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Research Article

On a Nonlinear Wave Equation of Kirchhoff-Carrier Type: Linear Approximation and Asymptotic Expansion of Solution in a Small Parameter

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We consider the Robin-Dirichlet problem for a nonlinear wave equation of Kirchhoff-Carrier type. Using the Faedo-Galerkin method and the linearization method for nonlinear terms, the existence and uniqueness of a weak solution are proved. An asymptotic expansion of high order in a small parameter of a weak solution is also discussed.

1. Introduction

In this paper, we consider the following Robin-Dirichlet problem for a nonlinear wave equation of Kirchhoff-Carrier type:

$$u_{tt} - \frac{\partial}{\partial x}\left[\mu\left(x, t, \int_0^1 g\left(x, y, t, u\left(y, t\right), u_x\left(y, t\right)\right) dy\right) \cdot u_x\right] = f\left(x, t\right), \quad 0 < x < 1, \ 0 < t < T, \tag{1}$$

$$u_x\left(0, t\right) - h_0 u\left(0, t\right) = u\left(1, t\right) = 0, \tag{2}$$

$$u\left(x, 0\right) = \tilde{u}_0\left(x\right), \quad u_t\left(x, 0\right) = \tilde{u}_1\left(x\right), \tag{3}$$

where μ, f, g, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $h_0 \geq 0$ is a given constant.

Equation (1) can be considered as a general equation containing relatively some classical equations; for example, when $g(x, y, t, u, u_x) = u_x^2$, $f = 0$, (1) has a relation to the Kirchhoff wave equation:

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial y}\left(y, t\right)\right|^2 dy\right) u_{xx} \tag{4}$$

(see [1]). This equation is a generalization of the well-known D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. The parameters in (4) have the following meanings: u is the lateral deflection, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In another case, with $g(x, y, t, u, u_x) = u^2$, $f = 0$, (1) contains the form of Carrier equation. In [2], Carrier established the equation modeling the vibration of an elastic string when the changes in tension are not small:

$$\rho h u_{tt} - \left(1 + \frac{EA}{LT_0}\int_0^L u^2 dx\right) u_{xx} = 0, \tag{5}$$

where $u(x,t)$ is the x-derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross section of a string, L is the length of a string, ρ is the density of a material. Therefore, it is clear that (1) considered here contains (4) and (5) as special cases.

Moreover, with various boundary conditions, the particular forms of (1) have been extensively studied by many authors; for example, we refer to [3–15] and the references given therein. In these works, many interesting results about existence, regularity, asymptotic behavior, asymptotic expansion, and decay of solutions were obtained.

Cavalcanti et al., in [4–7], investigated a series of four papers in which the results of existence, global existence, exponential or uniform decay rates, and asymptotic behavior for Kirchhoff-Carrier models are considered.

In [10], the unique existence and asymptotic expansion of solutions of (1) with $\mu = 1$ associated with the boundary conditions

$$u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0 \tag{6}$$

and the initial conditions are also studied.

In [15], de Lima Santos studied the asymptotic behavior of solutions of (1) with $f(x,t) \equiv 0$, $\mu = \mu(t)$, associated with the Dirichlet boundary condition at $x = 0$ and a boundary condition of memory type at $x = 1$; that is, $u(1,t) + \int_0^t g(t-s)\mu(s)u_x(1,s)ds = 0,\ t > 0$.

In [3], Beilin investigated the existence and uniqueness of a generalized solution for the following wave equation with an integral nonlocal condition

$$\begin{aligned} &u_{tt} - \Delta u + c(x,t)u = f(x,t), \\ &\qquad (x,t) \in \Omega \times (0,T), \\ &\frac{\partial u}{\partial \eta} + \int_0^t \int_\Omega k(x,\xi,\tau)u(\xi,\tau)\,d\xi\,d\tau = 0, \\ &\qquad (x,t) \in \partial\Omega \times [0,T), \\ &u(x,0) = \tilde{u}_0(x), \\ &u_t(x,0) = \tilde{u}_1(x), \\ &\qquad x \in \Omega, \end{aligned} \tag{7}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary, η is the unit outward normal on $\partial\Omega$, and $f, \tilde{u}_0, \tilde{u}_1$, $k(x,\xi,\tau)$ are given functions. Nonlocal conditions come up when values of the function on the boundary are connected to values inside the domain. There are various types of nonlocal boundary conditions of integral form for hyperbolic, parabolic, or elliptic equations; the ones were introduced in [3].

The well-posedness and optimal decay rate estimates of the energy associated with the Kirchhoff-Carrier problem with memory

$$\begin{aligned} &u'' - M\left(\|\nabla u(t)\|_2^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds = 0, \\ &\qquad \text{in } \Omega \times \mathbb{R}_+, \\ &u = 0, \quad \text{on } \Gamma \times \mathbb{R}_+, \\ &u(x,0) = u_0(x), \\ &u_t(x,0) = u_1(x), \\ &\qquad \text{in } \Omega, \end{aligned} \tag{8}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with a smooth boundary $\partial\Omega := \Gamma$, are proved in [8].

In [11], the following nonlinear wave equation with initial conditions and boundary conditions of two-point type has been investigated:

$$\begin{aligned} &u_{tt} - \frac{\partial}{\partial x}\left(\mu(x,t)u_x\right) + f(u,u_t) = F(x,t), \\ &\qquad 0 < x < 1,\ 0 < t < T. \end{aligned} \tag{9}$$

In [12], by combining the linearization method for the nonlinear term, the Faedo-Galerkin method, and the weak compact method, the existence of a unique weak solution of an initial and boundary value problem for nonlinear wave equation $u_{tt} - (\partial/\partial x)(\mu(x,t,u,\|u_x\|^2)u_x) = F(x,t,u,u_x,u_t)$ with the nonhomogeneous boundary conditions is proved.

Very recently, in [13, 14], with the same method used in [12], the authors proved the results of existence and uniqueness for the wave equations with nonlinear sources containing the nonlocal terms. In [13], the linearization method together with Taylor's expansion is used for both of the source term and the nonlinear integral in it. These techniques have not been used before.

In the same spirit of [10–14], we establish the local existence and uniqueness for prob. (1)–(3) by using the Faedo-Galerkin method and the weak compact method. These results are presented in Section 3. In Section 4, the perturbed solution $u_\varepsilon(x,t)$ is approximated by the polynomial of $N+1$ degree in a small parameter ε for the following perturbed equation:

$$\begin{aligned} &u_{tt} - \frac{\partial}{\partial x}\left(\mu_\varepsilon[u](x,t)u_x\right) = f(x,t), \\ &\qquad 0 < x < 1,\ 0 < t < T, \end{aligned} \tag{10}$$

associated with (2), (3), where

$$\begin{aligned} \mu_\varepsilon[u](x,t) &= \mu\left(x,t,\int_0^1 g[u](x,y,t)\,dy\right) \\ &\quad + \varepsilon\mu_1\left(x,t,\int_0^1 g_1[u](x,y,t)\,dy\right), \\ g[u](x,y,t) &= g(x,y,t,u(y,t),u_x(y,t)), \\ g_1[u](x,y,t) &= g_1(x,y,t,u(y,t),u_x(y,t)). \end{aligned} \tag{11}$$

2. Preliminaries

Put $\Omega = (0,1)$ and denote the usual function spaces used in this paper by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2, $\|\cdot\|_X$ is the norm in the Banach space X, and X' is the dual space of X.

We denote $L^p(0,T;X)$, $1 \le p \le \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty, \quad \text{for } 1 \le p < \infty, \tag{12}$$
$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \nabla u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $(\partial u/\partial t)(x,t)$, $(\partial^2 u/\partial t^2)(x,t)$, $(\partial u/\partial x)(x,t)$, $(\partial^2 u/\partial x^2)(x,t)$, respectively.

With $\in C^k([0,1]^2 \times \mathbb{R}_+ \times \mathbb{R}^2)$, $g = g(x,y,t,z_1,z_2)$, we put $D_1 g = \partial g/\partial x$, $D_2 g = \partial g/\partial y$, $D_3 g = \partial g/\partial t$, $D_{i+3} g = \partial g/\partial z_i$, with $i = 1,2$ and $D^\beta g = D_1^{\beta_1} \cdots D_5^{\beta_5} g$; $\beta = (\beta_1,\ldots,\beta_5) \in \mathbb{Z}_+^5$, $|\beta| = \beta_1 + \cdots + \beta_5 = k$, $D^{(0,\ldots,0)} g = g$.

Similarly, with $\mu = \mu(x,t,z)$, we also put $D_1\mu = \partial\mu/\partial x$, $D_2\mu = \partial\mu/\partial t$, $D_3\mu = \partial\mu/\partial z$.

We shall use the following norm on H^1:

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \tag{13}$$

We put

$$V = \left\{v \in H^1 : v(1) = 0\right\}, \tag{14}$$

$$a(u,v) = \int_0^1 u_x(x)\, v_x(x)\, dx + h_0 u(0)\, v(0), \quad \forall u, v \in V. \tag{15}$$

V is a closed subspace of H^1 and on V three norms $\|v\|_{H^1}$, $\|v_x\|$, and $\|v\|_a = \sqrt{a(v,v)}$ are equivalent norms.

We have the following lemmas, the proofs of which are straightforward and hence we omit the details.

Lemma 1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \le \sqrt{2}\, \|v\|_{H^1} \quad \forall v \in H^1, \tag{16}$$

where $\|v\|_{C^0(\overline{\Omega})} = \sup_{x\in[0,1]}|v(x)|$ (see [16]).

Lemma 2. *Let $h_0 \ge 0$. The imbedding $V \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \le \|v_x\| \le \|v\|_a,$$
$$\frac{1}{\sqrt{2}}\|v\|_{H^1} \le \|v_x\| \le \|v\|_a \le \sqrt{1+h_0}\, \|v\|_{H^1}, \tag{17}$$

for all $v \in V$.

Lemma 3. *Let $h_0 \ge 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by (15) is continuous on $V \times V$ and coercive on V.*

Lemma 4. *Let $h_0 \ge 0$. Then there exists the Hilbert orthonormal base $\{\widetilde{w}_j\}$ of L^2 consisting of the eigenfunctions $\widetilde{w}_j$ corresponding to the eigenvalues λ_j such that*

$$\lim_{j\to+\infty} \lambda_j = +\infty, \quad 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots,$$
$$a\left(\widetilde{w}_j, v\right) = \lambda_j \left\langle \widetilde{w}_j, v\right\rangle \quad \forall v \in V, \ j = 1,2,\ldots. \tag{18}$$

Furthermore, the sequence $\{\widetilde{w}_j/\sqrt{\lambda_j}\}$ is also a Hilbert orthonormal base of V with respect to the scalar product $a(\cdot,\cdot)$.

On the other hand, we also have $\widetilde{w}_j$ satisfying the following boundary value problem:

$$-\Delta\widetilde{w}_j = \lambda_j \widetilde{w}_j, \quad in \ (0,1),$$
$$\widetilde{w}_{jx}(0) - h_0\widetilde{w}_j(0) = \widetilde{w}_j(1) = 0, \quad \widetilde{w}_j \in V \cap C^\infty\left(\overline{\Omega}\right). \tag{19}$$

The proof of Lemma 4 can be found in ([17], p.87, Theorem (7.7)), with $H = L^2$ and $V, a(\cdot,\cdot)$ as defined by (14), (15).

Remark 5. The weak formulation of the initial-boundary value problem (1)–(3) can be given in the following manner: Find $u \in \widetilde{W} = \{u \in L^\infty(0,T;V\cap H^2) : u_t \in L^\infty(0,T;V),\ u_{tt} \in L^\infty(0,T;L^2)\}$, such that u satisfies the following variational equation:

$$\langle u_{tt}(t), w\rangle + A[u](t;u(t),w) = \langle f(t), w\rangle, \tag{20}$$

for all $w \in V$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \widetilde{u}_0,$$
$$u_t(0) = \widetilde{u}_1, \tag{21}$$

where, for each $w \in \widetilde{W}$, $\{A[w](t;\cdot,\cdot)\}_{0\le t\le T}$ is the family of symmetric bilinear forms on $V \times V$ defined by

$$A[w](t;u,v) = \langle \mu[w](t)\, u_x, v_x\rangle + h_0\mu[w](0,t)\, u(0)\, v(0), \tag{22}$$

for all $u, v \in V$, $0 \le t \le T$, with $h_0 \ge 0$ being given constant, and

$$\mu[w](x,t) = \mu\left(x,t,\int_0^1 g[w](x,y,t)\, dy\right),$$
$$g[w](x,y,t) = g(x,y,t,w(y,t), w_x(y,t)). \tag{23}$$

3. The Existence and Uniqueness

Let $T^* > 0$. We make the following assumptions:

(H_1) $(\widetilde{u}_0, \widetilde{u}_1) \in (V \cap H^2) \times V$ satisfying the condition $\widetilde{u}_{0x}(0) - h_0\widetilde{u}_0(0) = 0$.

