Research Article

Global Output Feedback Stabilization of Nonlinear Systems with a Time-Varying Power and Unknown Output Function

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This paper studies the problem of global output feedback stabilization for a class of nonlinear systems with a time-varying power and unknown output function. For nonlinear systems with a time-varying power and unknown continuous output function, by constructing a new nonlinear reduced-order observer together with adding a power integrator method, a new function to determine the maximal open sector Ω of output function is given. As long as output function belongs to any closed sector included in Ω , it is shown that the equilibrium point of the closed-loop system can be guaranteed globally uniformly asymptotically stable by an output feedback controller.

1. Introduction

Consider nonlinear systems with the unknown output function

$$\begin{aligned}\dot{x}_i(t) &= [x_{i+1}(t)]^{p(t)} + \phi_i(t, x_1(t), \dots, x_i(t)), \\ i &= 1, \dots, n-1, \\ \dot{x}_n(t) &= [u(t)]^{p(t)} + \phi_n(t, x_1(t), \dots, x_n(t)), \\ y(t) &= h(x_1(t)),\end{aligned}\tag{1}$$

where $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $y \in \mathbb{R}$ are the unmeasurable state, control input, and output, respectively. For a real constant $\varrho > 0$, $[\cdot]^\varrho$ is defined as $[\cdot]^\varrho = \text{sgn}(\cdot) \cdot |\cdot|^\varrho$. The time-varying power $p(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous bounded function satisfying $1 \leq \underline{p} \leq p(t) \leq \bar{p}$ with two constants $\underline{p}$ and $\bar{p}$. For $i = 1, \dots, n$, $\phi_i(\cdot) : \mathbb{R}^+ \times \mathbb{R}^i \rightarrow \mathbb{R}$ are continuous in the first argument and locally Lipschitz with respect to the rest variables with $\phi_i(t, 0) = 0$. Output function $h(\cdot)$ is an unknown continuous function with $h(0) = 0$.

Over the past decade, with the help of adding a power integrator method, homogeneous domination method, and recursive observer design, there exist some interesting results on state/output feedback design of high-order nonlinear systems, whose powers are known constant ratios of odd integers; see [1–16] and the references therein.

In recent years, some interesting results have been achieved on output feedback design of nonlinear systems with known constant powers and unknown output function. For the nonlinear systems (1) with $p(t) = 1$, when $h(\cdot)$ is a continuous differentiable function and its derivative with known upper and lower bounds, global output feedback stabilization and finite-time output feedback stabilization have been achieved in [17–19] and [20], respectively. When the derivative of $h(\cdot)$ is with unknown upper bound, [21] achieved semi-global output feedback control. Furthermore, when $h(\cdot)$ is extended to be only a continuous function, with the help of the given maximal sector region of output function, [22] achieved output feedback stabilization. Lately, in [23], a new design and analysis method for high-order nonlinear systems with unknown continuous output function was proposed based on adding

a power integrator method and homogeneous domination method.

According to some practice [24–26], it is well-known that the timely deteriorated performance of system often results in different running data, which usually identify the powers of system. Therefore, the powers of system are usually not fixed and can be varying with a suitable bound even in the same working condition. For example, as a practical second-order dynamic model of reduced-order boiler-turbine unit, two typical different powers $p(t) = 1.072$ and $p(t) = 1.031$ have been identified in [25] and [26], respectively. For nonlinear systems with a time-varying power, [27, 28] achieved global state feedback stabilization based on interval homogeneous domination approach. As far as we know, [29] is the first paper to study the output feedback stabilization of nonlinear systems (1) with $h(x_1) = x_1$ by the revamped method of adding a power integrator together with the recursive nonlinear observer design. Naturally, an interesting problem is put forward: *For more general nonlinear systems (1) with $h(x_1)$ being an unknown continuous function, can we design an output feedback controller?*

In this paper, we make an attempt to handle this problem. Some essential technical difficulties in control design will be inevitably produced: (i) Compared with [29], since output function is unknown, we construct a new nonlinear reduced-order observer without using the unmeasurable state x_1 . (ii) Compared with [23], a new function to determine the maximal open sector Ω of output function is given since the power of system is time-varying. As long as output function belongs to any closed sector included in Ω , the equilibrium point of the closed-loop system is globally uniformly asymptotically stable under the constructed output feedback controller.

This paper is organized as follows. Section 2 gives some preliminaries. The design and analysis of output feedback controller are given in Section 3, following a simulation example in Section 4. Section 5 concludes this paper.

2. Mathematical Preliminaries

Some notations, definitions, and lemmas are to be used throughout this paper.

In this paper, the argument of function will be omitted whenever no confusion can arise from the context. R , R^+ , and R^n denote the set of real numbers, the set of all nonnegative real numbers, and the real n -dimensional space. For constant β_i , $i = 1, \dots, n$, let $\beta_{i \sim j} = \prod_{k=i}^j \beta_k$, $1 \leq i < j \leq n$, and $\beta_{i \sim i} = \beta_i$, $\beta_{i \sim (i-1)} = 1$, $\beta_{0 \sim i} = 0$. For integers a, b and constant Λ_i , $\sum_{i=a}^b \Lambda_i = 0$ when $a > b$.

Definition 1 (see [30]). A function $h(\cdot) : R \rightarrow R$ is said to belong to the sector $[\rho_1, \rho_2]$ if $(h(s) - \rho_1 s)(h(s) - \rho_2 s) \leq 0$, where ρ_1 and ρ_2 are constants with $\rho_1 < \rho_2$. If the inequality is strict, we write the sector as (ρ_1, ρ_2) .

The following lemmas will serve as the basis for the development of output feedback controller. Lemmas 2–5 are used to enlarge inequalities. Lemmas 6 and 7 are Lyapunov

stability theorem for the global uniformly asymptotically stable of the closed-loop system.

Lemma 2 (see [31]). Let $p(t)$ be a real-valued function of $t \in R^+$ satisfying $0 < p(t) < +\infty$. For any $x_i \in R$, $i = 1, \dots, n$,

$$\begin{aligned} & (|x_1| + \dots + |x_n|)^{p(t)} \\ & \leq \max \{n^{p(t)-1}, 1\} (|x_1|^{p(t)} + \dots + |x_n|^{p(t)}). \end{aligned} \quad (2)$$

Lemma 3 (see [32]). Let $p(t), q(t)$ be positive real-valued functions of $t \in R^+$ and $\gamma(x, y)$ be a positive real-valued function of $x, y \in R$. For any $x, y \in R$,

$$\begin{aligned} & |x|^{p(t)} |y|^{q(t)} \\ & \leq \frac{p(t)}{p(t) + q(t)} \gamma(x, y) |x|^{p(t)+q(t)} \\ & \quad + \frac{q(t)}{p(t) + q(t)} \gamma^{-p(t)/q(t)}(x, y) |y|^{p(t)+q(t)}. \end{aligned} \quad (3)$$

Lemma 4 (see [33]). Let $p(t)$ be a real-valued function of $t \in R^+$ satisfying $p(t) \geq 1$. For any $x, y \in R$,

$$\begin{aligned} & |[x]^{p(t)} - [y]^{p(t)}| \\ & \leq p(t) (2^{p(t)-3} + 1) (|x - y|^{p(t)} + |x - y| |y|^{p(t)-1}), \end{aligned} \quad (4)$$

where $|y|^{p(t)-1} := 0$ if $y = 0$ and $p(t) = 1$.

