Research Article

Refined Estimates and Generalizations of Inequalities Related to the Arctangent Function and Shafer's Inequality

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We give some sharper refinements and generalizations of inequalities related to Shafer's inequality for the arctangent function, stated in Theorems 1, 2, and 4 in Mortici and Srivastava, 2014, by C. Mortici and H.M. Srivastava.

1. Introduction

Inverse trigonometric functions play an important role and have many applications in engineering [1–4]. In particular, the arctangent function and various related inequalities have been studied and effectively applied to problems in fundamental sciences and many areas of engineering, such as electronics, mechanics, and aeronautics [3, 5, 6]; see also [7].

Various approximations of the arctangent function can be found in [4–6, 8–21]; see also [22, 23]. One of the inequalities that attracted attention of many authors is Shafer's inequality [9]:

$$\frac{3x}{1 + 2\sqrt{1 + x^2}} < \arctan x, \quad (1)$$

which holds for $x > 0$; see also [10–12].

Recently, in [8], Mortici and Srivastava proved the following results, cited here as Statements 1, 2, and 3, related to the above inequality. These results are the starting point of our research.

Statement 1 (Theorem 1, [8]). For every $x > 0$, the following two-sided inequality holds:

$$\begin{aligned} \frac{3x}{1 + 2\sqrt{1 + x^2}} + a(x) &< \arctan x \\ &< \frac{3x}{1 + 2\sqrt{1 + x^2}} + b(x), \end{aligned} \quad (2)$$

where $a(x) = (1/180)x^5 - (13/1512)x^7$ and $b(x) = (1/180)x^5$.

Statement 2 (Theorem 2, [8]). For every $x > 0$, it is asserted that

$$\frac{3x + c(x)}{1 + 2\sqrt{1 + x^2}} < \arctan x < \frac{3x + d(x)}{1 + 2\sqrt{1 + x^2}}, \quad (3)$$

where $c(x) = (1/60)x^5 - (17/840)x^7$ and $d(x) = (1/60)x^5$.

Statement 3 (Theorem 4, [8]). For every $x > 0$, it is asserted that

$$-\frac{1}{12}x^3 < \arctan x - \frac{2x}{1 + \sqrt{1 + x^2}} < -\frac{1}{12}x^3 + \frac{3}{40}x^5. \quad (4)$$

The main results of this paper are refined estimates and generalizations of the inequalities given in Statements 1, 2, and 3. Although inequalities (2), (3), and (4) hold for $x > 0$, considering them in a neighborhood of zero is of primary importance, as noted in [8].

2. Main Results

First, let us recall some well-known power series expansions that will be used in our proofs.

For $|x| \leq 1$,

$$\arctan x = \sum_{m=0}^{\infty} (-1)^m A(m) x^{2m+1}, \quad (5)$$

where

$$A(m) = \frac{1}{2m+1}. \quad (6)$$

For $|x| \leq 1$,

$$\sqrt{1+x^2} = 1 + \sum_{m=0}^{\infty} (-1)^m K(m) x^{2m+2}, \quad (7)$$

where

$$K(m) = \frac{(2m)!}{m!(m+1)!2^{2m+1}}. \quad (8)$$

The following power series expansion holds:

$$\frac{3x}{1+2\sqrt{1+x^2}} = \sum_{m=0}^{\infty} B(m) x^{2m+1}, \quad (9)$$

where $|x| \leq \sqrt{3}/2$, with $B(0) = 1$ and $B(1) = -1/3$, and for $m \geq 2$,

$$B(m) = \frac{(-1)^m 4^{m-1}}{3^m} \left(1 - 8 \sum_{i=2}^m \frac{(2i-2)!}{(i-1)!i!2^{2i-1}} \left(\frac{3}{4} \right)^i \right). \quad (10)$$

Power series coefficients are calculated by applying Cauchy's product to the power series expansions arising from the following transformation of the corresponding function:

$$\frac{3x}{1+2\sqrt{1+x^2}} = -\frac{x(1-2\sqrt{1+x^2})}{1+(4/3)x^2}. \quad (11)$$

