Research Article

Three-Dimensional Coupled NLS Equations for Envelope Gravity Solitary Waves in Baroclinic Atmosphere and Modulational Instability

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Envelope gravity solitary waves are an important research hot spot in the field of solitary wave. And the weakly nonlinear model equations system is a part of the research of envelope gravity solitary waves. Because of the lack of technology and theory, previous studies tried hard to reduce the variable numbers and constructed the two-dimensional model in barotropic atmosphere and could only describe the propagation feature in a direction. But for the propagation of envelope gravity solitary waves in real ocean ridges and atmospheric mountains, the three-dimensional model is more appropriate. Meanwhile, the baroclinic problem of atmosphere is also an inevitable topic. In the paper, the three-dimensional coupled nonlinear Schrödinger (CNLS) equations are presented to describe the evolution of envelope gravity solitary waves in baroclinic atmosphere, which are derived from the basic dynamic equations by employing perturbation and multiscale methods. The model overcomes two disadvantages: (1) baroclinic problem and (2) propagation path problem. Then, based on trial function method, we deduce the solution of the CNLS equations. Finally, modulational instability of wave trains is also discussed.

1. Introduction

According to media reported, “6.1 the Oriental Star cruise ship capsized event” was caused by the violence rainstorm attack which was brought by a sudden squall lines process. Unfortunately, the forecast for the severe weather such as squall lines is very difficult. As we know, the nonlinearity concentration of the gravity waves makes the energy assemble together and forms disastrous weather phenomena, such as squall lines and rainstorm. So constructing the theoretical model of gravity waves suits the real atmosphere condition and analyzing the formation mechanism of disastrous weather phenomena based on the theoretical model has significant scientific meaning and application value.

The first discovery of solitary waves was attributed to Russell [1]; then, the study of solitary wave continued to deepen. There was vast research literature devoted to the solitary waves. Solitary waves in the westerly shear flow were first found by Long [2]. Afterward, Benny [3] amplified the conclusions and got a conclusion that velocity and amplitude of solitary waves were related. Recently, a variety of equation models which described the solitary waves, such as ILW-Burgers equation and ZK-Burgers equation, were discussed by Yang et al. [4, 5]. Meanwhile the generation and evolution of solitary waves in different topography condition and different fluid depths were discussed. In recent years, in the solitary waves community, gravity solitary wave as a rising star had been paid more and more attention by many scientists. In the 60s, Long [2] proposed that the amplitude of atmospheric gravity wave satisfied KdV equation and got the solutions of amplitude. Then a lot of researchers obtained nonlinear KdV equations of gravity waves from the basic
dynamic equation. From the original equation of two layers of \( f \)-plane, under the shearing basic flow, the famous KdV equation was derived out by Li [6], and he pointed out that the nonlinear characteristic of gravity waves was the generation mechanism of squall lines. S. K. Liu and S. D. Liu [7] based on nonlinear atmospheric motion equation deduced out nonlinear gravity wave solutions and found that gravitational wave amplitude with propagation speed was proportional. Later, Boussinesq equation was derived in Luo [8] to describe the algebraic gravity solitary wave in atmosphere. Additionally, two-dimensional dissipative nonlinear Schrödinger equation [9] was also obtained to reflect the evolution of envelope solitary Rossby waves.

Therefore, for better understanding the characteristic of solitary waves deeply, it is desired to get the exact solutions [10, 11] of soliton equations. Up to now, people have obtained several sorts of nonlinear evolution equations and studied their properties, including the boundary value problem [12–14], Hamiltonian structure [15, 16], integrable systems [17, 18], and conservation law [19]. In the process of solving nonlinear evolution equations, many solution methods are found, like Darboux transformation method [20–22], Hirota method [23], homogeneous balance method [24], Jacob elliptic function method [25], symmetry method [26], trial function method [27], the alternative variational-asymptotic method [28–31], and so on [32, 33]. Biswas et al. [34–36] have also found some other methods of solving the nonlinear partial equation, but we noticed that they have not been applied to solve the three-dimensional CNLS equations.

Through the analysis of previous research on the gravity wave we can get the following two points:

(1) In order to simplify the calculation, the former researches studied the propagation of gravity solitary waves in two-dimension space and barotropic atmosphere [37]. But for the envelope of gravity waves in the real atmosphere, it was not sufficient to consider only two dimensions. And the initial equation was a simple model in barotropic atmosphere

\[
\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) u - v = -\frac{\partial P}{\partial x},
\]

\[
\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) v + u = -\frac{\partial P}{\partial y},
\]

\[
\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \phi + \lambda^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0,
\]

where \( \lambda = L/L_0 \), \( L_0 = c_s^0/f_0 \), \( c_s^0 = g^+ H_0 \), and \( g^+ = g((\rho - \rho')/\rho) \). These variables were simplified. Therefore, the three-dimensional model in baroclinic atmosphere is more in line with the actual atmospheric conditions.

(2) Previous studies have focused on the solution of gravity waves and the dynamic characteristics of the wave equation mainly based on the numerical calculation from the original equation. As we all know, many of the variables in the original equation are simplified and dimensional; therefore, numerical simulation directly based on the original equation is usually imprecise and difficult. So it is necessary for us to find the appropriate model and solve it to study the evolution of the waves.

In this paper, overcoming the limitations of calculations and using the appropriate method we obtain a new model. The paper will be organized as follows: (1) using multiscale analysis and turbulence method, from the basic dynamic equations of multivariable in baroclinic environment, we derived out the gravity wave model in Section 2, which is a coupled nonlinear Schrödinger equation (CNLS). Not only is the model three-dimensional and more suited to describe the feature of two envelope gravity solitary waves in a plane, specially, but also it is a coupled model and can show the interaction process between two waves. (2) Based on the CNLS model, using the trial function method to solve the equations, the analytical solution was obtained in Section 3. By observing the structures of the solution, the evolution characteristics of gravity solitary waves was obtained in Section 4. (3) Finally, modulational instability of a uniform three-dimensional gravity waves trains was also discussed.

