied.

In this paper, a fractional-order epidemic model with time delay is studied. We investigate stability and bifurcation of the model with respect to basic reproduction number R_0, fractional-order α and time delay τ. We provide theoretical analysis, using the eigenvalues method and linearization techniques and bifurcation method. The model is depicted as follows:

$$\begin{aligned} D^{\alpha}S(t) &= rS(t)\left(1-\frac{S(t)}{k}\right)-\beta S(t-\tau)I(t-\tau) \\ &\quad + \mu S(t), \\ D^{\alpha}I(t) &= \beta S(t)I(t)-dI(t), \end{aligned} \tag{2}$$

where $\alpha \in (0,1]$, $\tau \geq 0$, and α is in the sense of Caputo fractional derivatives. $S(t) = \phi_1(t)$, $I(t) = \phi_2(t)$, and $t \in [-\tau, 0]$.

The corresponding linearized system of (2) at any equilibrium point (S^*, I^*) is defined as

$$\begin{aligned} D^{\alpha}S(t) &= \left(r+\mu-\frac{2rS^*}{k}\right)S(t)-\beta I^* S(t-\tau) \\ &\quad - \beta S^* I(t-\tau), \\ D^{\alpha}S(t) &= \beta I^* S(t)+(\beta S^*-d)I(t). \end{aligned} \tag{3}$$

Taking Laplace transform [33] on both sides of (3), one obtains the characteristic matrix as follows:

$$\Delta(s) = \begin{pmatrix} s-r-\mu-\dfrac{2rS^*}{k}+\beta I^* e^{-s\tau} & \beta e^{-s\tau}S^* \\ -\beta I^* & s-\beta S^*+d \end{pmatrix}. \tag{4}$$

The properties of eigenvalues of characteristic equation $\det(\Delta(s))$ indicate the stability of system (2).

The rest of the paper is organized as follows. In Section 2, some necessary definitions and notions are presented. In Section 3, stability and Hopf bifurcation of the equilibrium point are analyzed. Numerical simulations are given in Section 4 and some conclusions are given in Section 5.

2. Preliminary

There are three main definitions of fractional-order differential, that is, Riemann-Liouville, Grünwald-Letnikov, and Caputo's definitions. This paper is based on Caputo's definition.

Definition 1 (see [34]). The Caputo fractional derivative with order α of a continuous function $f: R^+ \longrightarrow R$ is defined as follows:

$$D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi)\,d\xi, & n-1<\alpha<n, \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n, \end{cases} \tag{5}$$

where $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$.

Lemma 2 (see [35]). *Considering the fractional differential system with the Caputo derivative,*

$$D^{\alpha}X = AX, \tag{6}$$

where $\alpha \in (0,1)$, $X \in R^n$, and $A \in R^{n\times n}$. The characteristic equation of system (6) is $\det|s^{\alpha}I - A| = 0$. If the real parts of all the eigenvalues of A are negative, then the zero solution to system (6) is locally asymptotically stable.

Lemma 3 (see [33]). *Considering the fractional delayed differential system with the Caputo derivative,*

$$\begin{aligned} D^{\alpha}X(t) &= AX(t)+BX(t-\tau), \\ X(t) &= \Phi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{7}$$

where $\alpha \in (0,1]$, $X \in R^n$, $A, B \in R^{n\times n}$, and $\Phi(t) \in R_+^{n\times n}$. The characteristic equation of the system (7) is $\det|s^{\alpha}I - A - Be^{-s\tau}| = 0$. If all the roots of the characteristic equation have negative real parts, then the zero solution of system (7) is locally asymptotically stable.

3. Main Results

3.1. Basic Production Number and the Existence of the Equilibrium Point. Following from [16], system (2) has a disease-free equilibrium point $E_0 = (S^0, I^0) = (k(r+\mu)/r, 0)$ and the basic reproduction number for the model is $R_0 = k\beta(r+\mu)/rd$. Endemic equilibrium point is $E_1 = (S^1, I^1) = (d/\beta, (k\beta(r+\mu)-rd)/k\beta^2)$. Obviously, $I^1 = (k\beta(r+\mu)-rd)/k\beta^2 = (R_0-1)rd/k\beta^2$. Then we know that model (2) has an endemic equilibrium point $E_1(d/\beta, (k\beta(r+\mu)-rd)/k\beta^2)$ when $R_0 > 1$.

