

## Research Article

# Stochastic Responses of Lightly Nonlinear Vibroimpact System with Inelastic Impact Subjected to External Poisson White Noise Excitation

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A procedure for analyzing stationary responses of lightly nonlinear vibroimpact system with inelastic impact subjected to external Poisson white noise excitation is proposed. First, the original vibroimpact system is transformed to a new system without velocity jump in terms of the Zhuravlev nonsmooth coordinate transformation and the Dirac delta function. Second, the averaged generalized Fokker-Planck-Kolmogorov (FPK) equation for transformed system under parametric excitation of Poisson white noise is derived by stochastic averaging method. Third, the averaged generalized FPK equation is solved by using the perturbation technique and inverse transformation of the Zhuravlev nonsmooth coordinate transformation to obtain the approximately stationary solutions for response probability density functions of original vibroimpact system. Last, analytical and numerical results for two typical lightly nonlinear vibroimpact systems are presented to assess the effectiveness of the proposed method. It is found that they are in good agreement and the proposed method is quite effective.

## 1. Introduction

Vibroimpact system, as a class of typical nonsmooth system [1], has attracted much attention in the past few decades due to its existence in various engineering applications [2, 3]. Certain applications or problems involved impacts include vibratory pile drivers, heat exchanger tube wear in nuclear power stations, large gaps in labyrinth seals for rotating shafts of high-pressure pumps, and a moored body with an inextensible mooring line [4–7]. There are two commonly used methods to model impacts. One is the classical or hard impact model [1], and the other is the Hertz contact model or soft impact model [8, 9]. The classical impact model consists of two parts: the ordinary motion described by a differential equation and the impact condition. In the Hertz contact model, the contact force between two elastic bodies is assumed to be proportional to the  $3/2$  (sometimes with different values) power of the relative displacement between them. By means of the two models, many interesting dynamical behaviors were observed such as chaos motion and

bifurcations [10, 11], Hopf bifurcation [12], grazing bifurcation [13, 14], torus bifurcation [15], chattering bifurcation [16, 17], sticking motion [18, 19], and border collision bifurcation [20].

As is known to all, random excitations widely exist in vibroimpact systems. Their effects usually can not be neglected and sometimes are even quite large [21]. Therefore, studies on the prediction of stochastic responses of vibroimpact systems are naturally important but difficult due to the presence of random excitations and nonsmooth factors. In recent years, some methods [22–27] have been developed to explore the response probability density function (PDF) which is an important characteristic of a stochastic system. For the classical impact model, Dimentberg et al. [28, 29] studied stochastic response of linear vibroimpact system under Gaussian white noise. Feng et al. [30, 31] explored the response PDFs of Duffing-Van der Pol vibroimpact system under independent and correlated Gaussian white noises, respectively. Zhu [32] obtained the response PDFs of a vibro-impact Duffing system with a randomly varying damping

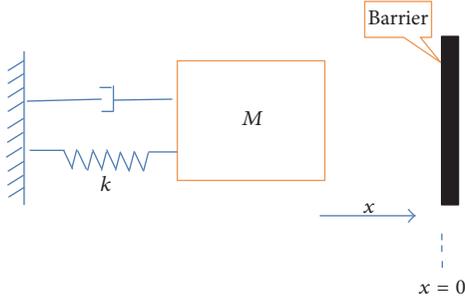


FIGURE 1: Vibroimpact system with a rigid barrier on right side.

term. For the Hertz contact model, Huang et al. [33] investigated stationary responses of multi-degree-of-freedom vibroimpact system under white noise excitations. Xu et al. [34] analyzed random vibration problems of vibroimpact system with inelastic impact. Although stochastic responses of vibroimpact system have been reported, the above studies are limited to responses of vibroimpact systems subject to Gaussian white noise. In practice, many random excitations are essentially non-Gaussian, for example, those due to sea wave [35] and wind [36]. The Poisson white noise is the most used model for non-Gaussian random excitations [37–40]. However, the research on stochastic responses of vibroimpact system subject to Poisson white noise has been rarely addressed [41] and is far from enough. This paper is devoted to presenting a procedure to predict the stochastic responses of lightly nonlinear vibroimpact system subject to external Poisson white noise excitation. In the following parts, the Zhuravlev nonsmooth coordinate transformation [22], the stochastic averaging method [42], and the perturbation technique [43] are applied in succession to obtain the stationary solutions of response PDFs. Two examples are also presented to assess the effectiveness of the procedure. Comparison between results obtained from the proposed method and those from Monte-Carlo simulation shows that they are in good agreement. Therefore, the proposed procedure is quite effective.

## 2. Problem Statement

Without loss of generality, a simple mass-spring-damper model (Figure 1) for single-degree-of-freedom lightly nonlinear vibroimpact system with inelastic impact subjected to external Poisson white noise excitation is considered, and the corresponding equation of motion can be expressed as

$$\ddot{x} + \varepsilon^2 f(x, \dot{x}) + kx = \varepsilon \xi(t), \quad x < 0, \quad (1a)$$

$$\dot{x}_+ = -r\dot{x}_-, \quad x = 0, \quad (1b)$$

where  $\varepsilon > 0$  is a small parameter;  $\varepsilon^2 f(x, \dot{x})$  denotes lightly linear or nonlinear damping force;  $k$  is the stiffness coefficient;  $\xi(t)$  is Poisson white noise;  $\dot{x}_+$  and  $\dot{x}_-$  denote the rebound and impact velocity, respectively; the subscripts “+” and “-” refer to values of velocity just before and after impact;  $r$  is the restitution factor whose value is among zero and one. Particularly, when  $r$  equals one, the system reduces to

an elastic impact one.  $x = 0$  denotes that the rigid barrier is located at the system’s equilibrium position. Equation (1a) governs the system’s motion between impacts as  $x < 0$ . Equation (1b) describes the impact law, which means that the oscillator’s velocity has an instantaneous change when it impacts with the barrier at  $x = 0$ .

The Poisson white noise can be treated as the formal derivative of the compound Poisson process  $C(t)$ ; that is,

$$\xi(t) = \frac{dC(t)}{dt}, \quad (2)$$

in which

$$C(t) = \sum_{i=1}^{N(t)} Y_i U(t - t_i), \quad (3)$$

where  $N(t)$  is the Poisson counting process with a constant impulse arrival rate  $\lambda$  which also means the expected number of event occurrences per second.  $Y_i$  is a random variable representing the random magnitude of  $i$ th impulse arriving at time  $t_i$ .  $U(\cdot)$  is a unit step function. Assume that  $Y_i$  are independent identically distributed random variables with zero mean which are independent of the impulse arrival time  $t_i$ .  $C(t)$  is a process with independent increments, and the  $k$ th correlation function of whose increment process  $dC(t) = C(t + dt) - C(t)$  is as follows [44]:

$$\begin{aligned} R^{(k)}[dC(t_1), dC(t_2), \dots, dC(t_k)] \\ = \lambda E[Y^k] \delta(t_2 - t_1) \cdots \delta(t_k - t_1) dt_1 dt_2 \cdots dt_k. \end{aligned} \quad (4)$$

By the relation between the correlation function and the moment function of random processes, omitting the small quantity that higher than  $dt$ , the  $k$ th moment function can be derived as

$$E[(dC)^k] = \lambda E[Y^k] dt. \quad (5)$$

Given that the nonsmooth characteristic of vibroimpact system described by (1a) and (1b), the following coordinate transformation introduced by Zhuravlev [22] is used as

$$\begin{aligned} x &= x_1 = |y|, \\ \dot{x} &= x_2 = \dot{y} \operatorname{sgn} y, \\ \operatorname{sgn} y &= \begin{cases} 1, & y > 0 \\ 0, & y = 0 \\ -1, & y < 0. \end{cases} \end{aligned} \quad (6)$$

The relation between the original variables  $x, \dot{x}$  and the new variables  $y, \dot{y}$  can be simply represented by Figures 2 and 3.

