Research Article

A Triangle Algorithm of Padé-Type Approximant for Two-Dimensional Fredholm Integral Equations of the Second Kind

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The function-valued Padé-type approximation (2DFPTA) is used to solve two-dimensional Fredholm integral equation of the second kind. In order to compute 2DFPTA, a triangle recursive algorithm based on Sylvester identity is proposed. The advantage of this algorithm is that, in the process of calculating 2DFPTA to avoid the calculation of the determinant, it can start from the initial value, from low to high order, and gradually proceeds. Compared with the original two methods, the numerical examples show that the algorithm is effective.

1. Introduction

Consider a two-dimensional Fredholm integral equation of the second kind (2DF-II):

$$f(x, \lambda) = g(x) + \lambda \int_D K(x, y) f(y, \lambda) dy, \quad (1)$$

$$x, y \in D,$$

where D is a bounded domain, $x = (x_1, x_2)$, $y = (y_1, y_2)$, $dy = dy_1 dy_2$, K and f are given functions defined on D , and λ is a real parameter called a characteristic value. It is known that a sequence of approximations leading to an infinite power series can be yielded by successive substitution, which is an iterative procedure, from this kind of integral equation. Set $f(x, \lambda)$ to be analytic and meromorphic in a neighborhood $\bar{D}$ of the origin $\lambda = 0$. Let its Neumann series be given by

$$f(x, \lambda) = \sum_{i=0}^{\infty} c_i(x) \lambda^i, \quad (2)$$

in which $c_i(x)$ are continuous on D and $c_0(x) = g(x)$,

$$c_i(x) = \int_D K(x, y) c_{i-1}(y) dy, \quad \text{for } i = 1, 2, \dots \quad (3)$$

Let $c_i(x), c_j(x) \in L^2[a, b]$ and the inner product and the norm be defined by

$$(c_i, c_j) = \int_D c_i(x) c_j(x) dx, \quad (4)$$

$$\|c_i\| = (c_i, c_i)^{1/2} = \left(\int_D c_i^2(x) dx \right)^{1/2}. \quad (5)$$

Two-dimensional Fredholm integral equation of the second kind (2DF-II) is a useful tool to model many problems arising in fracture mechanics, transportation [1], 2D electromagnetic scattering [2], and computer graphics manipulations [3]. For domain D , some works treat the square case [4–10] and some are based on discrete Galerkin method [5], Monte Carlo methods [6], or piecewise approximating polynomials [7], Nytröm methods based on cubature rules obtained as the tensor product of two-variate Gaussian

rules [8, 10], and meshless method with complex factors [11]. Rational approximation has extensive application in engineering, technology and calculation [12–18]. As efficient rational approximating techniques, a generalized inverse function-valued Padé approximants had been used in solving integral equation [13, 14], and a function-valued Padé-type approximation method was used to solve one and two-dimensional Fredholm integral equation of the second kind [15–18].

In the paper, the function-valued Padé-type approximation (2DFPTA) is used to solve 2DF-II. In order to compute 2DFPTA, a recursive algorithm based on Sylvester identity is proposed. The remainder of this paper is organized as follows. In Section 2, we give the definition of 2DFPTA [18]. In Section 3, we apply Sylvester identity to propose a recursive algorithm to compute 2DFPTA. In Section 4, we compare our algorithm with the two algorithms in [4, 18].

2. Function-Valued Padé-Type Approximants and Its Convergence Analysis

In this section, we will give the definition of the function-valued Padé-type approximation [15–18]. Let $\phi^{(l)} : \mathbf{P} \rightarrow \mathbb{C}$ be a linear functional on the polynomial space $\mathbf{P}$, and define it by

$$\phi^{(l)}(t^i) = c_{l+i}(x_1, x_2), \quad i = 0, 1, \dots, l \in \mathbb{Z}, \quad (6)$$

where $\phi^{(l)}(t^i) = 0$ for $l + i < 0$. We can obtain from linear functional $\phi = \phi^{(0)}$ in (6) that

$$\begin{aligned} \phi((1 - t\lambda)^{-1}) &= \phi(1 + t\lambda + (t\lambda)^2 + \dots) \\ &= \sum_{i=0}^{\infty} c_i(x_1, x_2) \lambda^i = f(x, \lambda). \end{aligned} \quad (7)$$

Let

$$v_n(\lambda) = b_0 + b_1\lambda + \dots + b_n\lambda^n, \quad (8)$$

where $v_n \in \mathbf{P}_n$ is a scalar polynomial of degree n and assume $b_n \neq 0$.

Define the polynomial w with function-valued coefficient by

$$w_m(x, \lambda) = \phi\left(\frac{t^{m-n+1}v_n(t) - \lambda^{m-n+1}v_n(\lambda)}{t - \lambda}\right). \quad (9)$$

Note that ϕ acts on P and $w_m(x, \lambda)$ is a function-valued polynomial in λ of degree m .

Set

$$\tilde{v}_n(\lambda) = \lambda^n v_n(\lambda^{-1}) = \sum_{j=0}^n b_j \lambda^{n-j}. \quad (10)$$

$$\tilde{w}_m(x, \lambda) = \lambda^m w_m(x, \lambda^{-1}). \quad (11)$$

The polynomial v_n is called the generating polynomial of the Padé-type approximation (m/n) . It can be arbitrarily chosen and we have n degrees of freedom.