3.2. Stability of the Disease-Free Equilibrium Point

Theorem 4. *The disease-free equilibrium point E_0 of system (2) is locally asymptotically stable if $R_0 < 1$.*

Proof. The characteristic matrix of system (3) evaluated at the equilibrium point E_0 is

$$\Delta(s) = \begin{bmatrix} s+r+\mu & \beta e^{-s\tau}\dfrac{k}{r}(r+\mu) \\ 0 & s-\beta\dfrac{k}{r}(r+\mu)+d \end{bmatrix}, \tag{8}$$

and the characteristic equation is

$$(s^{\alpha}+r+\mu)\left(s^{\alpha}-\beta\frac{k}{r}(r+\mu)+d\right) = 0. \tag{9}$$

Let $s^{\alpha} = \lambda$; we can rewrite (9) as

$$(\lambda+r+\mu)\left(\lambda-\beta\frac{k}{r}(r+\mu)+d\right) = 0. \tag{10}$$

Clearly, $\lambda_1 = -r-\mu < 0$, $\lambda_2 = (k\beta(r+\mu)-rd)/r$. When $R_0 = k\beta(r+\mu)/rd < 1$, we get $\lambda_2 < 0$. According to Lemma 2, the disease-free equilibrium point E_0 is locally asymptotically stable. This completes the proof. $\square$

3.3. Stability of the Endemic Equilibrium Point. The characteristic matrix of system (3) evaluated at the equilibrium point E_1 is

$$\Delta(s) = \begin{pmatrix} s - \dfrac{k\beta(r+\mu)-2rd}{k\beta} + e^{-s\tau}\dfrac{k\beta(r+\mu)-rd}{k\beta} & e^{-s\tau}d \\ -\dfrac{k\beta(r+\mu)-rd}{k\beta} & s \end{pmatrix}, \quad (11)$$

from which we have the characteristic equation

$$s^{2\alpha} + a_1 s^\alpha + a_2 = 0, \quad (12)$$

where

$$a_1 = -\frac{k\beta(r+\mu)-2rd}{k\beta} + e^{-s\tau}\frac{k\beta(r+\mu)-rd}{k\beta},$$
$$a_2 = e^{-s\tau}d\frac{k\beta(r+\mu)-rd}{k\beta}. \quad (13)$$

Theorem 5. *When $\tau = 0$, the endemic equilibrium point E_1 of system (2) is locally asymptotically stable if $R_0 > 1$.*

Proof. Let $s^\alpha = \lambda$; we can rewrite (12) as

$$\lambda^2 + \frac{rd}{k\beta}\lambda + d\frac{k\beta(r+\mu)-rd}{k\beta} = 0. \quad (14)$$

If $R_0 > 1$, one obtains $d((k\beta(r+\mu)-rd)/k\beta) = d((R_0 - 1)rd/k\beta) > 0$. Obviously, the two roots of (14) are negative. According to Lemma 2, the endemic equilibrium point E^* is locally asymptotically stable. This completes the proof. □

When $\tau > 0$, (12) can be rewritten as

$$s^{2\alpha} + b_1 s^\alpha + b_2 e^{-s\tau} s^\alpha + b_3 e^{-s\tau} = 0, \quad (15)$$

where

$$b_1 = -\frac{k\beta(r+\mu)-2rd}{k\beta},$$
$$b_2 = \frac{k\beta(r+\mu)-rd}{k\beta}, \quad (16)$$
$$b_3 = d\frac{k\beta(r+\mu)-rd}{k\beta}.$$

Assume that (15) has a pair of pure imaginary roots $s_{1,2} = \pm i\omega$, $\omega > 0$ and then substitute $s_1 = i\omega$ into equation (15); one obtains

$$(i\omega)^{2\alpha} + b_1(i\omega)^\alpha + b_2 e^{-i\omega\tau}(i\omega)^\alpha + b_3 e^{-i\omega\tau} = 0, \quad (17)$$

and then separating the real and imaginary parts of (17) one has

$$\omega^{2\alpha}\cos\alpha\pi + b_1\omega^\alpha\cos\frac{\alpha\pi}{2} = -b_2\omega^\alpha\cos\left(\frac{\alpha\pi}{2} - \omega\tau\right) - b_3\cos\omega\tau,$$
$$\omega^{2\alpha}\sin\alpha\pi + b_1\omega^\alpha\sin\frac{\alpha\pi}{2} = -b_2\omega^\alpha\sin\left(\frac{\alpha\pi}{2} - \omega\tau\right) + b_3\sin\omega\tau. \quad (18)$$