Substituting (6) into (1a) and (1b), the transformation maps the domain  $x < 0$  of original phase plane  $(x, \dot{x})$  onto the whole phase plane  $(y, \dot{y})$  (see Figure 3). Since  $(d/dy)(\operatorname{sgn} y) = 0$  when  $y \neq 0$ , this implies also that  $\ddot{x}(t) = \ddot{y}(t) \operatorname{sgn} y$ . When  $y = 0$ , the impact/rebound condition for the transformed state variables is found to be  $\dot{y}_+ = r\dot{y}_-$

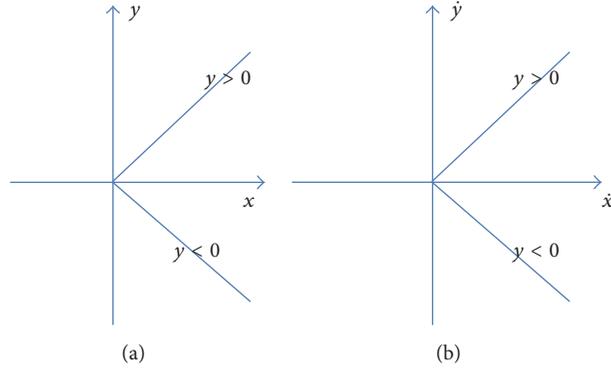
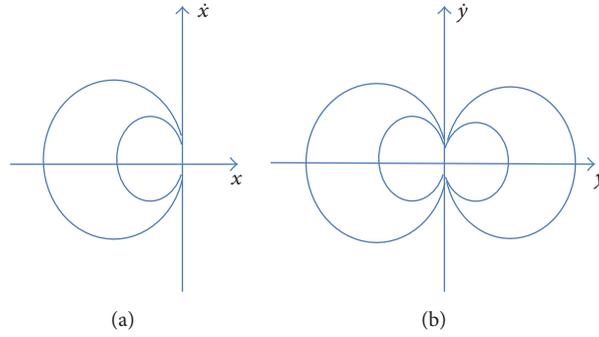

 FIGURE 2: Relation between the original variables and the new variables: (a)  $x$  and  $y$ ; (b)  $\dot{x}$  and  $\dot{y}$ .


FIGURE 3: Phase plane for vibroimpact system: (a) original vibroimpact system; (b) the transformed system.

according to Dimentberg and Menyailov [28]. Note that the identity  $(\operatorname{sgn} y)^2 = 1$ , and the original equation of motion can be replaced by

$$\ddot{y} + \varepsilon^2 f(y \operatorname{sgn} y, \dot{y} \operatorname{sgn} y) \operatorname{sgn} y + ky = \varepsilon \operatorname{sgn} y \xi(t), \quad (7a)$$

$$t \neq t_*,$$

$$\dot{y}_+ = r \dot{y}_-, \quad t = t_*, \quad (7b)$$

where  $t_*$  represents the time instant of impact corresponding to  $y = 0$ , that is,  $y(t_*) = 0$ . Also, the velocity jump becomes proportional to  $1 - r$  instead of  $1 + r$  for original one. Using the Dirac delta function, an additional impulsive term can be introduced as follows:

$$(\dot{y}_- - \dot{y}_+) \delta(t - t_*) = (1 - r) \dot{y} |\dot{y}| \delta(y). \quad (8)$$

According to [28], it is reasonable to treat (7b) as an additional impulse of (7a) as  $t = t_*$ . Therefore, (7a), (7b), and (8) can be combined into one new equation as

$$\ddot{y} + \varepsilon^2 f(y \operatorname{sgn} y, \dot{y} \operatorname{sgn} y) \operatorname{sgn} y + ky - (r - 1) \dot{y} |\dot{y}| \delta(y) = \varepsilon \operatorname{sgn} y \xi(t). \quad (9)$$

The above transformed equation may permit rigorous analytical study since there contains no velocity jump and the impact condition is taken place by an additional impulsive term.

### 3. Stochastic Responses

**3.1. Stochastic Averaging.** When the values of  $(1 - r)$  approaches to zero, the energy loss of impact is very small, and the system becomes a quasiconservative one as long as the damping coefficients and random excitations are weak. Therefore, the stochastic averaging method can be applied to study the response of the transformed system analytically. For simplicity,  $(1 - r)$  is assumed of order  $o(\varepsilon^2)$ . The transformed equation (9) is equivalent to a pair of first-order equations:

$$dy_1 = y_2 dt,$$

$$dy_2 = -\varepsilon^2 \tilde{f}(y_1, y_2) - ky_1 + (r - 1) y_2 |y_2| \delta(y_1) + \varepsilon \operatorname{sgn} y_1 dC(t), \quad (10)$$

where

$$\tilde{f}(y_1, y_2) = f(y_1 \operatorname{sgn} y_1, y_2 \operatorname{sgn} y_1) \operatorname{sgn} y_1. \quad (11)$$

The total energy of system is

$$H = \frac{1}{2} y_2^2 + \frac{1}{2} k y_1^2. \quad (12)$$

Substituting (12) into (10) yields the following Stratonovich stochastic differential equations about displacement  $y_1$  and energy  $H$  with parametric excitation of Poisson white noise:

$$dy_1 = \frac{\partial H}{\partial y_2} dt,$$

$$dH = -\varepsilon^2 \tilde{f}\left(y_1, \frac{\partial H}{\partial y_2}\right) \frac{\partial H}{\partial y_2} dt$$

$$\begin{aligned} &+ (r-1) \left(\frac{\partial H}{\partial y_2}\right)^2 \left|\frac{\partial H}{\partial y_2}\right| \delta(y_1) dt \\ &+ \varepsilon \operatorname{sgn} y_1 \frac{\partial H}{\partial y_2} \circ dC(t). \end{aligned} \quad (13)$$