2. Derivation of the Three-Dimensional CNLS Equations Group

Using the sum of disturbance pressure gradient force and buoyancy force, express the vertical pressure gradient force and gravity force, adopt Boussinesq approximation, and the basic dynamic equations of atmospheric motion are as follows:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + f v,
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} - fu,
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_0} \frac{\partial P}{\partial z} + g \frac{\theta}{\theta_0},
\]

\[
\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = 0;
\]

in the above equations, \( \theta_0 \) is the temperature of environmental flow field and \( \rho_0 \) is the density; \( \sigma = d\theta_0/dz \). Each variable above will be dimensionless. Let

\[
(x, y) = L (x', y'),
\]

\[
\zeta = D (\zeta'),
\]

\[
t = f^{-1} (t'),
\]

\[
(u, v) = U (u', v'),
\]

\[
w = \frac{U}{L} D (w'),
\]

\[
\theta = \delta \theta (\theta').
\]
where $H$ is the height of the homogeneous atmosphere, $P$ is the characteristics pressure of the ground, $\delta p_{x,y}$ is the horizontal pressure changes, and $\delta p_z$ is the pressure changes in the vertical direction. Substituting (3) into (2), get the dimensionless equations as follows:

$$\frac{\partial u'}{\partial t'} + \frac{U}{fL} \left( \frac{\partial u'}{\partial x'} + \nu \frac{\partial u'}{\partial y'} + \frac{\partial u'}{\partial z'} \right) = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x'} + \nu',
$$

$$\frac{\partial v'}{\partial t'} + \frac{U}{fL} \left( \frac{\partial v'}{\partial x'} + \nu \frac{\partial v'}{\partial y'} + \frac{\partial v'}{\partial z'} \right) = -\frac{1}{\rho_0} \frac{\partial p'}{\partial y'} - u',
$$

$$\frac{\partial w'}{\partial t'} + \frac{U}{fL} \left( \frac{\partial w'}{\partial x'} + \nu \frac{\partial w'}{\partial y'} + \frac{\partial w'}{\partial z'} \right) = -\frac{1}{\rho_0} \frac{\partial p'}{\partial z'} + \nu',$$

$$\frac{\partial \theta'}{\partial t'} + \frac{U}{fL} \left( \frac{\partial \theta'}{\partial x'} + \nu \frac{\partial \theta'}{\partial y'} + \frac{\partial \theta'}{\partial z'} \right) = -\frac{1}{\rho_0} \frac{\partial \theta}{\partial z'} + \theta',$$

Introducing parameter $\epsilon = f^2/N^2 (\epsilon \ll 1)$, where $N^2 = g\theta/\theta_0$. By varying, (4) transforms to the following:

$$\frac{\partial u'}{\partial t'} + u \frac{\partial u'}{\partial x'} + \nu \frac{\partial u'}{\partial y'} + \frac{\partial u'}{\partial z'} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x'} + \nu',
$$

$$\frac{\partial v'}{\partial t'} + u \frac{\partial v'}{\partial x'} + \nu \frac{\partial v'}{\partial y'} + \frac{\partial v'}{\partial z'} = -\frac{1}{\rho_0} \frac{\partial p}{\partial y'} - u',
$$

$$\frac{\partial w'}{\partial t'} + u \frac{\partial w'}{\partial x'} + \nu \frac{\partial w'}{\partial y'} + \frac{\partial w'}{\partial z'} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z'} + \nu',
$$

$$\frac{\partial \theta}{\partial t'} + u \frac{\partial \theta}{\partial x'} + \nu \frac{\partial \theta}{\partial y'} + \frac{\partial \theta}{\partial z'} = -\frac{1}{\rho_0} \frac{\partial \theta}{\partial z'} + \theta',$$

where $U, V, P, \Theta$ are the functions of $y, z$, where $U$ is the speed of the basic flow, $P$ is the air pressure, and $\Theta$ is the temperature field.

Substituting (8) and (9) into (6), the zero-order approximation of $\epsilon$ can be obtained:
\[ O(\epsilon^0) : \begin{cases} -\frac{1}{\rho_s} \frac{\partial P}{\partial y} - U = 0, \\
\frac{1}{\rho_s} \frac{\partial P}{\partial z} + \Theta = 0. \end{cases} \tag{10} \]

It is shown that basic flow is geostrophic equilibrium and static balance. For a basic flow we can further assume that

\[ \left| \frac{1}{\rho_s} \frac{\partial \rho_s}{\partial y} \right| \ll 1, \tag{11} \]

in the first formula in (10), derivation of \( z \), and the second formula in (10), derivation of \( y \), so that

\[ \frac{\partial U}{\partial z} = -\frac{\partial \Theta}{\partial y}. \tag{12} \]

Further, take the first-order approximation of \( \epsilon \) and introduce the new variables

\[ \begin{align*}
\rho_s u_0 &= u_0, \\
\rho_s v_0 &= v_0, \\
\rho_s w_0 &= w_0, \\
\rho_s \theta_0 &= \theta_0; \end{align*} \tag{13} \]

we have

\[ O(\epsilon^1) : \begin{cases} \frac{\partial u_0}{\partial t} + U \frac{\partial u_0}{\partial x} + V \frac{\partial u_0}{\partial y} + (U_y - 1) v_0 + U_z w_0 + \frac{\partial p_0}{\partial x} = 0, \\
\frac{\partial v_0}{\partial t} + U \frac{\partial v_0}{\partial x} + V \frac{\partial v_0}{\partial y} + \frac{\partial p_0}{\partial y} + U_v = 0, \\
\frac{\partial w_0}{\partial t} + U \frac{\partial w_0}{\partial x} + V \frac{\partial w_0}{\partial y} + \Theta v_0 + w_0 = 0, \\
\frac{\partial \theta_0}{\partial t} + U \frac{\partial \theta_0}{\partial x} + V \frac{\partial \theta_0}{\partial y} + \Theta v_0 + w_0 = 0, \\
\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0. \end{cases} \tag{14} \]

