Research Article

Extensions of Hölder’s Inequality via Pseudo-Integral

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Hölder’s inequality and its various extensions are playing very important roles in many branches of modern mathematics and physics. In this paper, we present some extensions of Hölder’s inequality via pseudo-integral. Moreover, some related Hölder type inequalities are also given.

1. Introduction

The classical Hölder’s inequality states that if $X$ is a measure space, with measure $\mu$, if $p > 0$, $q > 0$ with $1/p + 1/q = 1$, and if $f$ and $g$ are measurable nonnegative functions on $X$, with range in $(0, \infty)$, then

$$\int_X f gd\mu \leq \left( \int_X f^p d\mu \right)^{1/p} \left( \int_X g^q d\mu \right)^{1/q}. \tag{1}$$

As is well known, the above Hölder’s inequality plays an important role in many branches of modern mathematics and physics. For example, Agahi et al. [1–3] presented some interesting extensions of Hölder’s inequality (1) for decomposition integral, Sugeno integral, and pseudo-integral. Benmerrouche et al. [4] obtained a new technique based on Hölder’s integral inequality (1) which provides fundamental constraints on the QCD sum-rules. Steele et al. [5] employed the Hölder inequality to place lower bounds on the sum of $u$ and $d$ 1.0 GeV MS running masses. Tian et al. [6–13] gave some new properties, generalizations, refinements, and applications of Hölder’s inequalities. Wu [14, 15] presented some new sharpened and generalized versions of Hölder’s inequalities in classical real analysis. For more detail expositions, the interested reader may consult [16–22] and the references therein.

Among various extensions of (1), Agahi et al. in [3] established the following interesting Hölder’s inequality and reverse Hölder’s inequality for pseudo-integral (for details on pseudo-integral, see (23–25)).

**Theorem A.** Let $1 < p < \infty$ and $1/p + 1/q = 1$, and let $(X, \sigma)$ be a given measurable space. Suppose that $u, v : X \to [a, b]$ are two measurable functions and that a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is an increasing function. Then for any $\sigma$-$\oplus$-measure $m$, we have

$$\int_X (u \odot v) \odot dm \leq \left( \int_X u^{(p)} \odot dm \right)^{(1/p)} \odot \left( \int_X v^{(q)} \odot dm \right)^{(1/q)}. \tag{2}$$

**Theorem B.** Let $1 < p < \infty$ and $1/p + 1/q = 1$, and let $(X, \sigma)$ be a given measurable space. Suppose that $u, v : X \to [a, b]$ are two measurable functions and that a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a decreasing function. Then for any $\sigma$-$\odot$-measure $m$, we have

$$\int_X (u \odot v) \odot dm \geq \left( \int_X u^{(p)} \odot dm \right)^{(1/p)} \odot \left( \int_X v^{(q)} \odot dm \right)^{(1/q)}. \tag{3}$$
2. Extensions of Hölder’s Inequality via Pseudo-Integral

In this section, we shall give the following lemma firstly before we give our results.

Lemma 1 (see [26]). Let $X$ be a measure space, with measure $\mu$; let $1 < \lambda_1, \lambda_2, \ldots, \lambda_n < \infty$ with $\sum_{i=1}^{n}(1/\lambda_i) = 1$, and let $f_i$ $(i = 1, 2, \cdots, n)$ be measurable nonnegative functions on $X$, with range in $(0, \infty)$. Then

$$\int_X \left( \bigoplus_{i=1}^{n} f_i \right) d\mu \leq \int_X f^{1/\lambda_1} d\mu,$$

(4)

Next, we give the following extension of inequality (2).

Theorem 2. Let $1 < \lambda_1, \lambda_2, \ldots, \lambda_n < \infty$ and $\sum_{i=1}^{n}(1/\lambda_i) = 1$, and let $(X, \mathcal{A}, \mu)$ be a given measurable space. Suppose that $u_i : X \rightarrow [a, b]$ $(i = 1, 2, \cdots, n)$ are measurable functions and that a generator $g : [a, b] \rightarrow [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\circ$ is an increasing function.