It is known from (10) that $b_n \neq 0$ implies $\tilde{v}_n(0) \neq 0$. In view of (8), (9), and (11), $\tilde{w}_m(x, \lambda)$ is of the following form:

$$\tilde{w}_m(x, \lambda) = \sum_{j=0}^n b_j \lambda^{n-j} \sum_{i=0}^{m-n+j} c_i(x) \lambda^i = \sum_{i=0}^m \sum_{j=0}^n b_j \tilde{c}_i^j(x) \lambda^i, \quad (12)$$

in which $\tilde{c}_i^j(x) = c_{i-n+j}(x)$. Using (10) and (12), we get

$$\begin{aligned} \tilde{v}_n(\lambda) f(x, \lambda) - \tilde{w}_m(x, \lambda) \\ &= \left(\sum_{j=0}^n b_j \lambda^{n-j} \right) \left(\sum_{i=0}^{\infty} c_i(x) \lambda^i \right) \\ &\quad - \sum_{j=0}^n b_j \lambda^{n-j} \sum_{i=0}^{m-n+j} c_i(x) \lambda^i. \\ &= \sum_{i=0}^{\infty} \left(\sum_{j=0}^n c_{m-n+1+i+j}(x) \right) \lambda^{m+1+i} \\ &= \sum_{i=m+1}^{\infty} \left(\sum_{j=0}^n b_j \tilde{c}_i^j(x) \right) \lambda^i = O(\lambda^{m+1}). \end{aligned} \quad (13)$$

Definition 1. Let $\tilde{w}_m(x, \lambda)$ and $\tilde{v}_n(\lambda)$ be given by (12) and (10), respectively, and then the rational function

$$\begin{aligned} R_{m,n}(x, \lambda) &= \frac{\tilde{w}_m(x, \lambda)}{\tilde{v}_n(\lambda)} \\ &= \frac{\sum_{j=0}^n b_j \lambda^{n-j} \sum_{i=0}^{m-n+j} c_i(x) \lambda^i}{\sum_{j=0}^n b_j \lambda^{n-j}} \\ &= \frac{\sum_{i=0}^m \sum_{j=0}^n b_j \tilde{c}_i^j(x) \lambda^i}{\sum_{j=0}^n b_j \lambda^{n-j}} \end{aligned} \quad (14)$$

is defined as a two-dimensional function-valued Padé-type approximation (2DFPTA) with generating polynomial $v_n(\lambda)$ and is denoted by $(m/n)_f(x, \lambda)$.

According to the relations (9) and (11), we can from [19] obtain the following error formula with the linear functional form:

$$\begin{aligned} f(x, \lambda) - \left(\frac{m}{n} \right)_f(x, \lambda) &= \frac{\lambda^{m+1}}{\tilde{v}_n(\lambda)} \phi^{(m-n+1)} \left(\frac{v_n(t)}{1 - t\lambda} \right) \\ &= \frac{\lambda^{m+1}}{\tilde{v}_n(\lambda)} \phi^{(m-n+1)} \left[v_n(t) \right. \\ &\quad \cdot \left. \left(1 + t\lambda + \dots + t^{n-1} \lambda^{n-1} + \frac{t^n \lambda^n}{1 - t\lambda} \right) \right]. \end{aligned} \quad (15)$$

Note that the error formula (15) is disadvantageous to estimate error numerically, and it implies from (13) that the Padé-type approximation property is

$$f(x, \lambda) - \left(\frac{m}{n} \right)_f(x, \lambda) = O(\lambda^{m+1}), \quad (16)$$

as $\lambda \rightarrow 0$ for $\tilde{v}_n(\lambda) \neq 0$.

In order to compute conveniently, we choose $\max_{x \in D} |\sum_{j=0}^n b_j \tilde{c}_i^j(x)|$ as the maximum absolute value of the coefficient of λ^{m+1} on D in (13). We can also observe from (15) and (16) that if the coefficients are given, the order of the approximation is λ^{m+1} .

3. A Triangle Recursive Algorithm

of $(m/n)_f(x, \lambda)$

As the definition given in Section 2, we find that the key to calculating 2DFPTA is to compute its the generating polynomial $v_n(\lambda)$. If the generator polynomial $v_n(\lambda)$ is determined, we can compute the 2DFPTA of type (m, n) according to (10) and (12). In the section, we will apply Sylvester's identity [20] to propose a three-term recurrence formula for computing $\tilde{v}_n(\lambda)$ and establish a complete recursive algorithm for calculating 2DFPTA.