(H_2) $g \in C^2([0,1]^2 \times [0,T^*] \times \mathbb{R}^2)$.

(H_3) $\mu \in C^2([0,1] \times [0,T^*] \times \mathbb{R})$ and there exists a constant $\mu_0 > 0$ such that $\mu(x,t,z) \geq \mu_0$, for all $(x,t,z) \in [0,1] \times [0,T^*] \times \mathbb{R}$.

(H_4) $f, f' \in L^2(Q_{T^*})$.

For each $M > 0$ given, we set the constants $\overline{K}_M(g)$, $\overline{K}_0(M,g)$, $\widetilde{K}_M(\mu)$, $\widetilde{K}_0(M,\mu,g)$, as follows:

$$\begin{aligned} \overline{K}_M(g) &= \sum_{|\alpha| \leq 2} \overline{K}_0(M, D^\alpha g), \\ \widetilde{K}_M(\mu) &= \sum_{|\beta| \leq 2} \widetilde{K}_0(M, D^\beta \mu, g), \end{aligned} \tag{24}$$

where

$$\begin{aligned} \overline{K}_0(M,g) &= \sup_{(x,y,t,z_1,z_2) \in A_1(M)} |g(x,y,t,z_1,z_2)|, \\ \widetilde{K}_0(M,\mu,g) &= \sup_{(x,t,z) \in A_2(g,M)} |\mu(x,t,z)|, \\ A_1(M) &= \Big\{(x,t,y,z_1,z_2) : 0 \leq x,\ y \leq 1,\ 0 \leq t \\ &\quad \leq T^*,\ \max_{1 \leq i \leq 2} |z_i| \leq M\Big\}, \\ A_2(g,M) &= \Big\{(x,t,v) : 0 \leq x \leq 1,\ 0 \leq t \leq T^*,\ |v| \\ &\quad \leq \overline{K}_0(M,g)\Big\}. \end{aligned} \tag{25}$$

For every $T \in (0,T^*]$ and $M > 0$, we put

$$\begin{aligned} V_T = \Big\{ & v \in L^\infty(0,T; V \cap H^2) : v_t \in L^\infty(0,T;V),\ v_{tt} \\ & \in L^2(Q_T)\Big\}, \end{aligned} \tag{26}$$

in which $Q_T = \Omega \times (0,T)$.

Then V_T is a Banach space with respect to the norm

$$\begin{aligned} &\|v\|_{V_T} \\ &\quad = \max\left\{\|v\|_{L^\infty(0,T;V\cap H^2)}, \|v_t\|_{L^\infty(0,T;V)}, \|v_{tt}\|_{L^2(Q_T)}\right\}. \end{aligned} \tag{27}$$

(See Lions [18]) We also put

$$\begin{aligned} W(M,T) &= \Big\{v \in V_T : \|v\|_{V_T} \leq M\Big\}, \\ W_1(M,T) &= \Big\{v \in W(M,T) : v_{tt} \in L^\infty(0,T;L^2)\Big\}. \end{aligned} \tag{28}$$

Now, we establish the recurrent sequence $\{u_m\}$. The first term is chosen as $u_0 \equiv \widetilde{u}_0$, and supposing that

$$u_{m-1} \in W_1(M,T), \tag{29}$$

we associate problem (1) with the following problem.

Find $u_m \in W_1(M,T)$ $(m \geq 1)$ satisfying the linear variational problem

$$\begin{aligned} &\langle u_m''(t), v\rangle + A_m(t; u_m(t), v) = \langle f(t), v\rangle, \\ &\qquad \forall v \in V, \\ &u_m(0) = \widetilde{u}_0, \\ &u_m'(0) = \widetilde{u}_1, \end{aligned} \tag{30}$$

where

$$\begin{aligned} A_m(t;u,v) &= A[u_{m-1}](t;u,v) \\ &= \langle \mu_m(t) u_x, v_x\rangle + h_0 \mu_m(0,t) u(0) v(0), \\ &\qquad \forall u, v \in V, \\ \mu_m(x,t) &= \mu\left(x, t, \int_0^1 g[u_{m-1}](x,y,t)\, dy\right), \\ g[u_{m-1}](x,y,t) &= g(x,y,t,u_{m-1}(y,t), \nabla u_{m-1}(y,t)). \end{aligned} \tag{31}$$

Theorem 6. *Suppose that (H_1)–(H_4) hold. Then, there exist positive constants $M, T > 0$ such that the recurrent sequence $\{u_m\}$ is defined by (29)–(31).*

Proof. The proof consists of several steps.

Step 1 (the Faedo-Galerkin approximation (introduced by Lions [18])). Consider the basis $\{w_j\}$ for V as in Lemma 4. Approximate solution of (29)–(31) problem which will be found in form

$$u_m^{(k)}(t) = \sum_{j=1}^k c_{mj}^{(k)}(t) w_j, \tag{32}$$

where the coefficients $c_{mj}^{(k)}(t)$ satisfy the system of linear differential equations

$$\begin{aligned} &\langle \ddot{u}_m^{(k)}(t), w_j\rangle + A_m(t; u_m^{(k)}(t), w_j) = \langle f(t), w_j\rangle, \\ &\qquad 1 \leq j \leq k, \\ &u_m^{(k)}(0) = \widetilde{u}_{0k}, \\ &\dot{u}_m^{(k)}(0) = \widetilde{u}_{1k}, \end{aligned} \tag{33}$$

where

$$\begin{aligned} \widetilde{u}_{0k} &= \sum_{j=1}^k \alpha_j^{(k)} w_j \longrightarrow \widetilde{u}_0 \quad \text{strongly in } V \cap H^2, \\ \widetilde{u}_{1k} &= \sum_{j=1}^k \beta_j^{(k)} w_j \longrightarrow \widetilde{u}_1 \quad \text{strongly in } V. \end{aligned} \tag{34}$$

The system of (33) can be rewritten in form

$$\begin{gathered}\ddot{c}_{mj}^{(k)}(t)+\sum_{i=1}^{k}A_{ij}^{(m)}(t)\,c_{mi}^{(k)}(t)=f_j(t),\\ c_{mj}^{(k)}(0)=\alpha_j^{(k)},\\ \dot{c}_{mj}^{(k)}(0)=\beta_j^{(k)},\\ 1\le j\le k,\end{gathered}\tag{35}$$

where

$$\begin{gathered}A_{ij}^{(m)}(t)=A_m\left(t;w_i,w_j\right),\\ f_j(t)=\left\langle f(t),w_j\right\rangle,\\ 1\le i,\ j\le k.\end{gathered}\tag{36}$$

By (29), it is not difficult to prove that system (35), (36) has a unique solution $c_{mj}^{(k)}(t)$, $1\le j\le k$ on interval $[0,T]$, so let us omit the details (see [19]).

Step 2 (a priori estimates). First, we need the following lemma.

Lemma 7. *Putting* $\mu^*=\widetilde{K}_M(\mu)[1+(1+2M)\overline{K}_M(g)]$, *one has*

(i) $\left|A_m(t;u,v)\right|\le\widetilde{K}(\mu)\,\|u\|_a\,\|v\|_a$

$$\forall u,v\in V,\ 0\le t\le T^*,$$

(ii) $A_m(t;v,v)\ge\mu_0\,\|v\|_a^2\quad\forall v\in V,\ 0\le t\le T^*,$

(iii)
$$\begin{aligned}&\frac{\partial A_m}{\partial t}(t;u,v)\\ &=\left\langle\mu_m'(t)\,u_x,v_x\right\rangle+h_0\mu_m'(0,t)\,u(0)\,v(0),\\ &\qquad\forall u,v\in V,\end{aligned}\tag{37}$$

(iv) $\left|\dfrac{\partial A_m}{\partial t}(t;v,v)\right|\le\mu^*\,\|v\|_a^2\quad\forall v\in V,\ 0\le t\le T^*,$

(v)
$$\begin{aligned}&\frac{d}{dt}A_m\left(t;u_m^{(k)}(t),u_m^{(k)}(t)\right)\\ &=2A_m\left(t;u_m^{(k)}(t),\dot{u}_m^{(k)}(t)\right)\\ &\quad+\frac{\partial A_m}{\partial t}\left(t;u_m^{(k)}(t),u_m^{(k)}(t)\right).\end{aligned}$$

The proof of Lemma 7 is easy; hence we omit the details. Next, we put

$$S_m^{(k)}(t)=X_m^{(k)}(t)+Y_m^{(k)}(t)+\int_0^t\left\|\ddot{u}_m^{(k)}(s)\right\|^2ds,\tag{38}$$

where

$$\begin{gathered}X_m^{(k)}(t)=\left\|\dot{u}_m^{(k)}(t)\right\|^2+A_m\left(t;u_m^{(k)}(t),u_m^{(k)}(t)\right),\\ Y_m^{(k)}(t)=\left\|\dot{u}_m^{(k)}(t)\right\|_a^2+\left\|\sqrt{\mu_m(t)}\Delta u_m^{(k)}(t)\right\|^2.\end{gathered}\tag{39}$$

Then, it follows from (33), $(37)_{(\mathrm{iii}),(\mathrm{v})}$, (38), (39) that

$$\begin{aligned}&S_m^{(k)}(t)\\ &\quad=S_m^{(k)}(0)+2\left\langle\mu_{mx}(0)\,\widetilde{u}_{0kx},\Delta\widetilde{u}_{0k}\right\rangle\\ &\qquad+2\left\langle f(0),\Delta\widetilde{u}_{0k}\right\rangle\\ &\qquad+\int_0^t ds\int_0^1\mu_m'(x,s)\left|\Delta u_m^{(k)}(x,s)\right|^2dx\\ &\qquad+\int_0^t\frac{\partial A_m}{\partial t}\left(s;u_m^{(k)}(s),u_m^{(k)}(s)\right)ds\\ &\qquad+2\int_0^t\left\langle\frac{\partial}{\partial s}\left[\mu_{mx}(s)\,u_{mx}^{(k)}(s)\right],\Delta u_m^{(k)}(s)\right\rangle ds\\ &\qquad-2\left\langle\mu_{mx}(t)\,u_{mx}^{(k)}(t),\Delta u_m^{(k)}(t)\right\rangle\\ &\qquad+2\int_0^t\left\langle f(s),\dot{u}_m^{(k)}(s)\right\rangle ds\\ &\qquad+2\int_0^t\left\langle f'(s),\Delta u_m^{(k)}(s)\right\rangle ds\\ &\qquad-2\left\langle f(t),\Delta u_m^{(k)}(t)\right\rangle+\int_0^t\left\|\ddot{u}_m^{(k)}(s)\right\|^2ds\\ &\quad\equiv S_m^{(k)}(0)+2\left\langle\mu_{mx}(0)\,\widetilde{u}_{0kx},\Delta\widetilde{u}_{0k}\right\rangle\\ &\qquad+2\left\langle f(0),\Delta\widetilde{u}_{0k}\right\rangle+\sum_{j=1}^{8}I_j.\end{aligned}\tag{40}$$

We shall estimate the terms I_j on the right-hand side of (40) as follows.

First Term I_1. We note that

$$\begin{aligned}\mu_m'(x,t)&=D_2\mu\left[u_{m-1}\right]\\ &\quad+D_3\mu\left[u_{m-1}\right]\int_0^1\frac{\partial g\left[u_{m-1}\right]}{\partial t}(x,y,t)\,dy,\end{aligned}\tag{41}$$

where we use the notations

$$\begin{aligned}&D_i\mu\left[u_{m-1}\right]=D_i\mu\Bigg(x,t,\\ &\qquad\int_0^1 g\left(x,y,t,u_{m-1}(y,t),\nabla u_{m-1}(y,t)\right)dy\Bigg),\\ &\qquad i=1,2,3,\\ &\frac{\partial g\left[u_{m-1}\right]}{\partial t}(x,y,t)=D_3g\left[u_{m-1}\right]+D_4g\left[u_{m-1}\right]\\ &\qquad\cdot u_{m-1}'(y,t)+D_5g\left[u_{m-1}\right]\nabla u_{m-1}'(y,t),\\ &D_ig\left[u_{m-1}\right](x,y,t)=D_ig\left(x,y,t,u_{m-1}(y,t),\right.\\ &\qquad\left.\nabla u_{m-1}(y,t)\right),\quad i=1,\ldots,5.\end{aligned}\tag{42}$$

So, by (24), (25), and (41), we obtain

$$\left|\mu_m'(x,t)\right| \le \mu^*. \tag{43}$$

Hence,

$$\begin{aligned} I_1 &= \int_0^t ds \int_0^1 \mu_m'(x,s)\left|\Delta u_m^{(k)}(x,s)\right|^2 dx \\ &\le \frac{\mu^*}{\mu_0}\int_0^t S_m^{(k)}(s)\,ds. \end{aligned} \tag{44}$$

Second Term I_2. By Lemma 7 (ii) and (iv), we have

$$\begin{aligned} \left|I_2\right| &= \left|\int_0^t \frac{\partial A_m}{\partial t}\left(s; u_m^{(k)}(s), u_m^{(k)}(s)\right) ds\right| \\ &\le \mu^* \int_0^t \left\|u_m^{(k)}(t)\right\|_a^2 ds \le \frac{\mu^*}{\mu_0}\int_0^t S_m^{(k)}(s)\,ds. \end{aligned} \tag{45}$$

Third Term I_3. The Cauchy-Schwartz inequality leads to

$$\begin{aligned} \left|I_3\right| &= 2\left|\int_0^t \left\langle \frac{\partial}{\partial s}\left[\mu_{mx}(s)\,u_{mx}^{(k)}(s)\right], \Delta u_m^{(k)}(s)\right\rangle ds\right| \\ &\le \frac{2}{\sqrt{\mu_0}}\int_0^t J_m^{(k)}(s)\sqrt{S_m^{(k)}(s)}\,ds, \end{aligned} \tag{46}$$

where $J_m^{(k)}(s) = \|(\partial/\partial s)[\mu_{mx}(s)u_{mx}^{(k)}(s)]\|$.