It is easy to prove that sequence $\{B(m)\}_{m \in N_0}$ for $m \geq 1$ satisfies the recurrence equation:

$$\begin{aligned} B(m+1) + \frac{4}{3}B(m) &= (-1)^m \frac{(2m-1)!}{2^{2m-1}(m-1)!(m+1)!} \\ &= 2(-1)^m \frac{(2m-1)!!}{(2m+2)!!}. \end{aligned} \quad (12)$$

2.1. Refinements of the Inequalities in Statement 1. Before we proceed to Theorem 9, which represents an improvement and generalization of Statement 1, we need the following lemmas.

Lemma 4. Let $\beta(0) = 1$, $\beta(1) = -1/3$, and $\beta(m) = (3(-1)^m/2^{2m+1}) \sum_{k=0}^{\infty} ((2k+2m-1)!/(k+m-1)!(k+m+1)!(3/16)^k)$, for $m \geq 2$. The sequence $\{\beta(m)\}_{m \in N_0}$ for $m \geq 1$ satisfies the recurrence relation (12).

Proof. In the proof of this lemma we use the Wilf-Zeilberger method [24–26]. (The same approach we used in [27].)

The assertion is obviously true for $m = 1$.

Let $m \geq 2$ and

$$\begin{aligned} g(k, m) &= \frac{(2k+2m-1)!}{(k+m-1)!(k+m+1)!} \text{ and} \\ \phi(m) &= \frac{3(-1)^m}{2^{2m+1}}. \end{aligned} \quad (13)$$

Then we have

$$\beta(m) = \phi(m) \sum_{k=0}^{\infty} g(k, m) \left(\frac{3}{16} \right)^k. \quad (14)$$

Further we have

$$\begin{aligned} \beta(m+1) + \frac{4}{3}\beta(m) &= \sum_{k=0}^{\infty} \left(\phi(m+1)g(k, m+1) + \frac{4}{3}\phi(m)g(k, m) \right) \\ &\cdot \left(\frac{3}{16} \right)^k = \frac{(-1)^m}{2^{2m-1}} \\ &\cdot \sum_{k=0}^{\infty} \frac{(2k+2m+13)(2k+2m-1)!}{(8k+8m+16)(k+m-1)!(k+m+1)!} \\ &\cdot \left(\frac{3}{16} \right)^k. \end{aligned} \quad (15)$$

Consider now the sequence $\{S(m)\}_{m \in N, m \geq 2}$, where

$$\begin{aligned} S(m) &= \sum_{k=0}^{m-1} F(m, k) \text{ and} \\ F(m, k) &= \frac{(2m+2k+13)(2m+2k-1)!}{(8m+8k+16)(m+k-1)!(m+k+1)!} \left(\frac{3}{16} \right)^k. \end{aligned} \quad (16)$$

Consider the function (an algorithm for determining function $G(m, k)$ for a given function $F(m, k)$ is described in [24]. Note that the pair of discrete functions $(F(m, k), G(m, k))$ is the so-called Wilf-Zeilberger pair)

$$\begin{aligned} G(m, k) &= \frac{(-8k-8m-16)(2k+2m-1)!}{(8k+8m+16)(k+m-1)!(k+m+1)!} \left(\frac{3}{16} \right)^k, \end{aligned} \quad (17)$$

where $m \in N_0$ and $k \in N_0$. It is not hard to verify that functions $F(m, k)$ and $G(m, k)$ satisfy the following relation:

$$F(m, k) = G(m, k+1) - G(m, k). \quad (18)$$

If we sum both sides of (18) over all $k \in N_0$, we get the following relation:

$$S(m) = -G(m, 0). \quad (19)$$

Finally, as

$$G(m, 0) = -\frac{(2m-1)!}{(m-1)!(m+1)!} \quad (20)$$

we have

$$S(m) = \frac{(2m-1)!}{(m-1)!(m+1)!}. \quad (21)$$

Therefore from (15) and (21) we conclude that

$$\begin{aligned}\beta(m+1) + \frac{4}{3}\beta(m) &= \frac{(-1)^m}{2^{2m-1}} \cdot S(m) \\ &= \frac{(-1)^m}{2^{2m-1}} \cdot \frac{(2m-1)!}{(m-1)!(m+1)!}.\end{aligned}\quad (22)$$

