Research Article

Dynamical Behaviors of Coupled Memristor-Based Oscillators with Identical and Different Nonlinearities

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Received 28 May 2018; Accepted 2 September 2018; Published 21 October 2018

Academic Editor: Alessandro Lo Schiavo

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In this work, the interesting dynamics of coupled nonlinear memristor-based oscillators in ring configuration are explored. The mathematical models are derived to describe the possible cases of employing identical or different nonlinearities. Analytical and numerical techniques, involving perturbation methods, normal forms, phase portraits, and Lyapunov exponents are used to investigate various types of dynamical behaviors along with their stability regions in parameters space. The effects of time-delayed coupling on the proposed system are numerically studied. It is demonstrated that the coupled oscillators show rich dynamics including periodic orbits, quasiperiodicity, two-dimensional, and three-dimensional tori.

1. Introduction

Many applications in engineering, physics, chemistry, biology, and economy are modeled using nonlinear differential equations (DEs) where their exact forms of solutions are very difficult to obtain in general or they do not provide tools to understand the underlying dynamics. However, dynamical systems methods provide powerful techniques to study the qualitative behaviors of mathematical model solution rather than quantitative ones [1].

One of the interesting problems in nonlinear theory is to study the dynamics of systems involving coupled nonlinear oscillators. This problem appears in many applications in biology [2], neuroscience [3–6], chemistry [7, 8], and physics [9–11]. Researchers in the field of dynamical systems are interested in studying the stability of coupled oscillators systems with nonlinear properties since 1950s till now [12–15].

In 1970, Chua introduced the fourth electronic element called the memristor [16]. It is a two-terminal element that relates the magnetic flux $\phi$ between the terminals with the electric charge $q$ that passes through it [16, 17]. Studies on the dynamical properties, i.e., stability and bifurcation of the memristor-based oscillator circuit, have received increasing interest [18–21]. The main reason is that coupled memristor-based oscillatory circuits appear in many applications such as filtering, energy storage, energy transmission, isolation impedance transformation, learning networks that require a synapse-like function, secure communications, and image stabilization [22–31].

In [32], the stability of three nonlinear identical memristor-based oscillators coupled by inductors in ladder circuit configuration was studied. In this paper, we examine a more generalized case by considering coupled three memristor-based oscillators in ring configuration where nonidentical memristors are allowed. Two cases are studied where both identical and different nonlinearities are considered. The memristor used is of a flux-controlled type that has memductance function $w(\phi)$ representing the flux-dependent rate of change of charge

$$w(\phi) = \frac{dq(\phi)}{d\phi}. \quad (1)$$

The two cases studied are according to the definition of $q(\phi)$. In the first case the memristors are described by three third-order polynomials, whereas in the second case one memristor is described by third-order polynomial while the remaining memristors are characterized by fourth-order polynomials.

Time delays play an important role in mathematical modeling of real systems and studying synchronization such that their effects cannot be neglected in real life; see, for
example, [33–35]. Therefore, numerical investigations of
time-delayed coupling influences on coupled oscillators
dynamics are carried out in this work. The rest of the paper
is organized as follows: In Section 2, mathematical models
of all cases studied in this work are presented. In Section 3,
the multiple scales method is applied to acquire the normal
form for each system. In Section 4, various types of dynamical
behaviors that can be extracted from normal forms in
addition to their stability analysis are obtained. In Section 5,
numerical simulations are carried out to verify theoretical
results. The general discussion and main conclusion of this
work are presented in Section 6.

2. Ring Model Equations

In this section, we derive the fundamental equations describ-
ing the three ring coupled oscillators circuit. Figure 1 shows
the memristor-based oscillator circuit we adopt in this work.
Applying Kirchhoff’s voltage law (KVL) and Kirchhoff’s
current law (KCL) to the circuit in Figure 2, the model can
be described as follows:

\[
\begin{align*}
I_1 - I_3 + C_1 \dot{V}_1 + g_1 V_1 + I_{L_1} + I_{w_1} (V_1) &= 0, \\
I_2 - I_1 + C_2 \dot{V}_2 + g_2 V_2 + I_{L_2} + I_{w_2} (V_2) &= 0, \\
I_3 - I_2 + C_3 \dot{V}_3 + g_3 V_3 + I_{L_3} + I_{w_3} (V_3) &= 0, \\
I_1 - \frac{V_1 - V_2}{L_0} &= 0, \\
I_2 - \frac{V_2 - V_3}{L_0} &= 0, \\
I_3 - \frac{V_3 - V_1}{L_0} &= 0,
\end{align*}
\]

We consider two cases of system (2) according to the
equations describing memristors.