Lemma 5 (see [33]). Let $p(t)$ be a real-valued function of $t \in R^+$ satisfying $p(t) \geq 1$. For any $x, y \in R$,

$$-(x - y) ([x]^{p(t)} - [y]^{p(t)}) \leq -2^{1-p(t)} |x - y|^{p(t)+1}. \quad (5)$$

Lemma 6 (see [30]). Let $V : R^n \rightarrow R$ be a continuous positive definite and radially unbounded function defined on R^n ; then there exist class $\mathcal{K}_\infty$ functions π_1 and π_2 defined on $[0, +\infty)$ such that $\pi_1(\|x\|) \leq V(x) \leq \pi_2(\|x\|)$ for all $x \in R^n$.

Lemma 7 (see [30]). For the nonautonomous system $\dot{x} = f(t, x)$, let $x = 0$ be an equilibrium point of system and $V : R^+ \times R^n \rightarrow R^+$ be a continuously differentiable function such that $W_1(x) \leq V(t, x) \leq W_2(x)$ and $\partial V / \partial t + (\partial V / \partial x) f(t, x) \leq -W_3(x)$ hold for any $t \geq 0$ and $x \in R^n$, where $W_1(x), W_2(x), W_3(x)$ are continuous positive definite functions on R^n and $W_1(x)$ is radially unbounded. Then $x = 0$ is globally uniformly asymptotically stable.

3. Output Feedback Controller Design and Stability Analysis

3.1. Control Objective of This Paper. The objective of this paper is to construct an output feedback controller for system (1) such that the equilibrium point of the closed-loop system is globally uniformly asymptotically stable when the maximal open sector of output function is given.

Assumption 8. There is a known constant $c \geq 0$ such that for $i = 1, \dots, n$,

$$|\phi_i(t, x) - \phi_i(t, y)| \leq c \sum_{j=1}^N \left(\sum_{k=1}^i \left(|x_k - y_k|^{a_{k,j}(t)} \sum_{q=1}^i \left(|x_q|^{b_{k,j}(t)} + |y_q|^{b_{k,j}(t)} \right) \right) \right), \quad (6)$$

for all $x, y \in R^n$, where N is an arbitrary positive integer and the real-valued functions $a_{k,j}(t)$ and $b_{k,j}(t)$ satisfy the following relations:

$$a_{k,j}(t) + b_{k,j}(t) = p(t), \quad (7)$$

$$\underline{a}_{k,j} \leq a_{k,j}(t) \leq \bar{a}_{k,j}, \quad \underline{b}_{k,j} \leq b_{k,j}(t) \leq \bar{b}_{k,j},$$

for all $j = 1, \dots, N$ and $k = 1, \dots, i$, with $0 < \underline{a}_{k,j} \leq \bar{a}_{k,j}$ and $0 < \underline{b}_{k,j} \leq \bar{b}_{k,j}$. By Lemma 3 and Assumption 8, there exists a constant $\tilde{c} \geq 0$ such that

$$|\phi_i(t, x_1, \dots, x_i)| \leq \tilde{c} \left(|x_1|^{p(t)} + \dots + |x_i|^{p(t)} \right), \quad (8)$$

$i = 1, \dots, n.$

Remark 9. As far as we know, under Assumption 8, [29] is the first paper to study output feedback stabilization of nonlinear system (1) with $h(x_1) = x_1$. In this paper, we will consider system (1) with $h(x_1)$ being an unknown continuous function.

3.2. State Feedback Controller Design of System (1)

Step 1. Taking $\xi_1 = x_1$, $V_1(x_1) = (1/2)x_1^2$, it follows from (1) and (8) that

$$\begin{aligned} \dot{V}_1 &= x_1 [x_2]^{p(t)} + x_1 \phi_1 \\ &\leq x_1 \left([x_2]^{p(t)} - [x_2^*]^{p(t)} \right) + x_1 [x_2^*]^{p(t)} \\ &\quad + \tilde{c} |x_1|^{p(t)+1}. \end{aligned} \quad (9)$$

Choose the virtual controller $x_2^* = -\beta_1 x_1$ with $\beta_1 = (r_{1,1} + \tilde{c})^{1/p}$, where $r_{1,1} \geq 1$ is a constant to be designed. Due to $(r_{1,1} + \tilde{c})^{p(t)/p} \geq (r_{1,1} + \tilde{c})$, (9) becomes

$$\dot{V}_1 \leq -r_{1,1} |\xi_1|^{p(t)+1} + \xi_1 \left([x_2]^{p(t)} - [x_2^*]^{p(t)} \right). \quad (10)$$

Inductive Step. Suppose that at step $k-1$ ($k \geq 2$), there is a positive definite and radially unbounded Lyapunov function $V_{k-1}(\xi_1, \dots, \xi_{k-1}) = V_{k-2}(\xi_1, \dots, \xi_{k-2}) + (1/2)\xi_{k-1}^2$ and a set of virtual controllers $x_2^*, \dots, x_k^*$ defined by

$$\begin{aligned} \xi_1 &= x_1, \\ x_i^* &= -\beta_{i-1} \xi_{i-1}, \\ \xi_i &= x_i - x_i^*, \\ i &= 2, \dots, k, \end{aligned} \quad (11)$$

such that

$$\begin{aligned} \dot{V}_{k-1} &\leq -\sum_{i=1}^{k-2} \left(r_{i,i} - \tilde{r}_{i+1,1} - \sum_{j=i+1}^{k-1} \tilde{r}_{j,i} \right) |\xi_i|^{p(t)+1} \\ &\quad - r_{k-1,k-1} |\xi_{k-1}|^{p(t)+1} \\ &\quad + \xi_{k-1} \left([x_k]^{p(t)} - [x_k^*]^{p(t)} \right), \end{aligned} \quad (12)$$

where $\tilde{r}_{i+1,1}, \tilde{r}_{j,i}, j = i+1, \dots, k-1, i = 1, \dots, k-2$, are positive constants, $r_{1,1}, \dots, r_{k-1,k-1} \geq 1$ are constants to be designed, and $\beta_1, \dots, \beta_{k-1}$ are positive constants dependent on $r_{1,1}, \dots, r_{k-1,k-1}$. In what follows, we will show that (12) still holds at step k .

By (11), one has $x_k^* = -\sum_{i=1}^{k-1} \beta_{i \sim (k-1)} x_i$, which together with (1) lead to

$$\dot{\xi}_k = [x_{k+1}]^{p(t)} + \phi_k + \sum_{i=1}^{k-1} \beta_{i \sim (k-1)} \left([x_{i+1}]^{p(t)} + \phi_i \right). \quad (13)$$

Constructing $V_k(\xi_1, \dots, \xi_k) = V_{k-1}(\xi_1, \dots, \xi_{k-1}) + (1/2)\xi_k^2$, it follows from (12) and (13) that

$$\begin{aligned} \dot{V}_k &\leq -\sum_{i=1}^{k-2} \left(r_{i,i} - \tilde{r}_{i+1,1} - \sum_{j=i+1}^{k-1} \tilde{r}_{j,i} \right) |\xi_i|^{p(t)+1} \\ &\quad - r_{k-1,k-1} |\xi_{k-1}|^{p(t)+1} + \xi_{k-1} \left([x_k]^{p(t)} - [x_k^*]^{p(t)} \right) \\ &\quad + \xi_k \left([x_{k+1}]^{p(t)} + \phi_k \right. \\ &\quad \left. + \sum_{i=1}^{k-1} \beta_{i \sim (k-1)} \left([x_{i+1}]^{p(t)} + \phi_i \right) \right). \end{aligned} \quad (14)$$