Lemma 2 (Sylvester identity [20]). *Let*

$$A = \begin{pmatrix} a_{0,1} & a_{0,2} & \cdots & a_{0,n} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \end{pmatrix}, \quad (17)$$

$$B = \begin{pmatrix} a_{0,1} & a_{0,2} & \cdots & a_{0,n-1} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-1} \end{pmatrix},$$

and then it holds that

$$\begin{vmatrix} x_0 & x \\ u & A \end{vmatrix} |B| = |A| \begin{vmatrix} x_0 & x' \\ u' & B \end{vmatrix} - |A'| \begin{vmatrix} x' & x_n \\ B & u'' \end{vmatrix}, \quad (18)$$

where $A' = \begin{pmatrix} u' & B \\ a_{n-1,0} & w \end{pmatrix}$ and

$$\begin{aligned} x &= (x_1, x_2, \dots, x_n), \\ x' &= (x_1, x_2, \dots, x_{n-1}), \\ u &= (a_{0,0}, a_{1,0}, \dots, a_{n-1,0})^T, \\ u' &= (a_{0,0}, a_{1,0}, \dots, a_{n-2,0}), \\ u'' &= (a_{0,n}, a_{1,n}, \dots, a_{n-2,n})^T, \\ w &= (a_{n-1,1}, a_{n-1,2}, \dots, a_{n-1,n-1}). \end{aligned} \quad (19)$$

Set $b_0 = 1$, the generating polynomial $v_n(\lambda) = b_0 + b_1\lambda + \cdots + b_n\lambda^n$.

Lemma 3 (see [15]). *Let the determinant $|A_n| \neq 0$; then the generating polynomial $v_n(\lambda)$ exists and is unique and can be expressed as*

$$v_n(\lambda) = \frac{\left| \frac{1}{\eta} \overrightarrow{\lambda}^{(n)} A_n \right|}{|A_n|}, \quad (20)$$

where

$$A_n = \begin{pmatrix} (c_{m-n+1}, c_{m-n+2}) & (c_{m-n+1}, c_{m-n+3}) & \cdots & (c_{m-n+1}, c_{m+1}) \\ (c_{m-n+1}, c_{m-n+3}) & (c_{m-n+1}, c_{m-n+4}) & \cdots & (c_{m-n+1}, c_{m+2}) \\ \vdots & \vdots & \ddots & \vdots \\ (c_{m-n+1}, c_{m+1}) & (c_{m-n+1}, c_{m+2}) & \cdots & (c_{m-n+1}, c_{m+n}) \end{pmatrix}, \quad (21)$$

$$\overrightarrow{\lambda}^{(n)} = (\lambda, \lambda^2, \dots, \lambda^n),$$

$$\overrightarrow{\eta}^{(n)} = ((c_{m-n+1}, c_{m-n+1}), (c_{m-n+1}, c_{m-n+2}), \dots, (c_{m-n+1}, c_m))^T.$$

Definition 4. For any j th degree polynomial $v_j(\lambda) = b_0 + b_1\lambda + \cdots + b_j\lambda^j$, by the higher-order linear functional $\phi^{(l)}(\lambda^j) = c_{l+j}$, we define the vector inner product with l order as follows:

$$\begin{aligned} [c_l, v_j(\lambda)]_l &= (c_l, \phi^{(l)}(v_j(\lambda))) \\ &= (c_l, \phi^{(l)}(b_0 + b_1\lambda + \cdots + b_j\lambda^j)) \\ &= (c_l, b_0c_l + b_1c_{l+1} + \cdots + b_jc_{l+j}) \\ &= b_0(c_l, c_l) + b_1(c_l, c_{l+1}) + \cdots \\ &\quad + b_j(c_l, c_{l+j}), \end{aligned} \quad (22)$$

where $l = m - n + 1$, $j = 0, 1, 2, \dots$.

To discuss conveniently, we introduce the following notations:

$$G_k^{(i)} = \begin{vmatrix} \lambda^i & \lambda^{i+1} & \cdots & \lambda^{i+k} \\ [c_l, \lambda^i]_l & [c_l, \lambda^{i+1}]_l & \cdots & [c_l, \lambda^{i+k}]_l \\ \vdots & \vdots & \cdots & \vdots \\ [c_l, \lambda^{i+k-1}]_l & [c_l, \lambda^{i+k}]_l & \cdots & [c_l, \lambda^{i+2k-1}]_l \end{vmatrix}, \quad (23)$$

$$D_k^{(i)} = \begin{vmatrix} [c_l, \lambda^{i+1}]_l & [c_l, \lambda^{i+2}]_l & \cdots & [c_l, \lambda^{i+k}]_l \\ [c_l, \lambda^{i+2}]_l & [c_l, \lambda^{i+3}]_l & \cdots & [c_l, \lambda^{i+k+1}]_l \\ \vdots & \vdots & \cdots & \vdots \\ [c_l, \lambda^{i+k}]_l & [c_l, \lambda^{i+k+1}]_l & \cdots & [c_l, \lambda^{i+2k-1}]_l \end{vmatrix}. \quad (24)$$

Let

$$v_k^{(i)} = \frac{G_k^{(i)}}{D_k^{(i)}}, \quad (25)$$

where if $i = 0$, $k = n$ and

$$G_n^{(0)} = \begin{vmatrix} 1 & \vec{\lambda} \\ \vec{\eta}^{(n)} & A_n \end{vmatrix}, \quad (26)$$

$$D_n^{(0)} = |A_n|.$$

From (20) and (25), we conclude that

$$v_n^{(0)} = v_n(\lambda). \quad (27)$$

To compute $v_n^{(0)}$, we give the following theorem. According to Definition 4 and the j th degree polynomial $v_{k-1}^{(i)}(\lambda) = b_0 + b_1\lambda + \dots + b_j\lambda^j$, we gain the vector inner product with $l + k - 1$ order as follows:

$$\begin{aligned} [c_l, v_{k-1}^{(i)}(\lambda)]_{l+k-1} &= (c_l, \phi^{(l+k-1)}(v_{k-1}^{(i)}(\lambda))) \\ &= b_0(c_l, c_{l+k-1}) + b_1(c_l, c_{l+k}) + \dots \\ &\quad + b_j(c_l, c_{l+k-1+j}), \end{aligned} \quad (28)$$

where $l = m - n + 1$, $k = 1, 2, \dots, n$, $i = j - (k - 1)$.