2.1. Case 1. In this case, the nonlinearities of the three
memristors are described as follows:

\[
q_{w_i} (\phi_i) = m_i \phi_i^3 - n_i \phi_i, \quad m_i, n_i > 0, \ i = 1, 2, 3.
\]

By integrating both sides of each equation of system (2)
between times \( t_0 \) and \( t \) while using (3) and the relations

\[
\begin{align*}
q(t) &= \int_{-\infty}^{t} I(u) \, du, \\
\phi(t) &= \int_{-\infty}^{t} V(u) \, du,
\end{align*}
\]
we get

\[
\begin{align*}
\dot{\phi}_1 + \frac{q_1 \phi_1}{C_1} + \frac{q_{L_0}}{C_1} + \frac{m_1 \phi_1^3}{C_1} - n_1 \phi_1 + \frac{q_1 - q_3}{C_1 - C_1} &= 0, \\
\phi_2 + \frac{q_1 \phi_2}{C_2} + \frac{q_{L_0}}{C_2} + \frac{m_1 \phi_2^3}{C_2} - n_2 \phi_2 + \frac{q_1 + q_3}{C_2 + C_2} &= 0, \\
\phi_3 + \frac{q_1 \phi_3}{C_3} + \frac{q_{L_0}}{C_3} + \frac{m_1 \phi_3^3}{C_3} - n_3 \phi_3 + \frac{q_1 - q_3}{C_3 - C_3} &= 0,
\end{align*}
\]

\begin{equation}
\begin{align*}
q_1 - \phi_1 - \phi_2 &= 0, \\
q_2 - \phi_2 - \phi_3 &= 0, \\
q_3 - \phi_3 - \phi_1 &= 0,
\end{align*}
\end{equation}

By differentiating (6)–(8) and using (9)–(14), we obtain the following equations that depict our intended case:

\begin{equation}
\begin{align*}
\ddot{\phi}_j - \dot{\phi}_j \left( \frac{n_j - g_j}{C_j} - \frac{(3m_j) \phi_j^2}{C_j} \right) + \phi_j \left( \frac{1}{C_j L_j} \right) + 2 \frac{C_j L_j}{C_j L_0} - \sum_{k \neq j} \phi_k = 0, \\
&\quad j = 1, 2, 3.
\end{align*}
\end{equation}

**2.2. Case 2.** In this case, the nonlinearities of the three memristors are different and described as follows:

\[
\begin{align*}
q_{w_1}(\phi_1) &= m_1 \phi_1^3 - n_1 \phi_1, \\
q_{w_2}(\phi_2) &= h_2 \phi_2^4 + m_2 \phi_2^3 - n_2 \phi_2, \\
q_{w_3}(\phi_3) &= h_3 \phi_3^4 - n_3 \phi_3, \\
m_1, h_{\nu 1}, n_j &> 0, \quad i = 1, 2, \quad j = 1, 2, 3.
\end{align*}
\]

Using the same procedure as in Section 2.1, we obtain

\[
\begin{align*}
\ddot{\phi}_j - \dot{\phi}_j \left( \frac{n_j - g_j}{C_j} - \frac{(3m_j) \phi_j^2}{C_j} \right) + \phi_j \left( \frac{1}{C_j L_1} \right) + 2 \frac{L_1}{L_0} - \phi_2 + \phi_3 &= 0, \\
&\quad j = 1, 2, 3.
\end{align*}
\]

**3. Normal Forms**

The purpose of this section is to acquire the normal forms corresponding to each case of ring model derived in previous section. The powerful perturbation techniques are used to achieve this goal. First, systems (15) and (17) are written in perturbed form; then each system is reduced to an equivalent first-order system [36].

System (15) can be written in perturbed form as follows:

\[
\begin{align*}
\ddot{x}_j - \varepsilon \dot{x}_j &\left( a_j - \alpha_j x_j^2 \right) + \beta_j x_j - b_j \varepsilon \left( \sum_{k=1, k \neq j}^3 x_k \right) = 0, \\
&\quad j = 1, 2, 3.
\end{align*}
\]

And system (17) can be written in perturbed form as follows:

\[
\begin{align*}
\ddot{x}_1 - \varepsilon \dot{x}_1 &\left( a_1 - \alpha_1 x_1^2 \right) + \beta_1 x_1 - b_1 \varepsilon (x_2 + x_3) = 0, \\
\ddot{x}_2 - \varepsilon \dot{x}_2 &\left( a_2 - \alpha_2 x_2^2 - \alpha_3 x_2^2 \right) + \beta_2 x_2 - b_2 \varepsilon (x_1 + x_3) = 0, \\
\ddot{x}_3 - \varepsilon \dot{x}_3 &\left( a_3 - \alpha_3 x_3^2 \right) + \beta_3 x_3 - b_3 \varepsilon (x_1 + x_2) = 0.
\end{align*}
\]