By (8), (11), and Lemmas 2–4, we derive

$$\begin{aligned} &\xi_{k-1} \left([x_k]^{p(t)} - [x_k^*]^{p(t)} \right) \\ &\leq |\xi_{k-1}| \cdot \bar{p} \left(2^{\bar{p}-3} + 1 \right) \left(|\xi_k|^{p(t)} + |\xi_k| |x_k^*|^{p(t)-1} \right) \\ &\leq \tilde{r}_{k,1} |\xi_{k-1}|^{p(t)+1} + \lambda_{k,1} |\xi_k|^{p(t)+1}, \\ &\xi_k \left(\phi_k + \sum_{i=1}^{k-1} \beta_{i \sim (k-1)} \left([x_{i+1}]^{p(t)} + \phi_i \right) \right) \\ &\leq |\xi_k| \left(\bar{\beta}_1 |\xi_1|^{p(t)} + \bar{\beta}_2 |x_2|^{p(t)} + \dots + \bar{\beta}_k |x_k|^{p(t)} \right) \\ &\leq \sum_{i=1}^{k-1} \tilde{r}_{k,i} |\xi_i|^{p(t)+1} + \lambda_{k,2} |\xi_k|^{p(t)+1}, \end{aligned} \quad (15)$$

where $\bar{\beta}_i = \beta_{(i-1) \sim (k-1)} + \tilde{c}(1 + \sum_{j=i}^{k-1} \beta_{j \sim (k-1)})$, $\tilde{r}_{k,1}, \tilde{r}_{k,2}, \dots, \tilde{r}_{k,k-1}$ are positive constants, and $\lambda_{k,1}$ and $\lambda_{k,2}$ are positive constants

dependent on $\bar{r}_{k,1}$ and $\bar{r}_{k,1}, \dots, \bar{r}_{k,k-1}$, respectively. Substituting (15) and (16) into (14) leads to

$$\begin{aligned} \dot{V}_k \leq & -\sum_{i=1}^{k-1} \left(r_{i,i} - \bar{r}_{i+1,1} - \sum_{j=i+1}^k \bar{r}_{j,i} \right) |\xi_i|^{p(t)+1} \\ & + \xi_k \left([x_{k+1}]^{p(t)} - [x_{k+1}^*]^{p(t)} \right) + \xi_k [x_{k+1}^*]^{p(t)} \\ & + (\lambda_{k,1} + \lambda_{k,2}) |\xi_k|^{p(t)+1}. \end{aligned} \quad (17)$$

Choose the virtual controller $x_{k+1}^* = -\beta_k \xi_k$ with $\beta_k = (r_{k,k} + \lambda_{k,1} + \lambda_{k,2})^{1/p}$, where $r_{k,k} \geq 1$ is a constant to be designed. Due to $(r_{k,k} + \lambda_{k,1} + \lambda_{k,2})^{p(t)/p} \geq (r_{k,k} + \lambda_{k,1} + \lambda_{k,2})$, (17) becomes

$$\begin{aligned} \dot{V}_k \leq & -\sum_{i=1}^{k-1} \left(r_{i,i} - \bar{r}_{i+1,1} - \sum_{j=i+1}^k \bar{r}_{j,i} \right) |\xi_i|^{p(t)+1} \\ & - r_{k,k} |\xi_k|^{p(t)+1} + \xi_k \left([x_{k+1}]^{p(t)} - [x_{k+1}^*]^{p(t)} \right). \end{aligned} \quad (18)$$

This completes the inductive step.

Step n. The Lyapunov function $V_n(\xi) = V_{n-1}(\xi_1, \dots, \xi_{n-1}) + (1/2)\xi_n^2 = \sum_{i=1}^n (1/2)\xi_i^2$ and a positive constant λ give

$$\begin{aligned} \dot{V}_n \leq & -\sum_{i=1}^{n-1} \left(r_{i,i} - \bar{r}_{i+1,1} - \sum_{j=i+1}^n \bar{r}_{j,i} \right) |\xi_i|^{p(t)+1} \\ & + \lambda |\xi_n|^{p(t)+1} + \xi_n \left([u]^{p(t)} - [x_{n+1}^*]^{p(t)} \right) \\ & + \xi_n [x_{n+1}^*]^{p(t)}, \end{aligned} \quad (19)$$

where ξ_i is defined as in (11) for $i = 1, \dots, n$. Choose virtual controller $x_{n+1}^* = -\beta_n \xi_n$ with $\beta_n = (r_{n,n} + \lambda)^{1/p}$, where $r_{n,n} \geq 1$ is a constant to be designed. Due to $(r_{n,n} + \lambda)^{p(t)/p} \geq (r_{n,n} + \lambda)$, (19) becomes

$$\begin{aligned} \dot{V}_n \leq & -\sum_{i=1}^{n-1} \left(r_{i,i} - \bar{r}_{i+1,1} - \sum_{j=i+1}^n \bar{r}_{j,i} \right) |\xi_i|^{p(t)+1} \\ & - r_{n,n} |\xi_n|^{p(t)+1} + \xi_n \left([u]^{p(t)} - [x_{n+1}^*]^{p(t)} \right). \end{aligned} \quad (20)$$

3.3. Output Feedback Controller Design of System (1). For system (1), since output function is unknown, the state x_1 is exactly unknown. We construct the output-driven nonlinear reduced-order observer

$$\begin{aligned} \dot{\hat{z}}_i &= -l_i \left([\hat{x}_i]^{p(t)} + \hat{\phi}_{i-1} \right), \\ \hat{x}_i &= \hat{z}_i + l_i \hat{x}_{i-1}, \quad i = 2, \dots, n, \end{aligned} \quad (21)$$

where $\hat{x}_1 = y$, $\hat{\phi}_{i-1} := \phi_{i-1}(t, \hat{x}_1, \dots, \hat{x}_{i-1})$, $i = 2, \dots, n$, and the observer gains $l_i \geq 1$, $i = 2, \dots, n$, are constants to be determined.

Remark 10. Compared with the observer in [29], since the unknown output function $h(\cdot)$ causes x_1 to be exactly

unknown, a new output-driven nonlinear reduced-order observer (21) is designed without using x_1 but y to rebuild unmeasured states. Moreover, not all but the first $n - 1$ nonlinearities are used in the design of observer. For a second-order system Example 4.1 in [29], we can design an observer without using nonlinearity; see the simulation example in this paper for the details.

Based on $x_{n+1}^* = -\beta_n \xi_n = -\sum_{i=1}^n \beta_{i \sim n} x_i$, an output feedback controller is designed as

$$u = -\sum_{i=1}^n \beta_{i \sim n} \hat{x}_i = -\beta_{1 \sim n} y - \sum_{i=2}^n \beta_{i \sim n} \hat{x}_i. \quad (22)$$

Define the error

$$e_i = z_i - \hat{z}_i, \quad z_i = x_i - l_i x_{i-1}, \quad i = 2, \dots, n. \quad (23)$$

By (1), (21), and (23), one has

$$\begin{aligned} \dot{e}_i &= -l_i \left([x_i]^{p(t)} - [\hat{x}_i]^{p(t)} \right) + [x_{i+1}]^{p(t)} + \phi_i \\ &\quad - l_i \left(\phi_{i-1} - \hat{\phi}_{i-1} \right), \quad i = 2, \dots, n-1, \\ \dot{e}_n &= -l_n \left([x_n]^{p(t)} - [\hat{x}_n]^{p(t)} \right) + [u]^{p(t)} + \phi_n \\ &\quad - l_n \left(\phi_{n-1} - \hat{\phi}_{n-1} \right). \end{aligned} \quad (24)$$