After eliminating other variables in (14), we can get the equation about \( p_0 \), as follows:

\[ L_{y,z} \left( \frac{\partial p_0}{\partial x} \right) = 0, \tag{15} \]

where

\[ \Omega = U_y - 1 + U_z^2, \]

\[ \Omega_y = \frac{\partial \Omega}{\partial y}, \]

\[ \Omega_z = \frac{\partial \Omega}{\partial z}, \]

\[ L_{y,z} = \frac{\partial^2}{\partial y^2} - (U_y - 1) \frac{\partial^2}{\partial z^2} + 2U_z \frac{\partial^2}{\partial y \partial z} \]

\[ + \left[ U_{zz} - \frac{\Omega_y}{\Omega} - U_z \frac{\Omega_z}{\Omega} \right] \frac{\partial}{\partial y} \]

\[ + \left[ (U_y - 1) \frac{\Omega_z}{\Omega} - U_z \frac{\Omega_y}{\Omega} \right] \frac{\partial}{\partial z} \]

\[ - \frac{1}{U} \left[ U_{zz} - \frac{\Omega_y}{\Omega} - U_z \frac{\Omega_z}{\Omega} \right]. \tag{16} \]

Clearly, (15) is a variable separable equation; assume its solution is

\[ p_0 = \sum_{j=1}^{2} \tilde{p}_{0j}(y,z) A_j(T_1, T_2, X_1, X_2, Y) e^{i(k_j x - \omega_j t)}, \tag{17} \]

under a certain definite solution condition, we can get \( \tilde{p}_{0j} \); further, all the solutions of (14) can be obtained:

\[ \begin{align*}
u_0 &= \sum_{j=1}^{2} \tilde{v}_{0j}(y,z) A_j(T_1, T_2, X_1, X_2, Y) e^{i(k_j x - \omega_j t)} \\
w_0 &= \sum_{j=1}^{2} \tilde{w}_{0j}(y,z) A_j(T_1, T_2, X_1, X_2, Y) e^{i(k_j x - \omega_j t)} \\
\theta_0 &= \sum_{j=1}^{2} \tilde{\theta}_{0j}(y,z) A_j(T_1, T_2, X_1, X_2, Y) e^{i(k_j x - \omega_j t)}. \end{align*} \tag{18} \]

Taking the second-order approximation of \( \epsilon \) and introducing the new variables

\[ \begin{align*}
\rho_s u_1 &= u_1, \\
\rho_s v_1 &= v_1, \\
\rho_s w_1 &= w_1, \\
\rho_s \theta_1 &= \theta_1, \end{align*} \tag{19} \]

we have
\[ O(\epsilon^3) : \]
\[
\begin{align*}
\frac{\partial u_0}{\partial t} + U \frac{\partial u_0}{\partial x} + V \frac{\partial u_0}{\partial y} + (U_y - 1) v_1 + U_x w_1 + \frac{\partial p_0}{\partial x} &= - \left[ \frac{\partial u_0}{\partial t} + U \frac{\partial u_0}{\partial x} + V \frac{\partial u_0}{\partial y} + \frac{1}{\rho} \left( u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} + w_0 \frac{\partial u_0}{\partial z} \right) + \frac{\partial p_0}{\partial x} \right], \\
\frac{\partial v_1}{\partial t} + V \frac{\partial v_1}{\partial x} + u_1 + V_x v_1 + V_z w_1 + \frac{\partial p_0}{\partial y} &= - \left[ \frac{\partial v_1}{\partial t} + U \frac{\partial v_1}{\partial x} + V \frac{\partial v_1}{\partial y} + \frac{1}{\rho} \left( u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} + w_0 \frac{\partial u_0}{\partial z} \right) + \frac{\partial p_0}{\partial y} \right], \\
\frac{\partial p_0}{\partial z} - \theta_1 &= - \left( \frac{\partial u_0}{\partial t} + U \frac{\partial u_0}{\partial x} + V \frac{\partial u_0}{\partial y} \right), \\
\frac{\partial \theta_1}{\partial t} + U \frac{\partial \theta_1}{\partial x} + V \frac{\partial \theta_1}{\partial y} + \Theta_j v_1 + w_1 &= - \left[ \frac{\partial \theta_1}{\partial t} + U \frac{\partial \theta_1}{\partial x} + V \frac{\partial \theta_1}{\partial y} + \frac{1}{\rho} \left( u_0 \frac{\partial \theta_1}{\partial x} + v_0 \frac{\partial \theta_1}{\partial y} \right) \right], \\
\frac{\partial u_0}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} &= - \left( \frac{\partial u_0}{\partial X_1} + \frac{\partial v_1}{\partial Y} \right);
\end{align*}
\]

