Research Article

Finite-Time Boundedness and $H_{\infty}$ Control for Affine Switched Systems

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Received 6 October 2017; Revised 5 May 2018; Accepted 30 May 2018; Published 18 July 2018

Academic Editor: Guangming Xie

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For affine switched systems, the existence of multiple equilibria is related to subsystems owing to the affine terms, which makes asymptotic and finite-time stability analysis nontrivial. In this paper, the problems of finite-time boundedness (FTB) analysis and stabilization are addressed for affine switched systems, and several definitions and sufficient conditions are proposed to study FTB and $H_{\infty}$ performance. At first, the definition of FTB for affine switched systems is improved concerning the affine terms and multiple equilibria. Based on the FTB definition, sufficient conditions ensuring finite-time boundedness for affine switched systems under a prespecified state boundary are given. Then the results are extended to solve $H_{\infty}$ finite-time boundedness problem, in which the $H_{\infty}$ controllers are designed to guarantee the finite-time boundedness of affine switched system with $H_{\infty}$ performance. In our investigation, average dwell-time approach is employed to study the time-dependent constrained switching case. Finally, several numerical examples are given to illustrate the effectiveness of the proposed results.

1. Introduction

Switched systems are distinctive subclass of hybrid systems. They are composed of a family of continuous-time or discrete-time subsystems with a criterion that rules the switching among them. This switching rule can be classified as time-dependent, state-dependent, or time-state-dependent [1]. Since many physical processes possess switching nature, and many real-world applications resort to switching strategy to improve the control performance, the theory and application of switched systems have received a great attention during the recent decades. For more details on the recent results about the basic problems in stability and stabilization for switched systems, readers are referred to surveys [2–4] and books [1, 5] and the references cited therein.

The issue of stability analysis and stabilization is an important topic for switched dynamical systems [6–10]. Finding sufficient conditions ensuring the Lyapunov asymptotic stability dealing with infinite time interval has been the major concern for switched systems. Numerous published results discussed the asymptotic stability analysis and stabilization employing different variations of Lyapunov function [7, 11, 12]. Average dwell-time approach [13, 14] and Lie-algebraic condition technique [15, 16] are effective tools for analysis of switched systems. On the contrary, the finite-time behavior of dynamical systems is also of interest in many practical applications. It concerns that the states do not exceed a certain bound during a fixed time interval, e.g., to avoid saturations or excitation. The theory of finite-time stability (FTS) and finite-time boundedness (FTB) focuses on the transient response of dynamical systems over a finite-time interval, while asymptotic behavior is for infinite time. In the survey of recent development of this innovative theory, some necessary and sufficient conditions for finite-time stability and stabilization of continuous-time systems or discrete-time systems have been provided in [17, 18]. Based upon it, necessary and sufficient conditions for finite-time stability of systems with impulsive effects were obtained in [19, 20]. The authors [21, 22] applied FTS/FTB conceptions to switched systems and compared the conservativeness among different conditions. In [23], the mixed $H_{\infty}$/finite-time stability control problem was discussed. For quadratic input-output finite-time stability with an $H_{\infty}$ bound, [23] provided a
necessary and sufficient condition. Then the method was extended to robust $H_{\infty}$ controller and filter design for switched system with exogenous noise [24, 25]. It should be noted that finite-time stability and Lyapunov asymptotic stability are independent concepts: a Lyapunov asymptotic stable system may not fulfill FTS/FTB criteria since the transient response of a system may exceed the bound, and vice versa [26]. In many practical applications, switching is likely to occur in some short-time intervals, whereas for remaining long time no switching occurs. Since Lyapunov stability concerns with infinite time, it may not be influenced by such short-time switching. However, the boundedness of state may be affected by the switching. Hence, FTB criteria are needed to be considered for designing controller and switching laws during such applications.

Most of the existing literatures on stability issues of switched systems are based on the premise that all subsystems share a common equilibrium (typically the origin). On the other hand, for affine switched system, subsystems have different equilibria, so complex and interesting phenomena emerge. Almost all the practical hybrid systems can be modeled as affine switched systems. Many results like [27–30] analyzed interesting behaviors similar to those of asymptotically stable systems near an equilibrium for affine switched systems and depicted their real-world applications. Many extensions of the conventional stability concepts have been obtained for affine switched systems. S-Procedure method with the extensional state vector has been proposed in [31, 32] to analyze the asymptotic stability for continuous affine switched system. The relative results were extended to discrete affine switched systems in [33]. In [34], a method for designing switching rules driving the state of affine switched system to a desired equilibrium was investigated. Almost all the existing literatures on stability analysis of affine switched systems focused on the asymptotic stability. However, the boundedness of state for affine switched systems under constrained dwell-time switching is also of significant interest for affine switched systems. In FTB analysis, we also need to deal with affine terms leading to multiple equilibria for affine switched systems, but the investigation of this problem lacks researchers’ interest previously. Potential of affine switched systems theory and importance of finite-time transient behavior from the perspective of real-world applications are the major motivations for this investigation presented in this paper.

The main objective in this paper is to find sufficient conditions ensuring the FTB of affine switched systems by switching signal and feedback controllers design and to drive the state of affine switched system to the prescribed neighborhood of a desired equilibrium during a finite-time interval. Taking into account the influence of affine terms on FTB for affine switched system, we propose an innovative FTB concept. Based on this definition, sufficient conditions ensuring the affine switched system finite-time bounded are proposed. Specifically, with the prespecified state boundary, average dwell time and state-feedback controllers for each subsystem are determined to guarantee the finite-time boundedness. The paper [22] points out that the more information about switching signal we know, the less conservative results can be derived. We extend this idea to switched affine systems to further reduce the conservatism. Classifying subsystems into asymptotically stable and unstable systems, we get the less conservative results of finite-time boundedness for affine switched system with the help of additional information of switching signal. Then, results are extended to solve the FTB problem for $H_{\infty}$ controller design.

The rest of this paper is organized as follows. In Section 2, definitions of finite-time boundedness and $H_{\infty}$ finite-time boundedness for affine switched system are revisited. Based on these definitions, finite-time boundedness analysis and finite-time stabilization are presented in Section 3. Then in presence of exogenous signals, $H_{\infty}$ finite-time boundedness and the controllers design are investigated in Section 4. In Section 5, several numerical examples are presented to validate the proposed results. Conclusions are given in Section 6.

2. Preliminaries and Problem Formulation

For our investigation, we consider continuous-time affine switched system described as

$$\dot{x}(t) = A_1 x(t) + B_1 u(t) + b_1, \quad x(0) = x_0$$

(1)

$$y(t) = C_1 x(t)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^d$ is the measurement output, $A_1$, $B_1$, and $C_1$ are system matrices with appropriate dimensions, constants $b_1$ are affine terms, and $i(t) : \mathbb{R}^+ \rightarrow \{1, \ldots, m\}$ is switching signal. For notational simplicity, we use $i$ in place of $i(t)$.

Matrix variables $A_1$, $B_1$, and $b_1$ give rise to an equilibrium (stable or unstable) for each subsystem; assuming all $A_i$ to be nonsingular, we consider a given reference $x_r$ as the required equilibrium for the whole system, referred to as switched equilibrium. Without loss of generality, it is assumed that the desirable equilibrium is different from all the equilibria of subsystems. Now although the asymptotic stability of affine switched system may be achieved by other types of switching strategy such as min-switching and sliding method, the state will not exactly converge to $x_r$ under dwell-time constrained switching. The reason is that there always exist time interval (dwell time is always greater than zero) in which state must diverge from $x_r$. In our FTB investigation, we provide solution for boundedness of error state under dwell-time switching, which depicts the importance and innovation of our approach.

Here first we will extend the FTS and FTB concepts for affine switched systems keeping in view prescribed equilibrium $x_r$. In absence of control input, system (1) can be stated as

$$\dot{x}(t) = A_1 x(t) + b_1, \quad x(0) = x_0$$

(2)

**Definition 1.** Autonomous affine switched system (2) is said to be finite-time bounded with respect to $(\delta_x, \delta_{\omega}, \epsilon, R_{\omega}, R_{\omega}, T)$ if the following inequalities hold:
\[ (x_0 - x_r)^T R_x (x_0 - x_r) \leq \delta_x^2 \]
\[ k_{\text{max}}^T R_k k_{\text{max}} \leq \delta_k^2 \]
\[ t = 0 \]
\[ (x(t) - x_r)^T R_x (x(t) - x_r) \leq \varepsilon^2 \quad 0 < t \leq T \]
where \( k_{\text{max}} = \arg\max_{i = 1, \ldots, m} [k_i^T R_k k_i], k_i = A_i x_r + b_i, 0 \leq \delta_x < \varepsilon, \delta_k \geq 0, R_x > 0, R_k > 0, \text{and } T \in \mathbb{R}^+ \).