By using the substitutions

\[
\begin{align*}
x_j(t) &= \bar{y}_j(t) + y_j(t), \\
\dot{x}_j(t) &= i \sqrt{\beta_j} \left( y_j(t) - \bar{y}_j(t) \right),
\end{align*}
\]

\[
\begin{align*}
&\quad j = 1, 2, 3,
\end{align*}
\]

where \(\bar{y}_j(t)\) is the complex conjugate of \(y_j(t)\), systems (18) and (19) are reduced to the more simplified first-order systems of ODEs. More specifically, system (18) is reduced to the following system:

\[
\begin{align*}
\ddot{y}_j - \frac{1}{2} \varepsilon (y_j - \bar{y}_j) &\left( a_j - \alpha_j (y_j + \bar{y}_j)^2 \right) + \left( ib_j \varepsilon \sum_{k=1, k \neq j}^3 \bar{y}_k + y_k \right) \frac{2}{2 \sqrt{\beta_j}} - i \sqrt{\beta_j} y_j = 0, \\
&\quad j = 1, 2, 3.
\end{align*}
\]
And system (19) is reduced to the following system:

\[
y_1 - \frac{1}{2} \varepsilon (y_1 - \overline{y}_1) (a_1 - a_1 (\overline{y}_1 + y_1)^2) + \frac{(ib_1) \varepsilon (y_2 + \overline{y}_2 + y_2)}{2 \sqrt{\beta_1}} - i \sqrt{\beta_1} y_1 = 0,
\]

\[
y_2 - \frac{1}{2} \varepsilon (y_2 - \overline{y}_2) \cdot (-a_3 (\overline{y}_2 + y_2)^2 - a_2 (\overline{y}_2 + y_2)^2 + a_2) + \frac{(ib_2) \varepsilon (y_1 + \overline{y}_1 + y_1)}{2 \sqrt{\beta_2}} - i \sqrt{\beta_2} y_2 = 0,
\]

\[
y_3 - \frac{1}{2} \varepsilon (y_3 - \overline{y}_3) (a_3 - a_4 (\overline{y}_3 + y_3)^3) + \frac{(ib_3) \varepsilon (y_1 + \overline{y}_1 + y_1)}{2 \sqrt{\beta_3}} - i \sqrt{\beta_3} y_3 = 0.
\]

The normal forms corresponding to systems (21) and (22) are obtained by applying the method of multiple scales [36].

3.1. The Normal Form of Case 1 of Ring Model. Using the substitutions

\[
y_j (t; \varepsilon) = \sum_{k=0}^{2} e^{j \varepsilon k} y_{jk} (T_0, T_1, T_2),
\]

\[
T_{j-1} = e^{j-1} t,
\]

\[
\frac{d}{dt} = \sum_{k=0}^{2} D_k \varepsilon^k,
\]

\[
D_{j-1} = \partial_{T_{j-1}},
\]

in system (21) and equating coefficients of equal powers of \( \varepsilon \), we obtain

(Order \( \varepsilon^0 \))

\[
D_0 y_j - i \sqrt{\beta_j} y_j = 0, \quad j = 1, 2, 3.
\]

(Order \( \varepsilon^1 \))

\[
D_0 y_j - i \sqrt{\beta_j} y_j
= -D_1 y_j + \frac{(ib_j) (\sum_{k=1, k \neq j}^{3} y_{k0} + y_{k0})}{2 \sqrt{\beta_j}}.
\]

Solving system (24) of DEs we get

\[
y_{j0} = e^{i \sqrt{\beta_j} T_1 / A_{j0}} (T_1, T_2), \quad j = 1, 2, 3.
\]

By substituting these solutions in (25) we get the conditions needed to remove the secular terms in the following forms:

\[
D_1 A_{j0} = \frac{a_{j0} A_{j0} - 1}{2} \frac{a_j^2}{2} A_{j0} A_{j0}, \quad j = 1, 2, 3.
\]

Using (28) to solve system (25) and substituting from the results in (26), we get the following conditions to remove the secular terms

\[
D_2 A_{10} = \frac{i A_{10} \left( (\beta_1 - \beta_2) (\beta_1 - \beta_3) \left( -12 a_1 a_{10} A_{10} A_{10}^2 + 11 a_1^2 A_{10}^2 A_{10} + 2 a_1^2 \right) - 8 b_1 (b_1 (\beta_1 - \beta_2) + b_2 (\beta_1 - \beta_3)) \right)}{16 \sqrt{\beta_1} (\beta_1 - \beta_2) (\beta_1 - \beta_3)},
\]

\[
D_2 A_{20} = \frac{i A_{20} \left( (\beta_1 - \beta_2) (\beta_2 - \beta_3) \left( -12 a_2 a_{20} A_{20} A_{20}^2 + 11 a_2^2 A_{20}^2 A_{20} + 2 a_2^2 \right) - 8 b_2 (b_1 (\beta_1 - \beta_2) + b_1 (\beta_2 - \beta_3)) \right)}{16 \sqrt{\beta_2} (\beta_2 - \beta_1) (\beta_2 - \beta_3)}.
\]

\[
\frac{1}{2} \left( y_{j0} - y_{j0} \right) \left( a_j (y_{j0} + y_{j0})^2 - a_j \right),
\]

\[
\quad j = 1, 2, 3.
\]