We shall estimate the term $J_m^{(k)}(s)$ as follows.

By $S_m^{(k)}(t) \ge \|\dot{u}_{mx}^{(k)}(t)\|^2 + \|u_{mx}^{(k)}(t)\|^2$, we have

$$\begin{aligned} J_m^{(k)}(s) &= \left\|\frac{\partial}{\partial s}\left[\mu_{mx}(s)\,u_{mx}^{(k)}(s)\right]\right\| \\ &\le \left\|\mu_{mx}(s)\right\|_{C^0(\overline{\Omega})}\left\|\dot{u}_{mx}^{(k)}(s)\right\| \\ &\quad + \left\|\mu_{mx}'(s)\right\|\left\|u_{mx}^{(k)}(s)\right\|_{C^0(\overline{\Omega})} \\ &\le \left(\left\|\mu_{mx}(s)\right\|_{C^0(\overline{\Omega})} + \sqrt{\frac{1}{\mu_0}}\left\|\mu_{mx}'(s)\right\|\right)\sqrt{S_m^{(k)}(s)}. \end{aligned} \tag{47}$$

On the other hand, by $\mu_{mx}(x,t) = D_1\mu[u_{m-1}] + D_3\mu[u_{m-1}]\int_0^1 D_1 g[u_{m-1}](x,y,t)dy$, it implies that

$$\left\|\mu_{mx}(t)\right\|_{C^0(\overline{\Omega})} \le \widetilde{K}_M(\mu)\left(1 + \overline{K}_M(g)\right). \tag{48}$$

Similarly, from the following equality

$$\begin{aligned} \mu_{mx}'(x,t) &= \frac{\partial}{\partial t}\left[\frac{\partial \mu_m}{\partial x}(x,t)\right] = D_2D_1\mu[u_{m-1}] \\ &\quad + D_3D_1\mu[u_{m-1}]\int_0^1 \frac{\partial g[u_{m-1}]}{\partial t}(x,y,t)\,dy \\ &\quad + \left(D_2D_3\mu[u_{m-1}]\right. \\ &\quad \left. + D_3^2\mu[u_{m-1}]\int_0^1 \frac{\partial g[u_{m-1}]}{\partial t}(x,y,t)\,dy\right) \\ &\quad \cdot \int_0^1 D_1 g[u_{m-1}](x,y,t)\,dy + D_3\mu[u_{m-1}] \\ &\quad \cdot \int_0^1 \frac{\partial D_1 g[u_{m-1}]}{\partial t}(x,y,t)\,dy; \\ \frac{\partial g[u_{m-1}]}{\partial t}(x,y,t) &= D_3 g[u_{m-1}](x,y,t) \\ &\quad + D_4 g[u_{m-1}](x,y,t)\,u_{m-1}'(y,t) + D_5 g[u_{m-1}] \\ &\quad \cdot (x,y,t)\,\nabla u_{m-1}'(y,t); \\ \frac{\partial D_1 g[u_{m-1}]}{\partial t}(x,y,t) &= D_3D_1 g[u_{m-1}](x,y,t) \\ &\quad + D_4D_1 g[u_{m-1}](x,y,t)\,u_{m-1}'(y,t) \\ &\quad + D_5D_1 g[u_{m-1}](x,y,t)\,\nabla u_{m-1}'(y,t), \end{aligned} \tag{49}$$

we obtain that

$$\begin{aligned} \left\|\mu_{mx}'(t)\right\| &\le \widetilde{K}_M(\mu)\left[1\right. \\ &\quad \left. + \overline{K}_M(g)\int_0^1\left(1 + \left\|u_{m-1}'(t)\right\| + \left\|\nabla u_{m-1}'(t)\right\|\right)dy\right] \\ &\quad + \widetilde{K}_M(\mu)\overline{K}_M(g)\left[1\right. \\ &\quad \left. + \overline{K}_M(g)\int_0^1\left(1 + \left\|u_{m-1}'(t)\right\| + \left\|\nabla u_{m-1}'(t)\right\|\right)dy\right] \\ &\quad + \widetilde{K}_M(\mu)\overline{K}_M(g) \\ &\quad \cdot \int_0^1\left(1 + \left\|u_{m-1}'(t)\right\| + \left\|\nabla u_{m-1}'(t)\right\|\right)dy \\ &\le \widetilde{K}_M(\mu)\left[1 + 2(1+M)\overline{K}_M(g)\right. \\ &\quad \left. + 2(1+2M)\overline{K}_M^2(g)\right]. \end{aligned} \tag{50}$$

By (48) and (50), it follows from (47) that

$$J_m^{(k)}(s) \le \zeta_1(M)\sqrt{S_m^{(k)}(s)}, \tag{51}$$

where

$$\zeta_1(M) = \widetilde{K}_M(\mu)\left(1 + \overline{K}_M(g) + \sqrt{\frac{1}{\mu_0}}\left[1 + 2(1+M)\overline{K}_M(g) + 2(1+2M)\overline{K}_M^2(g)\right]\right). \tag{52}$$

Therefore, from (46) and (51), we obtain

$$I_3 \le \sqrt{\frac{2}{\mu_0}}\zeta_1(M)\int_0^t S_m^{(k)}(s)\,ds. \tag{53}$$

Fourth Term I_4. Applying the Cauchy-Schwartz inequality again, we have

$$\begin{aligned}|I_4| &= \left|-2\left\langle \mu_{mx}(t)u_{mx}^{(k)}(t), \Delta u_m^{(k)}(t)\right\rangle\right| \\ &\le \frac{1}{\beta}\left\|\mu_{mx}(t)u_{mx}^{(k)}(t)\right\|^2 + \frac{\beta}{\mu_0}S_m^{(k)}(t),\end{aligned} \tag{54}$$

for all $\beta > 0$. On the other hand, it follows from (51) that

$$\begin{aligned}&\left\|\mu_{mx}(t)u_{mx}^{(k)}(t)\right\| \\ &= \left\|\mu_{mx}(0)\nabla\widetilde{u}_{0k} + \int_0^t \frac{\partial}{\partial s}\left[\mu_{mx}(s)u_{mx}^{(k)}(s)\right]ds\right\| \\ &\le \left\|\mu_{mx}(0)\right\|_{C^0(\overline{\Omega})}\left\|\nabla\widetilde{u}_{0k}\right\| + \int_0^t J_m^{(k)}(s)\,ds \\ &\le \left\|\mu_{mx}(0)\right\|_{C^0(\overline{\Omega})}\left\|\nabla\widetilde{u}_{0k}\right\| + \zeta_1(M)\int_0^t \sqrt{S_m^{(k)}(s)}ds.\end{aligned} \tag{55}$$

Hence, we obtain from (54) and (55) that

$$\begin{aligned}|I_4| &\le \frac{\beta}{\mu_0}S_m^{(k)}(t) + \frac{2}{\beta}\left\|\mu_{mx}(0)\right\|_{C^0(\overline{\Omega})}^2\left\|\nabla\widetilde{u}_{0k}\right\|^2 \\ &\quad + \frac{2}{\beta}T\zeta_1^2(M)\int_0^t S_m^{(k)}(s)\,ds.\end{aligned} \tag{56}$$

Fifth Term I_5

$$\begin{aligned}|I_5| &= 2\left|\int_0^t \left\langle f(s), \dot{u}_m^{(k)}(s)\right\rangle ds\right| \\ &\le 2\int_0^t \|f(s)\|\left\|\dot{u}_m^{(k)}(s)\right\|ds \\ &\le T\|f\|_{L^\infty(0,T;L^2)}^2 + \int_0^t S_m^{(k)}(s)\,ds.\end{aligned} \tag{57}$$

Sixth Term I_6. Similarly, we obtain

$$\begin{aligned}|I_6| &= \left|2\int_0^t \left\langle f'(s), \Delta u_m^{(k)}(s)\right\rangle ds\right| \\ &\le \left\|f'\right\|_{L^2(Q_{T^*})}^2 + \frac{1}{\mu_0}\int_0^t S_m^{(k)}(s)\,ds.\end{aligned} \tag{58}$$

Seventh Term I_7. We have

$$\begin{aligned}|I_7| &= \left|-2\left\langle f(t), \Delta u_m^{(k)}(t)\right\rangle\right| \\ &\le \frac{1}{\beta}\|f(t)\|^2 + \beta\left\|\Delta u_m^{(k)}(t)\right\|^2 \\ &\le \frac{2}{\beta}\left(\|f(0)\|^2 + T\int_0^t \left\|f'(s)\right\|^2 ds\right) + \frac{\beta}{\mu_0}S_m^{(k)}(t) \\ &= \frac{2}{\beta}\left(\|f(0)\|^2 + T\left\|f'\right\|_{L^2(Q_{T^*})}^2\right) + \frac{\beta}{\mu_0}S_m^{(k)}(t), \\ &\qquad \forall\beta > 0.\end{aligned} \tag{59}$$

Eighth Term I_8. We note that $(33)_1$ can be rewritten as follows:

$$\begin{aligned}&\left\langle \ddot{u}_m^{(k)}(t), w_j\right\rangle - \left\langle \frac{\partial}{\partial x}\left(\mu_m(t)u_{mx}^{(k)}(t)\right), w_j\right\rangle \\ &\quad = \left\langle f(t), w_j\right\rangle, \quad 1 \le j \le k.\end{aligned} \tag{60}$$

Hence, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$ and integrating that

$$\begin{aligned}I_8 &= \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|^2 ds \\ &\le 2\int_0^t \left\|\frac{\partial}{\partial x}\left(\mu_m(s)u_{mx}^{(k)}(s)\right)\right\|^2 ds + 2\int_0^t \|f(s)\|^2 ds \\ &\le 2\int_0^t \left\|\frac{\partial}{\partial x}\left(\mu_m(s)u_{mx}^{(k)}(s)\right)\right\|^2 ds \\ &\quad + 2T\|f\|_{L^\infty(0,T^*;L^2)}^2.\end{aligned} \tag{61}$$

We estimate the term $\|(\partial/\partial x)(\mu_m(s)u_{mx}^{(k)}(s))\|^2$. By (48), we obtain

$$\begin{aligned}&\left\|\frac{\partial}{\partial x}\left(\mu_m(s)u_{mx}^{(k)}(s)\right)\right\|^2 \le \left(\left\|\mu_{mx}(s)u_{mx}^{(k)}(s)\right\| + \left\|\mu_m(s)\Delta u_m^{(k)}(s)\right\|\right)^2 \le 2\widetilde{K}_M(\mu) \\ &\cdot\left(\widetilde{K}_M(\mu)\left(1 + \overline{K}_M(g)\right)^2\left\|u_{mx}^{(k)}(s)\right\|^2 + \left\|\sqrt{\mu_m(s)}\Delta u_m^{(k)}(s)\right\|^2\right) \le 2\widetilde{K}_M(\mu)\left(1 + \widetilde{K}_M(\mu)\left(1 + \overline{K}_M(g)\right)^2\right)\left(\left\|u_{mx}^{(k)}(s)\right\|^2 + \left\|\sqrt{\mu_m(s)}\Delta u_m^{(k)}(s)\right\|^2\right) \le 2\widetilde{K}_M(\mu)\left(1 + \widetilde{K}_M(\mu)\left(1 + \overline{K}_M(g)\right)^2\right)\left(\frac{1+\mu_0}{\mu_0}\right)S_m^{(k)}(s).\end{aligned} \tag{62}$$