The corresponding Itô stochastic differential equations are derived using the converting rule for systems with Poisson white noise excitation proposed by Di Paola and Falsone [37]:

$$dy_1 = \frac{\partial H}{\partial y_2} dt,$$

$$\begin{aligned} dH = & -\varepsilon^2 \tilde{f}\left(y_1, \frac{\partial H}{\partial y_2}\right) \frac{\partial H}{\partial y_2} dt + (r-1) \left(\frac{\partial H}{\partial y_2}\right)^2 \left|\frac{\partial H}{\partial y_2}\right| \delta(y_1) dt + \varepsilon \operatorname{sgn} y_1 \frac{\partial H}{\partial y_2} dC(t) \\ &+ \frac{\varepsilon^2}{2!} \left(\operatorname{sgn} y_1 \frac{\partial H}{\partial y_2}\right) \frac{\partial}{\partial H} \left(\operatorname{sgn} y_1 \frac{\partial H}{\partial y_2}\right) [dC(t)]^2 \\ &+ \frac{\varepsilon^3}{3!} \left(\operatorname{sgn} y_1 \frac{\partial H}{\partial y_2}\right) \frac{\partial}{\partial H} \left[\left(\operatorname{sgn} y_1 \frac{\partial H}{\partial y_2}\right) \frac{\partial}{\partial H} \left(\operatorname{sgn} y_1 \frac{\partial H}{\partial y_2}\right)\right] [dC(t)]^3 \\ &+ \frac{\varepsilon^4}{4!} \left(\operatorname{sgn} y_1 \frac{\partial H}{\partial y_2}\right) \frac{\partial}{\partial H} \left\{\left(\operatorname{sgn} y_1 \frac{\partial H}{\partial y_2}\right) \frac{\partial}{\partial H} \left[\left(\operatorname{sgn} y_1 \frac{\partial H}{\partial y_2}\right) \frac{\partial}{\partial H} \left(\operatorname{sgn} y_1 \frac{\partial H}{\partial y_2}\right)\right]\right\} [dC(t)]^4 + \dots \end{aligned} \quad (14)$$

Give that

$$\begin{aligned} &\left(\operatorname{sgn} y_1 \frac{\partial H}{\partial y_2}\right) \frac{\partial}{\partial H} \left(\operatorname{sgn} y_1 \frac{\partial H}{\partial y_2}\right) \\ &= \frac{1}{2} \frac{\partial}{\partial H} \left[\left(\operatorname{sgn} y_1 \frac{\partial H}{\partial y_2}\right)^2\right] = 1. \end{aligned} \quad (15)$$

Equation (14) can be simplified as

$$\begin{aligned} dy_1 = & \frac{\partial H}{\partial y_2} dt, \\ dH = & -\varepsilon^2 \tilde{f}\left(y_1, \frac{\partial H}{\partial y_2}\right) \frac{\partial H}{\partial y_2} dt \\ &+ (r-1) \left(\frac{\partial H}{\partial y_2}\right)^2 \left|\frac{\partial H}{\partial y_2}\right| \delta(y_1) dt \\ &+ \varepsilon \operatorname{sgn} y_1 \frac{\partial H}{\partial y_2} dC(t) + \frac{\varepsilon^2}{2!} [dC(t)]^2. \end{aligned} \quad (16)$$

The generalized FPK equation associated with (16) is

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & -\frac{\partial}{\partial y_1} \left(\frac{\partial h}{\partial y_2} \rho\right) - \frac{\partial}{\partial h} \left\{\left(-\varepsilon^2 \tilde{f}\left(y_1, \frac{\partial h}{\partial y_2}\right) \frac{\partial h}{\partial y_2}\right. \right. \\ &+ (r-1) \left(\frac{\partial h}{\partial y_2}\right)^2 \left|\frac{\partial h}{\partial y_2}\right| \delta(y_1) + \varepsilon \operatorname{sgn} y_1 \frac{\partial h}{\partial y_2} E[Y] \end{aligned}$$

$$\begin{aligned} &+ \frac{\varepsilon^2}{2} \lambda E[Y^2] \left.\right\} \rho \left. + \frac{1}{2!} \frac{\partial^2}{\partial h^2} \left\{\left(\varepsilon^2 \left(\frac{\partial h}{\partial y_2}\right)^2 \lambda E[Y^2] \right. \right. \right. \\ &+ \varepsilon^3 (\operatorname{sgn} y_1)^3 \frac{\partial h}{\partial y_2} \lambda E[Y^3] + \frac{\varepsilon^4}{4} \lambda E[Y^4] \left.\right\} \rho \left. - \frac{1}{3!} \right. \\ &\cdot \frac{\partial^3}{\partial h^3} \left\{\left(\varepsilon^3 (\operatorname{sgn} y_1)^3 \left(\frac{\partial h}{\partial y_2}\right)^3 \lambda E[Y^3] \right. \right. \\ &+ \frac{3}{2} \varepsilon^4 \left(\frac{\partial h}{\partial y_2}\right)^2 \lambda E[Y^4] \left.\right\} \rho \left. + \frac{1}{4!} \frac{\partial^4}{\partial h^4} \left\{\varepsilon^4 \left(\frac{\partial h}{\partial y_2}\right)^4 \right. \right. \\ &\cdot \left. \lambda E[Y^4] \rho \left. + O(\varepsilon^5), \right. \end{aligned} \quad (17)$$

where  $\rho = \rho(y_1, h, t)$ .

Particularly, if the probability distribution of  $Y$  is symmetric, all odd order moments are zero. Thus, we have

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & -\frac{\partial}{\partial y_1} \left(\frac{\partial h}{\partial y_2} \rho\right) - \frac{\partial}{\partial h} \left\{\left(-\varepsilon^2 \tilde{f}\left(y_1, \frac{\partial h}{\partial y_2}\right) \frac{\partial h}{\partial y_2}\right. \right. \\ &+ (r-1) \left(\frac{\partial h}{\partial y_2}\right)^2 \left|\frac{\partial h}{\partial y_2}\right| \delta(y_1) + \frac{\varepsilon^2}{2} \lambda E[Y^2] \left.\right\} \rho \\ &+ \frac{1}{2!} \frac{\partial^2}{\partial h^2} \left\{\left(\varepsilon^2 \left(\frac{\partial h}{\partial y_2}\right)^2 \lambda E[Y^2] + \frac{\varepsilon^4}{4} \lambda E[Y^4] \right) \rho \right\} \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{3!} \frac{\partial^3}{\partial h^3} \left\{ \left( \frac{3}{2} \varepsilon^4 \left( \frac{\partial h}{\partial y_2} \right)^2 \lambda E[Y^4] \right) \rho \right\} + \frac{1}{4!} \\
 & \cdot \frac{\partial^4}{\partial h^4} \left\{ \varepsilon^4 \left( \frac{\partial h}{\partial y_2} \right)^4 \lambda E[Y^4] \rho \right\} + O(\varepsilon^5).
 \end{aligned} \tag{18}$$

The conditional probability density  $\rho(y_1 | h, t)$  given by Stratonovich [45] is

$$\rho(y_1 | h, t) = \begin{cases} \frac{\sqrt{k}}{2\pi\sqrt{2h - ky_1^2}} & \text{for } ky_1^2 < 2h, \\ 0 & \text{for } ky_1^2 > 2h. \end{cases} \tag{19}$$