Let

\[ Au_1 = \frac{\partial u_0}{\partial t} - U \frac{\partial u_0}{\partial x} - V \frac{\partial u_0}{\partial y} - \frac{1}{\rho} \left( u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} + w_0 \frac{\partial u_0}{\partial z} \right) - \frac{\partial p_0}{\partial X_1}, \]

\[ Av_1 = - \frac{\partial \theta_1}{\partial t} - U \frac{\partial \theta_1}{\partial x} - V \frac{\partial \theta_1}{\partial y} - \frac{1}{\rho} \left( u_0 \frac{\partial \theta_1}{\partial x} + v_0 \frac{\partial \theta_1}{\partial y} + w_0 \frac{\partial \theta_1}{\partial z} \right) - \frac{\partial p_0}{\partial Y}, \]

\[ Aw_1 = \frac{\partial w_1}{\partial t} - U \frac{\partial w_1}{\partial x} - V \frac{\partial w_1}{\partial y}, \]

\[ A\theta_1 = \frac{\partial \theta_1}{\partial t} - U \frac{\partial \theta_1}{\partial x} - V \frac{\partial \theta_1}{\partial y} - \frac{1}{\rho} \left( u_0 \frac{\partial \theta_1}{\partial x} + v_0 \frac{\partial \theta_1}{\partial y} \right), \]

\[ Ap_1 = \frac{\partial \theta_1}{\partial X_1} - \frac{\partial v_1}{\partial Y}. \]

Substitute (18) into (21)

\[ Au_1 = \sum_{j=1}^{2} \left[ \vec{w}_{0j} A_{jT_1} + \left( U \vec{w}_{0j} + \vec{p}_{0j} \right) A_{jX_1} + V \vec{w}_{0j} A_{jY} \right] e^{i(k_x x - \omega_j t)} - \left( \frac{1}{\rho} \right) \left( i k \vec{w}_{0j}^2 + \vec{v}_{0j} \mu_{0j} + \vec{w}_{0j} \omega_{0j} \right) |A| \]

\[ + \sum_{j=1}^{2} \left( U \vec{w}_{0j} \vec{\theta}_{0j} A_{jX_1} + V \vec{w}_{0j} \vec{\theta}_{0j} A_{jY} \right) e^{i(k_x x - \omega_j t)}, \]

\[ + e^{2i(k_x x - \omega_j t)} \]

After eliminating other variables in (20), the equation about \( p_1 \) can be obtained as follows:

\[ L_{12y,z} \left( \frac{\partial p_1}{\partial x} \right) = L_{11y,z} (Au_1) + L_{22y,z} (Av_1) \]

\[ + L_{32y,z} (Aw_1) + L_{42y,z} (A\theta_1), \]

where

\[ L_{11y,z} = \frac{\partial}{\partial y} U_{zz} + \frac{1}{U} + \frac{1}{\Omega} \left( \Omega_y + U_{zz} \Omega \right), \]

\[ L_{12y,z} = \frac{1}{U} \left( \frac{\partial}{\partial y} U_{zz} + \frac{1}{U} \frac{\partial}{\partial z} - \frac{1}{\Omega} \left( \Omega_y + U_{zz} \Omega \right) \right), \]
\begin{align*}
L_{3y,z} &= \frac{1}{U} U_z \frac{\partial}{\partial y} - \left( U_y - 1 \right) \frac{\partial}{\partial z} - \frac{1}{\Omega} U_z \Omega_y \\
&\quad - \frac{1}{\Omega} \left( U_y - 1 \right) \Omega_z, \\
L_{4y,z} &= U_z \frac{\partial}{\partial y} + \frac{U_z}{U} \left( U_y - 1 \right) \frac{\partial}{\partial z} - \frac{\Omega_y}{\Omega} U_z \\
&\quad - \frac{1}{U} \left( U_y - 1 \right) \Omega_z. \\
&\quad \quad \quad \quad (24)
\end{align*}

Similarly, (23) is also a variable separable equation; with comparison on both ends of (23), assume its basic solution is

\begin{align*}
p_i &= \sum_{j=1}^{2} \rho_{ij} (y, z) \left[ A_i e^{ik_1 x - \omega_1 t} + \left| A_i \right|^2 e^{2ik_1 x - \omega_1 t} \right], \\
&\quad \quad \quad \quad (25)
\end{align*}

further, we can get all solutions of (23):

\text{---}

\begin{align*}
o(\varepsilon^i) &:= \frac{1}{\varepsilon} \left[ \frac{\partial u_i}{\partial x} + U \frac{\partial u_i}{\partial X} + V \frac{\partial u_i}{\partial Y} + \frac{u_i}{\rho_1} \frac{\partial p_i}{\partial x} + \frac{u_i}{\rho_s} \frac{\partial u_i}{\partial Y} + w_i \frac{\partial u_i}{\partial z} \right] \\
&\quad + \frac{1}{\rho_1} \left( \frac{\partial u_i}{\partial X} + v_i \frac{\partial u_i}{\partial Y} + w_i \frac{\partial u_i}{\partial z} \right) + \frac{\partial p_i}{\partial x}.
\end{align*}

Let

\begin{align*}
A_{u_2} &= - \left[ \frac{\partial u_0}{\partial T_2} + U \frac{\partial u_0}{\partial X_2} + U \frac{\partial u_1}{\partial X_1} + V \frac{\partial u_1}{\partial Y} + \frac{u_1}{\rho_1} \frac{\partial p_1}{\partial x} + \frac{u_1}{\rho_s} \frac{\partial u_1}{\partial Y} + w_1 \frac{\partial u_1}{\partial z} \right] \\
&\quad + \frac{1}{\rho_1} \left( \frac{\partial u_0}{\partial X_2} + v_0 \frac{\partial u_0}{\partial Y} + u_0 \frac{\partial u_0}{\partial x} + v_1 \frac{\partial u_1}{\partial y} + w_0 \frac{\partial u_0}{\partial z} + \frac{\partial p_0}{\partial x} \right) + \frac{\partial p_1}{\partial X_1}, \\
A_{w_2} &= - \left[ \frac{\partial v_0}{\partial T_2} + U \frac{\partial v_0}{\partial X_2} + U \frac{\partial v_1}{\partial X_1} + V \frac{\partial v_1}{\partial Y} + \frac{v_1}{\rho_1} \frac{\partial p_1}{\partial x} + \frac{v_1}{\rho_s} \frac{\partial v_1}{\partial Y} + w_1 \frac{\partial v_1}{\partial z} \right] \\
&\quad + \frac{1}{\rho_1} \left( \frac{\partial v_0}{\partial X_2} + v_0 \frac{\partial v_0}{\partial Y} + u_0 \frac{\partial v_0}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + w_0 \frac{\partial v_0}{\partial z} + \frac{\partial p_0}{\partial x} \right) + \frac{\partial p_1}{\partial X_1}.
\end{align*}