Remark 2. Given equilibrium \( x_r \) and system (2), its tracking error system can be written as
\[ \dot{e}(t) = A_i e(t) + k_i \]
where \( e(t) = x(t) - x_r, k_i = b_i + A_i x_r \). According to Definition 1, we can conclude that affine switched system (2) is finite-time bounded with respect to \( (\delta_x, \delta_k, e, R_x, R_k, T) \) if \( e(t)^T R_x e(t) \leq \varepsilon^2 \) whenever \( e(t) \leq \delta_x^2 \) and \( k_{\text{max}}^T R_k k_{\text{max}} \leq \delta_k^2 \). The FTB criteria of affine switched systems ensure the state tracking the desired equilibrium \( x_r \) within the boundary \( \varepsilon \). In other words, it guarantees the error state \( e(t) \) tracking the origin in finite-time interval. Therefore, our study about FTB of affine switched systems can be turned into analyzing its corresponding tracking error system. Moreover, it is worth noting that FTTF theory for general switched systems is related to initial state \( x_{0i} \) \([35, 36]\); whereas for affine switched systems, we are concerned with \( x_0 \) as well as the desired equilibrium \( x_r \) and affine terms \( b_i \). Thus, in the Definition 1, the premise constraint conditions are extended to both initial state \( x_0 \) and \( k_i \) to analyze the FTB of affine switched systems, where \( k_i \) is related to the desired equilibrium \( x_r \) and affine terms \( b_i \).

Remark 3. With the state-feedback controller \( u(t) = K(x(t)) \), \( i \in I \), affine switched system (1) can be rewritten into the following closed-loop system:
\[ \dot{x}(t) = \overline{A}_i x(t) + b_i, \quad x(0) = x_0 \]
where \( \overline{A}_i = A_i + B_i K_i \) and the FTTF analysis method can be used directly. Similar to the significant impact of switching laws on asymptotic stability, the switching signals affect the finite-time boundedness of affine switched systems property significantly. Therefore, both switching signals and robust controllers should be designed during the FTTF analysis of affine switched systems.

On the other hand, external disturbances are inevitable to dynamical systems. We can state affine switched system with time-varying disturbance \( \omega(t) \) as
\[ \dot{x}(t) = A_i x(t) + B_i u(t) + G_i \omega(t) + b_i, \quad x(0) = x_0 \]
\( \omega(t) \) is assumed to be energy-bounded and hence for some scalar \( d > 0 \) it satisfies the inequality \( \int_0^T \omega^T(t) \omega(t) dt \leq d^2 \).

For simplifying FTTF analysis, following Definition 1 we can transform affine switched system (6) to its error tracking switched system as
\[ \dot{e}(t) = A_i e(t) + B_i u(t) + G_i \omega(t) + k_i, \quad x(0) = x_0 \]
\[ z(t) = C_i e(t) + D_i u(t) + D_{2i} \omega(t) \]
where \( e(t) = x(t) - x_r, k_i = A_i x_r + b_i, x_r \) is the desirable reference point, \( z(t) \in \mathbb{R}^q \) is the controlled output, and the switched equilibrium of system is moved to the origin accordingly. Considering state-feedback controller \( u(t) = K(x(t)) \), we derive the following closed-loop switched system:
\[ \dot{e}(t) = \overline{A}_i e(t) + G_i \omega(t) + k_i, \quad x(0) = x_0 \]
\[ z(t) = \overline{C}_i e(t) + D_{2i} \omega(t) \]
where \( \overline{A}_i = A_i + B_i K_i, \overline{C}_i = C_i + D_{1i} K_i \). Now we are able to state the following definition.

Definition 4. For affine switched system (7), considering state-feedback controller \( u(t) = K(x(t)) \) and \( H_{\infty} \) performance index \( \gamma > 0 \), if the following two conditions are satisfied:
(1) the closed-loop error tracking switched system (8) is finite-time bounded;
(2) under zero-initial condition, the controlled output \( z \) satisfies the inequality
\[ \int_0^T \varepsilon^2(t) z(t) dt < \gamma^2 \int_0^T \overline{\omega}^T(t) \overline{\omega}(t) dt \]
(9)
where \( \overline{\omega}(t) = [\omega^T(t) \ k_i^T]^T \), \( k_{\text{max}} = \arg\max_{i = 1, \ldots, m} [k_i^T R_k k_i] \), then \( u(t) \) is called 'finite-time \( H_{\infty} \) controller'.

Assuming \( u(t) = 0, k_i = 0 \) system (7) is expressed as
\[ \dot{e}(t) = A_i e(t) + G_i \omega(t), \quad x(0) = x_0 \]
\[ z(t) = C_i e(t) + D_{2i} \omega(t) \]
(10)
Now Definition 4 can be reduced to the following form. Switched system (10) is said to be \( H_{\infty} \) finite-time bounded with performance index \( \gamma \), if
(1) the error tracking switched system (10) is FTTF;
(2) under zero-initial condition, the controlled output \( z \) satisfies
\[ \int_0^T \varepsilon^2(t) z(t) dt < \gamma^2 \int_0^T \omega^T(t) \omega(t) dt \]
(11)

Based upon the above preliminaries we will focus on how to find sufficient conditions to ensure the finite-time boundedness of affine switched systems and address the \( H_{\infty} \) analysis and synthesis of piecewise linear state-feedback controllers resorting to LMI-based algorithms. The main problems we concern in this paper can be stated as follows.

Problem 5 (finite-time boundedness for affine switched systems). Given affine switched system (2), find sufficient conditions ensuring the finite-time boundedness with respect to \( (\delta_x, \delta_k, e, R_x, R_k, T) \).
Problem 6 (state-feedback stabilization under FTB). Given affine switched system (1), find set of static state-feedback controllers \( u(t) = K_i x(t) \) to ensure that the closed-loop system (5) is finite-time bounded with respect to \((\delta_\omega, \delta_e, \epsilon, R_e, R_w, T)\).

Problem 7 (\(H_\infty\) performance and controller design). Given affine switched system (8), analyze the \(H_\infty\) performance and design set of \(H_\infty\) controllers defined in Definition 4 to ensure the finite-time boundedness with respect to \((\delta_\omega, \delta_e, \epsilon, R_e, R_w, T)\) and reduce the effect of the exogenous signal \(\omega\) and \(k_i\) on the controlled output \(z\) to a prescribed level \(\gamma\).

3. Finite-Time Boundedness and State-Feedback Stabilization

In this section, Problems 5 and 6 are taken into consideration. Our main aim is to find sufficient conditions and state-feedback controllers to ensure the finite-time boundedness of affine switched system in the form of (2). For a finite-time interval \([0, T]\), we consider finite switchings \(k_{[0,T]}\). Each subsystem has an (stable or unstable) equilibrium point \(x_{e_i} = -A_i^{-1}b_i\). Regarding reference point \(x_e\) as an equilibrium point for the whole system called switched equilibrium and taking into account average dwell time, we will derive sufficient conditions ensuring finite-time boundedness.