Squaring and adding the two equations in (18), we obtain

$$\omega^{4\alpha} + 2b_1\cos\frac{\alpha\pi}{2}\omega^{3\alpha} + \left(b_1^2 - b_2^2\right)\omega^{2\alpha} - 2b_2b_3\cos\frac{\alpha\pi}{2}\omega^\alpha - b_3^2 = 0. \quad (19)$$

Denote $h(\omega) = \omega^{4\alpha} + 2b_1\cos(\alpha\pi/2)\omega^{3\alpha} + (b_1^2 - b_2^2)\omega^{2\alpha} - 2b_2b_3\cos(\alpha\pi/2)\omega^\alpha - b_3^2$, where $-b_3^2 < 0$; therefore (19) has one positive root at least. If $\omega_1, \omega_2, \omega_3, \omega_4$ are the roots of $h(\omega)$, we assume ω_k is positive. Substituting ω_k into (18), one obtains

$$\omega_k^{2\alpha}\cos\alpha\pi + b_2\omega_k^\alpha\cos\left(\frac{\alpha\pi}{2} - \omega\tau\right) = -b_1\omega_k^\alpha\cos\frac{\alpha\pi}{2} - b_3\cos\omega\tau,$$
$$\omega_k^{2\alpha}\sin\alpha\pi + b_2\omega_k^\alpha\sin\left(\frac{\alpha\pi}{2} - \omega\tau\right) = -b_1\omega_k^\alpha\sin\frac{\alpha\pi}{2} + b_3\sin\omega\tau. \quad (20)$$

Squaring and adding the two equations in (20), one obtains

$$\omega_k^{4\alpha} + 2b_2\cos\left(\frac{\alpha\pi}{2} + \omega_k\tau\right)\omega_k^{3\alpha} + \left(b_2^2 - b_1^2\right)\omega_k^{2\alpha} - 2b_2b_3\omega_k^\alpha\cos\left(\frac{\alpha\pi}{2} + \omega_k\tau\right) - b_3^2 = 0. \quad (21)$$

From (21), τ_k can be obtained

$$\tau_k^j = \frac{1}{\omega_k}\left[\cos^{-1}\left(\frac{\omega_k^{4\alpha} + \left(b_2^2 - b_1^2\right)\omega_k^{2\alpha} - b_3^2}{2b_2b_3\omega_k^\alpha - 2b_2\omega_k^{3\alpha}}\right) - \frac{\alpha\pi}{2} + 2j\pi\right], \quad j = 0, 1, 2, \ldots. \quad (22)$$

Theorem 6. *When $R_0 > 1$ and $h'(\omega) > 0$, the endemic equilibrium point E_1 of system (2) is locally asymptotically stable if $\tau < \tau_0$ and unstable if $\tau > \tau_0$, where $\tau_0 = \min\{\tau_k^j\}$.*

Proof. Let $F(s) = s^{2\alpha} + b_1 s^\alpha$ and $G(s) = b_2 s^\alpha + b_3$; (12) reduces to

$$F(s) + G(s)e^{-s\tau} = 0. \quad (23)$$

Denote $h(z) = z^{4\alpha} + 2b_1\cos(\alpha\pi/2)z^{3\alpha} + (b_1^2 - b_2^2)z^{2\alpha} - 2b_2b_3\cos(\alpha\pi/2)z^\alpha - b_3^2$; one has

$$h(\omega) = F(i\omega)\overline{F(i\omega)} - G(i\omega)\overline{G(i\omega)}. \quad (24)$$

Differentiating both sides of (24) with respect to ω, we obtain

$$\begin{aligned} h'(\omega) = i\left[F'(i\omega)\overline{F(i\omega)} - F(i\omega)\overline{F'(i\omega)}\right. \\ \left. - G'(i\omega)\overline{G(i\omega)} + G(i\omega)\overline{G'(i\omega)}\right]. \end{aligned} \tag{25}$$