(25)
Substituting from (28) and (29) in $A_{j_0} = D_j \varepsilon^2 A_{j_0} + D_j \varepsilon A_{j_0}$, where $j = 1, 2$ and 3, we obtain the normal form of system (21) which can be expressed in the following form:

$$
\begin{align*}
    r_{j_0} &= \frac{1}{2} \varepsilon r_{j_0} (a_j - \alpha_j r_{j_0}^2), \quad j = 1, 2, 3, \\
    \theta'_{10} &= \frac{\varepsilon^2}{16 \sqrt{\beta_1}} \left( 8b_2 (\beta_2 - \beta_3) - (\beta_1 - \beta_3) \left( -12a_1 \alpha_1 r_{10}^2 + 2a_1^2 + 11\alpha_1^2 r_{10}^4 \right) \right), \\
    \theta'_{20} &= \frac{\varepsilon^2}{16 \sqrt{\beta_2}} \left( 8b_2 (\beta_3 - \beta_4) - (\beta_2 - \beta_3) \left( -12a_3 \alpha_2 r_{20}^2 + 2a_3^2 + 11\alpha_2^2 r_{20}^4 \right) \right), \\
    \theta'_{30} &= \frac{\varepsilon^2}{16 \sqrt{\beta_3}} \left( (\beta_2 - \beta_3) \left( -12a_3 \alpha_3 r_{30}^2 + 2a_3^2 + 11\alpha_3^2 r_{30}^4 \right) + 8b_3 b_2 (\beta_3 - \beta_4) \right)
\end{align*}
$$

3.2. The Normal Form of Case 2 of Ring Model. Using the same procedure applied in Section 3.1, the following conditions are obtained to remove the secular terms of order $\varepsilon^1$ and the conditions needed to remove the secular terms of order $\varepsilon^2$ are given by

$$
\begin{align*}
    D_1 A_{10} &= \frac{a_1 A_{10}}{2} - \frac{1}{2} \alpha_1 A_{10}^2 A_{10}, \\
    D_1 A_{30} &= \frac{a_3 A_{30}}{2}, \\
    D_1 A_{20} &= \frac{a_2 A_{20}}{2}, \\
    D_2 A_{20} &= \frac{i A_{20} \left( (\beta_1 - \beta_2) \left( -60\alpha_3 a_2 A_{20} A_{30}^2 + 10a_2^2 + 55\alpha_2^2 \right) + 40b_2 b_2 \right)}{80 \sqrt{\beta_2}} (\beta_2 - \beta_1), \\
    D_2 A_{30} &= \frac{i A_{30} \left( (\beta_2 - \beta_3) \left( -60\alpha_2 a_3 A_{20} A_{30}^2 + 10a_3^2 + 55\alpha_3^2 \right) + 40b_2 b_2 \right)}{80 \sqrt{\beta_3}} (\beta_3 - \beta_2) \\
\end{align*}
$$

So, we get the following normal form:

$$
\begin{align*}
    r'_{10} &= \frac{1}{2} \varepsilon r_{10} (a_1 - \alpha_1 r_{10}^2), \\
    r'_{20} &= \frac{1}{2} \varepsilon r_{20} (a_2 - \alpha_2 r_{20}^2), \\
    r'_{30} &= \frac{1}{2} a_3 \varepsilon r_{30}, \\
    \theta'_{10} &= \frac{\varepsilon^2}{16 \sqrt{\beta_1}} \left( 8b_2 b_2 (\beta_2 - \beta_3) \left( -12a_1 \alpha_1 r_{10}^2 + 2a_1^2 + 11\alpha_1^2 r_{10}^4 \right) \right).
\end{align*}
$$
\[
\theta_{20}' = \frac{\varepsilon^2 \left( (\beta_1 - \beta_2) \left( -60a_3\alpha_2 r_{120}^2 + 10a_2^2 + 56\alpha_2^2 r_{120}^6 + 55\alpha_2^2 r_{120}^4 \right) + 40b_2 b_3 \right)}{80\sqrt{\beta_2} (\beta_2 - \beta_1)},
\]
\[
\theta_{30}' = -\frac{\varepsilon^2 \left( (\beta_2 - \beta_3) \left( a_1 (\beta_1 - \beta_3) (a_2 - \alpha_3 r_{30}^2) + 4b_2 b_3 \right) + 4b_2 b_3 (\beta_1 - \beta_3) \right)}{8\sqrt{\beta_3} (\beta_3 - \beta_1) (\beta_3 - \beta_2)}.
\]

(33)

4. Stability Analysis of Equilibrium Solutions

In this section, the stability analysis of the equilibrium solutions is carried out using the normal forms obtained in (30) and (33).

4.1. Stability Analysis of Case 1 of Ring Model. The Jacobian matrix of system (30) is expressed as

\[
J = \begin{pmatrix}
\frac{1}{2} \varepsilon (a_1 - r_{10}^2 a_1) - \varepsilon r_{10}^2 a_1 & 0 & 0 \\
0 & \frac{1}{2} \varepsilon (a_2 - r_{20}^2 a_2) - \varepsilon r_{20}^2 a_2 & 0 \\
0 & 0 & \frac{1}{2} \varepsilon (a_3 - r_{30}^2 a_3) - \varepsilon r_{30}^2 a_3
\end{pmatrix}.
\]

(34)

System (30) has the following equilibrium solutions.