Theorem 8. Affine switched system (2) is finite-time bounded with respect to \((\delta_\epsilon, \delta_e, \epsilon, R_e, R_w, T)\), if there exist positive definite matrices \(P_i\), scalars \(\alpha, \beta > 0, \xi \geq 0\), such that

\[
\alpha R_e < P_i < \beta R_e \tag{12a}
\]

\[
e^{[(\xi/\alpha)T + k_{[0,T]}]m(\beta/\alpha)} (\beta \delta_e^2 + T \delta \delta_e^2) - \alpha \xi^2 < 0 \tag{12b}
\]

\[
V(t) < \phi(t, t_0) V(t_0) \tag{12c}
\]

**Proof.** Consider the error tracking switched system (4), let \(\mathcal{R} = \text{diag}(R_e, R_w), \eta_i = \left[e^T(t) k_i^T\right]^T\). We choose piecewise Lyapunov function \(V_i(t) = e^T(t) P_i e(t)\). From condition (12b) we have

\[
\dot{V}_i(t) = e^T(t) P_i e(t) + e^T(t) P_i \dot{e}(t) = e^T(t) P_i e(t) \leq \xi \eta_i^T \mathcal{R} \eta_i \tag{13}
\]

Employing (12a) we derive

\[
\dot{V}_i(t) < \xi (1 - \epsilon) e^T(t) P_i e(t) + \alpha k_i^T R_w k_i \leq \xi \alpha V_i + \xi \delta e^2 \tag{14}
\]

Let \(\forall t > 0, t_0 < t_1 < \cdots < t_k\) be the switching instant of switched system. For overall system we can write \(V(t) = \sum_{i=1}^{k} \theta_i V_i(t), \theta_i \in [0, 1]\). Now from inequality (14),

\[
V(t) < \phi(t, t_k) V(t_k) + \xi \delta e^2 \int_{t_k}^{t} \phi(t, \tau) d\tau \tag{15}
\]

where \(\phi(t, \tau) = \exp(\xi \alpha^{-1}(t - \tau)) < \exp(\xi \alpha^{-1}T), T\) denotes the finite-time interval. Accordingly, the Lyapunov inequality in single step satisfies

\[
V(t_k+1) < \phi(t_{k+1}, t_k) V(t_k) + \xi \delta e^2 \int_{t_k}^{t_{k+1}} \phi(t, \tau) d\tau \tag{16}
\]

Suppose system switches from mode \(i\) to \(j\) at some instant \(t_k\); then from condition (12a),

\[
\frac{V(t_j)}{V(t_k)} < \frac{e^{t_j} (t_k) P_j e(t_k)}{e^{t_k} (t_k) P_j e(t_k)} < \frac{\beta e^{t_j}(t_k) R_e e(t_k)}{\alpha e^{t_k}(t_k) R_e e(t_k)} = \frac{\beta}{\alpha} \tag{17}
\]

It is evident that \(\beta/\alpha > 1\) and following (16) iteratively we can derive easily that

\[
V(t_k) < \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \phi(t, t_0) V(t_0) \tag{18}
\]

Applying (15) and (18) we deduce

\[
V(t) < \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} \phi(t, t_0) V(t_0) + \sum_{n=1}^{k} \left(\frac{\beta}{\alpha}\right)^{k_{[0,T]}} e^{(\xi/\alpha)T} V(t_0) \tag{19}
\]

On the other hand, from condition (12a), we have

\[
V(t) > e^T(t) P_e(t) > \alpha e^T(t) R_e e(t) \tag{20}
\]
Using the fact that $V(t_0) = e_0^T P e_0 < \beta \delta^2$ in order to ensure the finite-time boundedness of switched system (4), i.e., $e^T(t) R e(t) \leq \dot{\epsilon}^2$, the following condition should be satisfied:

$$\alpha e^T(t) R e(t) < \left( \frac{\beta}{\alpha} \right)^{k_{[\alpha,\gamma]}} e^{(\xi/\alpha)T} \beta \delta^2$$

$$+ T \left( \frac{\beta}{\alpha} \right)^{k_{[\alpha,\gamma]}} \xi \delta^2 e^{(\xi/\alpha)T} < \alpha \dot{\epsilon}^2$$

(21)

which can be rewritten as condition (12c). Therefore, we get $(x(t) - x_s)^T R_x(x(t) - x_s) \leq \dot{\epsilon}^2$ and we conclude that the affine switched system (2) is finite-time bounded which completes the proof.

**Remark 9.** When other parameters are fixed, condition (12c) can be described by average dwell time as [37]

$$\tau_a \geq \tau^*_a = T \ln \left( \frac{\beta}{\alpha} \left( \frac{\dot{\epsilon}^2}{\beta \delta^2 + T \xi \delta^2} \right) \right)$$

(22)

where $\tau^*_a = T/k_{[0,T]}$. In other words, the average dwell time $\tau_a$ should be chosen large enough to ensure that inequality (22) is satisfied, which is necessary to guarantee the finite-time boundedness of affine switched system (2). Moreover, assuming $R_x = I$, from (12a) and (19) we deduce

$$\sqrt{\alpha} \|e(t)\| < \sqrt{e^T(t) P e(t)}$$

$$< \sqrt{\left( \frac{\beta}{\alpha} \right)^{k_{[\alpha,\gamma]}} e^{(\xi/\alpha)T} \beta \delta^2} e(t_0)$$

$$+ \sqrt{T \left( \frac{\beta}{\alpha} \right)^{k_{[\alpha,\gamma]}} \xi \delta^2 e^{(\xi/\alpha)T}}$$

(23)

When $t \rightarrow \infty$, $T \rightarrow \infty$ and the term $\sqrt{T(\beta/\alpha)^{k_{[\alpha,\gamma]}}} \xi \delta^2 e^{(\xi/\alpha)T}$ on the right side of (23) will become infinite, which explains that the affine switched system (2) is not ultimately bounded, which illustrates FTB and ultimately boundedness are independent concepts.

**Remark 10.** Once the state bound $\dot{\epsilon}$ is not ascertained, the minimum value $\epsilon_{min}$ of $\dot{\epsilon}$ is of interest, which can be found through optimization problem $\min \theta$ subject to (12a) and (12b). If we fix the parameter $\xi$ and let $\alpha = 1$, $\beta = 6\alpha$, the optimization problem becomes

$$\min_{\delta \geq 1} \theta$$

$$\text{s.t.} \ R_x < P_i < \theta R_x$$

$$[A_i^T P_i + P_i A_i \ P_i] [e(t)] < \xi$$

(24)

Then $\epsilon_{min} = \sqrt{\theta^2 \delta^2 + T \xi \delta^2}$ can be derived with the optimized value $\theta$.

It is evident that smaller value of $\epsilon$ gives rise to less conservative FTB conditions. In Theorem 8, the parameter $\xi$ indicates the asymptotic stability property of each subsystem. It is well known that when $\xi = 0$ in condition (12b), this condition can be regarded as Lyapunov function condition which ensures each subsystem to be asymptotic stable; whereas when $\xi > 0$, the condition that $V(t)$ must be negative is relaxed in FTB sense, and $\dot{V}(t)$ just should be no greater than $\dot{V}(t) < \xi^{-1} V_i + \xi \delta^2$ to guarantee the boundedness of state in finite-time interval $[0,T]$. The parameter $\alpha > 0$ in condition (12b) covers both the asymptotic stable and unstable subsystems. Now let subsystems $A_1, \cdots, A_m$ be asymptotic stable and $A_{r+1}, \cdots, A_m$ be unstable, and $T^\star$, $T^\dagger$ denote the total activation time for stable and unstable subsystems during $[0,T]$. Then the less conservative results about FTB of affine switched system can be obtained in the following corollary.

**Corollary 11.** Switched system (2) is finite-time bounded (FTB) with respect to $(\delta, \delta_w, \epsilon, R_x, R_w, T)$, if there exist a set of positive definite symmetric matrices $P_i, i \in I$, scalars $\alpha > 0$, $\beta > 0$, and $\xi > 0$ such that the following conditions are satisfied:

$$\alpha R_x < P_i < \beta \delta$$

(25a)

$$[A_i^T P_i + P_i A_i \ P_i] \leq \begin{cases} 0 & i \leq r \\ \xi^+ \begin{bmatrix} R_x & 0 \\ 0 & R_w \end{bmatrix} & i > r \end{cases}$$

(25b)

$$\epsilon^{(\xi/\alpha)T + k_{[\alpha,\gamma]}} \ln(\beta/\alpha) (\beta \delta^2 + T i \xi \delta^2) - \alpha \dot{\epsilon}^2 < 0$$

(25c)

**Proof.** Consider the error tracking switched system (4), let $\mathcal{R} = \text{diag}(R_x, R_w)$, $\eta_i = [\epsilon^T(t) \ k_i^T]^T, i \in I$; we choose piecewise Lyapunov function $V_i(t) = \epsilon^T(t) P_i e(t)$.