□

Corollary 5. Given that the sequences $\{B(m)\}_{m \in N_0}$ and $\{\beta(m)\}_{m \in N_0}$ satisfy the same recurrence relation and as they agree for $m = 0$ and $m = 1$, we conclude that

$$B(m) = \beta(m), \quad \text{for } m \in N_0. \quad (23)$$

Let us introduce the notation:

$$C(m) = A(m) - B(m), \quad (24)$$

where $C(0) = C(1) = 0$, and for $m \geq 2$ the following holds:

$$\begin{aligned}C(m) &= \frac{1}{2m+1} \\ &\quad - \frac{4^{m-1}}{3^m} \left(1 - 8 \sum_{i=2}^m \frac{(2i-2)!}{(i-1)!i!2^{2i-1}} \left(\frac{3}{4}\right)^i \right).\end{aligned}\quad (25)$$

Thus, we have the power series expansion:

$$\begin{aligned}f(x) &= \arctan x - \frac{3x}{2 + \sqrt{1+x^2}} \\ &= \sum_{m=0}^{\infty} (-1)^m C(m) x^{2m+1},\end{aligned}\quad (26)$$

for $m \in N_0$ and $|x| \leq \sqrt{3}/2$.

Let us introduce the notation:

$$\begin{aligned}\beta^+(m) &= \frac{3}{2^{2m+1}} \sum_{k=0}^{\infty} \frac{(2k+2m-1)!}{(k+m-1)!(k+m+1)!2^{2k}} \left(\frac{3}{4}\right)^k, \\ &\quad m \in N_0\end{aligned}\quad (27)$$

$$\beta_1(m) = \sum_{k=0}^{+\infty} \frac{\prod_{n=0}^{k-1} (m+1/2+n)}{\prod_{n=0}^{k-1} (m+2+n)} \left(\frac{3}{4}\right)^k,$$

$$\beta_1^k(m) = \frac{\prod_{n=0}^{k-1} (m+1/2+n)}{\prod_{n=0}^{k-1} (m+2+n)}, \quad m \in N_0.$$

Lemma 6. For $m \in N_0$ the following holds:

$$\beta^+(m) = \frac{3(2m-1)!!}{2(2m+2)!!} \beta_1(m). \quad (28)$$

Proof.

$$\begin{aligned}\frac{3}{2^{2m+1}} \frac{(2m+2k-1)!}{(m+k-1)!(m+k+1)!2^{2k}} &= \frac{3}{2} \frac{1}{(2m+2)!!} \\ &\cdot \frac{(2m+2k-1)!!(m+1)!}{(m+k+1)!2^k} = \frac{3(2m-1)!!}{2(2m+2)!!} \\ &\cdot \frac{(2m+1)(2m+3)\dots(2m+2k-1)}{(m+1)!(m+2)(m+3)\dots(m+k+1)} \\ &\cdot \frac{(m+1)!}{2^k} = \frac{3(2m-1)!!}{2(2m+2)!!} \frac{\prod_{n=0}^{k-1} (m+1/2+n)}{\prod_{n=0}^{k-1} (m+2+n)}.\end{aligned}\quad (29)$$

□

Lemma 7. For $m \in N_0$ the following holds:

$$\begin{aligned}\frac{8(m+2)}{2m+13} &= \sum_{k=0}^{\infty} \left(\frac{m+1/2}{m+2}\right)^k \left(\frac{3}{4}\right)^k < \beta_1(m) \\ &< \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k = 4.\end{aligned}\quad (30)$$