For the Lyapunov function $U(e) = \sum_{i=2}^n (1/2)e_i^2$, it follows from (21), (23), and (24) that

$$\begin{aligned} \dot{U} &= -\sum_{i=2}^n l_i e_i \left([z_i + l_i x_{i-1}]^{p(t)} - [\hat{z}_i + l_i x_{i-1}]^{p(t)} \right) \\ &\quad - \sum_{i=2}^n l_i e_i \left([\hat{z}_i + l_i x_{i-1}]^{p(t)} - [\hat{z}_i + l_i \hat{x}_{i-1}]^{p(t)} \right) \\ &\quad + \sum_{i=2}^n e_i [x_{i+1}]^{p(t)} + e_n [u]^{p(t)} + \sum_{i=2}^n e_i \phi_i \\ &\quad - \sum_{i=2}^n l_i e_i \left(\phi_{i-1} - \hat{\phi}_{i-1} \right). \end{aligned} \quad (25)$$

By (23), Lemma 5, and the fact that $1 \leq \underline{p} \leq p(t) \leq \bar{p}$, we obtain for $i = 2, \dots, n$,

$$\begin{aligned} & -l_i e_i \left([z_i + l_i x_{i-1}]^{p(t)} - [\hat{z}_i + l_i x_{i-1}]^{p(t)} \right) \\ &= -l_i \left((z_i + l_i x_{i-1}) - (\hat{z}_i + l_i x_{i-1}) \right) \\ &\quad \cdot \left([z_i + l_i x_{i-1}]^{p(t)} - [\hat{z}_i + l_i x_{i-1}]^{p(t)} \right) \\ &\leq -2^{1-\bar{p}} l_i |e_i|^{p(t)+1}. \end{aligned} \quad (26)$$

Next, we give the estimate of the others on the right-hand side of (25) by Propositions 11–14, whose proofs are included in the Appendix.

Proposition 11. *There is a positive constant b_1 , and positive constants $c_{1,i}$, $i = 2, \dots, n$, dependent on b_1 such that*

$$\begin{aligned} & \sum_{i=2}^{n-1} e_i [x_{i+1}]^{p(t)} + \sum_{i=2}^n e_i \phi_i \\ & \leq b_1 \sum_{i=1}^n |\xi_i|^{p(t)+1} + \sum_{i=2}^n c_{1,i} |e_i|^{p(t)+1}. \end{aligned} \quad (27)$$

Proposition 12. *There is a positive constant b_2 , a positive constant α_n dependent on b_2 , and positive constants α_i , $i = 1, \dots, n-1$, dependent on $b_2, l_{i+1}, \dots, l_n$ such that*

$$\begin{aligned} & - \sum_{i=2}^n l_i e_i \left([\tilde{z}_i + l_i x_{i-1}]^{p(t)} - [\tilde{z}_i + l_i \hat{x}_{i-1}]^{p(t)} \right) \\ & \leq b_2 \sum_{i=1}^n |\xi_i|^{p(t)+1} + \alpha_n |e_n|^{p(t)+1} \\ & \quad + \sum_{i=2}^{n-1} \alpha_i (l_{i+1}, \dots, l_n) |e_i|^{p(t)+1} \\ & \quad + \alpha_1 (l_2, \dots, l_n) |x_1 - \hat{x}_1|^{p(t)+1}. \end{aligned} \quad (28)$$

Proposition 13. *There is a positive constant b_3 , a positive constant γ_n dependent on b_3 , and positive constants γ_i , $i = 1, \dots, n-1$, dependent on $b_3, l_{i+1}, \dots, l_n$ such that*

$$\begin{aligned} & - \sum_{i=2}^n l_i e_i (\phi_{i-1} - \hat{\phi}_{i-1}) \\ & \leq b_3 \sum_{i=1}^n |\xi_i|^{p(t)+1} + \gamma_n |e_n|^{p(t)+1} \\ & \quad + \sum_{i=2}^{n-1} \gamma_i (l_{i+1}, \dots, l_n) |e_i|^{p(t)+1} \\ & \quad + \gamma_1 (l_2, \dots, l_n) |x_1 - \hat{x}_1|^{p(t)+1}. \end{aligned} \quad (29)$$

Proposition 14. *There is a positive constant b_4 , a positive constant σ_n dependent on b_4 , and positive constants σ_i , $i = 1, \dots, n-1$, dependent on $l_{i+1}, \dots, l_n$ such that*

$$\begin{aligned} e_n [u]^{p(t)} & \leq b_4 |\xi_n|^{p(t)+1} + \sigma_n |e_n|^{p(t)+1} \\ & \quad + \sum_{i=2}^{n-1} \sigma_i (l_{i+1}, \dots, l_n) |e_i|^{p(t)+1} \\ & \quad + \sigma_1 (l_2, \dots, l_n) |x_1 - \hat{x}_1|^{p(t)+1}. \end{aligned} \quad (30)$$

To estimate the term $\xi_n ([u]^{p(t)} - [x_{n+1}^*]^{p(t)})$ in (20), we give the following proposition whose proof is also included in the Appendix.

Proposition 15. *There is a positive constant b_5 , a positive constant ω_n dependent on b_5 , and positive constants ω_i , $i = 1, \dots, n-1$, dependent on $b_5, l_{i+1}, \dots, l_n$ such that*

$$\begin{aligned} & \xi_n \left([u]^{p(t)} - [x_{n+1}^*]^{p(t)} \right) \\ & \leq b_5 |\xi_n|^{p(t)+1} + \omega_n |e_n|^{p(t)+1} \\ & \quad + \sum_{i=2}^{n-1} \omega_i (l_{i+1}, \dots, l_n) |e_i|^{p(t)+1} \\ & \quad + \omega_1 (l_2, \dots, l_n) |x_1 - \hat{x}_1|^{p(t)+1}. \end{aligned} \quad (31)$$

Define $T(\xi, e) = V_n(\xi) + U(e) = \sum_{i=1}^n (1/2) \xi_i^2 + \sum_{i=2}^n (1/2) e_i^2$. By (20), (25), (26), and Propositions 11–15, one obtains

$$\begin{aligned} \dot{T} & \leq - \sum_{i=1}^{n-1} \left(r_{i,i} - \tilde{r}_{i+1,1} - \sum_{j=i+1}^n \bar{r}_{j,i} - (b_1 + b_2 + b_3) \right) \\ & \quad \cdot |\xi_i|^{p(t)+1} - (r_{n,n} - (b_1 + b_2 + b_3 + b_4 + b_5)) \\ & \quad \cdot |\xi_n|^{p(t)+1} - (2^{1-\bar{p}} l_n - \theta_n) |e_n|^{p(t)+1} \\ & \quad - \sum_{i=2}^{n-1} (2^{1-\bar{p}} l_i - \theta_i (l_{i+1}, \dots, l_n)) |e_i|^{p(t)+1} \\ & \quad + \theta_1 (l_2, \dots, l_n) |x_1 - \hat{x}_1|^{p(t)+1}, \end{aligned} \quad (32)$$

where $\theta_i = \alpha_i + \gamma_i + \sigma_i + \omega_i + c_{1,i}$ with $c_{1,1} = 0$, $i = 1, \dots, n$, it is easy to see that θ_n is a positive constant, and θ_i , $i = 1, \dots, n-1$, are positive constants dependent on $l_{i+1}, \dots, l_n$. For some positive constants $b_0, \tilde{b}_1, \dots, \tilde{b}_n$, design $r_{1,1}, \dots, r_{n,n}$ such that

$$\begin{aligned} r_{1,1} & \geq \max \left\{ \tilde{r}_{2,1} + \sum_{j=2}^n \bar{r}_{j,1} + b_1 + b_2 + b_3 + b_0 + \tilde{b}_1, 1 \right\}, \\ r_{i,i} & \geq \max \left\{ \tilde{r}_{i+1,1} + \sum_{j=i+1}^n \bar{r}_{j,i} + b_1 + b_2 + b_3 + \tilde{b}_i, 1 \right\}, \\ & i = 2, \dots, n-1, \\ r_{n,n} & \geq \max \{ b_1 + b_2 + b_3 + b_4 + b_5 + \tilde{b}_n, 1 \}, \end{aligned} \quad (33)$$

and for some positive constants $\tilde{d}_2, \dots, \tilde{d}_n$, choose $l_2, \dots, l_n$ as

$$\begin{aligned} l_n & \geq \max \{ 2^{\bar{p}-1} (\tilde{d}_n + \theta_n), 1 \}, \\ l_i & \geq \max \{ 2^{\bar{p}-1} (\tilde{d}_i + \theta_i (l_{i+1}, \dots, l_n)), 1 \}, \\ & i = n-1, \dots, 2. \end{aligned} \quad (34)$$