Then for any $\sigma$-$\mu$-measurable $m$, we have

$$\int_X \bigoplus_{i=1}^{n} (u_1 \circ u_2 \cdots \circ u_n) \circ dm \leq \left( \int_X u_1^{\lambda_1} \circ dm \right)^{1/\lambda_1} \circ \cdots \circ \left( \int_X u_n^{\lambda_n} \circ dm \right)^{1/\lambda_n},$$

(5)

Proof. From Lemma 1 we obtain

$$\int_X \prod_{i=1}^{n} (g \circ u_i) \circ dm \leq \prod_{i=1}^{n} \left( \int_X (g \circ u_i)^{1/\lambda_i} \circ dm \right)^{\lambda_i}.$$

(6)

If pseudo-operations are generated by a monotone and continuous function $g : [a, b] \rightarrow [0, \infty]$, then $x \circ y = g^{-1}(g(x) \cdot g(y))$. Therefore

$$x_1 \circ x_2 \circ \cdots \circ x_n = g^{-1}(g(x_1) \cdot g(x_2) \cdots g(x_n)).$$

(9)

From inequality (8) and the pseudo-integral for a measurable function $f : X \rightarrow [a, b]$ given by

$$\int_X f \circ dm = g^{-1} \left( \int_X (g \circ f) \circ (g \circ m) \right),$$

(10)

we have

$$\int_X \bigoplus_{i=1}^{n} (u_1 \circ u_2 \cdots \circ u_n) \circ dm \leq \left( \int_X g^{-1} \left( \prod_{i=1}^{n} (g \circ u_i) \right) \circ dm \right)^{1/\lambda_1} \circ \cdots \circ \left( \int_X g^{-1} \left( \prod_{i=1}^{n} (g \circ u_i)^{1/\lambda_i} \right) \circ dm \right)^{1/\lambda_n}.$$
Therefore by inequalities (11) and (12), we obtain immediately the desired inequality (5).

Example 3. If we set \([a, b] = [0, \infty]\) and \(g(x) = x^\alpha\) for some \(\alpha \in [1, \infty]\), then from the definition of pseudo-operations [23–25], we find that \(x \odot y = \sqrt[\lambda]{x^\alpha + y^\alpha}\) and \(x \odot y = xy\). Therefore we obtain the following Hölder type inequality:

\[
\begin{align*}
\ln \int_{[c, d]} e^{f_1(x)+f_2(x)+\cdots+f_n(x)} \, dx \\
\leq \sum_{i=1}^{n} \ln \left( \int_{[c, d]} e^{\lambda_i f_i(x)} \, dx \right),
\end{align*}
\]

that is

\[
\int_{[c, d]} e^{f_1(x)+f_2(x)+\cdots+f_n(x)} \, dx \leq \sum_{i=1}^{n} \left( \int_{[c, d]} e^{\lambda_i f_i(x)} \, dx \right)^{1/\lambda_i}.
\]

By the same methods as in the proof of Theorem 2, we can obtain the following generalization of inequality (3).

**Theorem 5.** Let \(1 < \lambda_1, \lambda_2, \ldots, \lambda_n < \infty\) and \(\sum_{i=1}^{n} (1/\lambda_i) = 1\), and let \((X, \mathfrak{A}, \mu)\) be a given measurable space. Suppose that \(u_i : X \to [a, b]\) \((i = 1, 2, \ldots, n)\) are measurable functions and a generator \(g : [a, b] \to [0, \infty]\) of the pseudo-addition \(\oplus\) and the pseudo-multiplication \(\odot\) is a decreasing function. Then for any \(\alpha \oplus\)-measure \(m\), we have

\[
\int_X (u_1 \oplus u_2 \odot \cdots \odot u_n) \, dm \\
\leq \left( \int_X u_1^{(\lambda_1)} \odot dm \right)^{(1/\lambda_1)} \odot \left( \int_X u_2^{(\lambda_2)} \odot dm \right)^{(1/\lambda_2)} \odot \cdots \odot \left( \int_X u_n^{(\lambda_n)} \odot dm \right)^{(1/\lambda_n)}.
\]

Next we consider the Case: \(\oplus = \sup\) and \(\odot = g^{-1}(g(x)g(y))\).