Therefore, by Lemma 7 (ii), (61) and (62), we obtain

$$I_8 \le 2T\|f\|_{L^\infty(0,T^*;L^2)}^2 + \zeta_2(M)\int_0^t S_m^{(k)}(s)\,ds, \tag{63}$$

where

$$\zeta_2(M) = 4\widetilde{K}_M(\mu)\left(\frac{1+\mu_0}{\mu_0}\right)\left[1+\widetilde{K}_M(\mu)\left(1+\overline{K}_M(g)\right)^2\right]. \tag{64}$$

Choosing $\beta > 0$, with $2\beta/\mu_0 \le 1/2$, it follows from (40), (44), (45), (53), (56)–(59), and (63) that

$$S_m^{(k)}(t) \le \widetilde{C}_0^{(k)} + 2T\left(\frac{2}{\beta}\left\|f'\right\|^2_{L^2(Q_{T^*})} + 3\left\|f\right\|^2_{L^\infty(0,T^*;L^2)}\right) + \widetilde{C}_1(M,T)\int_0^t S_m^{(k)}(s)\,ds, \tag{65}$$

where

$$\begin{aligned}\widetilde{C}_0^{(k)} &= \widetilde{C}_0^{(k)}(\mu, f, g, \widetilde{u}_{0k}, \widetilde{u}_{1k}) = 2S_m^{(k)}(0) \\ &\quad + 4\left\langle \mu_{mx}(0)\widetilde{u}_{0kx}, \Delta\widetilde{u}_{0k}\right\rangle + 4\left\langle f(0), \Delta\widetilde{u}_{0k}\right\rangle \\ &\quad + \frac{4}{\beta}\left\|\mu_{mx}(0)\right\|^2_{C^0(\overline{\Omega})}\left\|\nabla\widetilde{u}_{0k}\right\|^2 + \frac{4}{\beta}\left\|f(0)\right\|^2 \\ &\quad + 2\left\|f'\right\|^2_{L^2(Q_{T^*})},\end{aligned} \tag{66}$$

$$\widetilde{C}_1(M,T) = 2\left[1+\frac{1+2\mu^*}{\mu_0}+\frac{2}{\beta}T\zeta_1^2(M) + \sqrt{\frac{2}{\mu_0}}\zeta_1(M)+\zeta_2(M)\right].$$

By means of the convergences in (34), we can deduce the existence of a constant $M > 0$ independent of k and m such that

$$\widetilde{C}_0^{(k)}(\mu, f, g, \widetilde{u}_{0k}, \widetilde{u}_{1k}) \le \frac{1}{2}M^2. \tag{67}$$

So, from $(66)_2$, we can choose $T \in (0, T^*]$, such that

$$\left[\frac{1}{2}M^2 + 2T\left(\frac{2}{\beta}\left\|f'\right\|^2_{L^2(Q_{T^*})} + 3\left\|f\right\|^2_{L^\infty(0,T^*;L^2)}\right)\right] \cdot \exp\left(T\widetilde{C}_1(M,T)\right) \le M^2, \tag{68}$$

$$k_T = 2\sqrt{T}M\widetilde{K}_M(\mu)\overline{K}_M(g)\left[1+\sqrt{2}\left(2+\overline{K}_M(g)\right)\right] \cdot\left(1+\frac{1}{\sqrt{\mu_0}}\right)\exp\left(\frac{T(\mu_0+\mu^*)}{2\mu_0}\right) < 1. \tag{69}$$

Finally, it follows from (65), (67), and (68) that

$$S_m^{(k)}(t) \le M^2\exp\left(-T\widetilde{C}_1(M,T)\right) + \widetilde{C}_1(M,T)\int_0^t S_m^{(k)}(s)\,ds. \tag{70}$$

By using Gronwall's Lemma, we deduce from (70) that

$$S_m^{(k)}(t) \le M^2\exp\left(-T\widetilde{C}_1(M,T)\right)\exp\left(t\widetilde{C}_1(M,T)\right) \le M^2, \tag{71}$$

for all $t \in [0,T]$, for all m and k. Therefore, we have

$$u_m^{(k)} \in W(M,T), \quad \forall m, k. \tag{72}$$

Step 3 (limiting process). From (72), we deduce the existence of a subsequence of $\{u_m^{(k)}\}$ still so denoted, such that

$$\begin{aligned} &u_m^{(k)} \longrightarrow u_m \quad \text{in } L^\infty\left(0,T;V\cap H^2\right) \text{weak}^*, \\ &\dot{u}_m^{(k)} \longrightarrow u_m' \quad \text{in } L^\infty(0,T;V) \text{ weak}^*, \\ &\ddot{u}_m^{(k)} \longrightarrow u_m'' \quad \text{in } L^2(Q_T) \text{ weak}, \\ &u_m \in W(M,T). \end{aligned} \tag{73}$$

Passing to limit in (33), we have u_m satisfying (30), (31) in $L^2(0,T)$. On the other hand, it follows from $(30)_1$ and $(73)_4$ that $u_m'' = (\partial/\partial x)(\mu_m(t)u_{mx}) + f \in L^\infty(0,T;L^2)$; hence $u_m \in W_1(M,T)$ and the proof of Theorem 6 is complete. □

We note that $W_1(T) = \{v \in L^\infty(0,T;V) : v' \in L^\infty(0,T;L^2)\}$ is a Banach space with respect to the norm (see Lions [18]).

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;V)} + \left\|v'\right\|_{L^\infty(0,T;L^2)}. \tag{74}$$

We use the result given in Theorem 6 and the compact imbedding theorems to prove the existence and uniqueness of a weak solution of prob. (1)–(3). Hence, we get the main result in this section as follows.

Theorem 8. *Let (H_1)–(H_4) hold. Then one has the following.*

(i) Prob. (1)–(3) has a unique weak solution $u \in W_1(M,T)$, where the constants $M > 0$ and $T > 0$ are chosen as in Theorem 6.

(ii) Furthermore, the recurrent sequence $\{u_m\}$ defined by (29)-(30) converges to the solution u of prob. (1)–(3) strongly in $W_1(T)$.

And one has the estimate

$$\left\|u_m - u\right\|_{W_1(T)} \le C_T k_T^m, \quad \forall m \in \mathbb{N}, \tag{75}$$

where the constant $k_T \in [0,1)$ is defined as in (69) and C_T is a constant depending only on T, h_0, f, g, μ, $\widetilde{u}_0$, $\widetilde{u}_1$, and k_T.

Proof.

(a) Existence of the Solution. We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{aligned}&\left\langle w_m''(t), w\right\rangle + A_{m+1}(t; w_m(t), w) \\ &\quad = -A_{m+1}(t; u_m(t), w) + A_m(t; u_m(t), w), \\ &\qquad\qquad \forall w \in V, \\ &w_m(0) = w_m'(0) = 0.\end{aligned} \tag{76}$$

Note that

$$\begin{aligned}
&\frac{d}{dt}A_{m+1}\left(t;w_m(t),w_m(t)\right)\\
&\quad=2A_{m+1}\left(t;w_m(t),w'_m(t)\right)\\
&\qquad+\frac{\partial A_{m+1}}{\partial t}\left(t;w_m(t),w_m(t)\right),\\
&A_{m+1}\left(t;u_m(t),w'_m(t)\right)-A_m\left(t;u_m(t),w'_m(t)\right)\\
&\quad=-\left\langle\frac{\partial}{\partial x}\left[\left(\mu_{m+1}(t)-\mu_m(t)\right)u_{mx}(t)\right],w'_m(t)\right\rangle.
\end{aligned}\tag{77}$$

Taking $w=w'_m(t)$ in $(76)_1$, after integrating in t, we get

$$\begin{aligned}
Z_m(t)&=\int_0^t\frac{\partial A_{m+1}}{\partial t}\left(s;w_m(s),w_m(s)\right)ds\\
&\quad+2\int_0^t\left\langle\frac{\partial}{\partial x}\left[\left(\mu_{m+1}(s)-\mu_m(s)\right)u_{mx}(s)\right],\right.\\
&\quad\left.w'_m(s)\right\rangle ds\equiv J_1+J_2,
\end{aligned}\tag{78}$$

where

$$\begin{aligned}
Z_m(t)&=\left\|w'_m(t)\right\|^2+A_{m+1}\left(t;w_m(t),w_m(t)\right)\\
&\geq\left\|w'_m(t)\right\|^2+\mu_0\left\|w_m(t)\right\|_a^2,
\end{aligned}\tag{79}$$

and the integrals on the right-hand side of (78) are estimated as follows.

First Integral J_1. By $(37)_{(\mathrm{iv})}$ and (79), we have

$$\begin{aligned}
\left|J_1\right|&\leq\int_0^t\left|\frac{\partial A_{m+1}}{\partial t}\left(s;w_m(s),w_m(s)\right)\right|ds\\
&\leq\frac{\mu^*}{\mu_0}\int_0^t Z_m(s)\,ds.
\end{aligned}\tag{80}$$

Second Integral J_2. By the inequalities

$$\begin{aligned}
&\left\|\Delta u_m(s)\right\|_{H^2}\leq\left\|u_m(s)\right\|_{H^2}\leq M,\\
&\left\|u_{mx}(s)\right\|_{C^0(\overline{\Omega})}\leq\sqrt{2}\left\|u_{mx}(s)\right\|_{H^1}\leq\sqrt{2}\left\|u_m(s)\right\|_{H^2}\\
&\quad\leq\sqrt{2}M,\\
&\left\|D_i\mu\left[u_m\right](s)\right\|_{C^0(\overline{\Omega})}\leq\widetilde{K}_M(\mu),\quad i=1,3,\\
&\left\|D_1g\left[u_m\right](s)\right\|_{C^0(\overline{\Omega})}\leq\overline{K}_M(g),\\
&\left\|\mu_{m+1}(s)-\mu_m(s)\right\|_{C^0(\overline{\Omega})}\\
&\quad\leq2\widetilde{K}_M(\mu)\overline{K}_M(g)\left\|\nabla w_{m-1}(s)\right\|\\
&\quad\leq2\widetilde{K}_M(\mu)\overline{K}_M(g)\left\|w_{m-1}\right\|_{W_1(T)},\\
&\left\|D_i\mu\left[u_m\right](s)-D_i\mu\left[u_{m-1}\right](s)\right\|_{C^0(\overline{\Omega})}\\
&\quad\leq2\widetilde{K}_M(\mu)\overline{K}_M(g)\left\|w_{m-1}\right\|_{W_1(T)},\quad i=1,3,\\
&\left\|D_1g\left[u_m\right](s)-D_1g\left[u_{m-1}\right](s)\right\|_{C^0(\overline{\Omega})}\\
&\quad\leq2\overline{K}_M(g)\left\|w_{m-1}\right\|_{W_1(T)}
\end{aligned}\tag{81}$$

and from the equation

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[\left(\mu_{m+1}(s)-\mu_m(s)\right)\nabla u_m(s)\right]=\left(\mu_{m+1}(s)\right.\\
&\quad\left.-\mu_m(s)\right)\Delta u_m(s)+\left(D_1\mu\left[u_m\right](s)-D_1\mu\left[u_{m-1}\right]\right.\\
&\quad\left.\cdot(s)\right)u_{mx}(s)+\left[\left(D_3\mu\left[u_m\right](s)-D_3\mu\left[u_{m-1}\right](s)\right)\right.\\
&\quad\left.\cdot\int_0^1D_1g\left[u_m\right](x,y,s)\,dy\right]u_{mx}(s)+\left[D_3\mu\left[u_{m-1}\right]\right.\\
&\quad\left.\cdot(s)\int_0^1\left(D_1g\left[u_m\right]-D_1g\left[u_{m-1}\right]\right)dy\right]u_{mx}(s),
\end{aligned}\tag{82}$$

we obtain that

$$\begin{aligned}
&\left\|\frac{\partial}{\partial x}\left[\left(\mu_{m+1}(s)-\mu_m(s)\right)\nabla u_m(s)\right]\right\|\leq2M\widetilde{K}_M(\mu)\\
&\quad\cdot\overline{K}_M(g)\left[1+\sqrt{2}\left(2+\overline{K}_M(g)\right)\right]\left\|w_{m-1}\right\|_{W_1(T)}.
\end{aligned}\tag{83}$$

This implies that

$$\begin{aligned}
&\left|J_2\right|\leq2\left|\int_0^t\left\langle\frac{\partial}{\partial x}\left[\left(\mu_{m+1}(s)-\mu_m(s)\right)\nabla u_m(s)\right],\right.\right.\\
&\quad\left.\left.w'_m(s)\right\rangle ds\right|\leq4TM^2\widetilde{K}_M^2(\mu)\overline{K}_M^2(g)\left[1\right.\\
&\quad\left.+\sqrt{2}\left(2+\overline{K}_M(g)\right)\right]^2\left\|w_{m-1}\right\|_{W_1(T)}^2+\int_0^tZ_m(s)\,ds.
\end{aligned}\tag{84}$$