By (33) and (34), (32) becomes

$$\begin{aligned} \dot{T} \leq & -b_0 |\xi_1|^{p(t)+1} - \sum_{i=1}^n \tilde{b}_i |\xi_i|^{p(t)+1} - \sum_{i=2}^n \tilde{d}_i |e_i|^{p(t)+1} \\ & + \theta_1 (l_2, \dots, l_n) |x_1 - \hat{x}_1|^{p(t)+1}. \end{aligned} \quad (35)$$

3.4. Stability and Convergence Analysis

Theorem 16. *If Assumption 8 holds for system (1), there is a maximal open sector $\Omega = (1 - \bar{\rho}, 1 + \bar{\rho})$ with $\bar{\rho}$ being a positive constant, as long as the unknown output function $h(\cdot)$ belongs to any closed sector included in Ω , under the output feedback controller (21) and (22):*

(i) *the solutions of the closed-loop system (1), (21), (22) are well-defined on $[0, +\infty)$,*

(ii) *the equilibrium point $x = 0$ is globally uniformly asymptotically stable.*

Proof. (i) We firstly give the choice of $r_{1,1}, \dots, r_{n,n}, \beta_1, \dots, \beta_n$. At Step 1, for any given $\tilde{r}_{2,1}, \tilde{r}_{2,1}, \dots, \tilde{r}_{n,1}, b_1, b_2, b_3, b_0, \tilde{b}_1$, by (33), we can choose $r_{1,1}$; then β_1 can be calculated. At step 2, for any given $\tilde{r}_{3,1}, \tilde{r}_{3,2}, \dots, \tilde{r}_{n,2}, b_1, b_2, b_3, \tilde{b}_2$, by (33), we can choose $r_{2,2}$; then β_2 can be calculated when $\lambda_{2,1}, \lambda_{2,2}$ have been obtained by (15) and (16). One by one, at Step n, for any given $b_1, b_2, b_3, b_4, b_5, \tilde{b}_n$, by (33), we can choose $r_{n,n}$; then β_n can be calculated.

Secondly, we give the choice of $l_n, \dots, l_2$. For the given b_1, b_2, b_3, b_4, b_5 , we can get $c_{1,n}, \alpha_n, \gamma_n, \sigma_n, \omega_n$. Then for any given $\tilde{d}_n$, by (34), we can choose l_n . Next, for the given b_1, b_2, b_3, b_5 , one get $c_{1,n-1}, \alpha_{n-1}(l_n), \gamma_{n-1}(l_n), \sigma_{n-1}(l_n), \omega_{n-1}(l_n)$. Then for any given $\tilde{d}_{n-1}$, l_{n-1} can be chosen by (34). In turn, we can get $l_{n-2}, \dots, l_2$.

Finally, we give the sector of output function. Denote $\kappa = (\tilde{r}_{2,1}, \dots, \tilde{r}_{n,1}, \tilde{r}_{2,1}, \dots, \tilde{r}_{n,1}, \dots, \tilde{r}_{n,n-2}, \tilde{r}_{n,n-1}, b_1, \dots, b_5, b_0, \tilde{b}_1, \dots, \tilde{b}_n, \tilde{d}_2, \dots, \tilde{d}_n)$, $\Omega_l = \{l = (l_2, \dots, l_n) \mid r_{1,1} \geq \max\{\tilde{r}_{2,1} + \sum_{j=2}^n \tilde{r}_{j,1} + b_1 + b_2 + b_3 + b_0 + \tilde{b}_1, 1\}, r_{i,i} \geq \max\{\tilde{r}_{i+1,1} + \sum_{j=i+1}^n \tilde{r}_{j,i} + b_1 + b_2 + b_3 + \tilde{b}_i, 1\}, i = 2, \dots, n-1, r_{n,n} \geq \max\{b_1 + b_2 + b_3 + b_4 + b_5 + \tilde{b}_n, 1\}, l_n \geq \max\{2^{\bar{p}-1}(\tilde{d}_n + \theta_n), 1\}, l_i \geq \max\{2^{\bar{p}-1}(\tilde{d}_i + \theta_i(l_{i+1}, \dots, l_n)), 1\}, i = n-1, \dots, 2\}$, and

$$\begin{aligned} \rho = \rho(l_2, \dots, l_n) = \min & \left\{ \left(\frac{b_0}{2\theta_1(l_2, \dots, l_n)} \right)^{1/(\bar{p}+1)}, \right. \\ & \left. \left(\frac{b_0}{2\theta_1(l_2, \dots, l_n)} \right)^{1/(\underline{p}+1)} \right\}. \end{aligned} \quad (36)$$

In fact, one can determine the supremum of ρ ; that is, select constant

$$\bar{\rho} = \sup_{\kappa \in R^+, l \in \Omega_l} \rho(l_2, \dots, l_n), \quad (37)$$

as the supremum of ρ . Hence, the maximal open sector of output function is $\Omega = (1 - \bar{\rho}, 1 + \bar{\rho})$. When output function

$h(x_1)$ in (1) belongs to any closed sector $[1 - \bar{\rho} + \epsilon, 1 + \bar{\rho} - \epsilon] \subset \Omega$ with $0 < \epsilon < \bar{\rho}$, by Definition 1,

$$\begin{aligned} & (h(x_1) - (1 - \bar{\rho} + \epsilon)x_1)(h(x_1) - (1 + \bar{\rho} - \epsilon)x_1) \\ & \leq 0 \implies \\ & (h(x_1) - (1 + \bar{\rho} - \epsilon)x_1)x_1 \leq 0 \\ & \leq (h(x_1) - (1 - \bar{\rho} + \epsilon)x_1)x_1 \implies \\ & (1 - \bar{\rho} + \epsilon)x_1^2 \leq h(x_1)x_1 \leq (1 + \bar{\rho} - \epsilon)x_1^2 \implies \\ & -(\bar{\rho} - \epsilon)x_1^2 \leq (h(x_1) - x_1)x_1 \leq (\bar{\rho} - \epsilon)x_1^2 \implies \\ & |x_1 - h(x_1)| \leq (\bar{\rho} - \epsilon)|x_1|, \end{aligned} \quad (38)$$

from which one has

$$\begin{aligned} |x_1 - \hat{x}_1|^{p(t)+1} &= |\xi_1 - h(\xi_1)|^{p(t)+1} \\ &\leq (\bar{\rho} - \epsilon)^{p(t)+1} |\xi_1|^{p(t)+1}. \end{aligned} \quad (39)$$

With the help of $\underline{p} + 1 \leq p(t) + 1 \leq \bar{p} + 1$, (36), and (37), we derive

$$(\bar{\rho} - \epsilon)^{p(t)+1} < \rho^{p(t)+1} \leq \rho^{\bar{p}+1} + \rho^{\underline{p}+1} \leq \frac{b_0}{\theta_1(l_2, \dots, l_n)}. \quad (40)$$