Theorem 5 (three-term recurrence formula). *Let $D_k^{(i)}D_{k-1}^{(i)} \neq 0$ for all $k = 1, 2, \dots, n$, $i = 0, 1, 2, \dots, n$. Then it holds that*

$$v_k^{(i)} = v_{k-1}^{(i)} - \frac{(c_l, \phi^{(l+k-1)}(v_{k-1}^{(i)}))}{(c_l, \phi^{(l+k-1)}(v_{k-1}^{(i+1)}))} v_{k-1}^{(i+1)}, \quad (29)$$

where the initial value is $v_0^{(i)} = \lambda^i$.

Proof. Set $x_0 = \lambda^i$, $x = (\lambda^{i+1}, \lambda^{i+2}, \dots, \lambda^{i+k})$, $u = ([c_1, \lambda^i]_1, [c_2, \lambda^i]_2, \dots, [c_k, \lambda^i]_k)$ and $|A| = D_k^{(i)}$, $|B| = D_{k-1}^{(i)}$. Substitute them into the formula (18) and we conclude that

$$G_k^{(i)} D_{k-1}^{(i)} = G_{k-1}^{(i)} D_k^{(i)} - G_{k-1}^{(i+1)} D_k^{(i-1)}. \quad (30)$$

Due to $D_k^{(i)}D_{k-1}^{(i)} \neq 0$, we have

$$\frac{G_k^{(i)}}{D_k^{(i)}} = \frac{G_{k-1}^{(i)}}{D_{k-1}^{(i)}} - \frac{G_{k-1}^{(i+1)} D_k^{(i-1)}}{D_{k-1}^{(i)} D_k^{(i)}}. \quad (31)$$

By (25) we get $G_{k-1}^{(i+1)} = D_{k-1}^{(i+1)} v_{k-1}^{(i+1)}$ and substitute it into the above formula; then

$$\frac{G_k^{(i)}}{D_k^{(i)}} = \frac{G_{k-1}^{(i)}}{D_{k-1}^{(i)}} - \frac{D_k^{(i-1)} D_{k-1}^{(i+1)} v_{k-1}^{(i+1)}}{D_{k-1}^{(i)} D_k^{(i)}}, \quad (32)$$

that is,

$$v_k^{(i)} = v_{k-1}^{(i)} - \frac{D_k^{(i-1)} D_{k-1}^{(i+1)}}{D_{k-1}^{(i)} D_k^{(i)}} v_{k-1}^{(i+1)}. \quad (33)$$

According to the higher-order linear functional and nature of the determinant, we have

$$(c_l, \phi^{(l+k-1)}(v_{k-1}^{(i)})) = \frac{(-1)^{k-1} D_k^{(i-1)}}{D_{k-1}^{(i)}}, \quad (34)$$

and

$$(c_l, \phi^{(l+k-1)}(v_{k-1}^{(i+1)})) = \frac{(-1)^{k-1} D_k^{(i)}}{D_{k-1}^{(i+1)}}. \quad (35)$$

Dividing (34) by (35), we deduce that

$$\frac{(c_l, \phi^{(l+k-1)}(v_{k-1}^{(i)}))}{(c_l, \phi^{(l+k-1)}(v_{k-1}^{(i+1)}))} = \frac{D_k^{(i-1)} D_{k-1}^{(i+1)}}{D_k^{(i)} D_{k-1}^{(i)}}. \quad (36)$$

From (33) and (36), we obtain that

$$v_k^{(i)} = v_{k-1}^{(i)} - \frac{(c_l, \phi^{(l+k-1)}(v_{k-1}^{(i)}))}{(c_l, \phi^{(l+k-1)}(v_{k-1}^{(i+1)}))} v_{k-1}^{(i+1)}. \quad (37)$$

□

Example 6. Let us choose an example of $n = 6$ to illustrate the process to generate $v_6(\lambda)$. By the three-term recurrence relation (16), the calculation program of $v_6(\lambda)$ can be arranged according to the following bottom-down triangle:

$$\begin{array}{ccccccc} v_0^{(0)} & & & & & & \\ & \searrow & & & & & \\ v_0^{(1)} & \longrightarrow & v_1^{(0)} & & & & \\ & \searrow & & \searrow & & & \\ v_0^{(2)} & \longrightarrow & v_1^{(1)} & \longrightarrow & v_2^{(0)} & & \\ & \searrow & & \searrow & & \searrow & \\ v_0^{(3)} & \longrightarrow & v_1^{(2)} & \longrightarrow & v_2^{(1)} & \longrightarrow & v_3^{(0)} \\ & \searrow & & \searrow & & \searrow & \\ v_0^{(4)} & \longrightarrow & v_1^{(3)} & \longrightarrow & v_2^{(2)} & \longrightarrow & v_3^{(1)} & \longrightarrow & v_4^{(0)} \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ v_0^{(5)} & \longrightarrow & v_1^{(4)} & \longrightarrow & v_2^{(3)} & \longrightarrow & v_3^{(2)} & \longrightarrow & v_4^{(1)} & \longrightarrow & v_5^{(0)} \\ & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\ v_0^{(6)} & \longrightarrow & v_1^{(5)} & \longrightarrow & v_2^{(4)} & \longrightarrow & v_3^{(3)} & \longrightarrow & v_4^{(2)} & \longrightarrow & v_5^{(1)} & \longrightarrow & v_6^{(0)} \end{array} \quad (38)$$