By means of the formulation  $\rho(y_1 | h, t) = \rho(y_1, h, t)/\rho(h, t)$  and integrating (18) with respect to  $y_1$ , the following averaged generalized FPK equation for probability density  $\rho(h, t)$  is obtained:

$$\begin{aligned}
 & \frac{\partial \rho(h, t)}{\partial t} \\
 & = \frac{\partial}{\partial h} \left\{ \left( \varepsilon^2 F(h) + G(h) - \frac{\varepsilon^2}{2} \lambda E[Y^2] \right) \rho(h, t) \right\} \\
 & + \frac{1}{2!} \frac{\partial^2}{\partial h^2} \left\{ \left( \varepsilon^2 \lambda E[Y^2] h + \frac{\varepsilon^4}{4} \lambda E[Y^4] \right) \rho(h, t) \right\} \\
 & - \frac{1}{3!} \frac{\partial^3}{\partial h^3} \left\{ \frac{3\varepsilon^4}{2} h \lambda E[Y^4] \rho(h, t) \right\} \\
 & + \frac{1}{4!} \frac{\partial^4}{\partial h^4} \left\{ \frac{3\varepsilon^4}{2} h^2 \lambda E[Y^4] \rho(h, t) \right\} + O(\varepsilon^5),
 \end{aligned} \tag{20}$$

where

$$F(h) = \oint \frac{\sqrt{k}}{2\pi} \tilde{f} \left( y_1, \frac{\partial h}{\partial y_2} \right) dy_1 \tag{21}$$

$$\begin{aligned}
 G(h) & = \oint - (r-1) \left( \frac{\partial h}{\partial y_2} \right) \left| \frac{\partial h}{\partial y_2} \right| \delta(y_1) \frac{\sqrt{k}}{2\pi} dy_1 \\
 & = -\frac{2\sqrt{k}(r-1)}{\pi} h.
 \end{aligned} \tag{22}$$

It is worth noting that the impulsive damping term should be averaged over a half period since there have two impacts in each period (see Figure 3).

**3.2. Approximately Stationary Solution.** The averaged generalized FPK (20) is too complicated to be solved analytically. Herein, we seek the approximately stationary solutions. Then, the left-hand side of (20) vanishes and the original equation can be rewritten as

$$\frac{dJ(h)}{dh} = 0, \tag{23}$$

where

$$\begin{aligned}
 J(h) & = \varepsilon^2 F(h) \rho(h) + G(h) \rho(h) \\
 & + \frac{\varepsilon^2 \lambda E[Y^2]}{2} h \frac{d\rho(h)}{dh} + \frac{\varepsilon^4 \lambda E[Y^4]}{8} h \frac{d^2 \rho(h)}{dh^2} \\
 & + \frac{\varepsilon^4 \lambda E[Y^4]}{16} \frac{d^3 \rho(h)}{dh^3} + o(\varepsilon^5).
 \end{aligned} \tag{24}$$

It is easy to know from (23) that  $J(h)$  is a constant. In another way, the probability current function satisfies  $J(h)|_{h \rightarrow \infty} = 0$ . Thus, we only need to solve the following equation to obtain the approximately stationary solutions:

$$\begin{aligned}
 & \varepsilon^2 F(h) \rho(h) + G(h) \rho(h) + \frac{\varepsilon^2 \lambda E[Y^2]}{2} h \frac{d\rho(h)}{dh} \\
 & + \frac{\varepsilon^4 \lambda E[Y^4]}{8} h \frac{d^2 \rho(h)}{dh^2} + \frac{\varepsilon^4 \lambda E[Y^4]}{16} \frac{d^3 \rho(h)}{dh^3} \\
 & + o(\varepsilon^5) = 0.
 \end{aligned} \tag{25}$$

The above equation can be solved by a perturbation technique [43]. The perturbation solution is assumed to be of the form

$$\rho(h) = \rho_0(h) + \varepsilon^2 \rho_1(h) + \varepsilon^4 \rho_2(h) + \dots, \tag{26}$$

where  $\rho(h)$  satisfies the boundary condition:

$$\begin{aligned}
 & \rho(h) \geq 0, \quad h \in (0, \infty), \\
 & \rho \rightarrow 0, \quad h \rightarrow \infty.
 \end{aligned} \tag{27}$$

Substituting (26) into (25) yields the following equations:

$$\begin{aligned}
 & F(h) \rho_0(h) + \frac{G(h)}{\varepsilon^2} \rho_0(h) + \frac{\lambda E[Y^2]}{2} h \frac{d\rho_0(h)}{dh} = 0, \\
 & F(h) \rho_1(h) + \frac{G(h)}{\varepsilon^2} \rho_1(h) + \frac{\lambda E[Y^2]}{2} h \frac{d\rho_1(h)}{dh} \\
 & = -\frac{\lambda E[Y^4]}{8} \left( h \frac{d^2 \rho_0(h)}{dh^2} + \frac{h^2}{2} \frac{d^3 \rho_0(h)}{dh^3} \right), \dots
 \end{aligned} \tag{28}$$

Generally speaking, the approximately stationary solution of  $\rho(h)$  can be obtained analytically and works well with the numerical simulation results up to order of  $\varepsilon^4$ . The higher terms can be neglected since their effects are relatively small.

The joint PDF for the variables  $y_1$  and  $y_2$  can be further obtained as

$$\begin{aligned}
 & \rho(y_1, y_2) = \rho(h) \rho(y_1 | h) y_2 \\
 & = \frac{\sqrt{k}}{2\pi} \rho(h) \Big|_{h=(1/2)y_2^2 + (1/2)ky_1^2}.
 \end{aligned} \tag{29}$$

Using the inverse transformation of formula (6), the joint PDF for original displacement  $x_1$  and velocity  $x_2$  is calculated as

$$\begin{aligned}
 & \rho(x_1, x_2) = \rho(y_1, y_2) \left| \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right| \\
 & = \rho(y_1, y_2) + \rho(-y_1, -y_2) = 2\rho(y_1, y_2).
 \end{aligned} \tag{30}$$

Thus, the marginal PDFs for displacement  $x_1$  and velocity  $x_2$  are easy to be derived, respectively:

$$\begin{aligned}\rho(x_1) &= \int_{-\infty}^{+\infty} \rho(x_1, s) ds, \\ \rho(x_2) &= \int_{-\infty}^0 \rho(s, x_2) ds.\end{aligned}\quad (31)$$

#### 4. Illustrative Examples

In this section, two examples are shown to illustrate the proposed procedure. In order to compare the analytical results and numerical results, the impulses magnitudes  $Y$  of Poisson white noise are assumed to be Gaussian distributed with zero mean. Thus, an interesting result can be used to simplify the practical computation.

$$E[Y^4] = 3(E[Y^2])^2. \quad (32)$$

**4.1. Rayleigh Oscillator.** In this example, a typical Rayleigh vibroimpact system under external Poisson white noise excitation is considered:

$$\ddot{x} + \varepsilon^2 (\dot{x}^2 - 1) \dot{x} + kx = \varepsilon \xi(t), \quad x < 0, \quad (33a)$$

$$\dot{x}_+ = -r\dot{x}_-, \quad x = 0. \quad (33b)$$

In this case,  $f(x, \dot{x}) = (\dot{x}^2 - 1)\dot{x}$ . The following transformed equation without velocity jump is obtained by means of the Zhuravlev transformation and Dirac delta function.

$$\begin{aligned}\ddot{y} + \varepsilon^2 (y^2 - 1) \dot{y} + ky - (r-1) \dot{y} |\dot{y}| \delta(y) \\ = \varepsilon \operatorname{sgn} y \xi(t).\end{aligned}\quad (34)$$