Differentiating both sides of (24) with respect to τ yields

$$\begin{aligned} &\left[F'(s) + G'(s)e^{-s\tau} - \tau G(s)e^{-s\tau}\right]\frac{ds}{d\tau} - sG(s)e^{-s\tau} \\ &= 0. \end{aligned} \tag{26}$$

From (26), one obtains

$$\begin{aligned} \frac{ds}{d\tau} &= \frac{sG(s)e^{-s\tau}}{F'(s) + G'(s)e^{-s\tau} - \tau G(s)e^{-s\tau}} \\ &= \frac{sG(s)}{F'(s)e^{s\tau} + G'(s) - \tau G(s)} \\ &= \frac{s\left(\overline{F'(s)}F(s) + \overline{G'(s)}G(s) - \tau|G(s)|^2\right)}{\left|F'(s)e^{s\tau} + G'(s) - \tau G(s)\right|^2}. \end{aligned} \tag{27}$$

With $s = i\omega_k$, the above equality becomes

$$\begin{aligned} \operatorname{Re}\left(\frac{ds}{d\tau}\right)\bigg|_{s=i\omega_k} &= \frac{\operatorname{Re}\left(s\left(\overline{F'(s)}F(s) + \overline{G'(s)}G(s) - \tau|G(s)|^2\right)\right)}{\left|F'(s)e^{s\tau} + G'(s) - \tau G(s)\right|^2}\Bigg|_{s=i\omega_k} \\ &= \frac{i\omega_k\left(\overline{F'(i\omega_k)}F(i\omega_k) - F'(i\omega_k)\overline{F(i\omega_k)} + \overline{G'(i\omega_k)}G(i\omega_k) - G'(i\omega_k)\overline{G(i\omega_k)}\right)}{2\left|F'(i\omega_k)e^{i\omega_k\tau} + G'(i\omega_k) - \tau G(i\omega_k)\right|^2} \\ &= \frac{\omega_k h'(\omega_k)}{2\left|F'(i\omega_k)e^{i\omega\tau} + G'(i\omega_k) - \tau G(i\omega_k)\right|^2}. \end{aligned} \tag{28}$$

When $h'(\omega_k) > 0$, one gets $\operatorname{Re}(ds/d\tau)|_{s=i\omega_k} > 0$. Hence, the endemic equilibrium point E_1 of system (2) is locally asymptotically stable if $\tau < \tau_0$ and unstable if $\tau > \tau_0$. This completes the proof. □

Remark 7. It is worth noting that there will be some future directions to apply our main results to more complex ones like models with time varying delay [36] and models with perturbed parameters [37] or to study the Hopf bifurcation of models with discrete and distributed delays [38].

4. Numerical Simulations

In this section, we give some numerical simulations for system (2) by using the method mentioned in [39, 40].

In Figure 1, we select parameters as $\alpha = 0.98$, $k = 5$, $\beta = 0.1$, $\mu = 0.5$, $\gamma = 2$, and $d = 0.82$, with initial conditions $S_0 = 3.5$, $I_0 = 2$. After calculation, one obtains disease-free equilibrium point $E_0 = (6.25, 0)$ and $R_0 = 0.7622 < 1$. In (a), we take $\tau = 0$, and in (b) we take $\tau = 10$. According to Theorem 4, the disease-free equilibrium point of system (2) is locally asymptotically stable when $R_0 < 1$.