Combining (78), (80), and (84), we obtain

$$\begin{aligned}
Z_m(t)&\leq4TM^2\widetilde{K}_M^2(\mu)\overline{K}_M^2(g)\\
&\quad\cdot\left[1+\sqrt{2}\left(2+\overline{K}_M(g)\right)\right]^2\left\|w_{m-1}\right\|_{W_1(T)}^2\\
&\quad+\frac{\mu_0+\mu^*}{\mu_0}\int_0^tZ_m(s)\,ds.
\end{aligned}\tag{85}$$

Using Gronwall's Lemma, we deduce from (85) that

$$\left\|w_m\right\|_{W_1(T)}\leq k_T\left\|w_{m-1}\right\|_{W_1(T)},\quad\forall m\in\mathbb{N},\tag{86}$$

where $k_T \in (0,1)$ is defined as in (69), which implies that

$$\left\|u_m - u_{m+p}\right\|_{W_1(T)} \le \left\|u_0 - u_1\right\|_{W_1(T)} \left(1 - k_T\right)^{-1} k_T^m, \quad \forall m, p \in \mathbb{N}. \tag{87}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \longrightarrow u \quad \text{strongly in } W_1(T). \tag{88}$$

Note that $u_m \in W_1(M,T)$; then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{aligned} &u_{m_j} \longrightarrow u \quad \text{in } L^\infty\left(0,T;V\cap H^2\right) \text{weak}^*, \\ &u'_{m_j} \longrightarrow u' \quad \text{in } L^\infty(0,T;V) \text{ weak}^*, \\ &u''_{m_j} \longrightarrow u'' \quad \text{in } L^2\left(Q_T\right) \text{weak}, \\ &u \in W(M,T). \end{aligned} \tag{89}$$

We also note that

$$\begin{aligned} &\left|\mu_m(x,t) - \mu[u](x,t)\right| \\ &\quad \le 2\widetilde{K}_M(\mu)\overline{K}_M(g)\left\|u_{m-1} - u\right\|_{W_1(T)}, \\ &\qquad \text{a.e. } (x,t) \in Q_T. \end{aligned} \tag{90}$$

Hence, from (88) and (90), we obtain

$$\mu_m \longrightarrow \mu[u] \quad \text{strongly in } L^\infty\left(Q_T\right). \tag{91}$$

On the other hand, for all $v \in V$, we have

$$\begin{aligned} &\left|A_m(t;u_m,v) - A[u](t;u,v)\right| \le (1+h_0)\widetilde{K}_M(\mu) \\ &\quad \cdot \left[2\overline{K}_M(g) M \left\|u_{m-1} - u\right\|_{W_1(T)} + \left\|u_m - u\right\|_{W_1(T)}\right] \\ &\quad \cdot \left\|v_x\right\|. \end{aligned} \tag{92}$$

Hence

$$\int_0^T \left(A_m(t;u_m,v) - A[u](t;u,v)\right)\phi(t)\,dt \longrightarrow 0 \quad \forall v \in V,\ \forall \phi \in L^1(0,T). \tag{93}$$

Finally, passing to limit in (30)-(31) as $m = m_j \to \infty$, it is implied from (88), $(89)_{1,3}$, and (93) that there exists $u \in W(M,T)$ satisfying the equation

$$\left\langle u''(t), w\right\rangle + A[u](t;u(t),w) = \left\langle f(t), w\right\rangle \tag{94}$$

for all $w \in V$ and the initial conditions

$$\begin{aligned} u(0) &= \widetilde{u}_0, \\ u'(0) &= \widetilde{u}_1. \end{aligned} \tag{95}$$

On the other hand, from the assumptions (H_2)–(H_4) we obtain from $(89)_4$, (93), and (94) that

$$u'' = \frac{\partial}{\partial x}\left(\mu[u](t)u_x\right) + f \in L^\infty\left(0,T;L^2\right), \tag{96}$$

and thus we have $u \in W_1(M,T)$. The existence result follows.

(b) Uniqueness of the Solution. Let $u_1, u_2 \in W_1(M,T)$ be two weak solutions of prob. (1)–(3). Then $u = u_1 - u_2$ satisfies the variational problem

$$\begin{aligned} &\left\langle u''(t), w\right\rangle + A[u_1](t;u(t),w) \\ &\quad = -A[u_1](t;u_2(t),w) + A[u_2](t;u_2(t),w), \\ &\qquad \forall w \in V, \\ &u(0) = u'(0) = 0, \end{aligned} \tag{97}$$

where

$$\begin{aligned} &A[u_i](t;u,w) \\ &\quad = \left\langle \mu[u_i](t)u_x, w_x\right\rangle + h_0\mu[u_i](0,t)u(0)w(0), \\ &\qquad u,w \in V, \\ &\mu[u_i](x,t) \\ &\quad = \mu\left(x,t,\int_0^1 g\left(x,y,t,u_i(y,t),\nabla u_i(y,t)\,dy\right)\right), \\ &\qquad i = 1,2. \end{aligned} \tag{98}$$

We take $w = u'$ in $(97)_1$ and integrate in t to get

$$\begin{aligned} Z(t) &= \int_0^t \frac{\partial A[u_1]}{\partial t}(s;u(s),u(s))\,ds \\ &\quad + 2\int_0^t \left\langle \frac{\partial}{\partial x}\left[\left(\mu[u_1](s) - \mu[u_2](s)\right)\nabla u_2(s)\right], \right. \\ &\quad \left. u'(s)\right\rangle ds, \end{aligned} \tag{99}$$

where $Z(t) = \|u'(t)\|^2 + A[u_1](t;u(t),u(t))$.

Put $\widetilde{Z}_M = \mu^*/\mu_0 + (2/\sqrt{\mu_0})M\widetilde{K}_M(\mu)\overline{K}_M(g)[1+\sqrt{2}(2+\overline{K}_M(g))]$, and then it follows from (99) that

$$Z(t) \le \widetilde{Z}_M \int_0^t Z(s)\,ds. \tag{100}$$

By Gronwall's Lemma, we deduce $Z(t) = 0$; that is, $u_1 \equiv u_2$. Theorem 8 is proved completely. □

4. Asymptotic Expansion of the Solution with respect to a Small Parameter

In this section, let (H_1)–(H_4) hold. We make more the following assumptions:

(H_2') $g_1 \in C^2([0,1]^2 \times \mathbb{R}_+ \times \mathbb{R}^2)$.

(H_3') $\mu_1 \in C^2([0,1]\times\mathbb{R}_+\times\mathbb{R})$ and $\mu_1(x,t,z) \geq 0$, for all $(x, t, z) \in [0,1]\times\mathbb{R}_+\times\mathbb{R}$.

We consider the following perturbed problem, where ε is a small parameter such that $|\varepsilon| \leq 1$:

$$\begin{aligned} u_{tt} - \frac{\partial}{\partial x}\left[\mu_\varepsilon[u](x,t)\, u_x\right] &= f(x,t), \\ &\qquad 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) &= u(1,t) = 0, \\ u(x,0) &= \tilde{u}_0(x), \\ u_t(x,0) &= \tilde{u}_1(x), \end{aligned} \tag{P_ε}$$

where

$$\begin{aligned} \mu_\varepsilon[u](x,t) &= \mu\left(x,t,\int_0^1 g[u](x,y,t)\,dy\right) \\ &\quad + \varepsilon\mu_1\left(x,t,\int_0^1 g_1[u](x,y,t)\,dy\right), \\ g[u](x,y,t) &= g(x,y,t,u(y,t),u_x(y,t)), \\ g_1[u](x,y,t) &= g_1(x,y,t,u(y,t),u_x(y,t)). \end{aligned} \tag{101}$$

By Theorem 8, problem (P_ε) has a unique weak solution u_ε depending on ε, satisfying $u_\varepsilon \in W_1(M,T)$, in which M, T are independent of ε; these constants are chosen as in (67), (68), and (69), with $\overline{K}_M(g)+\overline{K}_M(g_1)$, $\widetilde{K}_M(\mu)+\widetilde{K}_M(\mu_1)$ stand for $\overline{K}_M(g)$, $\widetilde{K}_M(\mu)$, respectively.

Moreover, we can prove that the limit u_0 in suitable function spaces of the family $\{u_\varepsilon\}$ as $\varepsilon \to 0$ is a unique weak solution of the problem (P_0) (corresponding to $\varepsilon = 0$) also satisfying $u_0 \in W_1(M,T)$.

We shall study the asymptotic expansion of the solution of the problem (P_ε) with respect to a small parameter ε.

We use the following notations. For a multi-index $\alpha = (\alpha_1,\ldots,\alpha_p) \in \mathbb{Z}_+^N$ and $x = (x_1,\ldots,x_N) \in \mathbb{R}^N$, we put

$$\begin{aligned} |\alpha| &= \alpha_1 + \cdots + \alpha_N, \\ \alpha! &= \alpha_1!\cdots\alpha_N!, \\ \alpha,\beta &\in \mathbb{Z}_+^N, \\ \alpha \leq \beta &\Longleftrightarrow \alpha_i \leq \beta_i, \quad \forall i = 1,\ldots,N, \\ x^\alpha &= x_1^{\alpha_1}\cdots x_N^{\alpha_N}. \end{aligned} \tag{102}$$

First, we need the following lemma.

Lemma 9. *Let $m, N \in \mathbb{N}$ and $x = (x_1,\ldots,x_N) \in \mathbb{R}^N$, $\varepsilon \in \mathbb{R}$. Then*

$$\left(\sum_{i=1}^N x_i\varepsilon^i\right)^m = \sum_{k=m}^{mN} P_k^{(m)}[N,x]\,\varepsilon^k, \tag{103}$$

where the coefficients $P_k^{(m)}[N,x]$, $m \leq k \leq mN$, depend on $x = (x_1,\ldots,x_N)$ defined by the formulas

$$\begin{aligned} &P_k^{(m)}[N,x] \\ &\quad = \begin{cases} x_k, & 1 \leq k \leq N,\ m = 1, \\ \displaystyle\sum_{\alpha\in A_k^{(m)}(N)} \frac{m!}{\alpha!}x^\alpha, & m \leq k \leq mN,\ m \geq 2, \end{cases} \\ &A_k^{(m)}(N) = \left\{\alpha \in \mathbb{Z}_+^N : |\alpha| = m,\ \sum_{i=1}^N i\alpha_i = k\right\}. \end{aligned} \tag{104}$$

Proof of Lemma 9. The proof of Lemma 9 is easy; hence we omit the details. □

Now, we assume that

(H_2^N) $g \in C^{N+2}([0,1]^2\times\mathbb{R}_+\times\mathbb{R}^2)$, $g_1 \in C^{N+1}([0,1]^2\times\mathbb{R}_+\times\mathbb{R}^2)$;

(H_3^N) $\mu \in C^{N+2}([0,1]\times\mathbb{R}_+\times\mathbb{R})$, $\mu_1 \in C^{N+1}([0,1]\times\mathbb{R}_+\times\mathbb{R})$, $\mu \geq \mu_0 > 0$ and $\mu_1 \geq 0$, for all $(x,t,z) \in [0,1]\times\mathbb{R}_+\times\mathbb{R}$.

We use notations $\mu[u](x,t) = \mu\left(x,t,\int_0^1 g[u](x,y,t)dy\right)$, $g[u](x,y,t) = g(x,y,t,u(y,t),u_x(y,t))$.