The corresponding Itô stochastic differential equation is derived as

$$\begin{aligned}dy_1 &= \frac{\partial H}{\partial y_2} dt, \\ dH &= -\varepsilon^2 \left( \left( \frac{\partial H}{\partial y_2} \right)^2 - 1 \right) \left( \frac{\partial H}{\partial y_2} \right)^2 dt \\ &\quad + (r-1) \left( \frac{\partial H}{\partial y_2} \right)^2 \left| \frac{\partial H}{\partial y_2} \right| \delta(y_1) dt \\ &\quad + \varepsilon \operatorname{sgn} y_1 \frac{\partial H}{\partial y_2} dC(t) + \frac{\varepsilon^2}{2} [dC(t)]^2.\end{aligned}\quad (35)$$

According to (20) and (21),  $F(h)$  is calculated as follows in the averaged generalized FPK equation for probability density  $\rho(h, t)$ :

$$F(h) = \oint -\frac{\sqrt{k}}{2\pi} \beta \left( \left( \frac{\partial h}{\partial y_2} \right)^2 - 1 \right) \frac{\partial h}{\partial y_2} dy_1 = \frac{3}{2} h^2 - h. \quad (36)$$

Using the perturbation technique, the approximately stationary solution for energy PDF is obtained:

$$\begin{aligned}\rho(h) \\ = \tilde{C} e^{-N} \left( 1 + \varepsilon^2 (Dh + Eh^2 + Fh^3 + Gh^4 + Mh^5) \right),\end{aligned}\quad (37)$$

where

$$\begin{aligned}A &= \lambda E[Y^2], \\ B &= \lambda E[Y^4], \\ C &= -1 - \frac{2(r-1)\sqrt{k}}{\pi\varepsilon^2}, \\ D &= \frac{3B}{4A^2} - \frac{BC^2}{A^3}, \\ E &= \frac{BC^3}{2A^4} - \frac{21BC}{8A^3}, \\ F &= \frac{3BC^2}{2A^4} - \frac{15B}{8A^3}, \\ G &= \frac{27BC}{16A^4}, \\ M &= \frac{27B}{40A^4}, \\ N &= \frac{2}{A} \left( \frac{3h^2}{4} + Ch \right).\end{aligned}\quad (38)$$

Furthermore, the joint PDF for original displacement  $x_1$  and velocity  $x_2$  can be calculated by substituting (37) and (38) into (29) and (30) as

$$\rho(x_1, x_2) = \frac{\sqrt{k}}{\pi} \rho(h) \Big|_{h=(1/2)y_2^2+(1/2)ky_1^2}. \quad (39)$$

To verify the effectiveness of the proposed procedure, analytical results and Monte-Carlo simulation results are shown in Figure 4 for PDFs of system energy with  $k = 1$ ,  $r = 0.99$ ,  $\lambda = 1$ , and  $E[Y^2] = 1$  and two different values of small parameter  $\varepsilon$  ( $\varepsilon = 0.1$  and  $0.2$ ). The solid line is the analytical results obtained by the proposed method while  $\bullet$  represents the Monte-Carlo simulation results based on the original system (33a) and (33b). It is obvious that the two results agree well with each other. In order to illustrate the effects of average arrival rate of external Poisson white noise on the accuracy of the approximately stationary solution, Figure 5 shows PDFs of system energy analytically and numerically with different values of  $\lambda$  ( $\lambda = 1$  and  $2$ ) when  $\varepsilon$  is fixed as  $0.1$ . It is seen that the proposed method works well.

To investigate the effects of restitution coefficient on the stationary response of vibroimpact system, numerical and analytical results are shown in Figures 6–8 with the parameters  $k = 1$ ,  $\varepsilon = 0.1$ ,  $\lambda = 1$ , and  $E[Y^2] = 1$  and different values of  $r$ . It is seen from Figure 6 that the stationary PDFs of total energy are shown for  $r = 0.97, 0.98, 0.99$ , and  $0.995$ . The PDFs of total energy present lower peaks with the increase of restitution coefficient. It is found that the values of peak are at the system's static equilibrium position when  $r = 0.97, 0.98$  while they do not stay there when  $r = 0.99, 0.995$ . This may mean that the steady state has a qualitative change with the increase of restitution coefficient. To show it more clearly, Figures 7 and 8 provide analytical and numerical results of

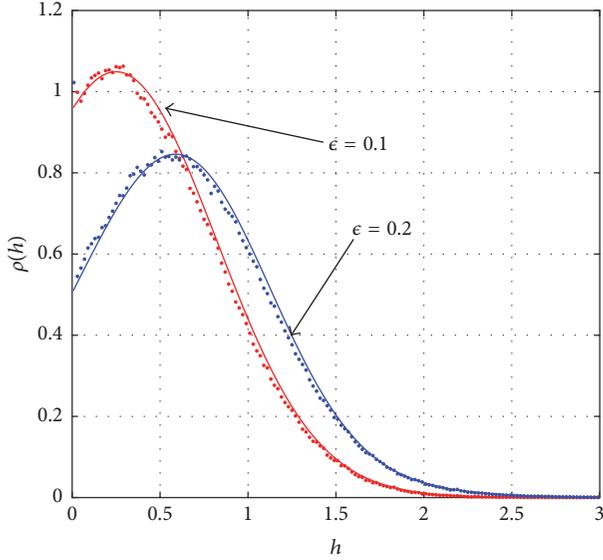


FIGURE 4: Stationary PDFs of total energy obtained numerically and analytically for different values of small parameter  $\epsilon$ .

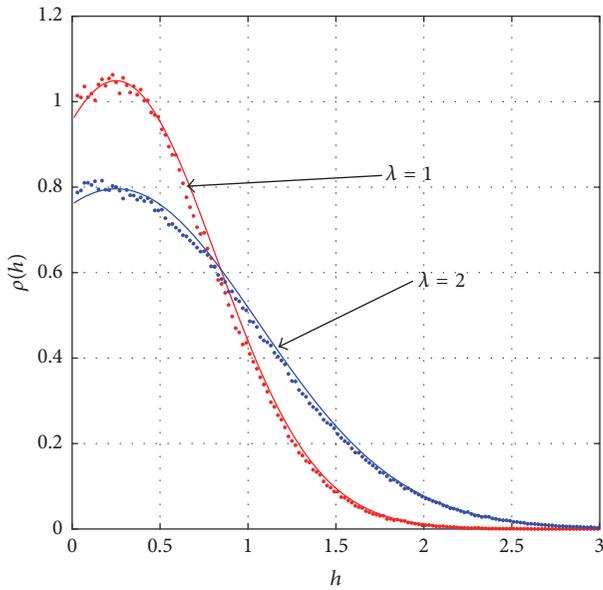


FIGURE 5: Stationary PDFs of total energy obtained numerically and analytically for different values of average arrival rate  $\lambda$ .

joint PDFs of the displacement  $x_1$  and velocity  $x_2$  for  $r = 0.98$  and  $r = 0.995$ , respectively. It is seen that the analytical results agree well with the numerical results. In Figure 7, the joint PDFs of the displacement  $x_1$  and velocity  $x_2$  have a singular peak which appears at the static equilibrium position. When  $r$  increases to  $r = 0.995$ , the joint PDFs present the shape of crater in Figure 8. This illuminates the occurrence of P-bifurcation according to the concept of stochastic bifurcation. In other words, the restitution coefficient  $r$  can induce the occurrence of P-bifurcation.