4.1.1. Equilibrium Point. The following equilibrium solution,

\[r_{i0} = 0, \quad i = 1, 2, 3,\]

(35)

is corresponding to an equilibrium point at origin. Evaluating the eigenvalues of Jacobian matrix (34) at equilibrium point (35), we get the stability region of this point in parameters space as follows:

\[a_1 < 0,\]
\[a_2 < 0,\]
\[a_3 < 0.\]

4.1.2. Periodic Solution I. The second equilibrium solution,

\[r_{10} = 0,\]
\[r_{20} = 0,\]
\[r_{30} = \sqrt{\frac{a_3}{\alpha_3}},\]

(37)

\[\theta_{30}' = -\frac{\varepsilon^2 \left( (\beta_2 - \beta_3) \left( (\beta_1 - \beta_3) \left( -12a_3\alpha_3 r_{30}^2 + 2a_2^3 + 11\alpha_2^2 r_{30}^4 \right) + 8b_2 b_3 \right) + 8b_2 b_3 (\beta_1 - \beta_3) \right)}{16\sqrt{\beta_3} (\beta_3 - \beta_1) (\beta_3 - \beta_2)},\]

(38)

corresponds to a periodic solution in the phase space of case 1 of ladder model. The second order approximate solution of this periodic orbit has an amplitude given by \(\sqrt{a_3/\alpha_3}\) and an angular frequency computed as

\[\sqrt{\beta_3},\]
\[\frac{\varepsilon^2 \left( (\beta_2 - \beta_3) \left( a_1 (\beta_1 - \beta_3) + 8b_2 b_3 \right) + 8b_2 b_3 (\beta_1 - \beta_3) \right)}{16\sqrt{\beta_3} (\beta_3 - \beta_1) (\beta_3 - \beta_2)}.\]

(39)

evaluating the eigenvalues of Jacobian matrix (34) at periodic solution I (37), we get its stability region in parameters space as follows:

\[a_1 < 0,\]
\[a_2 < 0,\]
\[a_3 > 0.\]
4.1.3. Periodic Solution II. The third equilibrium solution, 

\[ r_{10} = 0, \]
\[ r_{20} = \sqrt{\frac{a_2}{a_2}}, \]
\[ r_{30} = 0, \]
\[ \theta_{20} = -\frac{\epsilon^2 (8b_2 (b_3 (\beta_1 - \beta_2) + b_1 (\beta_1 - \beta_3)) - (\beta_1 - \beta_2) (\beta_1 - \beta_3) (-12a_2a_2r_{20}^2 + 2a_2^2 + 11a_2^2r_{20}^4))}{16\sqrt{\beta_2 (\beta_2 - \beta_1) (\beta_2 - \beta_3)}}, \]  

represents a periodic solution in the phase space of case I of ring model. The second order approximate solution of this periodic orbit has an amplitude given by \( \sqrt{\frac{a_2}{a_2}} \) and an angular frequency computed as 

\[ \sqrt{\beta_2} - \frac{\epsilon^2 (8b_2 (b_3 (\beta_1 - \beta_2) + b_1 (\beta_1 - \beta_3)) - (\beta_1 - \beta_2) (\beta_1 - \beta_3) (-12a_2a_2r_{20}^2 + 2a_2^2 + 11a_2^2r_{20}^4))}{16\sqrt{\beta_2 (\beta_2 - \beta_1) (\beta_2 - \beta_3)}}. \]  

Evaluating the eigenvalues of Jacobian matrix (34) at periodic solution II (40), we get its stability region in parameters space as follows:

\[ a_2 > 0, \]
\[ a_3 < 0. \]  

(42)

4.1.4. Periodic Solution III. The fourth equilibrium solution, 

\[ r_{10} = \sqrt{\frac{a_1}{a_1}}, \]
\[ r_{20} = 0, \]
\[ r_{30} = 0, \]
\[ \theta_{10} = \frac{\epsilon^2 (8b_1 (b_3 (\beta_1 - \beta_2) + b_2 (\beta_1 - \beta_3)) - (\beta_1 - \beta_2) (\beta_1 - \beta_3) (-12a_1a_1r_{10}^2 + 2a_1^2 + 11a_1^2r_{10}^4))}{16\sqrt{\beta_1 (\beta_1 - \beta_2) (\beta_1 - \beta_3)}}, \]  

corresponds to a periodic solution in the phase space of case 1 of ring model. The second order approximate solution of this periodic orbit has an amplitude given by \( \sqrt{\frac{a_1}{a_1}} \) and an angular frequency computed as 

\[ \sqrt{\beta_1} + \frac{\epsilon^2 (8b_1 (b_3 (\beta_1 - \beta_2) + b_2 (\beta_1 - \beta_3)) - (\beta_1 - \beta_2) (\beta_1 - \beta_3) (-12a_1a_1r_{10}^2 + 2a_1^2 + 11a_1^2r_{10}^4))}{16\sqrt{\beta_1 (\beta_1 - \beta_2) (\beta_1 - \beta_3)}}. \]  