Let u_0 be a unique weak solution of the problem (P_0) corresponding to $\varepsilon = 0$; that is,

$$\begin{aligned} u_0'' - \frac{\partial}{\partial x}\left(\mu[u_0]\,u_{0x}\right) &= f, \quad 0 < x < 1,\ 0 < t < T, \\ u_{0x}(0,t) - h_0 u_0(0,t) &= u_0(1,t) = 0, \\ u_0(x,0) &= \tilde{u}_0(x), \\ u_0'(x,0) &= \tilde{u}_1(x), \\ &\qquad u_0 \in W_1(M,T). \end{aligned} \tag{P_0}$$

Let us consider the sequence of the weak solutions u_k, $1 \leq k \leq N$, defined by the following problems:

$$\begin{aligned} u_k'' - \frac{\partial}{\partial x}\left(\mu[u_0]\,u_{kx}\right) &= \widetilde{F}_k[u_k], \\ &\qquad 0 < x < 1,\ 0 < t < T, \\ u_{kx}(0,t) - h_0 u_k(0,t) &= u_k(1,t) = 0, \\ u_k(x,0) &= u_k'(x,0) = 0, \\ &\qquad u_k \in W_1(M,T), \end{aligned} \tag{$\widetilde{P}_k$}$$

where $\widetilde{F}_k[u_k]$, $1 \leq k \leq N$, are defined by the formulas

$$\begin{aligned} \widetilde{F}_k &= \widetilde{F}_k[u_k] \\ &= \sum_{i=1}^k \frac{\partial}{\partial x}\left[\left(\rho_i[N,\mu,g] + \rho_{i-1}[N-1,\mu_1,g_1]\right)\right. \\ &\quad \left.\nabla u_{k-i}\right], \quad 1 \leq k \leq N, \end{aligned} \tag{105}$$

with $\rho_k[N,\mu,g]$, which are defined by the formulas

$$\rho_k[N,\mu,g] = \begin{cases} \mu[u_0], & k=0, \\ \sum_{j=1}^{k} \frac{1}{j!} D^j \mu[u_0] \Phi_k[j,N,g,u_0,\vec{u}], & 1 \le k \le N, \end{cases} \tag{106}$$

where $\vec{u} = (u_1,\ldots,u_N)$ and

$$\Phi_k[j,N,g,u_0,\vec{u}] = P_k^{(j)}[N,\vec{\Psi}[N,g,u_0,\vec{u},\nabla\vec{u}]] = \begin{cases} \overline{\Psi}_k[N,g,u_0,\vec{u},\nabla\vec{u}], & j=1, \\ \sum_{\alpha\in A_k^{(j)}(N)} \frac{j!}{\alpha!}\vec{\Psi}^{\alpha}[N,g,u_0,\vec{u},\nabla\vec{u}], & j \le k \le jN,\ j \ge 2, \end{cases} \tag{107}$$

with $\vec{\Psi}[N,g,u_0,\vec{u},\nabla\vec{u}] = (\overline{\Psi}_1[N,g,u_0,\vec{u},\nabla\vec{u}],\ldots,\overline{\Psi}_N[N,g,u_0,\vec{u},\nabla\vec{u}])$, are defined by

$$\begin{aligned} \overline{\Psi}_k[N,g,u_0,\vec{u},\nabla\vec{u}] &= \sum_{1\le|\beta|\le k} \frac{1}{\beta!}\int_0^1 D^\beta g[u_0]\Psi_k[\beta,N,\vec{u},\nabla\vec{u}]\,dy, \\ &\qquad 1 \le k \le N, \\ \Psi_k[\beta,N,\vec{u},\nabla\vec{u}] &= \sum_{(i,j)\in\widetilde{A}(\beta,N),i+j=k} P_i^{(\beta_1)}[N,\vec{u}]P_j^{(\beta_2)}[N,\nabla\vec{u}], \\ \widetilde{A}(\beta,N) &= \{(i,j)\in\mathbb{Z}_+^2 : \beta_1 \le i \le N\beta_1,\ \beta_2 \le j \le N\beta_2\}. \end{aligned} \tag{108}$$

Then, we have the following theorem.

Theorem 10. *Let (H_1), (H_4), (H_2^N), and (H_3^N) hold. Then there exist constants $M>0$ and $T>0$ such that, for every $0\le\varepsilon\le1$, the problem (P_ε) has a unique weak solution $u_\varepsilon \in W_1(M,T)$ satisfying the asymptotic estimation up to order $N+1$ as follows:*

$$\left\| u_\varepsilon - \sum_{k=0}^{N} u_k\varepsilon^k \right\|_{W_1(T)} \le C_T\varepsilon^{N+1}, \tag{109}$$

where the functions u_k, $0\le k\le N$, are the weak solutions of the problems (P_0), $(\widetilde{P}_k)$, $1\le k\le N$, respectively, and C_T is a constant depending only on $N, T, \mu, \mu_1, f, g, g_1, u_k, 0\le k\le N$.

In order to prove Theorem 10, we need the following lemmas.

Lemma 11. *Let $\rho_k[N,\mu,g]$, $1\le k\le N$, be the functions defined by the formulas (106)–(108). Putting $h=\sum_{k=0}^N u_k\varepsilon^k$, then one has*

$$\mu[h] = \sum_{k=0}^{N}\rho_k[N,\mu,g]\varepsilon^k + \varepsilon^{N+1}\widehat{R}_N[\mu,g,u_0,\vec{u},\varepsilon], \tag{110}$$

with $\|\widehat{R}_N[\mu,g,u_0,\vec{u},\varepsilon]\|_{L^\infty(0,T;L^2)} \le C$, where C is a constant depending only on N, T, μ, g, u_k, $0\le k\le N$.

Proof of Lemma 11. In the case of $N=1$, the proof of (110) is easy; hence we omit the details, which we only prove with $N\ge2$. Putting $h=u_0+\sum_{k=1}^N u_k\varepsilon^k \equiv u_0+h_1$, we rewrite as follows:

$$\begin{aligned} \mu[h](x,t) &= \mu\left(x,t,\int_0^1 g(x,y,t,u_0+h_1,\nabla u_0+\nabla h)\,dy\right) \\ &= \mu\left(x,t,\int_0^1 g[u_0](x,y,t)\,dy+\xi\right), \end{aligned} \tag{111}$$

where $\xi = \int_0^1(g[u_0+h_1](x,y,t)-g[u_0](x,y,t))dy$ and $\overline{g}[u_0](x,t)=\int_0^1 g(x,y,t,u_0(y,t),\nabla u_0(y,t))dy$.

By using Taylor's expansion of the function $\mu(x,t,\int_0^1 g[u_0](x,y,t)dy+\xi)$ around the point $[u_0]\equiv(x,t,\int_0^1 g[u_0](x,y,t)dy)$ up to order $N+1$, we obtain

$$\begin{aligned} \mu[h](x,t) &= \mu\left(x,t,\int_0^1 g[u_0](x,y,t)\,dy+\xi\right) \\ &= \mu[u_0] + \sum_{j=1}^{N}\frac{1}{j!}D_3^j\mu[u_0]\xi^j \\ &\quad + R_N[\mu,u_0,h_1,\xi], \end{aligned} \tag{112}$$

where

$$\begin{aligned} R_N[\mu,u_0,h_1,\xi] &= \frac{1}{N!}\xi^{N+1}\int_0^1(1-\theta)^N D_3^{N+1}\mu \\ &\quad \cdot\left(x,t,\int_0^1 g[u_0](x,y,t)\,dy+\theta\xi\right)d\theta \\ &= \varepsilon^{N+1}\widehat{R}_N^{(1)}[\mu,u_0,h_1,\xi,\varepsilon]. \end{aligned} \tag{113}$$

Similarly, with $g[h]$, by using Taylor's expansion of the function $g[h](x,y,t)=g(x,y,t,u_0+h_1,\nabla u_0+\nabla h_1)$ around the point $[u_0]\equiv(x,y,t,u_0,\nabla u_0)$ up to order $N+1$, we obtain

$$\begin{aligned} g[h](x,y,t) &= g[u_0] \\ &\quad + \sum_{1\le|\beta|\le N}\frac{1}{\beta!}D^\beta g[u_0]h_1^{\beta_1}(\nabla h_1)^{\beta_2} \\ &\quad + R_N[g,u_0,h_1,\varepsilon], \end{aligned} \tag{114}$$

where

$$\begin{aligned} R_N[g,u_0,h_1,\varepsilon] &= \sum_{|\beta|=N+1}\frac{N+1}{\beta!}\int_0^1(1-\theta)^N \\ &\quad \cdot D^\beta g(x,y,t,u_0+\theta h_1,\nabla u_0+\theta\nabla h_1) \\ &\quad \cdot h_1^{\beta_1}(\nabla h_1)^{\beta_2}\,d\theta = \varepsilon^{N+1}R_N^{(1)}[\varepsilon,g,u_0,h_1], \end{aligned} \tag{115}$$

$\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^2, |\beta| = \beta_1 + \beta_2, \beta! = \beta_1!\beta_2!, D^\beta g = D_4^{\beta_1} D_5^{\beta_2} g$, $D^\beta g[u_0] = D^\beta g(x, y, t, u_0, \nabla u_0)$.

Note, by formula (103), we get

$$h_1^{\beta_1} = \left(\sum_{k=1}^{N} u_k \varepsilon^k\right)^{\beta_1} = \sum_{i=\beta_1}^{N\beta_1} P_i^{(\beta_1)}\left[N, \overrightarrow{u}\right] \varepsilon^i, \tag{116}$$

where $\overrightarrow{u} = (u_1, \ldots, u_N)$.

Similarly, with $(\nabla h_1)^{\gamma_2}$, we also have

$$(\nabla h_1)^{\beta_2} = \left(\sum_{k=1}^{N} \nabla u_k \varepsilon^k\right)^{\beta_2} = \sum_{j=\beta_2}^{N\beta_2} P_j^{(\beta_2)}\left[N, \nabla\overrightarrow{u}\right] \varepsilon^j, \tag{117}$$

where $\nabla\overrightarrow{u} = (\nabla u_1, \ldots, \nabla u_N)$.

Hence, we deduce from (116), (117) that

$$\begin{aligned} h_1^{\beta_1} (\nabla h_1)^{\beta_2} = {} & \sum_{k=|\beta|}^{N} \Psi_k\left[\beta, N, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \varepsilon^k \\ & + \sum_{k=N+1}^{N|\beta|} \Psi_k\left[\beta, N, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \varepsilon^k, \end{aligned} \tag{118}$$

where

$$\begin{aligned} & \Psi_k\left[\beta, N, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \\ & \quad = \sum_{(i,j)\in\widetilde{A}(\beta,N), i+j=k} P_i^{(\beta_1)}\left[N, \overrightarrow{u}\right] P_j^{(\beta_2)}\left[N, \nabla\overrightarrow{u}\right], \\ & \widetilde{A}(\beta, N) \\ & \quad = \left\{(i, j) \in \mathbb{Z}_+^2 : \beta_1 \le i \le N\beta_1,\ \beta_2 \le j \le N\beta_2\right\}. \end{aligned} \tag{119}$$

Hence, it follows from (114), (118) that

$$\begin{aligned} & g[h] \\ & \quad = g[u_0] \\ & \qquad + \sum_{1\le|\beta|\le N} \frac{1}{\beta!} D^\beta g[u_0] \sum_{k=|\beta|}^{N} \Psi_k\left[\beta, N, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \varepsilon^k \\ & \qquad + \sum_{1\le|\beta|\le N} \frac{1}{\beta!} D^\beta g[u_0] \sum_{k=N+1}^{N|\beta|} \Psi_k\left[\beta, N, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \varepsilon^k \\ & \qquad + \varepsilon^{N+1} R_N^{(1)}\left[\varepsilon, g, u_0, h_1\right] \\ & \quad = g[u_0] \\ & \qquad + \sum_{k=1}^{N} \sum_{1\le|\beta|\le k} \frac{1}{\beta!} D^\beta g[u_0] \Psi_k\left[\beta, N, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \varepsilon^k \\ & \qquad + \varepsilon^{N+1} R_N^{(2)}\left[\varepsilon, g, u_0, h_1, \overrightarrow{u}\right], \end{aligned} \tag{120}$$

where

$$\begin{aligned} & \varepsilon^{N+1} R_N^{(2)}\left[\varepsilon, g, u_0, h_1, \overrightarrow{u}\right] \\ & \quad = \sum_{1\le|\beta|\le N} \frac{1}{\beta!} D^\beta g[u_0] \sum_{k=N+1}^{N|\beta|} \Psi_k\left[\beta, N, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \varepsilon^k \\ & \qquad + \varepsilon^{N+1} R_N^{(1)}\left[\varepsilon, g, u_0, h_1\right]. \end{aligned} \tag{121}$$

Therefore

$$\begin{aligned} \xi &= \int_0^1 \left(g[h](x, y, t) - g[u_0](x, y, t)\right) dy \\ &= \sum_{k=1}^{N} \left(\sum_{1\le|\beta|\le k} \frac{1}{\beta!} \int_0^1 D^\beta g[u_0] \Psi_k\left[\beta, N, \overrightarrow{u}, \nabla\overrightarrow{u}\right] dy\right) \\ &\quad \cdot \varepsilon^k + \varepsilon^{N+1} \int_0^1 R_N^{(2)}\left[\varepsilon, g, u_0, h_1, \overrightarrow{u}\right] dy \\ &= \sum_{k=1}^{N} \overline{\Psi}_k\left[N, g, u_0, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \varepsilon^k \\ &\quad + \varepsilon^{N+1} R_N^{(3)}\left[\varepsilon, g, u_0, h_1, \overrightarrow{u}\right], \end{aligned} \tag{122}$$

with

$$\begin{aligned} & \overline{\Psi}_k\left[N, g, u_0, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \\ & \quad = \sum_{1\le|\beta|\le k} \frac{1}{\beta!} \int_0^1 D^\beta g[u_0] \Psi_k\left[\beta, N, \overrightarrow{u}, \nabla\overrightarrow{u}\right] dy, \\ & \qquad\qquad 1 \le k \le N, \\ & R_N^{(3)}\left[\varepsilon, g, u_0, h_1, \overrightarrow{u}\right] = \int_0^1 R_N^{(2)}\left[\varepsilon, g, u_0, h_1, \overrightarrow{u}\right] dy. \end{aligned} \tag{123}$$