From condition (25b), we get

$$\dot{V}_i(t) = \epsilon^T(t) P_i e(t) + \epsilon^T(t) P_i \dot{e}(t)$$

$$= \begin{bmatrix} e(t) \ k_i \end{bmatrix}^T \begin{bmatrix} A_i^T P_i + P_i A_i \ P_i \end{bmatrix} [e(t)]$$

$$< \begin{cases} 0 & i \leq r \\ \xi^+ \eta_i^T \mathcal{R} \eta_i & i > r \end{cases}$$

(26)

$$\dot{V}_i(t) < \begin{bmatrix} \xi^+ \dot{\epsilon} + \xi \delta^2 & i > r \end{bmatrix}$$
Let \( \forall t > 0, t_0 < \cdots < t_k \) be the switching instant of switched system, and \( V(t) = \sum_{i \in I} \theta_i V_i(t) = \sum_{i \in I} \theta_i V_i(t) \), with \( \theta_i \in [0, 1] \). From inequalities (25a) and (26), we have

\[
V(t) = \begin{cases} 
\beta \alpha^{-1} V(t_k) & i(t_k) \leq r \\
\beta \alpha^{-1} \phi(t, t_k) V(t_k) + \xi^* \delta^2 \omega_1^{-1} \int_{t_k}^t \phi(t, \tau) d\tau & i(t_k) > r
\end{cases} 
\] (27)

where \( \phi(t, r) = \exp(\xi^* \alpha^{-1}(t-r)) < \exp(\xi^* \alpha^{-1}T^+) \). Following (27) iteratively

\[
V(t) < \left( \beta \over \alpha \right)^{k_{t, t_k}} e^{\xi^* T^+} V(t_0) + T^+ \left( \beta \over \alpha \right)^{k_{t, t_k}} \xi^* \delta^2 \omega_1^{-1} e^{\xi^* T^+} < \alpha \varepsilon^2
\] (28)

By the same proof line in Theorem 8, we know that in order to ensure the finite-time boundedness of switched system (4), i.e., \( e^T(t) R_e e(t) \leq \varepsilon^2 \), the following condition should be satisfied:

\[
\alpha e^T(t) R_e e(t) < V(t)
\]

\[
< \left( \beta \over \alpha \right)^{k_{t, t_k}} e^{\xi^* T^+} V(t_0) + T^+ \left( \beta \over \alpha \right)^{k_{t, t_k}} \xi^* \delta^2 \omega_1^{-1} e^{\xi^* T^+} < \alpha \varepsilon^2
\] (29)

Since \( V(t_0) = \epsilon^T P_e \epsilon \leq \beta \varepsilon^2 \), we have

\[
\left( \beta \over \alpha \right)^{k_{t, t_k}} e^{\xi^* T^+} \beta \varepsilon^2 + T^+ \left( \beta \over \alpha \right)^{k_{t, t_k}} \xi^* \delta^2 \omega_1^{-1} e^{\xi^* T^+} < \alpha \varepsilon^2
\] (30)

which can be rewritten as (25c). Hence, \((x(t) - x_\star)^T R_e (x(t) - x_\star) \leq \varepsilon^2 \) and proof is complete.

**Remark 12.** Similar to the optimization problem of state bound \( \varepsilon \) described in Remark 10, the optimal value \( \varepsilon_{\text{min}} \) can be found according to \( \min(\beta / \alpha)^{k_{t, t_k}} e^{\xi^* T^+} (\beta \delta^2 + T^+ \xi^* \delta^2 \omega_1^{-1}) \) subject to (25a) and (25b). We fix the parameter \( \xi^* \) and let \( \alpha = 1, \beta = \theta \alpha \), the optimization problem becomes

\[
\min_{\theta \geq 1} \theta \\
\text{s.t.} \quad R_e < P_i < \theta R_e
\] (31)

Then the minimum \( \varepsilon_{\text{min}} = \sqrt{\theta \alpha^{-2} e^{\xi^* T^+} (\beta \delta^2 + T^+ \xi^* \delta^2 \omega_1^{-1})} \) can be derived with the optimized value \( \theta \). Since \( T^+ \leq T \), comparing the value of the optimal state bound \( \varepsilon_{\text{min}} \) in

**Theorem 8 and Corollary 11,** we know that, by classifying subsystems into asymptotically stable and unstable, the FTB conditions derived in Corollary II are less conservative than that in Theorem 8.

Constituting state-feedback controller of the form \( u(t) = K_x x(t) \), affine switched system (1) can be transformed into the closed-loop form of (5) and Definition 1 of FTB can be used directly. Now we will consider problem-2 to provide sufficient conditions for finite-time state-feedback stabilization.

**Theorem 13.** For affine switched system (1) holding Definition 1, if there exist state-feedback controllers \( u(t) = K_x x(t) \), positive definite matrices \( Q_i \), matrices \( X_i \), and scalars \( \theta \geq 1 \), \( \xi \geq 0 \) such that

\[
\begin{bmatrix}
- Q_i & Q_i \\
* & - R_e^{-1}
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
- \theta R_e & I \\
* & - Q_i
\end{bmatrix} < 0
\] (32a)

\[
\begin{bmatrix}
A_i Q_i + Q_i A_i^T + B_i X_i + X_i^T B_i^T - \xi Q_i & I \\
I & - \xi R_e
\end{bmatrix} < 0
\] (32b)

\[
e^{\xi T + k_{t, t_k} T \eta_i} \left( \theta \delta^2 + T^+ \xi \delta^2 \omega_1^{-1} \right) - \varepsilon^2 < 0
\] (32c)

then closed-loop system (5) is FTB with respect to \( (\delta, \delta \omega, \varepsilon, R_e, R_e, T) \) with \( K_i = X_i Q_i^{-1} \).

**Proof.** Assume \( x_\star \) is the switched equilibrium point of affine switched system (1). Applying coordinate transformation we can get its corresponding error tracking switched system as

\[
\dot{e}(t) = A_i e(t) + B_i u(t) + k_i
\] (33)

where \( k_i = A_i x_\star + b_i \) and \( e(t) = x(t) - x_\star \). Then under the state-feedback controllers \( u(t) = K_e e(t) \), the closed-loop error system can be written as

\[
\dot{e}(t) = \overline{A} e(t) + k_i
\] (34)

where \( \overline{A}_i = A_i + B_i K_i \). From Remark 2, we know that FTB analysis and finite-time control can be realized employing tracking error system. Hence, we consider the closed-loop error system (34) here to design the controllers stabilizing the system (1) in finite-time interval.

Let \( \mathcal{R} = \text{diag}(R_e, R_e) \), \( \eta_i = e^{T(t) k_i^T} \), we choose piecewise Lyapunov function \( V_i(t) = e^{T(t) P_e(t)} \) for each
subsystem; then the derivative of $V_i$ along the solution of system (34) is described as

$$V_i(t) = e^T(t) P_i e(t) + e^T(t) P_i \dot{e}(t) = \begin{bmatrix} e(t) \end{bmatrix}^T k_i$$

$$\begin{bmatrix} A_i^T P_i + P_i A_i \end{bmatrix} e(t) + \begin{bmatrix} e(t) \end{bmatrix} = \begin{bmatrix} e(t) \end{bmatrix}^T$$

(35)

Letting $Q_i = P_i^{-1}$, pre- and postmultiplying (32b) by $\text{diag}(P_i, I)$ we get

$$\begin{bmatrix} P_i A_i + A_i^T P_i + P_i B_i K_i + K_i^T B_i^T P_i - \xi P_i P_i & -\xi R_{\omega} \end{bmatrix} < 0$$

(36)

Due to condition (32a) and Schur’s complement formula [38], we deduce

$$\begin{bmatrix} -Q_i & Q_i \\ * & -R_{\omega}^{-1} \end{bmatrix} < 0 \implies R_{\omega} < P_i,$$

$$\begin{bmatrix} -\theta R_{\omega} & I \\ * & -Q_i \end{bmatrix} < 0 \implies P_i < \theta R_{\omega},$$

(37)

Now using (36), from (35) we can derive

$$V_i(t) < \eta_i^T \begin{bmatrix} \xi P_i & 0 \\ 0 & \xi \bar{R}_{\omega} \end{bmatrix} \eta_i = \xi \bar{e}^T(t) P_i e(t) + \xi k_i^T R_{\omega} k_i$$

$$< \xi V_i + \xi \bar{e}^2$$

(38)