When computing $v_k^{(i)}$ we mainly use the left polynomial $v_{k-1}^{(i+1)}$, the polynomial $v_{k-1}^{(i)}$ in the top left corner in the table, and their relation (29). It is not difficult to find that, in every column, the number of polynomial is in decline when k is bigger and bigger.

It is noticed that $v_k^{(i)}$ can also be arranged by another pattern as follows:

$$\begin{array}{ccccccc}
1 & \lambda & \lambda^2 & \lambda^3 & \lambda^4 & \lambda^5 & \lambda^6 \\
\downarrow \checkmark & \downarrow \checkmark & \downarrow \checkmark & \downarrow \checkmark & \downarrow \checkmark & \downarrow \checkmark & \downarrow \checkmark \\
v_1^{(0)} & v_1^{(1)} & v_1^{(2)} & v_1^{(3)} & v_1^{(4)} & v_1^{(5)} & \\
\downarrow \checkmark & \downarrow \checkmark & \downarrow \checkmark & \downarrow \checkmark & \downarrow \checkmark & & \\
v_2^{(0)} & v_2^{(1)} & v_2^{(2)} & v_2^{(3)} & v_2^{(4)} & & \\
\downarrow \checkmark & \downarrow \checkmark & \downarrow \checkmark & \downarrow \checkmark & & & \\
v_3^{(0)} & v_3^{(1)} & v_3^{(2)} & v_3^{(3)} & & & \\
\downarrow \checkmark & \downarrow \checkmark & \downarrow \checkmark & & & & \\
v_4^{(0)} & v_4^{(1)} & v_4^{(2)} & & & & \\
\downarrow \checkmark & \downarrow \checkmark & & & & & \\
v_5^{(0)} & v_5^{(1)} & & & & & \\
\downarrow \checkmark & & & & & & \\
v_6^{(0)} & & & & & &
\end{array} \quad (39)$$

When computing $v_k^{(i)}$ we mainly use the above polynomial $v_{k-1}^{(i)}$, the polynomial $v_{k-1}^{(i+1)}$ in the top-right corner in the table, and their relation (29). Similarly, the number of polynomials is in decline when k is bigger and bigger in every row. In this way, we can recursively get $v_6(\lambda) = v_6^{(0)}$.

From the three-term recurrence relation (29), we now build a complete recursive algorithm for calculating 2DFPTA. The advantage of this algorithm is that in the process of calculating 2DFPTA to avoid the calculation of the determinant, it can start from the initial value, from low to high order, and is gradually calculated. And its specific computational steps are as follows.

Algorithm 7 (Sylvester-type triangle algorithm).

- (1) Set the initial value as $v_0^{(i)} = \lambda^i$, $i = 0, 1, \dots, n$ compute (c_i, c_j) by (4).
- (2) For $k = 1, 2, \dots, n-1$ compute by (29) in turn $v_k^{(0)}, v_k^{(1)}, \dots, v_k^{(n-k)}$.
- (3) Compute $v_n(\lambda) = v_n^{(0)} = v_{n-1}^{(0)} - ((c_l, \phi^{(l+k-1)}(v_{n-1}^{(0)})) / (c_l, \phi^{(l+k-1)}(v_{n-1}^{(1)}))) v_{n-1}^{(1)}$.
- (4) $v_n(\lambda)$ is written as $v_n(\lambda) = b_0 + b_1 \lambda + \dots + b_n \lambda^n$ ($b_n = 1$) and the denominator polynomial of function-valued Padé-type approximation is

$$\tilde{v}_n(\lambda) = \lambda^n v_n(\lambda^{-1}) = \sum_{i=0}^n b_i \lambda^{n-i}. \quad (40)$$

- (5) From (12), numerator polynomial of $(m/n)_f(x, \lambda)$ is

$$\tilde{w}_m(x, \lambda) = \sum_{j=0}^n b_j \lambda^{n-j} \sum_{i=0}^{m-n+j} c_i(x) \lambda^i = \sum_{i=0}^m \sum_{j=0}^n b_j \tilde{c}_i^j(x) \lambda^i. \quad (41)$$

- (6) In view of Definition 1, there exists $(m/n)_f(x, \lambda)$ such that

$$\left(\frac{m}{n}\right)_f(x, \lambda) = \frac{\tilde{w}_m(x, \lambda)}{\tilde{v}_n(\lambda)}. \quad (42)$$

Example 8. Consider the second Fredholm integral equation

$$\begin{aligned}
f(x_1, x_2, \lambda) = & x_1 \cos(x_2) + \frac{4}{3} \sin(x_1) - \left(1 + \frac{4}{3} \right. \\
& \cdot \sin(1) \Big) x_2 + \lambda \int_{-1}^1 \int_{-1}^1 (y_2 \sin(x_1) + x_2 y_1) \\
& \cdot f(y_1, y_2, \lambda) dy_1 dy_2, \quad x_1, x_2 \in [-1, 1].
\end{aligned} \quad (43)$$

Its kernel is $K(x, t) = (y_2 \sin(x_1) + x_2 y_1)$.