On the other hand, we also have

$$\begin{aligned} \xi^j &= \left(\sum_{k=1}^{N} \overline{\Psi}_k\left[N, g, u_0, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \varepsilon^k \right. \\ &\qquad \left. + \varepsilon^{N+1} R_N^{(3)}\left[\varepsilon, g, u_0, h_1, \overrightarrow{u}\right]\right)^j \\ &= \left(\sum_{k=1}^{N} \overline{\Psi}_k\left[N, g, u_0, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \varepsilon^k\right)^j \\ &\quad + \varepsilon^{N+1} R_N^{(4)}\left[\varepsilon, j, g, u_0, h_1, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \\ &= \sum_{k=j}^{jN} P_k^{(j)}\left[N, \overrightarrow{\Psi}\left[N, g, u_0, \overrightarrow{u}, \nabla\overrightarrow{u}\right]\right] \varepsilon^k \\ &\quad + \varepsilon^{N+1} R_N^{(4)}\left[\varepsilon, j, g, u_0, h_1, \overrightarrow{u}, \nabla\overrightarrow{u}\right] \\ &\equiv \sum_{k=j}^{jN} \Phi_k\left[j, N, g, u_0, \overrightarrow{u}\right] \varepsilon^k \\ &\quad + \varepsilon^{N+1} R_N^{(4)}\left[\varepsilon, j, g, u_0, h_1, \overrightarrow{u}, \nabla\overrightarrow{u}\right], \end{aligned} \tag{124}$$

where

$$\Phi_k\left[j,N,g,u_0,\vec{u}\right]=P_k^{(j)}\left[N,\vec{\Psi}\left[N,g,u_0,\vec{u},\nabla\vec{u}\right]\right]$$
$$=\begin{cases}\overline{\Psi}_k\left[N,g,u_0,\vec{u},\nabla\vec{u}\right], & j=1,\\ \displaystyle\sum_{\alpha\in A_k^{(j)}(N)}\frac{j!}{\alpha!}\vec{\Psi}^{\alpha}\left[N,g,u_0,\vec{u},\nabla\vec{u}\right], & j\le k\le jN,\ j\ge 2,\end{cases}\quad(125)$$
$$A_k^{(j)}(N)=\left\{\alpha\in\mathbb{Z}_+^N:|\alpha|=j,\ \sum_{i=1}^{N}i\alpha_i=k\right\},$$

with $\vec{\Psi}[N,g,u_0,\vec{u},\nabla\vec{u}]=(\overline{\Psi}_1[N,g,u_0,\vec{u},\nabla\vec{u}],\ldots,\overline{\Psi}_N[N,g,u_0,\vec{u},\nabla\vec{u}])$, which are defined by (108).

Decompose the sum $\sum_{k=j}^{jN}$ into the sum of two sums $\sum_{k=j}^{N}$ and $\sum_{k=N+1}^{jN}$; therefore, we deduce from (124) that

$$\xi^j=\sum_{k=j}^{N}\Phi_k\left[j,N,g,u_0,\vec{u}\right]\varepsilon^k+\varepsilon^{N+1}R_N^{(5)}\left[\varepsilon,j,g,u_0,\vec{u}\right],\quad(126)$$

where

$$\varepsilon^{N+1}R_N^{(5)}\left[\varepsilon,j,g,u_0,\vec{u}\right]=\sum_{k=N+1}^{jN}\Phi_k\left[j,N,g,u_0,\vec{u}\right]\varepsilon^k+\varepsilon^{N+1}R_N^{(4)}\left[\varepsilon,j,g,u_0,h_1,\vec{u},\nabla\vec{u}\right].\quad(127)$$

Hence, it follows from (112) and (124) that

$$\begin{aligned}\mu[h]&=\mu[u_0]+\sum_{1\le j\le N}\frac{1}{j!}D^j\mu[u_0]\sum_{k=j}^{N}\Phi_k\left[j,N,g,u_0,\vec{u}\right]\varepsilon^k\\&+\varepsilon^{N+1}\left(\sum_{1\le j\le N}\frac{1}{j!}D^j\mu[u_0]R_N^{(5)}\left[\varepsilon,j,g,u_0,h_1,\vec{u}\right]+\widehat{R}_N^{(1)}[\mu,u_0,h_1,\xi,\varepsilon]\right)=\mu[u_0]\\&+\sum_{k=1}^{N}\left(\sum_{1\le j\le k}\frac{1}{j!}D^j\mu[u_0]\Phi_k\left[j,N,g,u_0,\vec{u}\right]\right)\varepsilon^k\\&+\varepsilon^{N+1}\widehat{R}_N\left[\mu,g,u_0,\vec{u},\varepsilon\right]=\sum_{k=0}^{N}\rho_k[N,\mu,g]\varepsilon^k\\&+\varepsilon^{N+1}\widehat{R}_N\left[\mu,g,u_0,\vec{u},\varepsilon\right],\end{aligned}\quad(128)$$

where $\rho_k[N,\mu,g]$, $1\le k\le N$, are defined by (106)–(108) and

$$\widehat{R}_N\left[\mu,g,u_0,\vec{u},\varepsilon\right]=\sum_{1\le j\le N}\frac{1}{j!}D^j\mu[u_0]R_N^{(5)}\left[\varepsilon,j,g,u_0,h_1,\vec{u}\right]+\widehat{R}_N^{(1)}[\mu,u_0,h_1,\xi].\quad(129)$$

By the boundedness of the functions u_k, ∇u_k, $1\le k\le N$, in the function space $L^\infty(0,T;L^2)$, we obtain from (113), (115), (121), $(123)_2$, (124), and (127) that $\|\widehat{R}_N[\mu,g,u_0,\vec{u},\varepsilon]\|_{L^\infty(0,T;L^2)}\le C$, where C is a constant depending only on N, T, μ, g, u_k, $1\le k\le N$. Thus, Lemma 11 is proved. □

Remark 12. Lemma 11 is a generalization of the formula contained in ([9], p.262, formula (4.38)) and it is useful to obtain the following Lemma 13. These lemmas are the key to establish the asymptotic expansion of the weak solution u_ε of order $N+1$ in a small parameter ε as follows.

Let $u=u_\varepsilon\in W_1(M,T)$ be the unique weak solution of problem (P_ε). Then $v=u_\varepsilon-\sum_{k=0}^{N}u_k\varepsilon^k\equiv u_\varepsilon-h$ satisfies the problem

$$\begin{aligned}&v''-\frac{\partial}{\partial x}\left(\mu_\varepsilon[v+h]v_x\right)=\frac{\partial}{\partial x}\left[\left(\mu_\varepsilon[v+h]-\mu_\varepsilon[h]\right)h_x\right]+E_\varepsilon(x,t),\\&\qquad 0<x<1,\ 0<t<T,\\&v_x(0,t)-h_0v(0,t)=v(1,t)=0,\\&v(x,0)=v'(x,0)=0,\end{aligned}\quad(130)$$

where

$$E_\varepsilon(x,t)=\frac{\partial}{\partial x}\left[\left(\mu[h]-\mu[u_0]+\varepsilon\mu_1[h]\right)h_x\right]-\sum_{k=1}^{N}\widetilde{F}_k\varepsilon^k.\quad(131)$$

Then, we have the following lemma.

Lemma 13. *Let (H_1), (H_4), (H_2^N), and (H_3^N) hold. Then there exists a constant C_* such that*

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)}\le C_*\varepsilon^{N+1},\quad(132)$$

where C_* is a constant depending only on N, T, μ, μ_1, g, g_1, u_k, $1\le k\le N$.

Proof of Lemma 13. In the case of $N=1$, the proof of Lemma 13 is easy; hence we omit the details, which we only prove with $N\ge 2$.

By using formula (110) for the function $\mu_1[h]$ we obtain

$$\mu_1[h] = \sum_{k=0}^{N-1} \rho_k[N-1, \mu_1, g_1]\varepsilon^k + \varepsilon^N \widehat{R}_{N-1}[\mu_1, g_1, u_0, \overrightarrow{u}, \varepsilon], \tag{133}$$

where $\|\widehat{R}_{N-1}[\mu_1, g_1, u_0, \overrightarrow{u}, \varepsilon]\|_{L^\infty(0,T;L^2)} \le C$, with C being a constant depending only on $N, T, \mu_1, g_1, u_k, 0 \le k \le N$.

By (133), we rewrite $\varepsilon\mu_1[h]$ as follows:

$$\varepsilon\mu_1[h] = \sum_{k=1}^{N} \rho_{k-1}[N-1, \mu_1, g_1]\varepsilon^k + \varepsilon^{N+1}\widehat{R}_{N-1}[\mu_1, g_1, u_0, \overrightarrow{u}, \varepsilon]. \tag{134}$$

Hence, we deduce from (110) and (134) that

$$\begin{aligned}(\mu[h] - \mu[u_0] + \varepsilon\mu_1[h])h_x &= \sum_{k=1}^{N} \nabla u_0 (\rho_k[N, \mu, g] \\ &\quad + \rho_{k-1}[N-1, \mu_1, g_1])\varepsilon^k \\ &\quad + \sum_{k=2}^{2N}\left(\sum_{i,j=1,i+j=k}^{N} (\rho_i[N, \mu, g] + \rho_{i-1}[N-1, \mu_1, g_1])\right. \\ &\quad \left.\nabla u_j\right)\varepsilon^k + \varepsilon^{N+1}\widetilde{R}_N^{(1)}[\mu, g, \mu_1, g_1, u_0, \overrightarrow{u}, \varepsilon],\end{aligned} \tag{135}$$

where

$$\begin{aligned}&\widetilde{R}_N^{(1)}[\mu, g, \mu_1, g_1, u_0, \overrightarrow{u}, \varepsilon] \\ &\quad = \left(\widehat{R}_N[\mu, g, u_0, \overrightarrow{u}, \varepsilon] + \widehat{R}_{N-1}[\mu_1, g_1, u_0, \overrightarrow{u}, \varepsilon]\right)h_x.\end{aligned} \tag{136}$$

We decompose the sum $\sum_{i=2}^{2N}$ into the sum of two sums $\sum_{i=2}^{N}$ and $\sum_{i=N+1}^{2N}$; therefore, we deduce from (135) that

$$\begin{aligned}(\mu[h] - \mu[u_0] + \varepsilon\mu_1[h])h_x &= \sum_{k=1}^{N} \nabla u_0 (\rho_k[N, \mu, g] \\ &\quad + \rho_{k-1}[N-1, \mu_1, g_1])\varepsilon^k \\ &\quad + \sum_{k=2}^{N}\left(\sum_{i,j=1,i+j=k}^{N} (\rho_i[N, \mu, g] + \rho_{i-1}[N-1, \mu_1, g_1])\right. \\ &\quad \left.\nabla u_j\right)\varepsilon^k \\ &\quad + \sum_{k=N+1}^{2N}\left(\sum_{i,j=1,i+j=k}^{N} (\rho_i[N, \mu, g] + \rho_{i-1}[N-1, \mu_1, g_1])\right. \\ &\quad \left.\nabla u_j\right)\varepsilon^k + \varepsilon^{N+1}\widetilde{R}_N^{(1)}[\mu, g, \mu_1, g_1, u_0, \overrightarrow{u}, \varepsilon] \\ &= \sum_{k=1}^{N}\left[\sum_{i=1}^{k} (\rho_i[N, \mu, g] + \rho_{i-1}[N-1, \mu_1, g_1])\nabla u_{k-i}\right]\varepsilon^k \\ &\quad + \varepsilon^{N+1}\widetilde{R}_N^{(2)}[\mu, g, \mu_1, g_1, u_0, \overrightarrow{u}, \varepsilon],\end{aligned} \tag{137}$$

where

$$\begin{aligned}&\varepsilon^{N+1}\widetilde{R}_N^{(2)}[\mu, g, \mu_1, g_1, u_0, \overrightarrow{u}, \varepsilon] \\ &\quad = \sum_{k=N+1}^{2N}\left(\sum_{i,j=1,i+j=k}^{N} (\rho_i[N, \mu, g] + \rho_{i-1}[N-1, \mu_1, g_1])\right. \\ &\quad \left.\nabla u_j\right)\varepsilon^k + \varepsilon^{N+1}\widetilde{R}_N^{(1)}[\mu, g, \mu_1, g_1, u_0, \overrightarrow{u}, \varepsilon].\end{aligned} \tag{138}$$

Combining (105), (131), and (137) leads to

$$E_\varepsilon(x,t) = \varepsilon^{N+1}\frac{\partial}{\partial x}\widetilde{R}_N^{(2)}[\mu, g, \mu_1, g_1, u_0, \overrightarrow{u}, \varepsilon]. \tag{139}$$

By the boundedness of the functions u_k, ∇u_k, $1 \le k \le N$, in the function space $L^\infty(0,T;H^1)$, we obtain from (110), (136), (138), and (139) that

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \le C_*\varepsilon^{N+1}, \tag{140}$$

where C_* is a constant depending only on $N, T, \mu, \mu_1, g, g_1, u_k, 1 \le k \le N$.