Proof. The statement immediately follows from the inequalities:

$$\left(\frac{m+1/2}{m+2}\right)^k < \beta_1^k(m) < 1. \quad (31)$$

□

Lemma 8. For $m \in N_0$ the following holds:

$$\beta_1(m+1) > \beta_1(m) \frac{m+2}{m+1/2} \left(1 - \frac{3}{2} \frac{1}{m+2}\right). \quad (32)$$

Proof.

$$\begin{aligned}\beta_1^k(m+1) &= \beta_1^k(m) \frac{(m+2)(m+1/2+k)}{(1/2+m)(m+2+k)} \\ &= \beta_1^k(m) \frac{m+2}{m+1/2} \left(1 - \frac{3}{2} \frac{1}{m+2+k}\right) \\ &> \beta_1^k(m) \frac{m+2}{m+1/2} \left(1 - \frac{3}{2} \frac{1}{m+2}\right).\end{aligned}\quad (33)$$

□

Theorem 9. For the real analytic function

$$f(x) = \arctan x - \frac{3x}{2 + \sqrt{1+x^2}} \quad (34)$$

the following inequalities hold for $k \in N$ and $x \in (0, \sqrt{3}/2]$:

$$\begin{aligned}\sum_{m=0}^{2k+1} (-1)^m C(m) x^{2m+1} &< f(x) \\ &< \sum_{m=0}^{2k} (-1)^m C(m) x^{2m+1}.\end{aligned}\quad (35)$$

where $C(0) = C(1) = 0$, and for $m \geq 2$ the following holds:

$$C(m) = \frac{1}{2m+1} - \frac{4^{m-1}}{3^m} \left(1 - 8 \sum_{i=2}^m \frac{(2i-2)!}{(i-1)! 2^{2i-1}} \left(\frac{3}{4} \right)^i \right). \quad (36)$$

Proof. We will prove that the sequence $\{C(m)\}_{m \in \mathbb{N}_0}$ is positive and monotonically decreasing and tends to zero as m tends to infinity. We will use Lemmas 7 and 8.

$$\begin{aligned} C(m) &= \frac{1}{2m+1} - \frac{3}{2} \frac{(2m-1)!!}{(2m+2)!!} \beta_1(m) \\ &> \frac{1}{2m+1} - 4 \frac{3}{2} \frac{(2m-1)!!}{(2m+2)!!} \\ &= \frac{1}{2m+1} \left(1 - 6 \frac{(2m+1)!!}{(2m+2)!!} \right). \end{aligned} \quad (37)$$

It is easy to verify that $6((2m+1)!!/(2m+2)!!) < 1$ for $m \geq 11$; therefore $C(m) > 0$. Let us note that $0 < C(m) < 1/(2m+1)$, so we can conclude that $\lim_{m \rightarrow +\infty} C(m) = 0$.

Let us now prove that $\{C(m)\}_{m \in \mathbb{N}_0}$ is a monotonically decreasing sequence.

$$\begin{aligned} C(m+1) - C(m) &= \frac{-2}{(2m+1)(2m+3)} - \frac{3}{2} \\ &\cdot \frac{(2m+1)!!}{(2m+4)!!} \beta_1(m+1) + \frac{3}{2} \frac{(2m-1)!!}{(2m+2)!!} \beta_1(m) \\ &= \frac{-2}{(2m+1)(2m+3)} + \frac{3}{2} \frac{(2m-1)!!}{(2m+2)!!} \left(\beta_1(m) \right. \\ &\left. - \frac{2m+1}{2m+4} \beta_1(m+1) \right) < \frac{-2}{(2m+1)(2m+3)} + \frac{3}{2} \\ &\cdot \frac{(2m-1)!!}{(2m+2)!!} \left(\beta_1(m) - \frac{2m+1}{2m+4} \right. \\ &\cdot \frac{m+2}{m+1/2} \left(1 - \frac{3}{2} \frac{1}{m+2} \right) \beta_1(m) \Big) \\ &= \frac{-2}{(2m+1)(2m+3)} + \beta_1(m) \frac{9}{4} \\ &\cdot \frac{(2m-1)!!}{(m+2)(2m+2)!!} < \frac{-2}{(2m+1)(2m+3)} + 4 \frac{9}{4} \\ &\cdot \frac{(2m-1)!!}{(m+2)(2m+2)!!} = \frac{-2}{(2m+1)(2m+3)} \left(1 \right. \\ &\left. - 9 \frac{(2m+3)!!}{(2m+4)!!} \right). \end{aligned} \quad (38)$$