**4.2. Van der Pol Oscillator.** As an example, a typical Van der Pol vibroimpact system under external Poisson white noise excitation is considered:

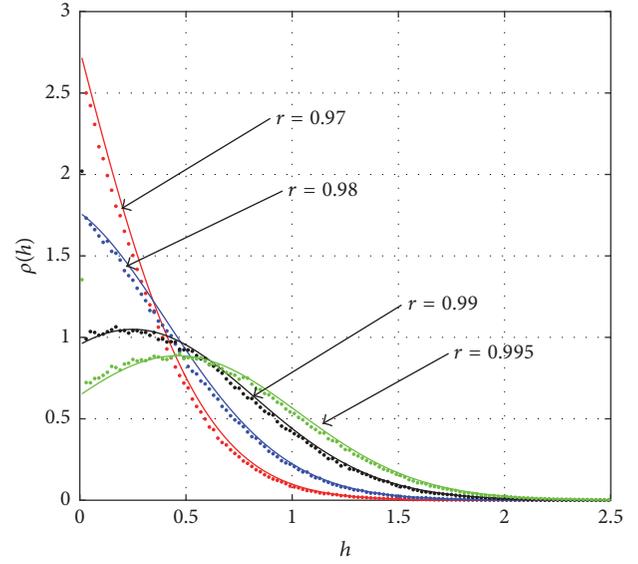


FIGURE 6: Stationary PDFs of total energy obtained numerically and analytically for different values of restitution coefficient  $r$ .

$$\ddot{x} - \epsilon^2 \beta (1 - x^2) \dot{x} + kx = \epsilon \xi(t), \quad x < 0, \quad (40a)$$

$$\dot{x}_+ = -r \dot{x}_-, \quad x = 0. \quad (40b)$$

Based on (9), the transformed system without velocity is obtained as

$$\begin{aligned} \ddot{y} - \epsilon^2 \beta (1 - y^2) \dot{y} + ky - (r - 1) \dot{y} | \dot{y} | \delta(y) \\ = \epsilon \operatorname{sgn} y \xi(t). \end{aligned} \quad (41)$$

The Itô stochastic differential equation about displacement  $y_1$  and energy  $H$  is derived as

$$\begin{aligned} dy_1 &= \frac{\partial H}{\partial y_2} dt, \\ dH &= \epsilon^2 \beta (1 - y_1^2) \left( \frac{\partial H}{\partial y_2} \right)^2 dt \\ &\quad + (r - 1) \left( \frac{\partial H}{\partial y_2} \right)^2 \left| \frac{\partial H}{\partial y_2} \right| \delta(y_1) dt \\ &\quad + \epsilon \operatorname{sgn} y_1 \frac{\partial H}{\partial y_2} dC(t) + \frac{\epsilon^2}{2} [dC(t)]^2. \end{aligned} \quad (42)$$

In the averaged generalized FPK (20),  $F(h)$  here is calculated as

$$F(h) = \oint -\frac{\sqrt{k}}{2\pi} \beta (1 - y_1^2) \frac{\partial h}{\partial y_2} dy_1 = \frac{\beta}{2k} h^2 - \beta h. \quad (43)$$

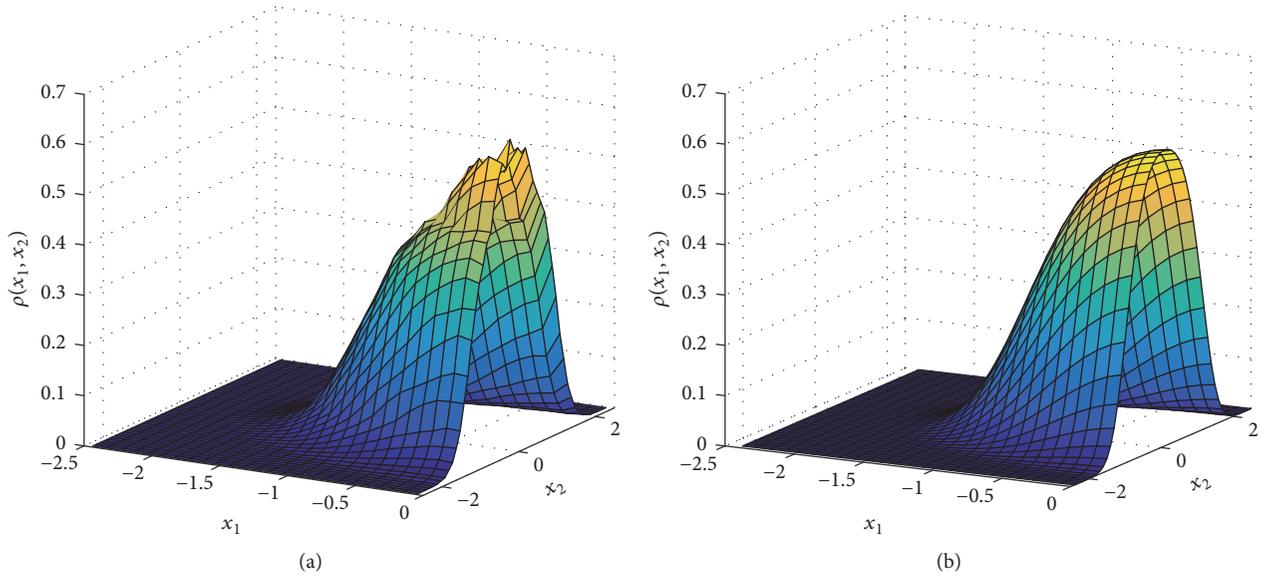


FIGURE 7: Joint PDFs of the displacement  $x_1$  and velocity  $x_2$  for  $r = 0.98$ : (a) Monte-Carlo results; (b) analytical results.

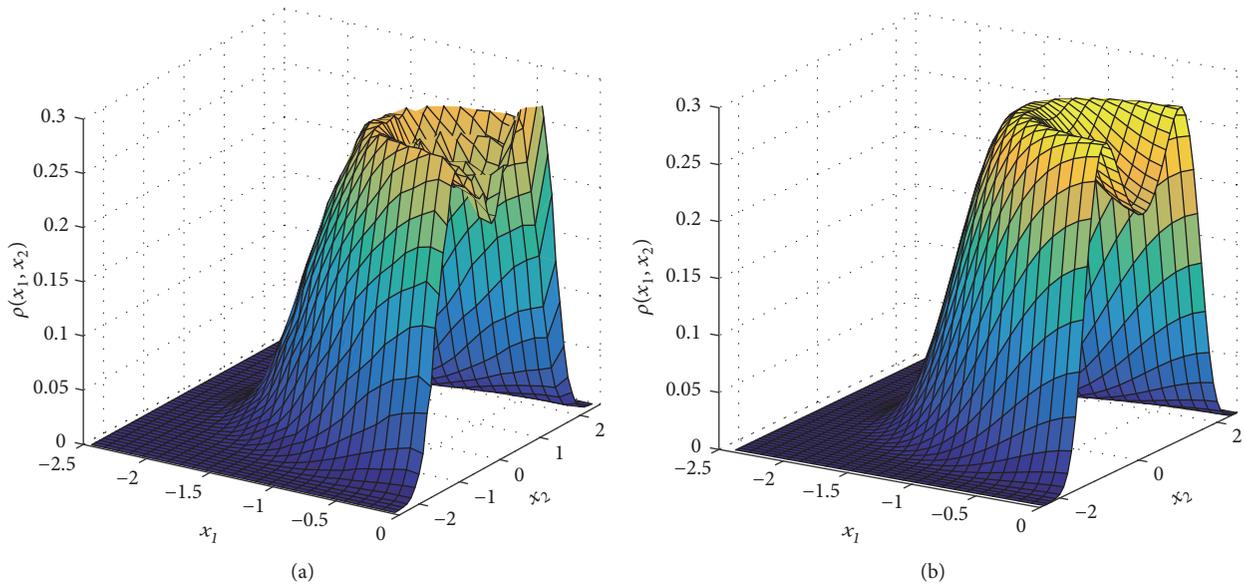


FIGURE 8: Joint PDFs of the displacement  $x_1$  and velocity  $x_2$  for  $r = 0.995$ : (a) Monte-Carlo results; (b) analytical results.