Evaluating the eigenvalues of Jacobian matrix (34) at periodic solution III (43), we get its stability region in parameters space as follows:

\[ a_2 < 0, \]
\[ a_3 < 0. \]  

(45)
4.1.5. 2D Torus I. The fifth equilibrium solution,

\[ r_{10} = \sqrt{\frac{a_1}{\alpha_1}} \]
\[ r_{20} = \sqrt{\frac{a_2}{\alpha_2}} \]
\[ r_{30} = 0, \]
\[ \dot{\theta}_{10} = \frac{\varepsilon^2 \left( 8b_1 (b_3 (\beta_1 - \beta_2) + b_2 (\beta_1 - \beta_3)) - (\beta_1 - \beta_2) (\beta_1 - \beta_3) \left( -12a_1 \alpha_1 r_{10}^2 + 2a_1^2 + 11\alpha_1^2 r_{10}^4 \right) \right)}{16 \sqrt{\beta_1} (\beta_1 - \beta_2) (\beta_1 - \beta_3)}, \]
\[ \dot{\theta}_{20} = -\frac{\varepsilon^2 \left( 8b_2 (b_3 (\beta_1 - \beta_2) + b_1 (\beta_1 - \beta_3)) - (\beta_1 - \beta_2) (\beta_2 - \beta_3) \left( -12a_2 \alpha_2 r_{20}^2 + 2a_2^2 + 11\alpha_2^2 r_{20}^4 \right) \right)}{16 \sqrt{\beta_2} (\beta_2 - \beta_1) (\beta_2 - \beta_3)}, \]

corresponds to a 2D torus in the phase space of case 1 of ring model. The second order approximate solution of this 2D torus has two fundamental angular frequencies given by (44) and (41). Evaluating the eigenvalues of Jacobian matrix (34) at 2D torus I (46), we get the stability region in parameters space as follows:

\[ a_1 > 0, \]
\[ a_2 > 0, \]
\[ a_3 < 0. \]

4.1.6. 2D Torus II. The sixth equilibrium solution,

\[ r_{10} = \sqrt{\frac{a_1}{\alpha_1}} \]
\[ r_{20} = 0, \]
\[ r_{30} = \sqrt{\frac{a_3}{\alpha_3}}, \]
\[ \dot{\theta}_{10} = \frac{\varepsilon^2 \left( 8b_1 (b_3 (\beta_1 - \beta_2) + b_2 (\beta_1 - \beta_3)) - (\beta_1 - \beta_2) (\beta_1 - \beta_3) \left( -12a_1 \alpha_1 r_{10}^2 + 2a_1^2 + 11\alpha_1^2 r_{10}^4 \right) \right)}{16 \sqrt{\beta_1} (\beta_1 - \beta_2) (\beta_1 - \beta_3)}, \]
\[ \dot{\theta}_{30} = -\frac{\varepsilon^2 \left( (\beta_2 - \beta_3) ((\beta_1 - \beta_3) (\beta_2 - \beta_3) (\beta_2 - \beta_3) (\beta_3 - \beta_2) \right)}{16 \sqrt{\beta_3} (\beta_3 - \beta_1) (\beta_3 - \beta_2)}, \]

\[ a_1 > 0, \]
\[ a_2 < 0, \]
\[ a_3 > 0. \]

4.1.7. 2D Torus III. The seventh equilibrium solution,

\[ r_{10} = 0, \]
\[ r_{20} = \sqrt{\frac{a_2}{\alpha_2}}, \]
corresponds to a 2D torus in the phase space of case 1 of ring model. The second order approximate solution of this 2D torus has two fundamental angular frequencies given by (41) and (38). Evaluating the eigenvalues of Jacobian matrix (34) at 2D torus III (50), we get the stability region in parameters space as follows:

\[
\begin{align*}
 r_{30} &= \frac{a_3}{\alpha_3} \\
 \theta'_0 &= \frac{a_3}{\beta_3} \left( \frac{8b_2 (b_3 (\beta_1 - \beta_2) + b_1 (\beta_3 - \beta_2)) - (\beta_1 - \beta_2) (\beta_3 - \beta_2)}{16 \sqrt{\beta_2 (\beta_2 - \beta_3)}} \right) \\
 \theta'_{30} &= \frac{a_3}{\beta_3} \left( \frac{(\beta_2 - \beta_3) \left( (\beta_3 - \beta_2) \left( -12b_1 a_3 r_{30}^2 + 2\alpha_3^2 + 11\alpha_3^2 r_{30}^4 \right) \right) + 8b_2 b_3 (\beta_2 - \beta_3)}{16 \sqrt{\beta_2 (\beta_2 - \beta_3)}} \right)
\end{align*}
\]

(50)