It is easy to prove that $C(m+1) - C(m) < 0$ for $m \geq 8$, i.e., the sequence is monotonically decreasing. Since $\{C(m)\}_{m \in \mathbb{N}_0}$ is positive for $m \geq 2$ and monotonically decreasing (for $m \geq 8$) and tends to zero, the same holds true for the sequence $\{C(m)x^{2m+1}\}_{m \in \mathbb{N}_0}$ for a fixed $x \in (0, \sqrt{3}/2]$ (noting that it is

decreasing for $m \geq 3$), so we can apply Leibniz's theorem for alternating series [28], thus proving the claim of Theorem 9:

$$\begin{aligned} \sum_{m=0}^{2k+1} (-1)^m C(m) x^{2m+1} &< f(x) \\ &< \sum_{m=0}^{2k} (-1)^m C(m) x^{2m+1}, \quad k \in \mathbb{N}. \end{aligned} \quad (39)$$

□

Examples. For $k = 1$ and $x \in (0, \sqrt{3}/2]$ we get Statement 1.

For $k = 2$ and $x \in (0, \sqrt{3}/2]$,

$$\begin{aligned} \frac{x^5}{180} - \frac{13x^7}{1512} + \frac{53x^9}{5184} - \frac{3791x^{11}}{342144} \\ < \arctan x - \frac{3x}{1+2\sqrt{1+x^2}} \\ < \frac{x^5}{180} - \frac{13x^7}{1512} + \frac{53x^9}{5184}. \end{aligned} \quad (40)$$

For $k = 3$ and $x \in (0, \sqrt{3}/2]$,

$$\begin{aligned} \frac{x^5}{180} - \frac{13x^7}{1512} + \frac{53x^9}{5184} - \frac{3791x^{11}}{342144} + \frac{55801x^{13}}{4852224} \\ - \frac{130591x^{15}}{11197440} < \arctan x - \frac{3x}{1+2\sqrt{1+x^2}} < \frac{x^5}{180} \\ - \frac{13x^7}{1512} + \frac{53x^9}{5184} - \frac{3791x^{11}}{342144} + \frac{55801x^{13}}{4852224}, \end{aligned} \quad (41)$$

etc.

2.2. Refinements of the Inequalities in Statement 2. We propose the following improvement and generalization of Statement 2.

Theorem 10. For every $x \in (0, 1]$ and $k \in \mathbb{N}$, it is asserted that

$$\begin{aligned} \frac{3x + \sum_{m=2}^{2k+1} (-1)^m E(m) x^{2m+1}}{1+2\sqrt{1+x^2}} &< \arctan x \\ &< \frac{3x + \sum_{m=2}^{2k} (-1)^m E(m) x^{2m+1}}{1+2\sqrt{1+x^2}}, \end{aligned} \quad (42)$$

where

$$\begin{aligned} E(m) &= \frac{3}{2m+1} \\ &- \sum_{i=0}^{m-1} \frac{(2m-2i-2)!}{2^{(2m-2i-2)} (2i+1) (m-i-1)! (m-i)!}. \end{aligned} \quad (43)$$

Examples. For $x \in (0, 1]$ and $k = 1$ we get inequality (3) from Statement 2.