By the same proof guidelines of Theorem 8, FTB conditions (32a) and (32c) of closed-loop error system (34) can be derived. Accordingly we get $(x(t) - x_i)^T R_i (x(t) - x_i) \leq \bar{e}^2$, which proves that the affine switched system (1) is finite-time bounded under state-feedback controllers $u(t) = K_i e(t)$. □

**Theorem 14.** Given autonomous robust switched system (10), if there exist positive definite matrices $P_i$, scalars $\alpha > 0$, $\beta > 0$, and $\xi \geq 0$ such that

$$\alpha R_{\omega} < P_i < \beta R_{\omega},$$

(39a)

$$\begin{bmatrix} A_i^T P_i + P_i A_i + C_i^T C_i & P_i G_i + C_i^T D_{2i} \\ * & -\gamma^2 I + D_{2i}^T D_{2i} \end{bmatrix}$$

$$< \xi \begin{bmatrix} R_{\omega} & 0 \\ * & R_{\omega} \end{bmatrix}$$

$$e^T(t) R_i e(t) + \xi \bar{e}^2 + \gamma^2 \bar{e}^T(t) \omega(t) + \gamma^2 \omega^T(t) \omega(t)$$

(39b)

then this system is finite-time bounded with $H_{\infty}$ performance with respect to $(\delta_{\omega}, \delta_{\omega e}, R_{\omega}, R_{\omega e}, T)$. Proof. Let $\mathcal{L} = \text{diag}(R_{\omega}, R_{\omega e}), \eta_i = \begin{bmatrix} e^T & \omega^T \end{bmatrix}^T$, and we opt Lyapunov function $V_i(t) = e^T(t) P_i e(t)$ and

$$\dot{V}_i(t) = e^T(t) P_i e(t) + e^T(t) \dot{e}(t)$$

(40)

Since $\begin{bmatrix} C_i^T C_i & C_i^T D_{2i} \\ * & D_{2i}^T D_{2i} \end{bmatrix} \succeq 0$, condition (39b) implies that

$$\begin{bmatrix} A_i^T P_i + P_i A_i & P_i G_i \\ * & -\gamma^2 I \end{bmatrix} < \xi \begin{bmatrix} R_{\omega} & 0 \\ * & R_{\omega} \end{bmatrix}$$

(41)

From (40) and (41), $\dot{V}_i(t) < \xi e^T(t) R_i e(t) + \xi \bar{e}^2 + \gamma^2 \omega^T(t) \omega(t)$ and together with condition (39a), we get

$$\dot{V}_i(t) < \frac{\xi}{\alpha} e^T(t) P_i e(t) + \xi \bar{e}^2 + \gamma^2 \omega^T(t) \omega(t)$$

(42)

Let $\forall t > 0, t_0 < \cdots < t_k$ be the switching instants, and $V(t) = \sum_{i=0}^{k} \theta_i V_i(t)$, $\theta_i \in [0, 1]$, where $\theta_i$ is the indication function for activated subsystem. From inequality (42), we have

$$V(t) < \phi(t, t_k) V(t_k)$$

$$+ \int_{t_k}^t \phi(t, r) \left[ \xi \bar{e}^2 + \gamma^2 \omega^T(r) \omega(r) \right] dr$$

(43)

where $\phi(t, r) = \exp((\xi/\alpha)(t-r)) < \exp((\eta/\alpha)T)$. Accordingly, the Lyapunov inequality in single step satisfies

$$V(t_{k+1})$$

$$< \phi(t_{k+1}, t_k) V(t_k)$$

$$+ \int_{t_k}^{t_{k+1}} \phi(t, r) \left[ \xi \bar{e}^2 + \gamma^2 \omega^T(r) \omega(r) \right] dr$$

(44)

4. $H_{\infty}$ Performance Analysis and Controller Design of Affine Switched Systems

Based upon FTB investigation of previous section, our main aim now is to design a set of $H_{\infty}$ controllers to solve Problem 7. As stated in Remark 2, finite-time $H_{\infty}$ control can be realized through tracking error system, and this will be the main focus in this section. For the sake of simplicity, we firstly consider the autonomous error switched system in the form of (10) assuming that $k_i = 0, u(t) = 0$ and the corresponding theorem is stated as follows; then we will show how to remove the assumption and extend the results to the general affine switched system with exogenous signal input.
Let system switch from mode $i$ to $j$ at instant $t_k (0 < t_k < T)$; then condition (39a) implies that

$$
\frac{V(t_k^+)}{V(t_k)} = \frac{V_j(t_k)}{V_i(t_k)} = \frac{e^T(t_k)P_je(t_k)}{e^T(t_k)P_ie(t_k)} < \frac{\beta e^T(t_k)R_e(t_k)}{\alpha e^T(t_k)R_e(t_k)} - \frac{\beta}{\alpha} \tag{45}
$$

Noting that $\beta/\alpha > 1$, following relation (44) iteratively, we can derive

$$
V(t_k) < \left(\frac{\beta}{\alpha}\right)^{k_{[0,\tau]}} \phi(t_k, t_0) V(t_0)
+ \sum_{n=1}^{k} \left(\frac{\beta}{\alpha}\right)^{-n} \phi(t_k, t_n)
\cdot \int_{t_{n-1}}^{t_n} \phi(t_n, \tau) \left[\xi \delta^2_\omega + y^2 \omega^T(\tau) \omega(\tau)\right] d\tau \tag{46}
$$

Applying (43) and (46), the following inequality is obtained:

$$
V(t) < \left(\frac{\beta}{\alpha}\right)^{k_{[0,\tau]}} \phi(t, t_0) V(t_0)
+ \sum_{n=1}^{k} \left(\frac{\beta}{\alpha}\right)^{-n} \phi(t, t_n)
\cdot \int_{t_{n-1}}^{t_n} \phi(t_n, \tau) \left[\xi \delta^2_\omega + y^2 \omega^T(\tau) \omega(\tau)\right] d\tau
\cdot e^{(\xi/\alpha)T} V(t_0)
+ \sum_{n=1}^{k} \left(\frac{\beta}{\alpha}\right)^{-n} \phi(t, t_n)
\cdot \int_{t_{n-1}}^{t_n} \phi(t_n, \tau) \left[\xi \delta^2_\omega + y^2 \omega^T(\tau) \omega(\tau)\right] d\tau
\cdot e^{(\xi/\alpha)T} V(t_0)
\cdot e^{(\xi/\alpha)T} \left[y^2 \int_{t_h}^{t} \omega^T(\tau) \omega(\tau) d\tau\right] \tag{47}
$$

Using the fact $\int_0^T \omega^T(\tau)\omega(\tau)d\tau \leq d^2$, (47) can be rewritten as

$$
V(t) < \left(\frac{\beta}{\alpha}\right)^{k_{[0,\tau]}} e^{(\xi/\alpha)T} V(t_0)
+ T \left(\frac{\beta}{\alpha}\right)^{k_{[0,\tau]}} \xi \delta^2_\omega e^{(\xi/\alpha)T} \tag{48}
$$

On the other hand, from condition (39a), we have

$$
V(t) = e^T(t)P_e(t) > \alpha e^T(t)R_e(t) \tag{49}
$$

Since $V(t_0) = e^T(t_0)P_e(t_0) < \beta e^T(t_0)R_e(t_0) \leq \beta \delta^2_\omega$, we conclude that in order to ensure FTB for system (10) such that $e^T(t)R_e(t) \leq e^2$, the following condition should be satisfied:

$$
\alpha e^T(t)R_e(t) < \left(\frac{\beta}{\alpha}\right)^{k_{[0,\tau]}} e^{(\xi/\alpha)T} \beta \delta^2_\omega
+ T \left(\frac{\beta}{\alpha}\right)^{k_{[0,\tau]}} \xi \delta^2_\omega e^{(\xi/\alpha)T}
+ \left(\frac{\beta}{\alpha}\right)^{k_{[0,\tau]}} y^2 \delta^2 e^{(\xi/\alpha)T} < \alpha e^2 \tag{50}
$$

which can be rewritten as condition (39c). Hence, FTB analysis for system (10) is completed. Considering $z^T(t)z(t) - y^2\omega^T(t)\omega(t) + \dot{V}(t)$, from (39a) and (39b) we deduce

$$
z^T(t)z(t) - y^2\omega^T(t)\omega(t) + \dot{V}(t) = \left[\begin{array}{c} e(t) \\ \omega(t) \end{array}\right]^T
\cdot \left[\begin{array}{c} A^T \hat{P}_1 + P_1A_1 + C^T \hat{C}_1 + P_2G_1 + C^T D_2 \\ D_2^2 - y^2I \end{array}\right] \omega(t)
< \left[\begin{array}{c} e(t) \\ \omega(t) \end{array}\right]^T R_e \left[\begin{array}{c} \hat{R}_e \\ \omega(t) \end{array}\right]
< \frac{\xi}{\alpha} V(t) + \xi \delta^2_\omega \Rightarrow \dot{V}(t) < \xi \alpha^{-1} V(t) + \xi \delta^2_\omega - z^T(t)z(t) + y^2\omega^T(t)\omega(t)
$$