4.1.8. 3D Torus. The eighth equilibrium solution,

\[
\begin{align*}
 r_{10} &= \frac{a_1}{\alpha_1} \\
 r_{20} &= \frac{a_2}{\alpha_2} \\
 r_{30} &= \frac{a_3}{\alpha_3} \\
 \theta'_0 &= \frac{a_3}{\beta_3} \left( \frac{8b_1 (b_3 (\beta_1 - \beta_2) + b_2 (\beta_1 - \beta_3)) - (\beta_1 - \beta_2) (\beta_1 - \beta_3)}{16 \sqrt{\beta_1 (\beta_1 - \beta_3)}} \right) \\
 \theta'_{20} &= \frac{a_3}{\beta_3} \left( \frac{8b_2 (b_3 (\beta_1 - \beta_2) + b_1 (\beta_3 - \beta_2)) - (\beta_1 - \beta_2) (\beta_2 - \beta_3)}{16 \sqrt{\beta_2 (\beta_2 - \beta_3)}} \right) \\
 \theta'_{30} &= \frac{a_3}{\beta_3} \left( \frac{(\beta_2 - \beta_3) \left( (\beta_1 - \beta_3) \left( -12a_1 a_3 r_{30}^2 + 2\alpha_3^2 + 11\alpha_3^2 r_{30}^4 \right) \right) + 8b_2 b_3 (\beta_2 - \beta_3)}{16 \sqrt{\beta_3 (\beta_3 - \beta_2)}} \right)
\end{align*}
\]

(52)

corresponds to a 3D torus in the phase space of case 1 of ring model. The second order approximate solution of this 3D torus has three fundamental angular frequencies given by (44), (41), and (38). Evaluating the eigenvalues of Jacobian matrix (34) at 3D torus (52), we get the stability region in parameters space as follows:

\[
\begin{align*}
 a_1 &> 0, \\
 a_2 &> 0, \\
 a_3 &> 0
\end{align*}
\]

(53)

4.2. Stability Analysis of Case 2 of Ring Model. The Jacobian matrix of system (33) is given by

\[
\begin{pmatrix}
\frac{1}{2} (a_1 - r_{10}^2 a_1) & -r_{10}^2 a_1 & 0 \\
0 & \frac{1}{2} (a_2 - r_{20}^2 a_2) & -r_{20}^2 a_2 \\
0 & 0 & \frac{a_3}{2}
\end{pmatrix}
\]

(54)

System (33) has the following equilibrium solutions.

4.2.1. Equilibrium Point. The equilibrium solution

\[
r_{10} = 0, \quad i = 1, 2, 3
\]

(55)

constitutes an equilibrium point at origin for case 2 of ring model. Evaluating the eigenvalues of Jacobian matrix (54) at
Figure 3: A stable equilibrium point of case 1 of ring model obtained when $\varepsilon = 0.1$, $a_1 = -1$, $a_2 = -0.5$, $a_3 = -1$, $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 2$, and $b_1 = b_2 = b_3 = \alpha_1 = \alpha_2 = \alpha_3 = 1$ in system (18).

equilibrium point (55), we get the following region of stability in parameters space:

\begin{align*}
a_1 &< 0, \\
a_2 &< 0, \\
a_3 &< 0.
\end{align*}

(56)

4.2.2. Periodic Solution I. The equilibrium solution, 

\begin{align*}
r_{10} &= 0, \\
r_{20} &= \sqrt{\frac{a_2}{a_2}}, \\
r_{30} &= 0, \\
\theta_{20} &= \frac{\varepsilon^2 \left( (\beta_1 - \beta_2) \left( -60a_2^2a_2r_{20}^2 + 10a_2^2 + 56a_2^2r_{20}^6 + 55a_2^2r_{20}^6 \right) + 40b_2 \right)}{80 \sqrt{\beta_2} (\beta_2 - \beta_1)}.
\end{align*}

(57)
constitutes a periodic solution in the phase space of case 2 of ring model. The second order approximate solution of this periodic orbit has an amplitude $\sqrt{a_2/\alpha_2}$ and an angular frequency given as

$$\sqrt{\beta_2 + \frac{\varepsilon^2 \left( a_2^2 (\beta_1 - \beta_2) \left( 56a_1\alpha_1^2 + 5\alpha_1^2 \right) + 40\alpha_1^2 b_1 b_2 \right)}{80\alpha_2 \sqrt{\beta_2 (\beta_2 - \beta_1)}}},$$

(58)

Evaluating the eigenvalues of Jacobian matrix (54) at periodic solution I (57), we get its stability region in parameters space as follows:

$$a_1 < 0,$$

(59)

$$a_2 > 0,$$

$$a_3 < 0.$$

4.2.3. Periodic Solution II. The equilibrium solution,

$$r_{10} = \frac{a_1}{\alpha_1},$$

$$r_{20} = 0,$$

$$r_{30} = 0,$$

$$\theta_{10} = \frac{\varepsilon^2 \left( 8b_1 b_2 - (\beta_1 - \beta_2) \left( -12a_1\alpha_1 r_{10}^2 + 2\alpha_1^2 + 11\alpha_1^2 r_{10}^4 \right) \right)}{16\sqrt{\beta_1 (\beta_1 - \beta_2)}},$$

constitutes a periodic solution in case 2 of ring model phase space. The second order approximate solution of this periodic orbit has an amplitude $\sqrt{a_1/\alpha_1}$ and an angular frequency given as
Figure 5: A stable 2D torus of case 1 of ring model obtained when $\varepsilon = 0.1$, $a_1 = 0.42$, $a_2 = 0.3$, $a_3 = -0.2$, $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 1$, and $b_1 = b_2 = b_3 = \alpha_1 = \alpha_2 = \alpha_3 = 1$ in system (18).