Lemma 13 is proved. □

Proof of Theorem 10. Consider the sequence $\{v_m\}$ defined by

$$\begin{aligned}&v_0 \equiv 0, \\ &v_m'' - \frac{\partial}{\partial x}(\mu_\varepsilon[v_{m-1} + h]v_{mx}) \\ &\quad = \frac{\partial}{\partial x}[(\mu_\varepsilon[v_{m-1} + h] - \mu_\varepsilon[h])h_x] + E_\varepsilon(x,t), \\ &\qquad\qquad 0 < x < 1,\ 0 < t < T, \\ &v_{mx}(0,t) - h_0 v_m(0,t) = v_m(1,t) = 0, \\ &v_m(x,0) = v_m'(x,0) = 0, \quad m \ge 1.\end{aligned} \tag{141}$$

By multiplying two sides of (141) with v_m' and after integration in t, we have

$$\begin{aligned}Z_m(t) &= 2\int_0^t \langle E_\varepsilon(s), v_m'(s)\rangle ds + \int_0^t \frac{\partial A_{m,\varepsilon}}{\partial t}(s; \\ &\quad v_m(s), v_m(s))ds \\ &\quad + 2\int_0^t \left\langle \frac{\partial}{\partial x}[(\mu_\varepsilon[v_{m-1} + h] - \mu_\varepsilon[h])h_x], \right. \\ &\quad \left. v_m'(s)\right\rangle ds,\end{aligned} \tag{142}$$

where

$$\begin{aligned}
&Z_m(t) = \left\|v_m'(t)\right\|^2 + A_{m,\varepsilon}\left(t; v_m(t), v_m(t)\right),\\
&A_{m,\varepsilon}(t; u, v)\\
&\quad = \left\langle \mu_{m,\varepsilon}(t) u_x, v_x\right\rangle + h_0 \mu_{m,\varepsilon}(0,t) u(0) v(0),\\
&\qquad\qquad u, v \in V,\\
&\mu_{m,\varepsilon}(x,t)\\
&\quad = \mu\left(x, t, \int_0^1 g\left[v_{m-1}+h\right](x,y,t)\,dy\right)\\
&\qquad + \varepsilon\mu_1\left(x, t, \int_0^1 g_1\left[v_{m-1}+h\right](x,y,t)\,dy\right),\\
&g\left[v_{m-1}+h\right](x,y,t)\\
&\quad = g\left(x,y,t,v_{m-1}+h, \nabla v_{m-1}+\nabla h\right),\\
&g_1\left[v_{m-1}+h\right](x,y,t)\\
&\quad = g_1\left(x,y,t,v_{m-1}+h, \nabla v_{m-1}+\nabla h\right).
\end{aligned} \tag{143}$$

By using Lemma 13, we deduce from (142) that

$$\begin{aligned}
&\left\|v_m'(t)\right\|^2 + \mu_0\left\|v_m(t)\right\|_a^2 \le TC_*^2\varepsilon^{2N+2}\\
&\quad + \int_0^t \left\|v_m'(s)\right\|^2 ds\\
&\quad + \int_0^t \frac{\partial A_{m,\varepsilon}}{\partial t}\left(s; v_m(s), v_m(s)\right) ds\\
&\quad + 2\int_0^t \left\|\frac{\partial}{\partial x}\left[\left(\mu_\varepsilon\left[v_{m-1}+h\right] - \mu_\varepsilon[h]\right)h_x\right]\right\|\\
&\quad \cdot \left\|v_m'(s)\right\| ds = TC_*^2\varepsilon^{2N+2} + \int_0^t \left\|v_m'(s)\right\|^2 ds\\
&\quad + \widehat{J}_1 + \widehat{J}_2.
\end{aligned} \tag{144}$$

We estimate the integrals on the right-hand side of (144) as follows.

Estimating $\widehat{J}_1$. We note that

$$\begin{aligned}
&\mu_{m,\varepsilon}'(x,t) = D_2\mu\left[v_{m-1}+h\right](x,t) + D_3\mu\left[v_{m-1}+h\right]\\
&\quad \cdot (x,t)\int_0^1 \left[D_3 g\left[v_{m-1}+h\right] + D_4 g\left[v_{m-1}+h\right]\right.\\
&\quad \cdot \left(v_{m-1}'+h'\right)(y,t) + D_5 g\left[v_{m-1}+h\right]\\
&\quad \cdot \left.\left(\nabla v_{m-1}'+\nabla h'\right)(y,t)\right] dy + \varepsilon D_2\mu_1\left[v_{m-1}+h\right]\\
&\quad \cdot (x,t) + \varepsilon D_3\mu_1\left[v_{m-1}+h\right](x,t)\\
&\quad \cdot \int_0^1 \left[D_3 g_1\left[v_{m-1}+h\right] + D_4 g_1\left[v_{m-1}+h\right]\right.\\
&\quad \cdot \left(v_{m-1}'+h'\right)(y,t) + D_5 g_1\left[v_{m-1}+h\right]\\
&\quad \cdot \left.\left(\nabla v_{m-1}'+\nabla h'\right)(y,t)\right] dy,
\end{aligned} \tag{145}$$

and we have

$$\left|\mu_{m,\varepsilon}'(x,t)\right| \le \overline{\zeta}_1, \tag{146}$$

with $\overline{\zeta}_1 = \widetilde{K}_{M_*}(\mu)[1+\overline{K}_{M_*}(g)(1+2M_*)] + \widetilde{K}_{M_*}(\mu_1)[1+\overline{K}_{M_*}(g_1)(1+2M_*)]$, $M_* = (N+2)M$.

It follows from (146) that

$$\begin{aligned}
\left|\widehat{J}_1\right| &\le \int_0^t \left|\frac{\partial A_{m,\varepsilon}}{\partial t}\left(s; v_m(s), v_m(s)\right)\right| ds\\
&\le \overline{\zeta}_1 \int_0^t \left\|v_m(s)\right\|_a^2 ds.
\end{aligned} \tag{147}$$

Estimating $\widehat{J}_2$. First, we need an estimation $\|(\partial/\partial x)[(\mu[v_{m-1}+h]-\mu[h])h_x]\|$.

We also note that

$$\begin{aligned}
&\left\|\mu\left[v_{m-1}+h\right] - \mu[h]\right\|_{C^0(\overline{\Omega})}\\
&\quad \le 2\widetilde{K}_{M_*}(\mu)\overline{K}_{M_*}(g)\left\|v_{m-1}\right\|_{W_1(T)},\\
&\left\|D_i\mu\left[v_{m-1}+h\right] - D_i\mu[h]\right\|\\
&\quad \le 2\widetilde{K}_{M_*}(\mu)\overline{K}_{M_*}(g)\left\|v_{m-1}\right\|_{W_1(T)}, \quad i = 1, 3,\\
&\left\|D_1 g\left[v_{m-1}+h\right] - D_1 g[h]\right\|\\
&\quad \le 2\overline{K}_{M_*}(g)\left\|v_{m-1}\right\|_{W_1(T)}.
\end{aligned} \tag{148}$$

From the equation

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[\left(\mu\left[v_{m-1}+h\right] - \mu[h]\right)h_x\right]\\
&\quad = \left(\mu\left[v_{m-1}+h\right] - \mu[h]\right)h_{xx}\\
&\quad + \left(D_1\mu\left[v_{m-1}+h\right] - D_1\mu[h]\right)h_x\\
&\quad + D_3\mu\left[v_{m-1}+h\right]\\
&\quad \cdot \left(\int_0^1 \left(D_1 g\left[v_{m-1}+h\right] - D_1 g[h]\right) dy\right)h_x\\
&\quad + \left(D_3\mu\left[v_{m-1}+h\right] - D_3\mu[h]\right)\left(\int_0^1 D_1 g[h]\,dy\right)\\
&\quad \cdot h_x,
\end{aligned} \tag{149}$$

it follows that

$$\begin{aligned}
&\left\|\frac{\partial}{\partial x}\left[\left(\mu\left[v_{m-1}+h\right] - \mu[h]\right)h_x\right]\right\|\\
&\quad \le d\left(\mu, g, M_*\right)\left\|v_{m-1}\right\|_{W_1(T)},
\end{aligned} \tag{150}$$

where $d(\mu, g, M_*) = 2M_*\widetilde{K}_{M_*}(\mu)\overline{K}_{M_*}(g)[1 + \sqrt{2}(2 + \overline{K}_{M_*}(g))]$.

Next, by $\mu_\varepsilon = \mu + \varepsilon\mu_1$, it follows that

$$\left\| \frac{\partial}{\partial x} \left[\left(\mu_\varepsilon \left[v_{m-1} + h \right] - \mu_\varepsilon \left[h \right] \right) h_x \right] \right\| \leq \overline{\zeta}_2 \left\| v_{m-1} \right\|_{W_1(T)}, \quad (151)$$

where $\overline{\zeta}_2 = d(\mu, g, M_*) + d(\mu_1, g_1, M_*)$.

By (151), we obtain

$$\begin{aligned} \widehat{J}_2 &= 2 \int_0^t \left\| \frac{\partial}{\partial x} \left[\left(\mu_\varepsilon \left[v_{m-1} + h \right] - \mu_\varepsilon \left[h \right] \right) h_x \right] \right\| \left\| v_m'(s) \right\| ds \\ &\leq T\overline{\zeta}_2^2 \left\| v_{m-1} \right\|_{W_1(T)}^2 + \int_0^t \left\| v_m'(s) \right\|^2 ds. \end{aligned} \quad (152)$$

Combining (144), (147), and (152), this leads to

$$\begin{aligned} Y_m(t) \leq{}& TC_*^2 \varepsilon^{2N+2} + T\overline{\zeta}_2^2 \left\| v_{m-1} \right\|_{W_1(T)}^2 \\ &+ \left(2 + \frac{\overline{\zeta}_1}{\mu_0} \right) \int_0^t Y_m(s)\, ds, \end{aligned} \quad (153)$$

where $Y_m(t) = \|v_m'(t)\|^2 + \mu_0 \|v_m(t)\|_a^2$.

By using Gronwall's Lemma, we deduce from (153) that

$$\left\| v_m \right\|_{W_1(T)} \leq \sigma_T \left\| v_{m-1} \right\|_{W_1(T)} + \delta_T(\varepsilon), \quad \forall m \geq 1, \quad (154)$$

where $\sigma_T = \eta_T \overline{\zeta}_2, \delta_T(\varepsilon) = C_* \eta_T \varepsilon^{N+1}$, $\eta_T = (1 + 1/\sqrt{\mu_0})\sqrt{T \exp[(1 + \overline{\zeta}_1/2\mu_0)T]}$.

We can assume that

$$\sigma_T < 1, \quad \text{with the suitable constant } T > 0. \quad (155)$$

We require the following lemma whose proof is immediate.

Lemma 14. *Let the sequence $\{\gamma_m\}$ satisfy*

$$\gamma_m \leq \sigma\gamma_{m-1} + \delta \quad \forall m \geq 1, \ \gamma_0 = 0, \quad (156)$$

where $0 \leq \sigma < 1$, $\delta \geq 0$ are the given constants. Then

$$\gamma_m \leq \frac{\delta}{(1 - \sigma)} \quad \forall m \geq 1. \quad (157)$$

Applying Lemma 14 with $\gamma_m = \|v_m\|_{W_1(T)}$, $\sigma = \sigma_T < 1$, $\delta = \delta_T(\varepsilon)$, it follows from (154) that

$$\left\| v_m \right\|_{W_1(T)} \leq \frac{\delta_T(\varepsilon)}{1 - \sigma_T} = C_T \varepsilon^{N+1}, \quad (158)$$

where C_T is a constant depending only on T.

On the other hand, the linear recurrent sequence $\{v_m\}$ defined by (141) converges strongly in the space $W_1(T)$ to the solution v of problem (130). Hence, letting $m \to +\infty$ in (158), we get

$$\|v\|_{W_1(T)} \leq C_T \varepsilon^{N+1}. \quad (159)$$

This implies (109).

The proof of Theorem 10 is complete. □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this article. They read and approved the final manuscript.

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