**Theorem 6.** Let \(1 < \lambda_1, \lambda_2, \ldots, \lambda_n < \infty\) with \(\sum_{i=1}^{n} (1/\lambda_i) = 1\). Suppose that \(\odot\) is represented by an increasing generator \(g\) and \(m\) is a complete sup-measure. Then for any functions \(u_i : X \to [a, b]\) \((i = 1, 2, \ldots, n)\), we have

\[
\int_X \left( \sup_{[a, b]} (u_1 \odot u_2 \odot \cdots \odot u_n) \right) \, dm \\
\leq \left( \int_X u_1^{(\lambda_1)} \odot dm \right)^{(1/\lambda_1)} \odot \left( \int_X u_2^{(\lambda_2)} \odot dm \right)^{(1/\lambda_2)} \odot \cdots \odot \left( \int_X u_n^{(\lambda_n)} \odot dm \right)^{(1/\lambda_n)}.
\]

Proof. Since \(\oplus = \sup\) and \(\odot = g^{-1}(g(x)g(y))\), by the pseudo-integral of the function \(f : \mathbb{R} \to [a, b]\) given by

\[
\int_{\mathbb{R}} f \odot dm = \sup \left\{ f(x) \odot \psi(x) \right\},
\]
we have
\[
\int_X \left( \bigcup_{\omega} u_1 \odot u_2 \odot \cdots \odot u_n \right) \odot dm = \sup_{x \in X} \left( \bigcup_{\omega} u_1(x) \odot u_2(x) \odot \cdots \odot u_n(x) \odot \psi(x) \right)
\]
\[
= g^{-1} \left\{ \sup_{x \in X} \left[ \left( \prod_{i=1}^{n} g(u_i(x)) \right) g(\psi(x)) \right] \right\},
\]
where \( \psi : X \rightarrow [a, b] \) is a density function related to \( m \).

Moreover
\[
\left( \int_X \left( \bigcup_{\omega} u_1 \odot u_2 \odot \cdots \odot u_n \right) \odot dm \right)^{(1/\lambda_i)}
\]
\[
= g^{-1} \left\{ \left[ \sup_{x \in X} \left( g(u_i(x)) \right)  \right] \right\}^{1/\lambda_i}
\]
\[
= g^{-1} \left\{ \sup_{x \in X} \left[ g(u_i(x)) \right]^{1/\lambda_i} \right\}
\]
\[
(i = 1, 2, \cdots, n).
\]

Consequently,
\[
\int_X \left( \bigcup_{\omega} u_1 \odot u_2 \odot \cdots \odot u_n \right) \odot dm
\]
\[
\leq \left( \int_X u_1 \odot dm \right) \odot \left( \int_X u_2 \odot dm \right) \odot \cdots \odot \left( \int_X u_n \odot dm \right),
\]
which implies
\[
\sup_{x \in X} \left[ \left( \bigcup_{\omega} u_1(x) \right) \odot u_1(x) \cdots \odot u_n(x) \odot \psi(x) \right]
\]
\[
\leq \inf \left\{ \sup_{x \in X} \left[ \left( \bigcup_{\omega} u_1(x) \right) \odot u_1(x) \cdots \odot u_n(x) \odot \psi(x) \right] \right\},
\]
\[
\sup_{x \in X} \left[ \left( \bigcup_{\omega} u_1(x) \right) \odot u_1(x) \cdots \odot u_n(x) \odot \psi(x) \right]
\]
\[
\leq \inf \left\{ \sup_{x \in X} \left[ \left( \bigcup_{\omega} u_1(x) \right) \odot u_1(x) \cdots \odot u_n(x) \odot \psi(x) \right] \right\},
\]
\[
(i = 1, 2, \cdots, n).
\]

\[\text{Example 8.}\] Let \([a, b] = [-\infty, +\infty]\), and let \( f_i, \psi \) be given real functions on \( X \). Suppose that \( g \) generating \( \odot \) is given by \( g(x) = e^x \). Then we have \( x \odot y = x + y \). Therefore from inequality (17) we obtain the following Hölder type inequality:
\[
\sup_{x \in X} \left( f_1(x) + f_2(x) + \cdots + f_n(x) + \psi(x) \right)
\]
\[
\leq \sum_{i=1}^{n} \frac{1}{\lambda_i} \sup_{x \in X} (\lambda_i \cdot u(x) + \psi(x)).
\]

\section{Conclusions}
Hölder’s inequality is playing a very important and basic role in different branches of modern mathematics such as classical real and complex analysis, numerical analysis, probability and statistics, fuzzy measure theory, qualitative theory of differential equations and their applications. It is also an indispensable and basic tool in engineering technology. Pseudo-integral is applicable in many of the fields of applied sciences which helps us to interpret many problems. In this paper we have presented some extensions of Hölder’s inequality via pseudo-integral. Moreover, we have obtained some related Hölder type inequalities. The obtained results may have some potential for applications. For example, we may improve some results on stochastic matrix in Markov chains and their applications.

\section*{Conflicts of Interest}
The authors declare that they have no conflicts of interest.

\section*{Authors’ Contributions}
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References