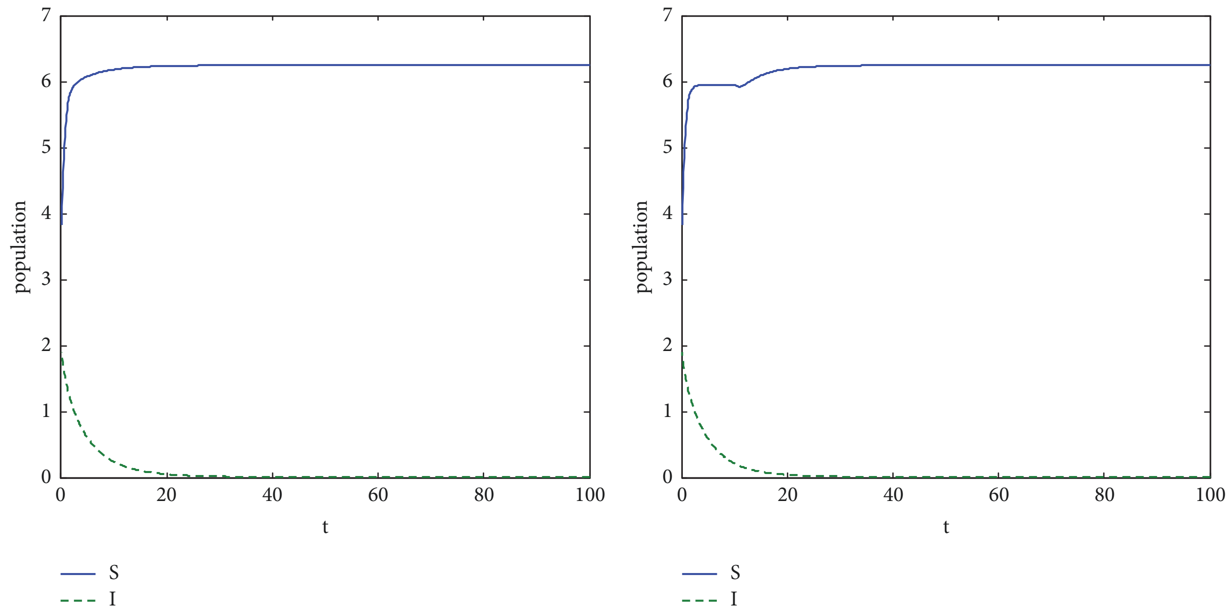
In Figure 2, the selected parameters are $\alpha = 0.98$, $\tau = 0$, $k = 8$, $\beta = 0.1$, $\mu = 0.5$, $\gamma = 4$, and $d = 0.7$, with initial conditions $S_0 = 4$, $I_0 = 3$. After calculation, one obtains endemic equilibrium $E_1 = (7, 10)$ and $R_0 = 1.2857 > 1$. According to Theorem 5, if $\tau = 0$ and $R_0 > 1$, the endemic equilibrium point of system (2) is locally asymptotically stable. The numerical simulation results are shown in Figure 2.

In Figures 3(a) and 3(b), we plotted the effect measure of immigration coefficient μ on susceptible and infected populations. The selected parameters are same as Figure 2 with initial conditions $S_0 = 6$, $I_0 = 3$. Values of μ are shown in the legend. From Figures 3(a) and 3(b), we observe that the number of susceptible individuals increases as μ increases at the beginning but is finally stable at the same fixed value. The number of infected individuals increases as μ increases. It shows that after the endemic formation, the number of the susceptible individuals increases as the number of floating population increases in the short term, but in the long run the number of susceptible individuals is the same, and only the number of infected individuals increases.

Figure 4 depicts the Hopf bifurcation of the endemic equilibrium. The parameters are taken as $k = 10$, $\beta = 0.1$, $r = 10$, $\mu = 0.6$, and $d = 0.94$, with initial conditions $S_0 = 12.4$, $I_0 = 9$. After calculation, one obtains $R_0 = 1.2857 > 1$, $h'(\omega) > 0$, and $E_1 = (9.4, 12)$. When $\alpha = 0.96$, $\tau_0 = 12.6480$ is calculated. In (a), we let $\tau = 12 < \tau_0$, and in (b) $\tau = 14 > \tau_0$. (a) and (b) show Hopf bifurcation occurs at τ_0. Then one selects different order $\alpha \in (0, 1]$, we get different time delay, and the results are shown in (c). Figure 4(c) shows that as the value of α becomes smaller, the stability domain becomes larger. When $\alpha = 0.9$, $\tau_0 = 15.1281$, and $\tau = 14 < \tau_0$, Figure 4(d) shows that the endemic equilibrium point becomes stable.

5. Conclusion

In this paper, a fractional-order epidemic model with time delay is studied and stability and bifurcation of the model are analyzed. The results show that when $R_0 < 1$, the disease-free equilibrium point is locally asymptotically stable for $\tau \geq 0$. And we get that when $R_0 > 1$ and $\tau = 0$, the endemic equilibrium point is locally asymptotically stable.