For $x \in (0, 1]$ and $k \geq 2$ inequality (42) refines inequality (3) from Statement 2 and we have the following new results:

(i) Taking $k = 2$ in (42) gives

$$\begin{aligned} & \frac{3x + (1/60)x^5 - (17/840)x^7 + (139/6720)x^9 - (8947/443520)x^{11}}{1 + 2\sqrt{1+x^2}} < \arctan x \\ & < \frac{3x + (1/60)x^5 - (17/840)x^7 + (139/6720)x^9}{1 + 2\sqrt{1+x^2}}. \end{aligned} \quad (44)$$

(ii) Taking $k = 3$ in (42) gives

$$\begin{aligned} & \frac{3x + (1/60)x^5 - (17/840)x^7 + (139/6720)x^9 - (8947/443520)x^{11} + (89279/4612608)x^{13} - (851677/46126080)x^{15}}{1 + 2\sqrt{1+x^2}} \\ & < \arctan x < \frac{3x + (1/60)x^5 - (17/840)x^7 + (139/6720)x^9 - (8947/443520)x^{11} + (89279/4612608)x^{13}}{1 + 2\sqrt{1+x^2}}, \end{aligned} \quad (45)$$

etc.

and

Proof of Theorem 10. Based on Cauchy's product of power series (7) and (5), the real analytical function,

$$f(x) = (1 + 2\sqrt{1+x^2}) \cdot \arctan x - 3x, \quad (46)$$

for $x \in (0, 1]$ has the following power series:

$$f(x) = \sum_{m=2}^{\infty} (-1)^m E(m) x^{2m+1}, \quad (47)$$

where

$$\begin{aligned} E(m) &= \frac{3}{2m+1} \\ &- \sum_{i=0}^{m-1} \frac{(2m-2i-2)!}{(2i+1)(m-i-1)!(m-i)!2^{(2m-2i-2)}}. \end{aligned} \quad (48)$$

We aim to show that sequence $\{E(m)\}_{m \in N, m \geq 2}$ decreases monotonically and that $\lim_{m \rightarrow +\infty} E(m) = 0$. It is easy to verify that sequence $\{E(m)\}_{m \in N, m \geq 2}$ satisfies the following recurrence relation:

$$\begin{aligned} & -2mE(m) + (2m+3)E(m+1) \\ &= \frac{1}{2m+1} - \frac{(m+1)(2m)!}{((m+1)!)^2 4^m}. \end{aligned} \quad (49)$$

Consider the sequence $\{e(m)\}_{m \in N, m \geq 2}$ where

$$e(m) = g(m) \cdot S(m) \quad (50)$$

and

$$S(m) = \sum_{j=1}^{m-1} \frac{(2j+2)! \left(((2j)!!)^2 (j+1) - (2j+1)! \right)}{2^{(2j+3)} (2j+1) ((j+1)!)^2 ((2j)!!)^2} \quad (51)$$

$$g(m) = \frac{m!(m-1)!2^{(2m+1)}}{(2m+1)!}. \quad (52)$$

It is easy to verify that sequence $\{e(m)\}_{m \in N, m \geq 2}$ satisfies the recurrence relation (49). Given that sequences $\{E(m)\}_{m \in N, m \geq 2}$ and $\{e(m)\}_{m \in N, m \geq 2}$ agree for $m = 2$ and $m = 3$, we conclude that

$$E(m) = e(m), \quad \text{for } m \in N, m \geq 2. \quad (53)$$

We prove that sequence $\{e(m)\}_{m \in N, m \geq 2}$ is a monotonically decreasing sequence and $\lim_{m \rightarrow +\infty} e(m) = 0$.

By the principle of mathematical induction, it follows that

$$(2j+1)! < ((2j)!!)^2 (j+1) \quad (54)$$

is true for all $j \in N$. Therefore $S(m) > 0$ for $m \geq 2$, i.e.,

$$e(m) > 0, \quad \text{for } m \geq 2. \quad (55)$$

To prove that $\{e(m)\}_{m \in N, m \geq 2}$ is a monotonically decreasing sequence, let us use the following notation:

$$S(m) = \sum_{j=1}^{m-1} h(j) \quad (56)$$

where

$$h(j) = \frac{(2j+2)! \left(((2j)!!)^2 (j+1) - (2j+1)! \right)}{2^{(2j+3)} (2j+1) ((j+1)!)^2 ((2j)!!)^2}. \quad (57)$$