In the following we use the triangle algorithm to yield $(2/2)_f(x, \lambda)$ of the integral equation.

Solution. The first few terms of the power series for $f(x, \lambda)$ are given by

$$\begin{aligned}
f(x, \lambda) = & c_0(x) + c_1(x) \lambda + c_2(x) \lambda^2 + c_3(x) \lambda^3 \\
& + c_4(x) \lambda^4 + \dots = x_1 \cos(x_2) + \frac{4}{3} \sin(x_1) - \left(1 \right. \\
& + \frac{4}{3} \sin 1 \Big) x_2 + (-2.829281750769593 \sin(x_1) \\
& + 2.728194267422565 x_2) \lambda \\
& + (3.637592356563420 \sin(x_1) \\
& - 3.408364188910562 x_2) \lambda^2 \\
& + (-4.544485585214082 \sin(x_1) \\
& + 4.382115538190247 x_2) \lambda^3 \\
& + (5.842820717586996 \sin(x_1) \\
& - 5.474626880638770 x_2) \lambda^4 + \dots.
\end{aligned} \quad (44)$$

Here $m = n = 1$, $l = m - n + 1 = 1$ and we set $d = l$.

Step 1. Set the initial value as $v_0^{(0)} = 1$, $v_0^{(1)} = \lambda$, $v_0^{(2)} = \lambda^2$, computing

$$\begin{aligned}
(c_1, c_1) &= 18.654952992399824, \\
(c_1, c_2) &= -23.623503547631628.
\end{aligned} \quad (45)$$

Step 2. Compute $v_1^{(0)}$ and $v_1^{(1)}$ by (29)

$$\begin{aligned}
v_1^{(0)} &= v_0^{(0)} - \frac{(c_1, \phi^{(1)}(v_0^{(0)}))}{(c_1, \phi^{(1)}(v_0^{(1)}))} v_0^{(1)} = 1 - \frac{(c_1, c_1)}{(c_1, c_2)} \lambda \\
&= 1 + 0.789677659572644 \lambda. \\
v_1^{(1)} &= v_0^{(1)} - \frac{(c_1, \phi^{(1)}(v_0^{(1)}))}{(c_1, \phi^{(1)}(v_0^{(2)}))} v_0^{(2)} = \lambda - \frac{(c_1, c_2)}{(c_1, c_3)} \lambda^2 \\
&= \lambda + 0.788390924640188 \lambda^2.
\end{aligned} \quad (46)$$

Step 3. Obtain the generating polynomial of $f(x, \lambda)$ function-value Padé-type approximant by (27)

$$\begin{aligned} v_2^{(0)} &= v_1^{(0)} - \frac{(c_1, \phi^{(2)}(v_1^{(0)}))}{(c_1, \phi^{(2)}(v_1^{(1)}))} v_1^{(1)} \\ &= 1 + 1.1752 \times 10^{-13} \lambda - 0.622574700198084 \lambda^2. \end{aligned} \quad (47)$$

Step 4. From (10) get the denominator of $f(x, \lambda)$ function-value Padé-type approximant

$$\begin{aligned} \tilde{v}_2(\lambda) &= \lambda^2 v_2(\lambda^{-1}) = b_0 \lambda^2 + b_1 \lambda + b_2 \\ &= -1.606232954345608 \lambda^2 - 1.89 \times 10^{-13} \lambda + 1 \end{aligned} \quad (48)$$

Step 5. Get the numerator of $f(x, \lambda)$ function-value Padé-type approximant by (12)

$$\begin{aligned} \tilde{w}_2(x, \lambda) &= (b_0 c_0 + b_1 c_1 + b_2 c_2) \lambda^2 + (b_1 c_0 + b_2 c_1) \lambda \\ &\quad + b_2 c_0 = a_2 \lambda^2 + a_1 \lambda + a_0 \end{aligned} \quad (49)$$

where

$$\begin{aligned} a_2 &= 1.495948417436477 \sin x_1 - 9.0779 \times 10^{-16} x_2 \\ &\quad - 1.606232954345608 x_1 \cos x_2 \\ a_1 &= -2.829281750769845 \sin x_1 \\ &\quad + 2.728194267422966 x_2 - 1.89 \\ &\quad \times 10^{-13} x_1 \cos x_2 \\ a_0 &= 1.333333333333333 \sin x_1 \\ &\quad - 2.121961313077195 x_2 + x_1 \cos x_2 \end{aligned} \quad (50)$$

Step 6. Obtain $f(x, \lambda)$ function-valued Padé-type approximant

$$\left(\frac{2}{2}\right)_f(x, \lambda) = \frac{a_2 \lambda^2 + a_1 \lambda + a_0}{b_0 \lambda^2 + b_1 \lambda + b_2}. \quad (51)$$

4. Numerical Results

In this section, we will consider the square domain case of $D = [\alpha, \beta] \times [\alpha, \beta]$. We apply Algorithm 7 for 2DFPTA to approximate the corresponding 2DF-II [11]. Compared with the methods in [4, 18], numerical experiments show that our method improves the calculation accuracy. Furthermore, our method can estimate the corresponding characteristic values of $f(x, \lambda)$ from the real roots of $\tilde{v}_n(\lambda)$.