From (39) and (40), (35) becomes

$$\dot{T} \leq -\sum_{i=1}^n \tilde{b}_i |\xi_i|^{p(t)+1} - \sum_{i=2}^n \tilde{d}_i |e_i|^{p(t)+1}. \quad (41)$$

Motivated by [29], with the help of $\underline{p} + p(t) + 1 \leq \underline{p} + \bar{p} + 1 \leq p(t) + 1 + \bar{p}$, one obtains

$$\begin{aligned} |\xi_i|^{p+\bar{p}+1} &\leq |\xi_i|^{p(t)+1} (|\xi_i|^{\underline{p}} + |\xi_i|^{\bar{p}}) \implies \\ |\xi_i|^{p(t)+1} &\geq \frac{|\xi_i|^{\bar{p}+1}}{1 + |\xi_i|^{\bar{p}-\underline{p}}}, \\ & i = 1, \dots, n, \\ |e_i|^{p+\bar{p}+1} &\leq |e_i|^{p(t)+1} (|e_i|^{\underline{p}} + |e_i|^{\bar{p}}) \implies \\ |e_i|^{p(t)+1} &\geq \frac{|e_i|^{\bar{p}+1}}{1 + |e_i|^{\bar{p}-\underline{p}}}, \\ & i = 2, \dots, n, \end{aligned} \quad (42)$$

from which (41) becomes

$$\dot{T} \leq -\left(\sum_{i=1}^n \frac{\tilde{b}_i |\xi_i|^{\bar{p}+1}}{1 + |\xi_i|^{\bar{p}-\underline{p}}} + \sum_{i=2}^n \frac{\tilde{d}_i |e_i|^{\bar{p}+1}}{1 + |e_i|^{\bar{p}-\underline{p}}} \right) =: -W(\xi, e). \quad (43)$$

It is easy to see that $W(\xi, e)$ is a continuous and positive function with respect to (ξ, e) .

FIGURE 1: The responses of the closed-loop system (46), (47) with $h(x_1) = x_1$.

By the transformations (11), system (1) can be transformed into a ξ -system:

$$\dot{\xi}_i(t) = f_i(t, \xi(t), u(t)), \quad i = 1, \dots, n, \quad (44)$$

where $f_i(\cdot)$ is a continuous function with $f_i(t, 0) = 0$. Denote $\bar{\omega} = (\xi_1, \dots, \xi_n, e_2, \dots, e_n)^\top$, by the existence and continuation of the solution, the solution $\bar{\omega}(t)$ of $\bar{\omega}$ -system (24), (44) is defined on $[0, t_m)$ with $0 < t_m \leq +\infty$.

Due to $T(\bar{\omega}) = \sum_{i=1}^n (1/2)\xi_i^2 + \sum_{i=2}^n (1/2)e_i^2$, it is obvious that $T(\bar{\omega})$ is a continuously differentiable, positive definite, and radially unbounded function. By Lemma 6, there are class $\mathcal{K}_\infty$ functions $\pi_1(\cdot)$ and $\pi_2(\cdot)$ such that

$$\pi_1(\|\bar{\omega}\|) \leq T(\bar{\omega}) \leq \pi_2(\|\bar{\omega}\|), \quad \forall \bar{\omega} \in R^{2n-1}. \quad (45)$$

Since $\pi_1(\cdot)$ is a class $\mathcal{K}_\infty$ function, then for any $\varepsilon > 0$, one can always find a $\delta = \delta(\varepsilon)$ with $\delta > \varepsilon > 0$ such that $\pi_2(\varepsilon) \leq \pi_1(\delta)$. If $\|\bar{\omega}(0)\| < \varepsilon$, by (43) and (45), $\pi_1(\|\bar{\omega}(t)\|) \leq T(\bar{\omega}(t)) \leq T(\bar{\omega}(0)) \leq \pi_2(\|\bar{\omega}(0)\|) \leq \pi_2(\varepsilon) \leq \pi_1(\delta)$ for all $t \in [0, t_m)$, which means that $\|\bar{\omega}(t)\| \leq \delta$ for all $t \in [0, t_m)$. Hence, t_m is not an escape time; that is, $\bar{\omega}(t)$ is well-defined on $[0, +\infty)$, so is $x(t)$.

(ii) Since $t_m = +\infty$, from (43), (45), and Lemma 7, we know that the equilibrium point $\bar{\omega} = 0$ of $\bar{\omega}$ -system (24), (44) is globally uniformly asymptotically stable. By the continuity of $x_i^*(\xi_{i-1})$ on ξ_{i-1} and $x_i^*(0) = 0$, it is easy to recursively prove that the equilibrium point $x = 0$ of the closed-loop system (1), (21), (22) is globally uniformly asymptotically stable. $\square$

Remark 17. Since the power of system is time-varying, a new function (36) is given to acquire its supremum, which is used to determine the maximal open sector of output function. Moreover, a rigorous analysis is given to establish the stability of the closed-loop system.

4. Simulation Example

Consider the second-order nonlinear system in [29]

$$\begin{aligned} \dot{x}_1 &= [x_2]^{2+\cos(t)}, \\ \dot{x}_2 &= [u]^{2+\cos(t)} + x_1^{1/3} |x_2|^{\cos(t)+5/3}, \\ y &= h(x_1). \end{aligned} \quad (46)$$

As discussed in [29], Assumption 8 holds and $\underline{p} = 1$, $\bar{p} = 3$. Following the design process as in Section 3 in this paper, we can design output feedback controller without using the nonlinearity $x_1^{1/3} |x_2|^{\cos(t)+5/3}$,

$$\begin{aligned} \dot{\hat{z}}_2 &= -l_2 [\hat{x}_2]^{p(t)}, \quad \hat{x}_2 = \hat{z}_2 + l_2 y, \\ u &= -\beta_1 \beta_2 y - \beta_2 \hat{x}_2. \end{aligned} \quad (47)$$

Choose $\kappa = (\tilde{r}_{2,1}, \tilde{r}_{2,1}, b_1, b_2, b_4, b_5, b_0, \tilde{b}_1, \tilde{b}_2, \tilde{d}_2) = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.4, 0.1, 0.1, 0.4)$. By choosing $r_{1,1} = 1, r_{2,2} = 1$, we obtain $\beta_1 = 1, \beta_2 = 9.63$, and choose $l = l_2 = 12$.

When $h(x_1) = x_1$ as in [29], it means that we do not need to give the sector of the output function. In simulation, under the output feedback controller (47), by selecting the initial conditions $x_1(0) = -0.1, x_2(0) = 0.3, \hat{z}_2(0) = 0$, Figure 1 demonstrates the effectiveness of control scheme.

When $h(x_1) \neq x_1$, the design method in [29] is inapplicable. Following the design process as in Section 3 in this paper, we can get a sector of output function, that is, $[1 - \rho, 1 + \rho]$ with $\rho = \rho(l_2) = \min\{(b_0/2\theta_1(l_2))^{1/(\bar{p}+1)}, (b_0/2\theta_1(l_2))^{1/(\underline{p}+1)}\} = 0.0131$. In simulation, we choose $h(x_1) = x_1 + 0.01|x_1|$ which is only continuous but not differentiable; $h(x_1)$ belongs to sector $[0.9869, 1.0131]$ obviously. By selecting the initial conditions $x_1(0) = 0.1, x_2(0) = -0.3, \hat{z}_2(0) = 0.01$, Figure 2 illustrates the validness of control scheme.

FIGURE 2: The responses of the closed-loop system (46), (47) with $h(x_1) = x_1 + 0.01|x_1|$.