\begin{align*}
u_1 &= \sum_{j=1}^{2} \tilde{u}_{ij} \left[ A_i e^{ik_1 x - \omega_1 t} + \left| A_i \right|^2 e^{2ik_1 x - \omega_1 t} \right], \\
\theta_1 &= \sum_{j=1}^{2} \tilde{\theta}_{ij} \left[ A_i e^{ik_1 x - \omega_1 t} + \left| A_i \right|^2 e^{2ik_1 x - \omega_1 t} \right].
\end{align*}

Similar to the above, introducing the new variables,

\begin{align*}
\rho_i u_2 &= u_2, \\
\rho_i v_2 &= v_2, \\
\rho_i w_2 &= w_2, \\
\rho_i \theta_2 &= \theta_2,
\end{align*}

the third-order approximation of \( \varepsilon \) transforms to

\begin{align*}
A_{\omega_2} &= - \left[ \frac{\partial u_0}{\partial T_1} + U \frac{\partial u_0}{\partial X_1} + V \frac{\partial u_0}{\partial Y} + \frac{u_0}{\rho_1} \frac{\partial p_1}{\partial x} + \frac{u_0}{\rho_s} \frac{\partial u_0}{\partial Y} + w_0 \frac{\partial u_0}{\partial z} \right] \\
&\quad + \frac{1}{\rho_1} \left( \frac{\partial u_0}{\partial X_1} + v_0 \frac{\partial u_0}{\partial Y} + u_0 \frac{\partial u_0}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + w_0 \frac{\partial u_0}{\partial z} + \frac{\partial p_0}{\partial x} \right) + \frac{\partial p_1}{\partial X_1}, \\
A_{\theta_2} &= - \left[ \frac{\partial \theta_0}{\partial T_2} + U \frac{\partial \theta_0}{\partial X_2} + U \frac{\partial \theta_1}{\partial X_1} + V \frac{\partial \theta_1}{\partial Y} + \frac{\theta_1}{\rho_1} \frac{\partial p_1}{\partial x} + \frac{\theta_0}{\rho_s} \frac{\partial \theta_1}{\partial Y} + w_1 \frac{\partial \theta_1}{\partial z} \right] \\
&\quad + \frac{1}{\rho_1} \left( \frac{\partial \theta_0}{\partial X_2} + v_0 \frac{\partial \theta_0}{\partial Y} + u_0 \frac{\partial \theta_0}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + w_0 \frac{\partial \theta_0}{\partial z} + \frac{\partial p_0}{\partial x} \right) + \frac{\partial p_1}{\partial X_1}.
\end{align*}

Substitute (18) and (26) into (29) and collecting the secular-producing terms proportional to \( e^{ik_1 x - \omega_1 t} \), we have
There are conditions for solution of (31),

\[
Au_2 = \sum_{j=1}^{2} \bar{u}_{0j} \left( A_{jT_2} + UA_{jX_2} \right) + U \bar{u}_{0j} A_{jX_2, X_2},
\]

\[
A_2 = \sum_{j=1}^{2} \bar{u}_{0j} \left( A_{jT_1} + UA_{jX_1} \right) + V \bar{v}_{ij} A_{jY},
\]

\[
A\theta_2 = \sum_{j=1}^{2} \bar{\theta}_{0j} \left( A_{jT_2} + UA_{jX_2} \right) + U \bar{\theta}_{1j} A_{jX_2, X_2},
\]

\[
Ap_2 = \sum_{j=1}^{2} \bar{u}_{0j} A_{jX_2} + \bar{u}_{1j} A_{jX_2, X_2} + \bar{v}_{ij} A_{jY}.
\]

After eliminating other variables in (28), the equation about \( p_2 \) can be obtained as follows:

\[
L_{y, z} \left( \frac{\partial p_2}{\partial x} \right) = L_{1y, z} (Au_1) + L_{2y, z} (Av_1) + L_{3y, z} (Au_1) + L_{4y, z} (A\theta_1).
\]

Introduce the following variables:

\[
x = \frac{1}{U} X_1 = \frac{1}{U} X_2,
\]

\[
t = T_1 = T_2.
\]

There are conditions for solution of (31),

\[
i \left( \frac{\partial A_1}{\partial t} + \frac{\partial A_1}{\partial x} \right) + \alpha_1 \frac{\partial^2 A_1}{\partial x^2} + \beta_1 \frac{\partial^2 A_1}{\partial y^2}
+ \gamma_1 \left( |A_1|^2 + |A_2|^2 \right) A_1 = 0,
\]

\[
i \left( \frac{\partial A_2}{\partial t} + \frac{\partial A_2}{\partial x} \right) + \alpha_2 \frac{\partial^2 A_2}{\partial x^2} + \beta_2 \frac{\partial^2 A_2}{\partial y^2}
+ \gamma_1 \left( |A_1|^2 + |A_2|^2 \right) A_2 = 0,
\]

where

\[
\alpha_1 = -i \left( U \bar{u}_{0j} + \bar{u}_{1j} \right), \]

\[
\beta_1 = -i \left( V \bar{v}_{ij} + \bar{v}_{ij} \right) \]

\[
y_1 = \frac{1}{\rho_1} \left[ 3k \bar{u}_{0j} \bar{u}_{1j} + k \bar{u}_{0j} \bar{v}_{ij} + 2k \bar{u}_{0j} \bar{v}_{ij} - i \left( \bar{u}_{0j} \bar{v}_{ij} \right) \right].
\]