Thus,  $\rho(h)$  is obtained analytically by solving the corresponding averaged generalized FPK equation as

$$\rho(h) = \bar{C}e^{-N} \left( 1 + \varepsilon^2 (Dh + Eh^2 + Fh^3 + Gh^4 + Mh^5) \right), \quad (44)$$

where

$$A = \lambda E [Y^2],$$

$$B = \lambda E [Y^4],$$

$$C = -\beta - \frac{2(r-1)\sqrt{k}}{\pi\varepsilon^2},$$

$$D = \frac{B\beta}{4A^2k} - \frac{BC^2}{A^3},$$

$$E = \frac{BC^3}{2A^4} - \frac{7BC\beta}{8A^3k},$$

$$F = \frac{BC^2\beta}{2A^4k} - \frac{5B\beta^2}{24A^3k^2},$$

$$G = \frac{3BC\beta^2}{16A^4k^2},$$

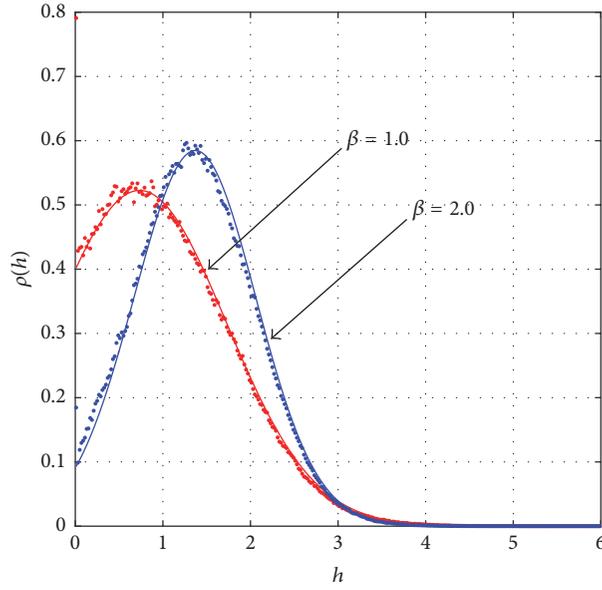


FIGURE 9: Stationary PDFs of total energy obtained numerically and analytically for different values of  $\beta$ .

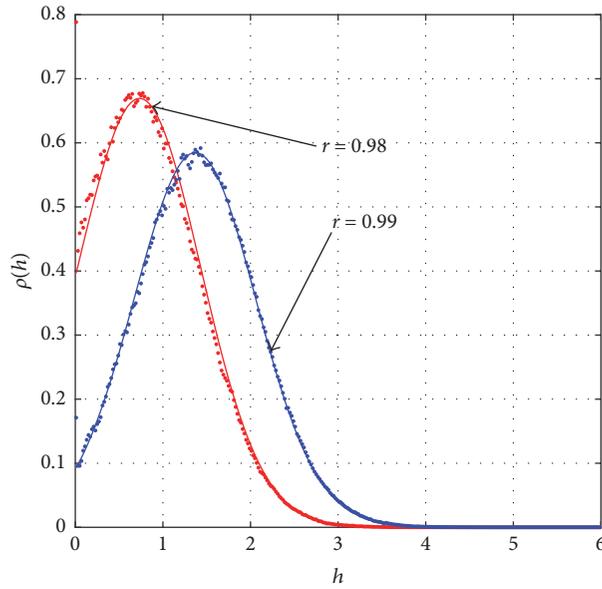


FIGURE 10: Stationary PDFs of total energy obtained numerically and analytically for different values of restitution coefficient  $r$ .

$$\begin{aligned}
 M &= \frac{B\beta^3}{40A^4k^3}, \\
 N &= \frac{2}{A} \left( \frac{\beta h^2}{4k} + Ch \right).
 \end{aligned}
 \tag{45}$$

Furthermore, the PDFs  $\rho(x_1, x_2)$ ,  $\rho(x_1)$ , and  $\rho(x_2)$  can be calculated following (29)–(31), respectively.

To demonstrate the effectiveness of the proposed method, numerical and analytical results of response PDFs are provided in Figures 9–13. The effects of damping coefficient on the accuracy of the energy PDFs are shown in Figure 9 with

$\varepsilon = 0.1$ ,  $r = 0.99$ ,  $\lambda = 1$ , and  $E[Y^2] = 1$  and two different values of  $\beta$  ( $\beta = 1.0$  and  $2.0$ ). Stationary PDFs of total energy obtained numerically and analytically for different values of restitution coefficient  $r$  are presented in Figure 10. It is clear from Figures 9 and 10 that the analytical results give powerful validation with Monte-Carlo simulation results. The joint PDFs of the displacement  $x_1$  and velocity  $x_2$  for different impulse intensities are shown numerically and analytically in Figures 11 and 12. The corresponding marginal PDFs  $\rho(x_1)$  and  $\rho(x_2)$  are provided in Figures 13(a) and 13(b), respectively. It is seen that the results obtained by using the proposed method and those from Monte-Carlo simulation are in good agreement. Additionally, the increase of impulse intensity

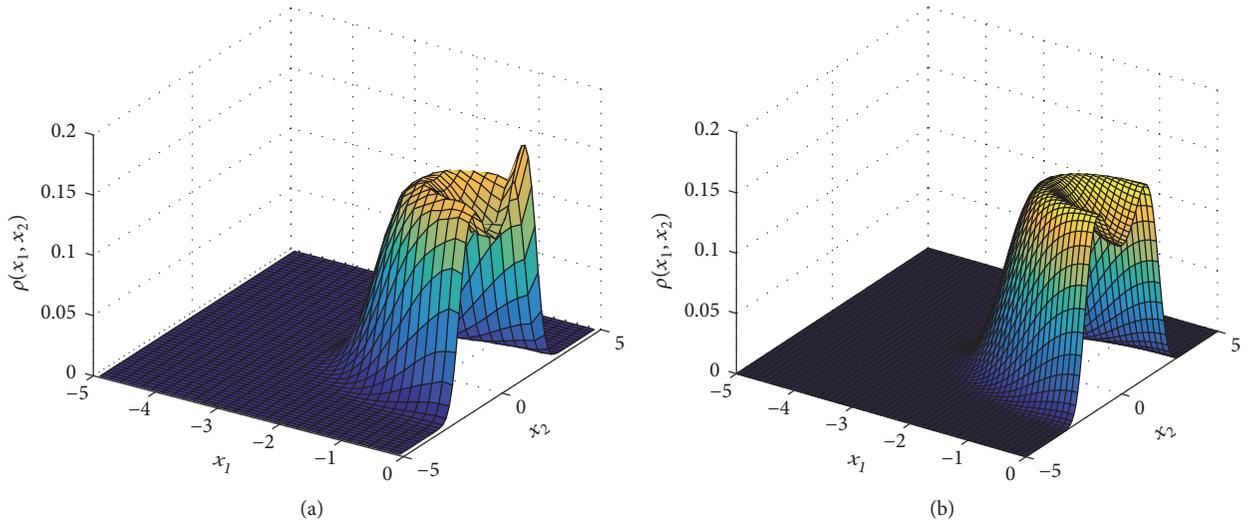


FIGURE 11: Joint PDFs of the displacement  $x_1$  and velocity  $x_2$  for  $E[Y^2] = 1.0$ : (a) Monte-Carlo results; (b) analytical results.