5. Conclusions

In this paper, output feedback stabilization of nonlinear systems with a time-varying power and unknown continuous output function is studied. By constructing a new nonlinear reduced-order observer together with adding a power integrator method, a new function to determine the maximal open sector Ω of output function is given. As long as output function belongs to any closed sector included in Ω , an output feedback controller is constructed to guarantee that the equilibrium point of the closed-loop system is globally uniformly asymptotically stable.

For the future work, an interesting problem is for more general nonlinear systems with time-varying powers, that is, nonlinear systems with different time-varying powers or under the weaker condition on nonlinearities, can we design a stable output feedback controller?

Appendix

Proof of Proposition 11. By (8), (11), and Lemmas 2 and 3, we obtain

$$\begin{aligned} & \sum_{i=2}^{n-1} e_i [x_{i+1}]^{p(t)} + \sum_{i=2}^n e_i \phi_i \\ & \leq \sum_{i=2}^{n-1} |e_i| |x_{i+1}|^{p(t)} \end{aligned}$$

$$\begin{aligned} & + \sum_{i=2}^n \bar{c} |e_i| \left(|x_1|^{p(t)} + \cdots + |x_i|^{p(t)} \right) \\ & \leq b_1 \sum_{i=1}^n |\xi_i|^{p(t)+1} + \sum_{i=2}^n c_{1,i} |e_i|^{p(t)+1}, \end{aligned} \quad (\text{A.1})$$

where $b_1 > 0$ is a constant, and $c_{1,i} > 0, i = 2, \dots, n$, are constants dependent on b_1 . $\square$

Proof of Proposition 12. By (11), (21), and (23), one has for $i = 2, \dots, n$,

$$\begin{aligned} x_i - \hat{x}_i &= e_i + l_i (x_{i-1} - \hat{x}_{i-1}) \\ &= e_i + \sum_{j=2}^{i-1} l_{(j+1) \sim i} e_j + l_{2 \sim i} (x_1 - \hat{x}_1), \end{aligned} \quad (\text{A.2})$$

$$\hat{z}_i + l_i x_{i-1} = x_i - e_i = \xi_i - \beta_{i-1} \xi_{i-1} - e_i, \quad (\text{A.3})$$

from which, by Lemmas 2–4, we derive

$$\begin{aligned} & -l_i e_i \left([\hat{z}_i + l_i x_{i-1}]^{p(t)} - [\hat{z}_i + l_i \hat{x}_{i-1}]^{p(t)} \right) \leq l_i |e_i| \\ & \quad \cdot \bar{p} (2^{\bar{p}-3} + 1) \left(|l_i (x_{i-1} - \hat{x}_{i-1})|^{p(t)} \right. \\ & \quad \left. + |l_i (x_{i-1} - \hat{x}_{i-1})| |\hat{z}_i + l_i x_{i-1}|^{p(t)-1} \right) \leq l_i |e_i| \\ & \quad \cdot \bar{p} (2^{\bar{p}-3} + 1) \left(|l_i e_{i-1} + l_{(i-1) \sim i} e_{i-2} + \cdots + l_{3 \sim i} e_2 \right. \end{aligned}$$

$$\begin{aligned}
& + l_{2 \sim i} (x_1 - \hat{x}_1) |^{p(t)} + |l_i e_{i-1} + l_{(i-1) \sim i} e_{i-2} + \dots \\
& + l_{3 \sim i} e_2 + l_{2 \sim i} (x_1 - \hat{x}_1) | |\xi_i - \beta_{i-1} \xi_{i-1} - e_i|^{p(t)-1} \\
& \leq \frac{b_2}{2} (|\xi_{i-1}|^{p(t)+1} + |\xi_i|^{p(t)+1}) + c_{2,i} |e_i|^{p(t)+1} \\
& + \sum_{j=2}^{i-1} \alpha_{i,j} (l_{j+1}, \dots, l_i) |e_j|^{p(t)+1} + \alpha_{i,1} (l_2, \dots, l_i) |x_1 \\
& - \hat{x}_1|^{p(t)+1}, \\
\end{aligned} \tag{A.4}$$

where $e_1 = 0$, $b_2 > 0$ is a constant, $c_{2,i} > 0$ is a constant dependent on b_2 , and $\alpha_{i,j} > 0$, $j = 1, \dots, i-1$, are constants dependent on $b_2, l_{j+1}, \dots, l_i$. From (A.4), it follows that

$$\begin{aligned}
& - \sum_{i=2}^n l_i e_i (\tilde{z}_i + l_i x_{i-1})^{p(t)} \\
& \leq b_2 \sum_{i=1}^n |\xi_i|^{p(t)+1} + \alpha_n |e_n|^{p(t)+1}
\end{aligned}$$

where $\alpha_j = \sum_{i=j+1}^n \alpha_{i,j} (l_{j+1}, \dots, l_i) + c_{2,j}$ with $c_{2,1} = 0$, $j = 1, \dots, n$, that is, $\alpha_n > 0$ is a constant dependent on b_2 , and $\alpha_j > 0$, $j = 1, \dots, n-1$, are constants dependent on $b_2, l_{j+1}, \dots, l_n$. $\square$

Proof of Proposition 13. By (A.2), Assumption 8, and Lemmas 2 and 3, one has

$$\begin{aligned}
|\phi_1(t, x_1) - \phi_1(t, \hat{x}_1)| & \leq c \sum_{j=1}^N (|x_1 - \hat{x}_1|^{a_{1,j}(t)} \\
& \cdot (|x_1|^{b_{1,j}(t)} + |x_1 - (x_1 - \hat{x}_1)|^{b_{1,j}(t)})) \\
& \leq \hat{c} \sum_{j=1}^N (|x_1 - \hat{x}_1|^{a_{1,j}(t)} |x_1|^{b_{1,j}(t)} + |x_1 - \hat{x}_1|^{p(t)}),
\end{aligned} \tag{A.6}$$

and for $i = 2, \dots, n$,

$$\begin{aligned}
|\phi_i(t, x_1, \dots, x_i) - \phi_i(t, \hat{x}_1, \dots, \hat{x}_i)| & \leq c \sum_{j=1}^N \left(\sum_{k=1}^i (|x_k - \hat{x}_k|^{a_{k,j}(t)} \sum_{q=1}^i (|x_q|^{b_{k,j}(t)} + |\hat{x}_q|^{b_{k,j}(t)})) \right) \leq c \sum_{j=1}^N (|x_1 - \hat{x}_1|^{a_{1,j}(t)} \\
& \cdot (|x_1|^{b_{1,j}(t)} + |x_1 - (x_1 - \hat{x}_1)|^{b_{1,j}(t)}) + |x_1 - \hat{x}_1|^{a_{1,j}(t)} \sum_{q=2}^i (|x_q|^{b_{1,j}(t)} \\
& + |x_q - (e_q + l_q e_{q-1} + l_{(q-1) \sim q} e_{q-2} + \dots + l_{3 \sim q} e_2 + l_{2 \sim q} (x_1 - \hat{x}_1))|^{b_{1,j}(t)}) \\
& + \sum_{k=2}^i \left(|e_k + l_k e_{k-1} + l_{(k-1) \sim k} e_{k-2} + \dots + l_{3 \sim k} e_2 + l_{2 \sim k} (x_1 - \hat{x}_1)|^{a_{k,j}(t)} (|x_1|^{b_{k,j}(t)} + |x_1 - (x_1 - \hat{x}_1)|^{b_{k,j}(t)}) \right. \\
& \left. + |e_k + l_k e_{k-1} + l_{(k-1) \sim k} e_{k-2} + \dots + l_{3 \sim k} e_2 + l_{2 \sim k} (x_1 - \hat{x}_1)|^{a_{k,j}(t)} \right. \\
& \cdot \sum_{q=2}^i (|x_q|^{b_{k,j}(t)} + |x_q - (e_q + l_q e_{q-1} + l_{(q-1) \sim q} e_{q-2} + \dots + l_{3 \sim q} e_2 + l_{2 \sim q} (x_1 - \hat{x}_1))|^{b_{k,j}(t)})) \leq \hat{c} \sum_{j=1}^N (|x_1 - \hat{x}_1|^{a_{1,j}(t)} \\
& \cdot \sum_{q=1}^i |x_q|^{b_{1,j}(t)} + \bar{\gamma}_{i,i} |e_i|^{p(t)} + \sum_{k=2}^{i-1} \bar{\gamma}_{i,k} (l_{k+1}, \dots, l_i) |e_k|^{p(t)} + \bar{\gamma}_{i,1} (l_2, \dots, l_i) |x_1 - \hat{x}_1|^{p(t)} \\
& \left. + \sum_{k=2}^i \left(|e_k + l_k e_{k-1} + l_{(k-1) \sim k} e_{k-2} + \dots + l_{3 \sim k} e_2 + l_{2 \sim k} (x_1 - \hat{x}_1)|^{a_{k,j}(t)} \sum_{q=1}^i |x_q|^{b_{k,j}(t)} \right) \right),
\end{aligned} \tag{A.7}$$