Example 9. Consider the same Fredholm integral equation with Example 8 [4, 18].

When $\lambda = 1$, the exact solution of (43) is

$$f(x_1, x_2, \lambda) = x_1 \cos(x_2) - x_2. \quad (52)$$

FIGURE 1: $\lambda = 1$, error of $(2/2)_f(x, \lambda)$ in Example 9.

By Algorithm 7, we get the following calculation results:

$$\begin{aligned} \tilde{v}_2(\lambda) &= 1 - 1.89 \times 10^{-13} \lambda \\ &\quad - 1.606232954345608 \lambda^2, \\ \left(\frac{2}{2}\right)_f(x_1, x_2, 1) &= -x_2 + x_1 \cos(x_2) + 5.8593 \\ &\quad \times 10^{-14} \sin(x_1) + 5.6 \\ &\quad \times 10^{-14} x_2, \end{aligned} \quad (53)$$

where when $\lambda = 1$ the maximum absolute error of $(2/2)_f(x_1, x_2, 1)$ caused by Algorithm 7 is shown in Figure 1. From Figure 1, we can observe that the error is 1.05×10^{-13} and in the process we just only employ the first $m+n = 4$ items of Neumann series (4), while in [4] the error was 8.7×10^{-5} when the first 5 items of Chebyshev series corresponding to this equation were used and in [18] the corresponding error was 5.3×10^{-13} .

By Algorithm 7, we also get that

$$\begin{aligned} \tilde{v}_3(\lambda) &= 1 + 0.99999999999785 \lambda \\ &\quad - 1.606232954345953 \lambda^2 \\ &\quad - 1.606232954345763 \lambda^3, \\ \tilde{v}_4(\lambda) &= 1 - 7.837686719857387 \lambda \\ &\quad - 2.094833520416517 \lambda^2 \\ &\quad + 12.589150695272894 \lambda^3 \\ &\quad + 0.784806330738950 \lambda^4, \end{aligned} \quad (54)$$

where their eigenvalue has estimates; see Table 1.

In the computational process of $\tilde{v}_2(\lambda)$, we get two possible characteristic values $\lambda_1 = 0.7890340298098$ and $\lambda_2 = -0.789034029809980$. After further computing we can

TABLE 1: Estimates of the characteristic values of $(n/n)_f(x, \lambda)$, $n = 1, 2, 3, 4$ in Example 9.

$(n/n)_f$	$(1/1)_f$	$(2/2)_f$	$(3/3)_f$	$(4/4)_f$
λ_1	0.789677659572644	0.789034029809863	0.789034029809820	0.789034029809775
λ_2		-0.789034029809980	-0.789034029809980	-0.789034029809980
λ_3			-0.999999999999958	-16.167681854028405
λ_4				0.126589673465619

TABLE 2: When $\lambda = 1$, the errors of $(2/2)_f(x, \lambda)$ and $(2/2)_u(x, \lambda)$ at corresponding points in Example 9.

x	$f(x, \lambda)$	$ f(x, \lambda) - (2/2)_u(x, \lambda) $	$ f(x, \lambda) - (2/2)_f(x, \lambda) $
(-0.1,0.1)	-1.9950e-001	2.9438e-014	2.4951e-016
(-0.2,0.2)	-3.9601e-001	5.8583e-014	4.4057e-016
(-0.3,0.3)	-5.8660e-001	8.7142e-014	5.1532e-016
(-0.4,0.4)	-7.6842e-001	1.1483e-013	4.1707e-016
(-0.5,0.5)	-9.3879e-001	1.4137e-013	9.0830e-017
(0.6,-0.6)	1.0952e+000	1.6650e-013	5.1608e-016
(0.7,-0.7)	1.2354e+000	1.8996e-013	1.4536e-015
(0.8,-0.8)	1.3574e+000	2.1153e-013	2.7682e-015
(0.9,-0.9)	1.4594e+000	2.3098e-013	4.5028e-015
(1.0,-1.0)	1.5403e+000	2.4813e-013	6.6960e-015

deduce that $\tilde{v}_3(\lambda)$ and $\tilde{v}_4(\lambda)$ both include the two real roots. Therefore we consider the two numbers as characteristic values of this equation.

In Table 2, $(2/2)_f(x, \lambda)$ is from Algorithm 7 in the paper and $(2/2)_u(x, \lambda)$ is from the method in [18], where

$$\left(\frac{2}{2}\right)_u(x_1, x_2, 1) = -x_2 + x_1 \cos(x_2) + 2.94875 \times 10^{-13} \sin(x_1). \quad (55)$$

We find that from the numerical results $(2/2)_f(x, \lambda)$ has better approximation effect compared with $(2/2)_u(x, \lambda)$ in [18].