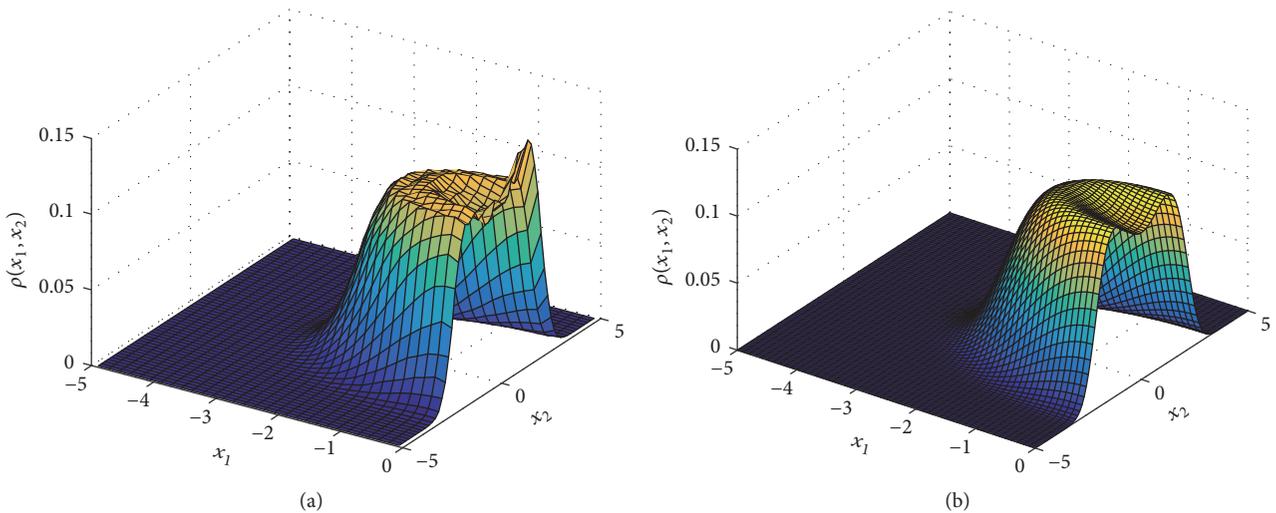


FIGURE 12: Joint PDFs of the displacement  $x_1$  and velocity  $x_2$  for  $E[Y^2] = 2.0$ : (a) Monte-Carlo results; (b) analytical results.

only effects the values of peaks of stationary response PDFs, and there is no occurrence of P-bifurcation.

## 5. Conclusions

The paper is devoted to presenting a solution procedure for predicting stationary responses of single-degree-of-freedom lightly nonlinear vibroimpact systems with inelastic impact subjected to external Poisson white noise excitations. The classical model with instantaneous impacts is used and the rigid barrier is located at the static equilibrium of the system. The solution procedure consists of three steps. First, the Zhuravlev nonsmooth coordinate transformation and the Dirac delta function are adopted to convert the original

vibroimpact system to a new system without velocity jump. Second, the Itô stochastic differential equations with correction terms are obtained and the averaged generalized Fokker-Planck-Kolmogorov (FPK) equation is derived by stochastic averaging method. Third, the perturbation method is utilized to solve the averaged generalized FPK equation and the inverse transformation of the Zhuravlev nonsmooth coordinate transformation is used to establish the PDFs for original vibroimpact system. Furthermore, two examples are presented to show the effectiveness of the proposed method. It is found that the analytical results obtained by the proposed method and the Monte-Carlo simulation results agree well. Therefore, the proposed procedure is quite promising and can be extended to more vibroimpact systems subject to Poisson white noise excitation.

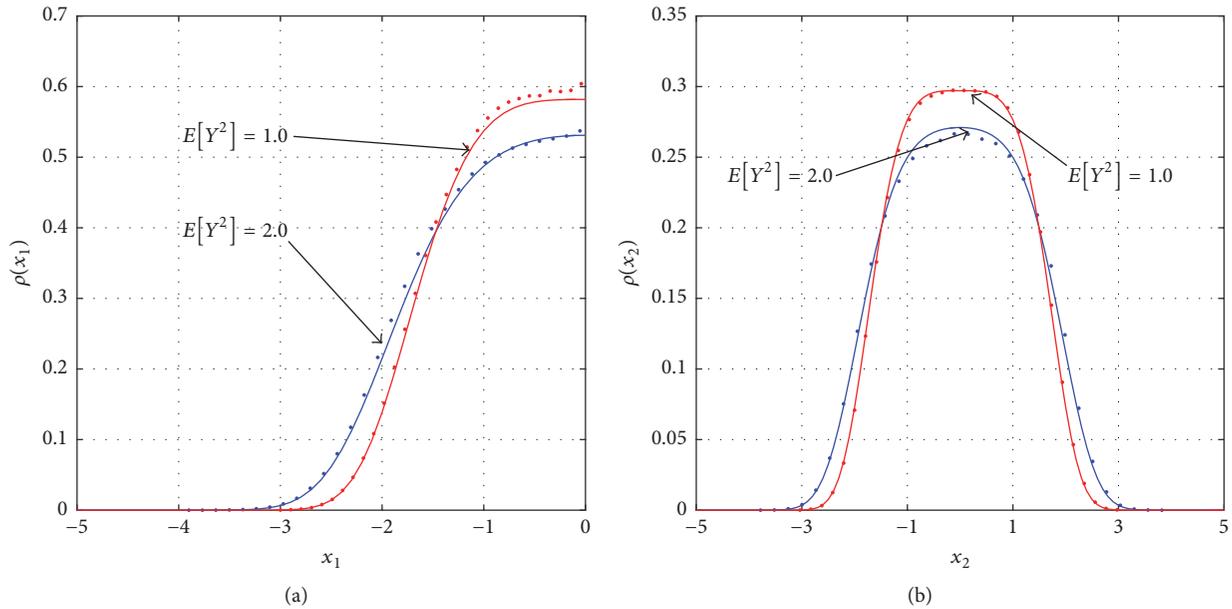


FIGURE 13: Stationary PDFs obtained numerically and analytically for different values of random impulse intensity  $E[Y^2]$ . -, analytical results; ·, Monte-Carlo results: (a) PDFs of displacement  $x_1$ ; (b) PDFs of velocity  $x_2$ .

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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