Integrating both sides of (51) and through iterations, we can deduce

$$
V(t_k) < \left(\frac{\beta}{\alpha}\right)^{k_{[0,\tau]}} \phi(t_k, t_0) V(t_0)
+ \sum_{n=1}^{k} \left(\frac{\beta}{\alpha}\right)^{-n} \phi(t_k, t_n)
\cdot \int_{t_{n-1}}^{t_n} \phi(t_n, \tau) \left[\xi \delta^2_\omega + y^2 \omega^T(\tau) \omega(\tau)\right] d\tau
\cdot \int_{t_{n-1}}^{t_n} \phi(t_n, \tau) \left[\xi \delta^2_\omega + y^2 \omega^T(\tau) \omega(\tau)\right] d\tau
\cdot e^{(\xi/\alpha)T} V(t_0)
\cdot e^{(\xi/\alpha)T} \left[y^2 \int_{t_h}^{t} \omega^T(\tau) \omega(\tau) d\tau\right] \tag{52}
$$
where $\phi(t, \tau) = \exp(\xi^{-1}(t - \tau)) < \exp(\xi^{-1}T)$. Then following the proof line of Theorem 8, we get

$$0 \leq V(t) < \left(\frac{\beta}{\alpha}\right)^{k_{\alpha, T}} e^{(\xi/\alpha)T} V(t_0) + T \left(\frac{\beta}{\alpha}\right)^{k_{\alpha, T}}$$

(53)

Since the zero-initial condition, we have $V(0) = 0$; thus,

$$\left(\frac{\beta}{\alpha}\right)^{k_{\alpha, T}} e^{(\xi/\alpha)T} \int_{t_0}^{t} \gamma^2 \omega^T (\tau) \omega (\tau) - z^T (\tau) z (\tau) d\tau > 0$$

(54)

Setting $t = T$, $t_0 = 0$, the $H_{\infty}$ performance condition (11) is satisfied which completes the proof.

**Remark 15.** The parameter $\gamma$ is $H_{\infty}$ performance index and its minimum value $\gamma_{\min}$ is often of interest from practical viewpoint; hence, we can state the optimization problem as

$$\min \gamma^2$$

s.t. $(39a), (39b), (39c)$

(55)

Similarly, fulfilling FTB criteria, minimum value of state bound $\epsilon_{\min}$ is also desired, which can be found as the optimization problem: $\min(\beta/\alpha)^{k_{\alpha, T}} e^{(\xi/\alpha)T} (\beta \delta_e^2 + T \tilde{\xi} \delta_e^2 + \gamma^2 \tilde{d}^2 \alpha^{-1})$ subject to (39a) and (39b). If we fix the parameter $\xi$ and let $\alpha = 1, \beta = \theta \alpha$, then we can state optimization problem as

$$\min_{\theta \geq 1} \theta$$

s.t. $R_e < P_i < \theta R_e$

$$\left[ A_i^T P_i + P_i A_i + C_i^T C_i + P_i G_i + C_i^T D_{2i} \right] \ast - \gamma^2 I + D_{2i}^T D_{2i}$$

(56)

$$< \xi \left[ \begin{array}{c} R_e \\ 0 \end{array} \right] \left[ \begin{array}{c} * \\ R_w \end{array} \right]$$

and $\epsilon_{\min} = \sqrt{\theta^{k_{\alpha, T}} e^{\xi T} (\theta \delta_e^2 + T \tilde{\xi} \delta_e^2 + \gamma^2 \tilde{d}^2)}$ is derived with the optimized value of $\theta$. We can adopt a convex combination of $\gamma_{\min}$ and $\epsilon_{\min}$ as $J(\rho) = \rho \gamma_{\min}^2 + (1 - \rho) \epsilon_{\min}^2, 0 \leq \rho \leq 1$ and a more general convex optimization problem can be stated as

$$\min J(\rho)$$

s.t. $R_e < P_i < \theta R_e$

$$\left[ A_i^T P_i + P_i A_i + C_i^T C_i + P_i G_i + C_i^T D_{2i} \right] \ast - \gamma^2 I + D_{2i}^T D_{2i}$$

$$< \xi \left[ \begin{array}{c} R_e \\ 0 \end{array} \right] \left[ \begin{array}{c} * \\ R_w \end{array} \right]$$

$$e^{(\xi/\alpha)T} (\beta \delta_e^2 + T \tilde{\xi} \delta_e^2 + \gamma^2 \tilde{d}^2 - \alpha \epsilon^2) < 0$$

(57)

Now we will extend the results to design the $H_{\infty}$ controllers, ensuring FTB of the closed-loop affine switched system (8). Different equilibria for subsystems exist because of the affine terms $k_i$, and hence stability analysis and $H_{\infty}$ control are not trivial. To solve this problem, a few results are available proposing extended state space method in [12, 31]. However, this approach seems conservative for system synthesis because the eigenvalues of the extended state matrices $\tilde{A}_e$ related to the affine terms are not exactly the same as for the original state matrices $A_i$. For state-dependent affine switched system, S-procedure method can be used to reduce the conservatism. However, for time-dependent affine switched systems, there are only few effective results. In our investigation, we redefine exogenous signal $\omega(t)$ as

$$\tilde{\omega}(t) = \left[ \begin{array}{c} \omega(t) \\ \xi \end{array} \right]$$

(58)

Hence, the closed-loop switched system (8) can be rewritten as

$$\dot{e}(t) = \tilde{A}_e e(t) + \tilde{G}_e \tilde{\omega}(t), \quad x(0) = x_0$$

$$z(t) = \tilde{C}_e e(t) + \tilde{D}_{2i} \tilde{\omega}(t), \quad \tilde{\omega}(0) = \tilde{\omega}_0$$

(59)

where $\tilde{G}_i = [G_i, 1], \tilde{D}_{2i} = [D_{2i}, 0]$. Since in the $H_{\infty}$ framework $\int_0^T \omega^T(t) \omega(t) dt \leq \tilde{d}^2$ holds, the proposed extension of the disturbance input is reasonable and we can design the $H_{\infty}$ controllers of the equivalent closed-loop error switched system (59) to ensure the finite-time $H_{\infty}$ bounded closedness for original affine switched system (8).