Example 10. Consider the following [4, 18]:

$$\begin{aligned} f(x_1, x_2, \lambda) &= x_1 e^{-x_2} + \left(\frac{1}{3}e^{-1} - \frac{7}{12}\right)x_1 - \frac{1}{2}x_2 \\ &+ \lambda \int_0^1 \int_0^1 (x_1 y_1 + x_2 e^{y_2}) f(y_1, y_2, \lambda) dy_1 dy_2, \\ x_1, x_2 &\in [0, 1]. \end{aligned} \quad (56)$$

When $\lambda = 1$, the exact solution of (56) is

$$f(x_1, x_2, \lambda) = x_1 e^{-x_2} + x_2. \quad (57)$$

By Algorithm 7, the calculation of Example 10 is shown as follows:

FIGURE 2: $\lambda = 1$, error of $(2/2)_f(x, \lambda)$ in Example 10.

$$\begin{aligned} \tilde{v}_2(\lambda) &= 1 - 1.333333333334607\lambda \\ &+ 0.118548104777520\lambda^2, \\ \left(\frac{2}{2}\right)_f(x_1, x_2, 1) &= -3.801972000573112 \times 10^{-15} x_1 \\ &+ 1.000000000000003 x_2 \\ &+ x_1 e^{-x_2}, \end{aligned} \quad (58)$$

where when $\lambda = 1$ the maximum absolute error of $(2/2)_f(x_1, x_2, 1)$ caused by Algorithm 7 is shown in Figure 2. From Figure 2, we can observe that the error is 3.7×10^{-15} and in the process we just only employ the first $m + n = 4$ items of Neumann series (4). While in [4] the error was 8.8×10^{-7}

TABLE 3: Estimates of the characteristic values of $(n/n)_f(x, \lambda)$, $n = 1, 2, 3, 4$ in Example 10.

$(n/n)_f$	$(1/1)_f$	$(2/2)_f$	$(3/3)_f$	$(4/4)_f$
λ_1	0.815393639551284	0.808054702319182	0.808054702319182	0.808054702319181
λ_2		10.439137615545988	10.439137615545846	10.439137615544714
λ_3			0.017857142857281	-0.0476584+0.4291069i
λ_4				-0.0476584-0.4291069i

TABLE 4: When $\lambda = 1$, the errors of $(2/2)_f(x, \lambda)$ and $(2/2)_u(x, \lambda)$ at corresponding points in Example 10.

x	$f(x, \lambda)$	$ f(x, \lambda) - (2/2)_u(x, \lambda) $	$ f(x, \lambda) - (2/2)_f(x, \lambda) $
(0.1,0.1)	1.9048e-001	9.9048e-016	8.0197e-017
(0.2,0.2)	3.6375e-001	1.9637e-015	1.6039e-016
(0.3,0.3)	5.2225e-001	2.9222e-015	2.4059e-016
(0.4,0.4)	6.6813e-001	3.8681e-015	3.2079e-016
(0.5,0.5)	8.0327e-001	4.8033e-015	4.0099e-016
(0.6,0.6)	9.2929e-001	5.7293e-015	4.8118e-016
(0.7,0.7)	1.0476e+000	6.6476e-015	5.6138e-016
(0.8,0.8)	1.1595e+000	7.5595e-015	6.4158e-016
(0.9,0.9)	1.2659e+000	8.4659e-015	7.2177e-016
(1.0,1.0)	1.3679e+000	9.3679e-015	8.0197e-016

when the first 5 items of Chebyshev series corresponding to this equation were used and in [18] the corresponding error was 5.5×10^{-15} .

From $\tilde{v}_2(\lambda)$ in (58), we get two possible characteristic values $\lambda_1 = 0.80805470231918$, $\lambda_2 = 10.43913761554$. By Algorithm 7, we also get that

$$\begin{aligned}
 \tilde{v}_3(\lambda) &= 1 - 0.428436108131838\lambda \\
 &\quad - 1.087981528825263\lambda^2 \\
 &\quad + 0.107273851063696\lambda^3, \\
 \tilde{v}_4(\lambda) &= 1 - 1.225739535044527\lambda \\
 &\quad - 0.111597656991711\lambda^2 \\
 &\quad + 0.128338192937178\lambda^3 \\
 &\quad - 0.010276622714483\lambda^4.
 \end{aligned} \tag{59}$$

Note that both $\tilde{v}_3(\lambda)$ and $\tilde{v}_4(\lambda)$ include the two real roots. Therefore, we consider the two real roots as the characteristic values of (56). For estimates of the characteristic values of $(n/n)_f(x, \lambda)$, $n = 1, 2, 3, 4$ in Example 10, see Table 3.

In Table 4, $(2/2)_f(x, \lambda)$ is from Algorithm 7 in the paper and $(2/2)_u(x, \lambda)$ is from the method in [18], where

$$\begin{aligned}
 \left(\frac{2}{2}\right)_u(x_1, x_2, 1) &= 1.000000000000009x_2 \\
 &\quad + 1.000000000000001x_1e^{-x_2}.
 \end{aligned} \tag{60}$$

From the numerical results in Table 4, $(2/2)_f(x, \lambda)$ has better approximation effect compared with $(2/2)_u(x, \lambda)$ in [18].

Data Availability

The [tables] data used to support the findings of this study are included within the article. The [figures] data used to support the findings of this study are included within the article. The [software codes] data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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