Consider the following equivalences for $m \geq 2$:

$$\begin{aligned} \frac{e(m+1)}{e(m)} &< 1 \iff \\ \frac{g(m+1)S(m+1)}{g(m)S(m)} &< 1 \iff \\ \frac{g(m+1)}{g(m)} \cdot \frac{S(m)+h(m)}{S(m)} &< 1 \iff \\ \frac{2m}{2m+3} \cdot \frac{S(m)+h(m)}{S(m)} &< 1 \iff \\ 2mh(m) &< 3S(m). \end{aligned} \quad (58)$$

Consider the last inequality. It is easy to verify that it is true for $m = 2$. Observing that

$$3S(m+1) = 3(S(m) + h(m)) = 3S(m) + 3h(m) \quad (59)$$

and using the induction hypothesis ($3S(m) > 2mh(m)$) for some positive integer $m \geq 2$, we conclude that

$$3S(m+1) > (2m+3)h(m) > 2(m+1)h(m). \quad (60)$$

Therefore, by the principle of mathematical induction, the inequality

$$2mh(m) < 3S(m) \quad (61)$$

is true for $m \geq 2$, i.e.,

$$\frac{e(m+1)}{e(m)} < 1, \quad \text{for } m \geq 2. \quad (62)$$

Let us further consider the positive addend of $S(m)$, i.e.,

$$\begin{aligned} S^+(m) &= \sum_{j=1}^{m-1} \frac{(2j+2)!((2j)!!)^2(j+1)}{2^{(2j+3)}(2j+1)((j+1)!)^2((2j)!!)^2} \\ &= \sum_{j=1}^{m-1} \frac{(2j)!}{4((2j)!!)^2}. \end{aligned} \quad (63)$$

By the principle of mathematical induction, it follows that

$$S^+(m) = \frac{m(2m)!}{2((2m)!!)^2} - \frac{1}{4}. \quad (64)$$

Finally, given that for $m \geq 2$

$$0 < e(m) \leq g(m) \cdot S^+(m) = \frac{1}{2m+1} \quad (65)$$

we have $\lim_{m \rightarrow +\infty} e(m) = 0$.

Finally, based on (53) we conclude that $\{E(m)\}_{m \in \mathbb{N}, m \geq 2}$ is a positive monotonically decreasing sequence and that it tends to zero. The same holds true for the sequence $\{E(m)x^{2m+1}\}_{m \in \mathbb{N}, m \geq 2}$ for a fixed $x \in (0, 1]$ so we can apply Leibniz's theorem for alternating series [28], thus proving the claim of Theorem 10. $\square$

2.3. Refinements of the Inequalities in Statement 3. We propose the following improvement and generalization of Statement 3.

Theorem 11. For every $x \in (0, 1]$ and $k \in \mathbb{N}$, it is asserted that

$$\begin{aligned} \sum_{m=1}^{2k-1} (-1)^m C(m) x^{2m+1} &< \arctan x - \frac{2x}{1 + \sqrt{1+x^2}} \\ &< \sum_{m=1}^{2k} (-1)^m C(m) x^{2m+1}, \end{aligned} \quad (66)$$

where

$$C(m) = \frac{1}{2m+1} - \frac{(2m-1)!!}{(m+1)!2^m}. \quad (67)$$

Examples. For $x \in (0, 1]$ and $k = 1$ we get inequality (4) from Statement 3.