**Theorem 16.** The closed-loop switched system (59) is FTB with $H_{\infty}$ performance $\gamma$ regarding $(\delta_e, \xi, \epsilon, R_e, R_w, T)$, if there exist constant state-feedback controller $u(t) = K_e x(t)$, positive definite matrices $Q_i$, matrices $W_i$, scalars $\theta \geq 1, \xi \geq 0$ such that

$$\begin{bmatrix} -Q_i & Q_i \\ * & -R_e^{-1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} -\theta R_e & I \\ * & -Q_i \end{bmatrix} < 0$$

(60a)
Employing (60a) and using Schur complement formula, we obtain

\[
\begin{bmatrix}
A_i Q_i + Q_i A_i^T + B_i W_i + W_i^T B_i^T - \xi Q_i & \bar{G}_i & Q_i C_i^T + W_i^T D_i^T \\
\bar{G}_i^T & -\gamma^2 I - \xi R_w & \bar{D}_i^T \\
C_i Q_i + D_i W_i & \bar{D}_i & -I
\end{bmatrix} < 0
\]

(60b)

Using Schur lemma, (62) can be rewritten as

\[
\begin{bmatrix}
A_i^T P_i + P_i A_i - \xi P_i + P_i \bar{G}_i & P_i \bar{G}_i & C_i^T \\
\bar{G}_i^T & -\gamma^2 I - \xi R_w & \bar{D}_i^T \\
C_i & \bar{D}_i & -I
\end{bmatrix} < 0
\]

(62)

Assuming \(Q_i = P_i^{-1}\), pre- and post-multiplying (60b) by \(\text{diag}(P_i, I, I)\),

\[
\begin{bmatrix}
A_i^T P_i + P_i A_i - \xi P_i + C_i^T C_i & P_i \bar{G}_i & C_i^T \\
\bar{G}_i^T & -\gamma^2 I - \xi R_w & \bar{D}_i^T \\
C_i & \bar{D}_i & -I
\end{bmatrix} < 0
\]

(63)

Since \(\begin{bmatrix} C_i^T C_i & \bar{D}_i \end{bmatrix} \geq 0\), we can get

\[
\begin{bmatrix}
A_i^T P_i + P_i A_i - \xi P_i + C_i^T C_i & P_i \bar{G}_i & C_i^T \\
\bar{G}_i^T & -\gamma^2 I - \xi R_w & \bar{D}_i^T \\
C_i & \bar{D}_i & -I
\end{bmatrix} < 0
\]

(64)

which implies that

\[
\dot{V}_i(t) < \xi \bar{G}_i^T(t) P_i \dot{e}(t) + \xi \bar{G}_i^T(t) P_i \bar{G}_i(t) + \gamma^2 \bar{G}_i^T(t) \bar{G}_i(t)
\]

(65)

Employing (60a) and using Schur complement formula,

\[
\begin{bmatrix}
-Q_i & Q_i \\
* & -R_e^{-1}
\end{bmatrix} < 0 \implies R_e < P_i,
\]

(66)

where \(K_i = W_i Q_i^{-1}\).

Proof. Let \(\mathcal{R} = \text{diag}(R_e, R_w)\) and \(\eta_i = \left[ e^T(t) \quad \bar{w}^T(t) \right]^T, i \in I\). Defining \(V_i(t) = e^T(t) P_i e(t)\) as before, for system (59) we can state that

\[
\dot{V}_i(t) = e^T(t) P_i \dot{e}(t) + e^T(t) P_i \bar{e}(t)
\]

\[
= \begin{bmatrix} e(t) \\ \bar{w}(t) \end{bmatrix}^T \begin{bmatrix} A_i^T P_i + P_i A_i - \xi P_i & P_i \bar{G}_i \\ \bar{G}_i^T & -\gamma^2 I - \xi R_w \end{bmatrix} \begin{bmatrix} e(t) \\ \bar{w}(t) \end{bmatrix} > 0
\]

(61)

Applying integration and iterations, and setting \(V_i(t_0) = 0\) under zero-initial conditions, we get

\[
0 < V(t)
\]

(67)

which illustrates that condition (9) is satisfied. We conclude that the affine switched system (59), and hence closed-loop affine switched system (8), is FTB with \(H_\infty\) performance under zero-initial conditions. From (60b),

\[
V(t) < 0
\]

(68)

Then setting \(t = T, t_0 = 0\), we obtain that

\[
\int_0^T y^2 \bar{w}^T(\tau) \bar{w}(\tau) - z^T(\tau) z(\tau) d\tau > 0
\]

(69)

which illustrates that condition (9) is satisfied. We conclude that the affine switched system (59), and hence closed-loop affine switched system (8), is FTB with \(H_\infty\) performance under zero-initial conditions. From (60b),

\[
V(t) < 0
\]

(67)

which guarantees the FTB of robust affine switched system can be obtained.

Following the proof guidelines of Theorem 14, condition (60c) which guarantees the FTB of robust affine switched system can be obtained.

Now we need to prove condition (9) for \(H_\infty\) performance under zero-initial conditions. From (60b),

\[
V(t) < 0
\]

(68)

which illustrates that condition (9) is satisfied. We conclude that the affine switched system (59), and hence closed-loop affine switched system (8), is FTB with \(H_\infty\) performance under zero-initial conditions. From (60b),

\[
V(t) < 0
\]

(67)

which illustrates that condition (9) is satisfied. We conclude that the affine switched system (59), and hence closed-loop affine switched system (8), is FTB with \(H_\infty\) performance under zero-initial conditions. From (60b),

\[
V(t) < 0
\]

(68)

which illustrates that condition (9) is satisfied. We conclude that the affine switched system (59), and hence closed-loop affine switched system (8), is FTB with \(H_\infty\) performance under zero-initial conditions. From (60b),

\[
V(t) < 0
\]

(67)
5. Numerical Examples

Example 1. Consider the affine switched system (2) with two modes of operation:

\[
A_1 = \begin{bmatrix} 0.01 & -2 \\ 1 & 0.02 \end{bmatrix},
A_2 = \begin{bmatrix} -0.1 & -1 \\ 3 & -0.1 \end{bmatrix}; \tag{70}
\]

\[
b_1 = \begin{bmatrix} -3.98 \\ -1.16 \end{bmatrix},
\]

\[
b_2 = \begin{bmatrix} -1.8 \\ -6.4 \end{bmatrix};
\]

\[
A_1 \text{ is unstable, } A_2 \text{ is Hurwitz stable, and eigenvalues } \lambda(A_1) = \{0.015 \pm 1.4142i\}, \lambda(A_2) = \{-0.1 \pm 1.732i\}. \text{ Assuming desired reference } x_r = [2, -2]^T, \text{ error tracking switched system will be}
\]

\[
A_1 = \begin{bmatrix} 0.01 & -2 \\ 1 & 0.02 \end{bmatrix},
A_2 = \begin{bmatrix} -0.1 & -1 \\ 3 & -0.1 \end{bmatrix}; \tag{71}
\]

\[
k_1 = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix},
\]

\[
k_2 = \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}.
\]

Evidently \(k_{\max} = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}\). Let \(\delta_\varepsilon = 1, \delta_\omega = 0.8, \varepsilon = 5.885, R_\varepsilon = R_\omega = I, \xi = 0.3, \text{ and } T = 5s.\) From the FTB condition (24), we get the average dwell time \(\tau^*_a = 1.5s\) to ensure the finite-time boundedness with respect to \(\varepsilon\), so that the switching signal \(S\) can be chosen as a periodical signal with \(T_s = 1.5s\), which implies that \(k_{[0,T]} = 3\) and \(T^* = 3s\) during the finite-time interval \([0, 5]\). Given the initial error state \(e(0) = [0.5, 0.8]^T\), the conditions \(\varepsilon_0 R_\varepsilon e_0 \leq \delta_\varepsilon^2\) and \(k_{\max}^T R_\omega k_{\max} \leq \delta_\omega^2\) are separately satisfied, then the error state trajectory of error affine switched system and the value of \(e^T R_\varepsilon e\) under the switching signal \(S\) are shown in Figure 1.

It is easy to see in Figure 1 that subject system is FTB with conditions (12a), (12b), and (12c) satisfied. Moreover, assuming \(\xi^* = \xi = 1\) and using optimization process (24) and (31), the optimal value \(\theta_{\min} = 1.0058, P_1 = \begin{bmatrix} 0.8425 & 0.1287 \\ 0.1287 & 1.2053 \end{bmatrix}, P_2 = \begin{bmatrix} 1.3718 & -0.0977 \\ -0.0977 & 1.0416 \end{bmatrix}\) can be obtained. Then substituting \(\theta_{\min}\) into (12c) and (25c) separately, we get

\[
\varepsilon_1_{\min} = 3.014,
\]

\[
\varepsilon_2_{\min} = 1.923 \tag{72}
\]

where \(\varepsilon_1_{\min}, \varepsilon_2_{\min}\) denote the minimum bound of state derived by Theorem 8 and Corollary 11. It is obvious that

Corollary II is less conservative than Theorem 8 since \(\varepsilon_{2\min} > \varepsilon_{1\min}\).