3. The Solutions of the Three-Dimensional CNLS Equations Group

In the past, Hui [38] gave a solitary wave solution of (2 + 1) dimensional nonlinear Schrödinger equation with the help of a space coordinate transformation. In this chapter, this method will be applied. Based on trial function method, we will discuss the solution of the CNLS equations group. The equations group is

\[
i \frac{\partial A_1}{\partial t} + \alpha_1 \frac{\partial^2 A_1}{\partial x^2} + \beta_1 \frac{\partial^2 A_1}{\partial y^2}
+ \left( \sigma_1 \cdot |A_1|^2 + r_{12} \cdot |A_2|^2 \right) A_1 = 0,
\]

\[
i \frac{\partial A_2}{\partial t} + \alpha_2 \frac{\partial^2 A_2}{\partial x^2} + \beta_2 \frac{\partial^2 A_2}{\partial y^2}
+ \left( \sigma_2 \cdot |A_2|^2 + r_{21} \cdot |A_1|^2 \right) A_2 = 0.
\]

If we further introduce

\[
Z = X \cos(\theta) + Y \cos(\theta) = KX + MY,
\]

(35) reduces to

\[
i \frac{\partial A_1}{\partial t} + \gamma_1 \frac{\partial^2 A_1}{\partial Z^2} + \left( \sigma_1 \cdot |A_1|^2 + r_{12} \cdot |A_2|^2 \right) A_1 = 0,
\]

\[
i \frac{\partial A_2}{\partial t} + \gamma_2 \frac{\partial^2 A_2}{\partial Z^2} + \left( \sigma_2 \cdot |A_2|^2 + r_{21} \cdot |A_1|^2 \right) A_2 = 0,
\]

where

\[
\gamma_1 = \alpha_1 \cos^2 \theta + \beta_1 \sin^2 \theta,
\]

\[
\gamma_2 = \alpha_2 \cos^2 \theta + \beta_2 \sin^2 \theta.
\]

By rescaling the spatial coordinate and both amplitudes, we can reduce the number of nontrivial coefficients of the CNLS system; the simplified system is

\[
i A_2 + \frac{|A_1|^2}{2} A + \frac{\gamma_1}{r_{12}} |B|^2 A = 0,
\]

\[
i B_2 + \frac{|B|^2}{2} B + \frac{\gamma_2}{r_{21}} |A|^2 B = 0,
\]
where the new unknowns and coefficients are

\[ A = \sqrt{\sigma_1} A_1, \]
\[ B = \sqrt{\sigma_2} A_2, \]
\[ \xi = \sqrt{\frac{1}{\gamma_1}} Z, \]
\[ \eta = \frac{\gamma_1}{\gamma_2}, \]
\[ \eta_{12} = \frac{\gamma_1\gamma_2}{\sigma_2}, \]
\[ \eta_{21} = \frac{\gamma_1\gamma_2}{\sigma_1}. \]

The steadily oscillating solutions of this are solutions and their spatially periodical generalizations. Solitons and cnoidal waves are obtained by assuming

\[ A_1 = \exp \left( i \mu^2 \gamma_1 t \right) u(Z), \]
\[ A_2 = \exp \left( i \rho^2 \gamma_2 t \right) v(Z), \]

where \( \mu, \rho \) and \( u, v \) are assumed to be real. Note that, in spite of the cross-modulation, the frequencies in the two modes are generally different. Substitute (41) into (39), a set of two coupled ODES equations' group can obtained and can be called SCNLSequations group

\[
\begin{align*}
    u_{ZZ} - \mu^2 u + \frac{\sigma_1}{\gamma_1} u^3 + \eta_{12} u v^2 &= 0, \\
    v_{ZZ} - \rho^2 v + \frac{\sigma_2}{\gamma_2} v^3 + \eta_{21} u^2 v &= 0.
\end{align*}
\]

(42)

By rescaling the spatial coordinate and both amplitudes, it is possible to reduce the number of nontrivial coefficients of the SCNLS system from six to three. The simplified system is

\[
\begin{align*}
    U_{\eta\eta} - U + 2U^3 + \eta_{12} V^2 U &= 0, \\
    V_{\eta\eta} - e^2 V + 2V^3 + \eta_{21} U^2 V &= 0,
\end{align*}
\]

(43)

where the new unknowns and coefficients are

\[
\begin{align*}
    U &= \frac{1}{\mu} \sqrt{\frac{\sigma_1}{\gamma_1}} u, \\
    V &= \frac{1}{\mu} \sqrt{\frac{\sigma_2}{\gamma_2}} v, \\
    e &= \frac{\rho}{\mu}, \\
    \eta_{12} &= \frac{2\gamma_2 \gamma_1}{\gamma_1 \sigma_2}, \\
    \eta_{21} &= \frac{2\gamma_1 \gamma_2}{\gamma_2 \sigma_1}.
\end{align*}
\]

(44)

To compute the solitary and cnoidal waves, it is always sufficient to solve this system of two ODES. However, to collide solitary waves it is necessary to shift the steadily oscillating solitary waves into a frame of reference such that they move at some translational velocity \( V \).