(a) (b)

Figure 1: Stability of disease-free equilibrium point: (a) $\tau = 0$; (b) $\tau = 10$.

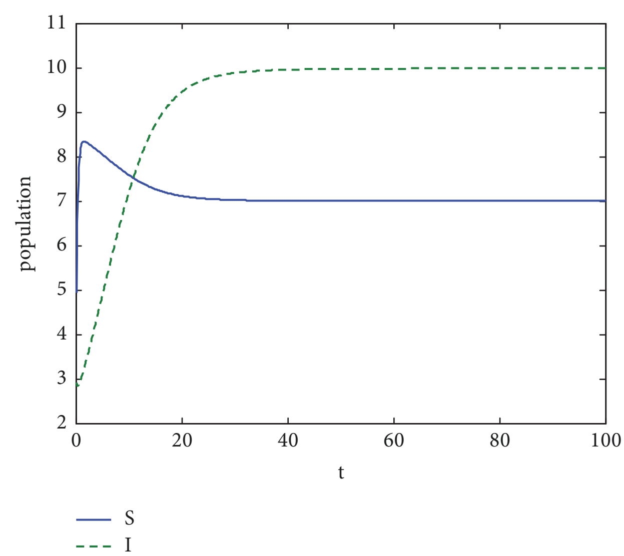


Figure 2: The stability of endemic equilibrium point when $\tau = 0$.

According to Theorem 6, when $R_0 > 1$ and $h'(\omega_k) > 0$, the stability of the endemic equilibrium point changes at bifurcation point τ_0. Some numerical simulations are given to verify the correctness of the theory, and stability region of model is related to the value of R_0, τ, and fractional-order θ.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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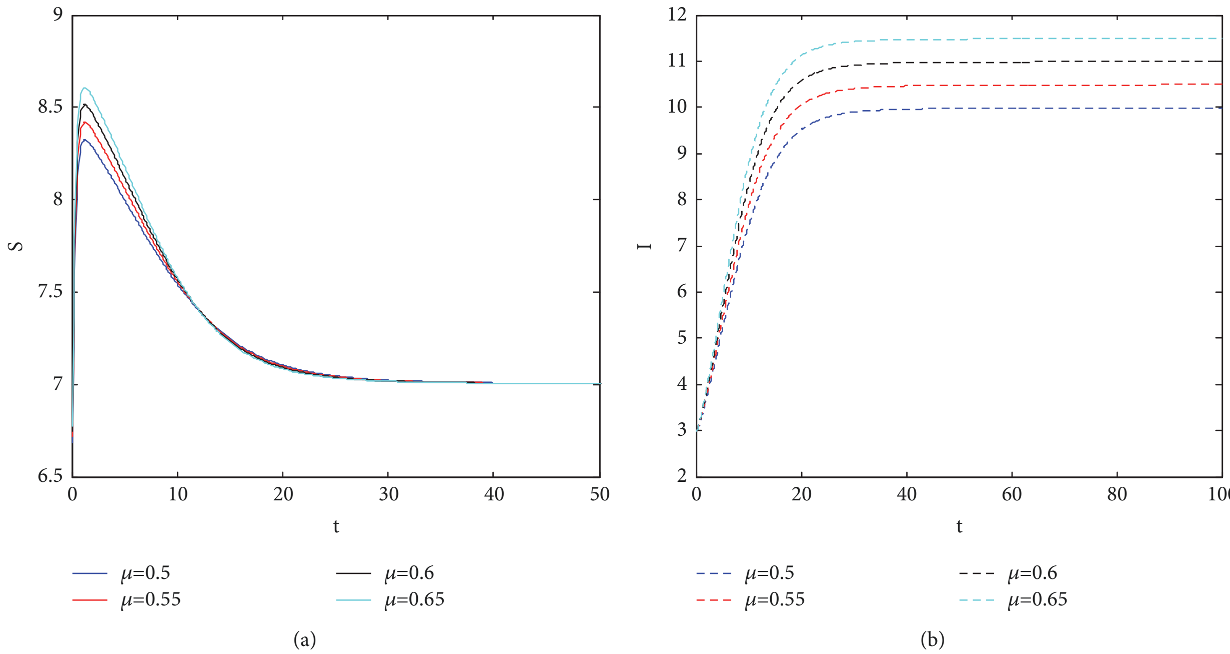

Figure 3: Effect of μ on S on I, respectively, when E_1 is stable.