where $\hat{c} > 0$ is a constant, $\bar{\gamma}_{i,i}$ is a positive constant, and $\bar{\gamma}_{i,k}, k = 1, \dots, i-1$, are positive constants dependent on $l_{k+1}, \dots, l_i$. From (11), (A.6), (A.7), and Lemmas 2 and 3, it follows that for $i = 2, \dots, n$,

$$\begin{aligned} & -l_i e_i (\phi_{i-1} - \hat{\phi}_{i-1}) \\ & \leq l_i |e_i| |\phi_{i-1}(t, x_1, \dots, x_{i-1}) - \phi_{i-1}(t, \hat{x}_1, \dots, \hat{x}_{i-1})| \\ & \leq \frac{b_3}{n} \sum_{j=1}^{i-1} |\xi_j|^{p(t)+1} + c_{3,i} |e_i|^{p(t)+1} \\ & \quad + \sum_{j=2}^{i-1} \gamma_{i,j} (l_{j+1}, \dots, l_i) |e_j|^{p(t)+1} \\ & \quad + \gamma_{i,1} (l_2, \dots, l_i) |x_1 - \hat{x}_1|^{p(t)+1}, \end{aligned} \quad (\text{A.8})$$

where $b_3 > 0$ is a constant, $c_{3,i} > 0$ is a constant dependent on b_3 , and $\gamma_{i,j} > 0, j = 1, \dots, i-1$, are constants dependent on $b_3, l_{j+1}, \dots, l_i$. Similar to (A.6), it follows from (A.8) that

$$\begin{aligned} & -\sum_{i=2}^n l_i e_i (\phi_{i-1} - \hat{\phi}_{i-1}) \\ & \leq b_3 \sum_{i=1}^n |\xi_i|^{p(t)+1} + \gamma_n |e_n|^{p(t)+1} \\ & \quad + \sum_{j=2}^{n-1} \gamma_j (l_{j+1}, \dots, l_n) |e_j|^{p(t)+1} \\ & \quad + \gamma_1 (l_2, \dots, l_n) |x_1 - \hat{x}_1|^{p(t)+1}, \end{aligned} \quad (\text{A.9})$$

where $\gamma_j = \sum_{i=j+1}^n \gamma_{i,j} (l_{j+1}, \dots, l_i) + c_{3,j}$ with $c_{2,1} = 0, j = 1, \dots, n$, that is, $\gamma_n > 0$ is a constant dependent on b_3 , and $\gamma_j > 0, j = 1, \dots, n-1$, are constants dependent on $b_3, l_{j+1}, \dots, l_n$. $\square$

Proof of Proposition 14. By (11), (22), and (A.2), one has

$$\begin{aligned} u &= \sum_{i=1}^n \beta_{i \sim n} (x_i - \hat{x}_i) - \sum_{i=1}^n \beta_{i \sim n} x_i \\ &= -\beta_n \xi_n + \bar{\sigma}_1 (l_2, \dots, l_n) (x_1 - \hat{x}_1) \\ & \quad + \sum_{i=2}^{n-1} \bar{\sigma}_i (l_{i+1}, \dots, l_n) e_i + \bar{\sigma}_n e_n, \end{aligned} \quad (\text{A.10})$$

where $\bar{\sigma}_i = \sum_{j=i}^n \beta_{j \sim n} l_{(i+1) \sim j}, i = 1, \dots, n$, that is, $\bar{\sigma}_n$ is a positive constant, and $\bar{\sigma}_i, i = 1, \dots, n-1$, are positive constants dependent on $l_{i+1}, \dots, l_n$. By (A.10) and Lemmas 2 and 3, we derive

$$\begin{aligned} e_n [u]^{p(t)} &\leq |e_n| |u|^{p(t)} \\ &\leq b_4 |\xi_n|^{p(t)+1} + \sigma_n |e_n|^{p(t)+1} \\ & \quad + \sum_{i=2}^{n-1} \sigma_i (l_{i+1}, \dots, l_n) |e_i|^{p(t)+1} \\ & \quad + \sigma_1 (l_2, \dots, l_n) |x_1 - \hat{x}_1|^{p(t)+1}, \end{aligned} \quad (\text{A.11})$$

where $b_4 > 0$ is a constant, $\sigma_n > 0$ is a constant dependent on b_4 , and $\sigma_i > 0, i = 1, \dots, n-1$, are constants dependent on $l_{i+1}, \dots, l_n$. $\square$

Proof of Proposition 15. By $x_{n+1}^* = \beta_n \xi_n$ and (A.10), one has

$$\begin{aligned} u - x_{n+1}^* &= \bar{\sigma}_1 (l_2, \dots, l_n) (x_1 - \hat{x}_1) \\ & \quad + \sum_{i=2}^{n-1} \bar{\sigma}_i (l_{i+1}, \dots, l_n) e_i + \bar{\sigma}_n e_n, \end{aligned} \quad (\text{A.12})$$

from which, by Lemmas 2–4, we derive

$$\begin{aligned} \xi_n ([u]^{p(t)} - [x_{n+1}^*]^{p(t)}) &\leq |\xi_n| \cdot \bar{p} (2^{\bar{p}-3} + 1) \\ & \quad \cdot (|u - x_{n+1}^*|^{p(t)} + |u - x_{n+1}^*| |x_{n+1}^*|^{p(t)-1}) \\ &\leq b_5 |\xi_n|^{p(t)+1} + \omega_n |e_n|^{p(t)+1} \\ & \quad + \sum_{i=2}^{n-1} \omega_i (l_{i+1}, \dots, l_n) |e_i|^{p(t)+1} + \omega_1 (l_2, \dots, l_n) \\ & \quad \cdot |x_1 - \hat{x}_1|^{p(t)+1}, \end{aligned} \quad (\text{A.13})$$

where $b_5 > 0$ is a constant, $\omega_n > 0$ is a constant dependent on b_5 , and $\omega_i > 0, i = 1, \dots, n-1$, are constants dependent on $b_5, l_{i+1}, \dots, l_n$. $\square$

Data Availability

The data supporting the conclusions of the manuscript are some open access articles that have been properly cited, and the readers can easily obtain these articles to verify the conclusions, replicate the analysis, and conduct secondary analyses. Therefore, we do not create a publicly available data repository.

Conflicts of Interest

The authors declare that they have no competing interests.

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