Example 2. Keeping in view autonomous error switched system (10), we consider this system:

\[
A_1 = \begin{bmatrix} 0.01 & -2 \\ 1 & 0.02 \end{bmatrix},
\]

\[
k_1 = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix},
\]

\[
G_1 = \begin{bmatrix} 0.25 \\ 0.01 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} 0.10 & 0.33 \end{bmatrix},
\]

![Figure 1: The state trajectories and the value of \(e^T R_\varepsilon e(t)\) under switching signal \(S\).](image)
with disturbance signal:

\[
\omega(t) = \begin{cases} 
8 & 0 \leq t \leq 5 \\
0 & \text{else}
\end{cases}
\]

(74)

which satisfies \(\int_0^T \omega^T(t)\omega(t)dt)^{1/2} = (\int_0^5 \omega^T(t)\omega(t)dt)^{1/2} = 8\sqrt{2}\). Let \(\delta_\varepsilon = 1\), \(\delta_\omega = 0.8\), \(\varepsilon = 21.758\), \(\gamma = 0.2\), and \(\xi = 0.3\). From FTB condition (39c), we get the average dwell time \(\tau^*_a = 1.5\) s to ensure FTB with respect to \(\varepsilon\). Then for the finite-time \(H_\infty\) performance, we should have

\[
\left(\int_0^T z^T(t)z(t)dt\right)^{1/2} < \gamma \left(\int_0^T \omega^T(t)\omega(t)dt\right)^{1/2}
\]

(75)

\[\approx 3.57\]

The simulation results with initial error state \(e(0) = [0.5, 0.8]^T\) are shown in Figure 2.

We observe in Figure 2 that the system is FTB, and the \(H_\infty\) performance satisfies

\[
\left(\int_0^T z^T(t)z(t)dt\right)^{1/2} \approx 2.18 < 3.57
\]

(76)

Thus, according to Definition 4, the autonomous robust error switched system can be regarded as finite-time \(H_\infty\) bounded. Moreover, using optimization procedure (56) we get \(\theta_{min} = 1.932\), \(P_{1,1} = \begin{bmatrix} 1.0165 & 0.0066 \\ 0.0066 & 1.9156 \end{bmatrix}\), \(P_{2,1} = \begin{bmatrix} 2.2218 & -0.2047 \\ -0.2047 & 1.0425 \end{bmatrix}\). Putting in (39c), we get \(\varepsilon_{min} = 6.314\).

**Example 3.** Consider the affine error switched system (7) with two modes of operation:

\[
A_2 = \begin{bmatrix} -0.1 & -1 \\ 3 & -0.1 \end{bmatrix},
\]

\[
k_2 = \begin{bmatrix} 0 \\ -0.2 \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} -1 \\ 0.5 \end{bmatrix},
\]

\[
G_2 = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix},
\]

\[
C_2 = [0.3 \ 0.01],
\]

\[
D_{21} = 0.05
\]

\[
A_2 = \begin{bmatrix} -0.1 & -1 \\ 3 & -0.1 \end{bmatrix},
\]

\[
k_2 = \begin{bmatrix} 0 \\ -0.2 \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} -1 \\ 0.5 \end{bmatrix},
\]

\[
G_2 = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix},
\]

\[
C_2 = [0.3 \ 0.01],
\]

\[
D_{21} = 0.05
\]

\[
\omega(t) = \begin{cases} 
8 & 0 \leq t \leq 5 \\
0 & \text{else}
\end{cases}
\]

(77)
which implies that \( (\int_{0}^{T \omega(t) \omega(t) dt})^{1/2} = (\int_{0}^{5 \omega(t) \omega(t) dt})^{1/2} = 8\sqrt{5} \) and \( k_{\text{max}} = [0.8]. \)

The objective in this example is to design a set of robust \( H_{\infty} \) controllers ensuring finite-time \( H_{\infty} \) boundedness of closed-loop error switched system with respect to \( (\delta_{\xi}, \delta_{\omega}, \varepsilon, R_{\xi}, R_{\omega}, T) \), where \( \delta_{\xi} = 1, \delta_{\omega} = 0.8, \varepsilon = 4.536, R_{\xi} = R_{\omega} = 1, \xi = 0.3, \) and \( T = 5s. \) Setting switching signal \( S \) as a periodical signal with \( T_{s} = 1.5s, \) based on Theorem 16, we calculate

\[
Q_{1} = \begin{bmatrix} 1.4665 & 0.0957 \\ 0.0957 & 1.3408 \end{bmatrix}, \\
W_{1} = [-1.2110 0.4831] \\
Q_{2} = \begin{bmatrix} 1.1617 & -0.2958 \\ -0.2958 & 1.7466 \end{bmatrix}, \\
W_{2} = [3.0403 0.9730] \tag{78}
\]

Then we can get the set of \( H_{\infty} \) controllers for each subsystem as

\[
K_{1} = W_{1}Q_{1}^{-1} = [-0.8533 0.4212], \\
K_{2} = W_{2}Q_{2}^{-1} = [2.8834 1.0454] \tag{79}
\]

Substitute controller gains into system (8), the closed-loop error switched system can be written as

\[
\begin{aligned}
\begin{bmatrix} a_{1} & b_{1} \\ c_{1} & d_{1} \end{bmatrix} &= \begin{bmatrix} -0.0753 & -1.9579 \\ 1.8533 & -0.4012 \end{bmatrix}, \\
k_{1} &= \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}, \\
G_{1} &= \begin{bmatrix} 0.25 \\ 0.01 \end{bmatrix}, \\
\begin{bmatrix} \bar{a}_{1} \\ \bar{c}_{1} \end{bmatrix} &= \begin{bmatrix} -0.0109 & 0.3848 \end{bmatrix}, \\
D_{21} &= 0.05 \\
\begin{bmatrix} a_{2} & b_{2} \\ c_{2} & d_{2} \end{bmatrix} &= \begin{bmatrix} -2.9834 & -2.0454 \\ 4.4417 & 0.4227 \end{bmatrix}, \\
k_{2} &= \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}, \\
G_{2} &= \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}, \\
\begin{bmatrix} \bar{a}_{2} \\ \bar{c}_{2} \end{bmatrix} &= \begin{bmatrix} 0.8767 & 0.2191 \end{bmatrix}, \\
D_{22} &= 0.028 \end{aligned} \tag{80}
\]

State responses under state-feedback controllers and switching signal \( S \) are shown in Figure 3. We can observe that the closed-loop system is FTB, and the \( H_{\infty} \) performance satisfies

\[
\begin{aligned}
\left( \int_{0}^{T} z^{T}(t)z(t)dt \right)^{1/2} &= 0.9366 \\
< \gamma \left( \int_{0}^{T} \omega^{T}(t)\omega(t)dt + k_{\text{max}}^{T}k_{\text{max}}dt \right)^{1/2} &= 3.596 \end{aligned} \tag{81}
\]

Thus, according to Definition 4, the given affine switched system can be regarded as finite-time \( H_{\infty} \) bounded under designed \( H_{\infty} \) controller gains.

6. Conclusion

In this paper, the problem of finite-time boundedness and finite-time \( H_{\infty} \) control for affine switched systems has been investigated. Several definitions and sufficient conditions for FTB and \( H_{\infty} \) performance are proposed. Based on the average dwell-time method, the FTB conditions of affine switched linear system with known state boundary are derived first in this investigation. To reduce the conservatism of FTB conditions, by classifying subsystems into asymptotically stable and unstable systems, we get the improved FTB conditions for affine switched system presented in Corollary 11. The conservatism of conditions under the two situations is compared. Then applying the finite-time boundedness analysis results, finite-time \( H_{\infty} \) performance is discussed. Finite-time \( H_{\infty} \) controllers are designed to ensure the corresponding closed-loop switched system FTB with \( H_{\infty} \) performance. Numerical examples are finally provided to validate our theoretical results. Many real-world systems concern with finite-time
and transient behavior; meanwhile, many engineering applications can be modeled as affine switched systems. Therefore, our theoretical results about finite-time boundedness of affine switched systems are supposed to have great potential in the application of practical switched systems. Furthermore, the proposed results in this paper can be extended to the nonlinear affine switched systems which will be considered in future work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (no. 51607151), Scientific Research Startup/Planning Project of Southwest Petroleum University (no. 2014QHZ027), and Young Scholars Development Fund of Southwest Petroleum University (no. 2015990110).

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