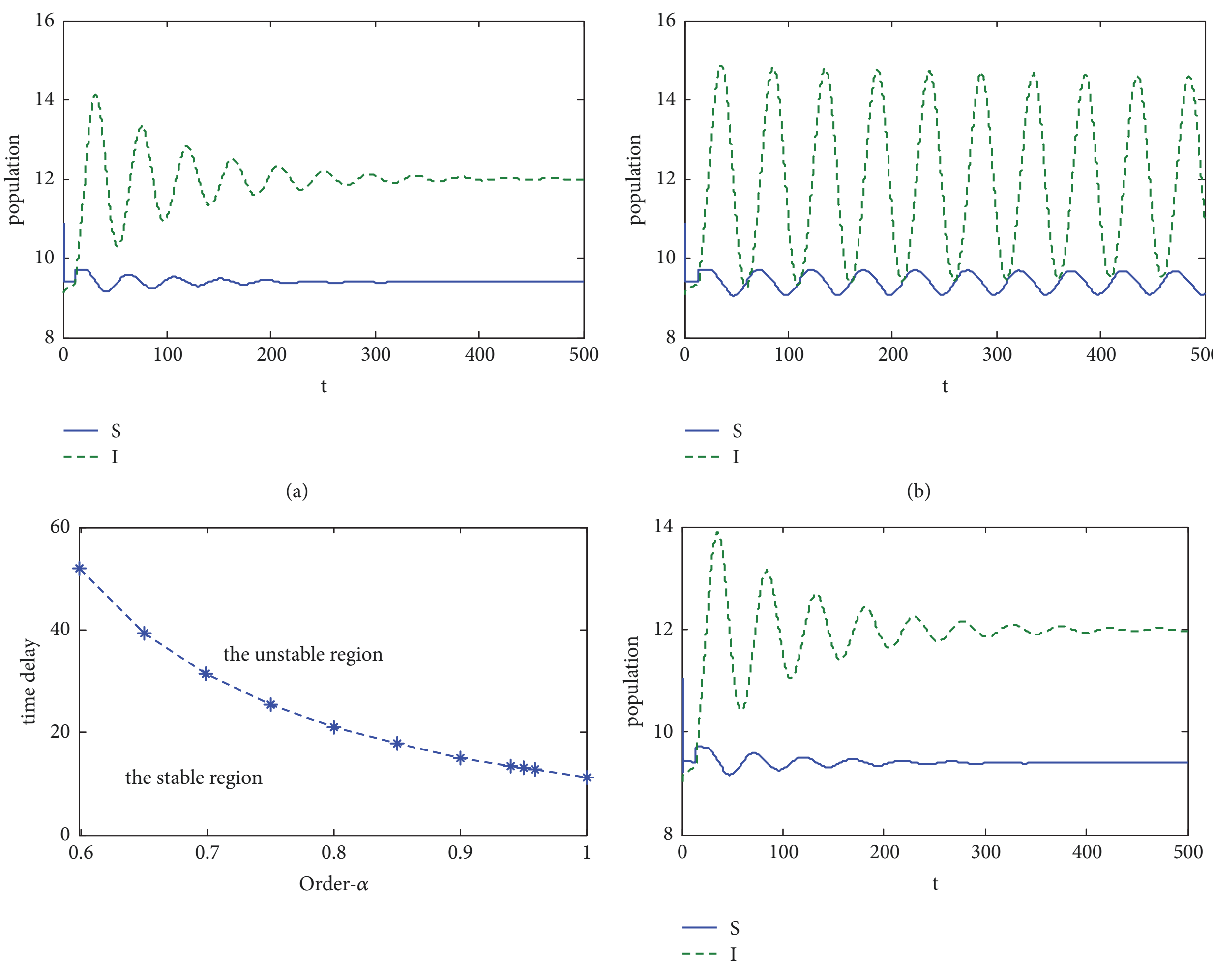

Figure 4: Hopf bifurcation of the endemic equilibrium point.

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