If \( A_1(X, T), A_2(X, T) \) are solutions to the CNLS equations group

\[
\begin{align*}
    i \frac{\partial A_1}{\partial T} + \gamma_1 \frac{\partial^2 A_1}{\partial X^2} + \left( \sigma_1 |A_1|^2 + r_{12} |A_2|^2 \right) A_1 &= 0, \\
    i \frac{\partial A_2}{\partial T} + \gamma_2 \frac{\partial^2 A_2}{\partial X^2} + \left( \sigma_2 |A_2|^2 + r_{21} |A_1|^2 \right) A_2 &= 0,
\end{align*}
\]

(45)

the another solution is

\[
\begin{align*}
    \tilde{A}_1 &= A_1 (Z - VT) \exp (i \eta_1 Z) \exp (-i \eta_1 Z^2 T), \\
    \tilde{A}_2 &= A_2 (Z - VT) \exp (i \eta_2 Z) \exp (-i \eta_2 Z^2 T),
\end{align*}
\]

(46)

where the velocity \( V \) is

\[ V = 2\gamma_1 \eta_1 = 2\gamma_2 \eta_2, \]

(47)

which implies that the constants \( \tilde{\xi}_1 \) and \( \tilde{\xi}_2 \) must be proportional as indicated set \( \mu = \rho \) and define

\[ u = \theta_1 g(Z, \mu^2), \]
\[ v = \theta_2 g(Z, \mu^2), \]

(48)

where \( g(Z, c) \) is a solution to

\[ g_{ZZ} - \mu^2 g + 2g^3 = 0. \]

(49)

Substitute (48) into the ODE system (42) and collecting terms in \( g^2 \) and imposing the condition that the coefficients must be equal 2, the \( 2 \times 2 \) matrix system is obtained

\[
\begin{pmatrix}
    \sigma_1 & \eta_{12} \\
    \eta_{21} & \sigma_2
\end{pmatrix}
\begin{pmatrix}
    \theta_1 \\
    \theta_2
\end{pmatrix} =
\begin{pmatrix}
    2 \\
    2
\end{pmatrix}.
\]

(50)

The solution is

\[ \theta_1^2 = 2 \frac{\gamma_1 \sigma_2 - \gamma_2 \gamma_{12}}{\sigma_1 \sigma_2 - \gamma_{12} \gamma_{21}}, \]
\[ \theta_2^2 = 2 \frac{\gamma_2 \sigma_1 - \gamma_1 \gamma_{21}}{\sigma_1 \sigma_2 - \gamma_{12} \gamma_{21}}. \]

(51)

The limiting solitary waves for both branches are

\[ g = \pm \mu \text{sech}(\mu Z). \]

(52)

So the solution of CNLS equations group is

\[
\begin{align*}
    A_1 &= \sqrt{2 \frac{\gamma_1 \sigma_2 - \gamma_2 \gamma_{12}}{\sigma_1 \sigma_2 - \gamma_{12} \gamma_{21}}} \mu \exp \left( i \mu^2 \gamma_1 t \right) \text{sech}(\mu Z), \\
    A_2 &= \sqrt{2 \frac{\gamma_2 \sigma_1 - \gamma_1 \gamma_{21}}{\sigma_1 \sigma_2 - \gamma_{12} \gamma_{21}}} \mu \exp \left( i \rho^2 \gamma_2 t \right) \text{sech}(\mu Z),
\end{align*}
\]

(53)

where \( \gamma_{ij}^*, \sigma_i^* \) are Landau coefficients and \( \gamma_{i2}^*, \gamma_{21}^* \) are interaction coefficients.
4. Energy Variation Characteristics of Coupled Envelope Gravity Waves

In this section we will discuss two waves energy variation characteristics in the process of interaction; the first equation of (33) is multiplied by \( A_1^* \); the conjugate equation is multiplied by \( A_1 \), subtracting the two types. In the same way, the second equation of (33) is multiplied by \( A_2^* \); the conjugate equation is multiplied by \( A_2 \), subtracting the two types, we can obtain the following equations group:

\[
\begin{align*}
  & i \frac{\partial}{\partial t} |A_1|^2 + \alpha_1 \frac{\partial}{\partial X} \left( A_1 \frac{\partial A_1}{\partial X} - A_1^* \frac{\partial A_1^*}{\partial X} \right) \\
  & + \beta_1 \frac{\partial}{\partial Y} \left( A_1 \frac{\partial A_1}{\partial Y} - A_1^* \frac{\partial A_1^*}{\partial Y} \right) = 0, \\
  & i \frac{\partial}{\partial t} |A_2|^2 + C_{g2} \frac{\partial}{\partial X} |A_2|^2 \frac{\partial}{\partial X} \\
  & + \alpha_2 \frac{\partial}{\partial Y} \left( A_2 \frac{\partial A_2}{\partial Y} - A_2^* \frac{\partial A_2^*}{\partial Y} \right) \\
  & + \beta_2 \frac{\partial}{\partial Y} \left( A_2 \frac{\partial A_2}{\partial Y} - A_2^* \frac{\partial A_2^*}{\partial Y} \right) = 0.
\end{align*}
\]

Equation (33) has infinite conservation law. Except the two kinds of energy conversation \( E_1 \) and \( E_2 \) above, (33) have four movement invariants:

\[
\begin{align*}
  E_3 &= \int_{-\infty}^{\infty} \left( A_1 \frac{\partial A_1}{\partial X} + \frac{\gamma_{12}^* A_2^*}{\gamma_{21}^*} \right) dX, \\
  E_4 &= \int_{-\infty}^{\infty} \left( A_1 \frac{\partial A_1}{\partial Y} + \frac{\gamma_{12}^* A_2^*}{\gamma_{21}^*} \right) dY, \\
  E_5 &= \int_{-\infty}^{\infty} \left( \alpha_1 \frac{\partial A_1}{\partial X}^2 + \frac{\gamma_{12}^* \alpha_2}{\gamma_{21}^*} \frac{\partial A_2}{\partial X}^2 - \frac{\alpha_1}{2} |A_1|^4 \right) dX, \\
  E_6 &= \int_{-\infty}^{\infty} \left( \alpha_1 \frac{\partial A_1}{\partial Y}^2 + \frac{\gamma_{12}^* \alpha_2}{\gamma_{21}^*} \frac{\partial A_2}{\partial Y}^2 - \frac{\alpha_1}{2} |A_1|^4 \right) dY.
\end{align*}
\]

Equations (58) to (61) are the momentum of two waves, so we can confirm the conservation of momentum.