For $x \in (0, 1]$ and $k \geq 2$ inequality (66) from Theorem 11 refines inequality (4) from Statement 3 and we have the following new results:

(i) Taking $k = 2$ in (66) gives

$$\begin{aligned} -\frac{1}{12}x^3 + \frac{3}{40}x^5 - \frac{29}{448}x^7 &< \arctan x - \frac{2x}{1 + \sqrt{1-x^2}} \\ &< -\frac{1}{12}x^3 + \frac{3}{40}x^5 - \frac{29}{448}x^7 + \frac{65}{1152}x^9. \end{aligned} \quad (68)$$

(ii) Taking $k = 3$ in (66) gives

$$\begin{aligned} -\frac{1}{12}x^3 + \frac{3}{40}x^5 - \frac{29}{448}x^7 + \frac{65}{1152}x^9 - \frac{281}{5632}x^{11} \\ < \arctan x - \frac{2x}{1 + \sqrt{1-x^2}} \\ < -\frac{1}{12}x^3 + \frac{3}{40}x^5 - \frac{29}{448}x^7 + \frac{65}{1152}x^9 - \frac{281}{5632}x^{11} \\ &\quad + \frac{595}{13312}x^{13}, \end{aligned} \quad (69)$$

etc.

Proof of Theorem 11. For $x \in (0, 1]$ the following power series expansion holds:

$$\begin{aligned} \arctan x - \frac{2x}{1 + \sqrt{1+x^2}} \\ = \arctan x + \frac{2(1 - \sqrt{1+x^2})}{x} \end{aligned} \quad (70)$$

$$= \sum_{m=1}^{\infty} (-1)^m C(m) x^{2m+1}$$

where

$$C(m) = \frac{1}{2m+1} - \frac{(2m-1)!!}{(m+1)!2^m}. \quad (71)$$

We prove that the sequence $\{C(m)\}_{m \in N}$ is positive and monotonically decreasing and tends to zero as m tends to infinity.

It is easy to verify that $(2m+2)!! > 2(2m+1)!!$ is true for $m \in N$. Thus, the following equivalences hold true for every $m \in N$:

$$\begin{aligned} C(m) > 0 &\iff \\ \frac{1}{2m+1} > \frac{(2m-1)!!}{(m+1)!2^m} &\iff \quad (72) \\ (2m+2)!! > 2(2m+1)!! &, \end{aligned}$$

and we conclude that $C(m) > 0$ for every $m \in N$.

Let us now prove that $\{C(m)\}_{m \in N}$ is a monotonically decreasing sequence. We have

$$\begin{aligned} C(m) - C(m+1) > 0 &\iff \\ \frac{2}{(2m+1)(2m+3)} - 3 \frac{(2m-1)!!}{(2m+2)!!(m+2)} > 0 &\iff \quad (73) \\ (2m+4)!! > 3(2m+3)!! &. \end{aligned}$$

As it is easy to show (by the principle of mathematical induction) that the last inequality holds true for $m \in N$, we may conclude that $\{C(m)\}_{m \in N}$ is a monotonically decreasing sequence.

Finally, as $0 < C(m) < 1/(2m+1)$, we conclude that $\lim_{m \rightarrow +\infty} C(m) = 0$.

Since $\{C(m)\}_{m \in N}$ is a positive monotonically decreasing sequence, and it tends to zero, the same holds true for the sequence $\{C(m)x^{2m+1}\}_{m \in N}$ for a fixed $x \in (0, 1]$. So we can apply Leibniz's theorem for alternating series [28] and thus prove the claim of Theorem 11. $\square$

3. Conclusion

In Theorems 9, 10, and 11 of this paper we proved some new inequalities related to Shafer's inequality for the arc-tangent function. These inequalities represent sharpening and generalization of the inequalities given in [8] (Theorems 1, 2, and 4).

Let us mention that it is possible to prove inequality (35), for any fixed $k \in N$ and $x \in (0, \sqrt{3}/2]$, by substituting $x = \tan t$ for $t \in (0, \pi/3]$ using the algorithms and methods (see also [29, 30]) developed in [31, 32]. Also, inequalities (42) and (66) for any fixed $k \in N$ and $x \in (0, 1]$ can be proved by substituting $x = \tan t$ for $t \in (0, \pi/4]$ using the algorithms and methods (see also [29, 30]) developed in [31, 32].

Conflicts of Interest

The authors would like to state that they do not have any conflicts of interest in the subject of this research.

Authors' Contributions

All the authors participated in every phase of the research conducted for this paper.

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