5. Modulation Instability of a Uniform Gravity Waves Trains in 3-Dimension Space

We will discuss the modulation instability for three-dimensional gravity wave trains in this section. Based on (53) we can obtain the solution of a uniform wave train. Initial solution of the equation can take the following form:

\[
A = A_0 \exp \left( i \sigma A_0^2 T \right);
\]

according to the studies of Yuen and Lake [39], the perturbed wave trains may be expressed as follows:

\[
A = \left[ A_0 + A_\omega \exp \left( i (KX + MY + \omega T) \right) \right] \exp \left( i \sigma A_0^2 T \right),
\]

where, in fact, \( A_0 \) denotes the amplitude of a uniform disturbance stream function, \( A_\omega \) is the amplitude of a sideband disturbance and is small enough compared with \( A_0 \) to permit linearization, \( \omega \) is eigenvalue to be determined, and its conjugate is \( \omega^* \). Informed by Craig [40], \( \pm K, \pm M \) denotes the prescribed zonal and meridional wave number perturbations in the \( X \) and \( Y \) directions. Here, we define \( \delta(\pm K, \pm M) = (\pm kp, \pm mq) \), in which \( p \ll 1 \) and \( q \ll 1 \) are required. Linearization with respect to \( A_0 \) satisfied the following eigenvalue equation:

\[
\omega^2 = \left( aK^2 + bM^2 \right) \left( aK^2 + bM^2 - 2\alpha_1 A_0^2 \right) \]

or

\[
\omega^2 = \frac{1}{\delta^2} \left[ \left( \alpha (pk)^2 + \beta (qm)^2 \right) \right] \]

\[
\omega^2 = \left( aK^2 + bM^2 - 2\alpha_1 \delta^2 A_0^2 \right).
\]
If \( B_0 = \delta A_0 \) and \( \omega^2 = R^2/\delta^4 \) are defined, we can get

\[
R^2 = \left[ \alpha(pk)^2 + \beta(qm)^2 \right] \\
\cdot \left[ \left( \alpha(pk)^2 + \beta(qm)^2 \right) - 2\sigma_1 B_0^2 \right];
\]

if \( p \) and \( q \) satisfy

\[
0 < \alpha(pk)^2 + \beta(qm)^2 < 2\sigma_1 B_0^2,
\]

the uniform gravity wave train is unstable.

When \( q = 0 \), the result resembles the instability diagram of Benjamin-Feir instability [41]. The case of \( q \neq 0 \) has not been studied yet. To obtain more detailed results, for gravity solitary waves, we only study the case where the fundamental number is 1; that is,

\[
k = \frac{1}{2.380\cos(\phi_0)}.
\]

where \( \phi_0 \) is the latitude. We will discuss the influence of \( q \) and latitude value on the instability regions. Taking \( Ly = 6, n_1 = 2 \) and \( B_0 = 0.5 \). In this case the instability growth rates \( 35 \Im R(35 \times \Im R) \) of a uniform gravity wave train at \( 30^\circ \mathrm{N}, 45^\circ \mathrm{N}, \) and \( 60^\circ \mathrm{N} \) for \( q = 0, 0.08, 0.24 \) are shown in Figures 1, 2, and 3.

From Figure 1 we know that the instability region is \(|p| \leq 0.38 \) for \( q = 0.08 \), \(|p| \leq 0.37 \) for \( q = 0.08 \), and \(|p| \leq 0.30 \) for \( q = 0.24 \) at the latitude \( 30^\circ \mathrm{N} \). This shows that the instability region of \( p \) will become narrower if \( q \) increase in the small value limit. Meanwhile, at latitudes \( 45^\circ \mathrm{N} \) and \( 60^\circ \mathrm{N} \), there are the same properties as those at latitude \( 30^\circ \mathrm{N} \), as shown in Figures 2 and 3.

In addition, as shown in the figures we can also note that the instability of \( p \) will become wider if the latitude increases for the same \( q \). It is worth noting that the higher the latitude, the more unstable the uniform gravity wave trains. This conclusion is valid for all small values of \( q \). In this paper, the effects of external forcing on the modulational instability of gravity wave trains are not discussed. This problem remains to be further studied.

6. Conclusions

In this paper, overcoming the limitations of calculations and using the appropriate method, we obtained a new model. Using the multiscale analysis and turbulence method, from the basic dynamic equations of multivariable in baroclinic environment, we derived out the gravity wave model, which is a coupled nonlinear Schrödinger equation (CNLS). Not only is the model three-dimensional and more suited to describe the feature of two envelope gravity solitary waves in a plane, specially, but also it is a coupled model and can show the interaction process between two waves. Based on
the CNLS model, using the trial function method to solve the equations, the analytical solution was obtained. By observing the structures of the solution, the evolution characteristics of gravity solitary waves were obtained. Finally, modulational instability of a uniform three-dimensional gravity waves trains were also discussed. This shows that the instability region of \( p \) will become narrower if \( q \) increase in the small value limit. Meanwhile, at latitudes 45°N and 60°N, there are the same properties as those at latitude 30°N. In this paper, the effects of external forcing on the modulational instability of gravity wave trains are not discussed. This problem remains to be further studied.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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