Figure 6: Lyapunov exponents of the equilibrium solution in Figure 5.
Figure 7: A stable 3D torus of case 1 of ring model obtained when $\epsilon = 0.1$, $a_1 = 0.4$, $a_2 = 0.4$, $a_3 = 0.5$, $\beta_1 = 1$, $\beta_2 = 3$, $\beta_3 = 2$, and $b_1 = b_2 = b_3 = \alpha_1 = \alpha_2 = \alpha_3 = 1$ in system (18).

Figure 8: Lyapunov exponents of the equilibrium solution in Figure 7.
Evaluating the eigenvalues of Jacobian matrix (54) at periodic solution II (60), we get its stability region in parametric space as follows:

\[
\sqrt{\beta_1 + \frac{\varepsilon^2 (a_1^2 (\beta_2 - \beta_1) + 8b_1b_2)}{16 \sqrt{\beta_1 (\beta_1 - \beta_2)}}},
\]

(61)

4.2.4.2 2D Torus. The equilibrium solution,
corresponds to a 2D torus in the phase space of case 2 of ring model. The second order approximate solution of this 2D torus has two fundamental angular frequencies given by (61) and (58). Evaluating the eigenvalues of Jacobian matrix (54) at 2D torus (63), we get the following stability region in parametric space:

\[
\begin{align*}
\dot{\theta}_{10} &= \frac{\varepsilon^2 \left( 8 b_1 b_2 - (\beta_1 - \beta_2) \left( -12 a_1 a_3 r_{10}^2 + 2 a_2^2 + 11 a_4^2 r_{10}^4 \right) \right)}{16 \sqrt{\beta_1} (\beta_1 - \beta_2)}, \\
\dot{\theta}_{20} &= \frac{\varepsilon^2 \left( (\beta_1 - \beta_2) \left( -60 a_3 a_4 r_{20}^2 + 10 a_2^2 + 56 a_5^2 r_{20}^6 + 55 a_5^2 r_{20}^4 \right) + 40 b_1 b_2 \right)}{80 \sqrt{\beta_2} (\beta_2 - \beta_1)}.
\end{align*}
\]

(63)

\[
\begin{align*}
& a_2 > 0, \\
& a_3 < 0.
\end{align*}
\]

(64)

5. Numerical Simulations

In this section, numerical simulations are carried out to verify the theoretical results obtained in Section 4.
Figure 11: A stable 2D torus of case 2 of ring model obtained when $\varepsilon = 0.1$, $a_1 = 0.45$, $a_2 = 0.3$, $a_3 = -0.2$, $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 1$, and $b_1 = b_2 = b_3 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ in system (19).

Figure 12: Lyapunov exponents of the equilibrium solution in Figure 11.
5.1. Numerical Simulation of Case 1 of Ring Model. Some numerical examples of system (18) are given to address the following scenarios. First, when parameters $a_1$, $a_2$, and $a_3$ have negative values, it is found that a stable equilibrium point at the origin exists as shown in Figure 3. This scenario always occurs for values of parameters satisfying conditions (36). Second, a stable periodic orbit appears in system dynamics when parameters $a_1$ and $a_2$ have negative values while $a_3$ is positive. The equilibrium point at the origin loses its stability and this scenario arises for values of parameters satisfying conditions in (39). Periodic orbits are still observed when parameters $a_1$, $a_2$, and $a_3$ satisfy conditions given in (42) and (45). Figure 4 shows the stable periodic orbit that emerges at $a_1 = -0.4$, $a_2 = -0.5$, and $a_3 = 0.7$. In third scenario, parameters $a_1$ and $a_2$ are positive while $a_3$ is negative. In this case, a 2D torus appears as shown in Figure 5. This agrees with previously obtained stability conditions in parameter space given by (47). The Lyapunov exponents spectrum plot associated with the 2D torus case is shown in Figure 6. Note that 2D tori can also be spotted in system dynamics when conditions (49) and (51) are satisfied. Finally, the parameters $a_1$, $a_2$, and $a_3$ are taken positive in numerical simulations. In particular, Figure 7 clarifies the occurrence of 3D torus in phase space of the system at $a_1 = 0.4$, $a_2 = 0.4$, and $a_3 = 0.5$. This case is corresponding to the conditions given in (53). The associated Lyapunov exponents spectrum is plotted and shown in Figure 8.

5.2. Numerical Simulation of Case 2 of Ring Model. The numerical examples of system (19) are shown in Figures 9–12 to illustrate the following scenarios. First, a stable equilibrium point at origin occurs when parameters $a_1$, $a_2$, and $a_3$ are all negative as shown in Figure 9. Setting the values of parameters to satisfy conditions (56) leads to this scenario. Second, when parameters $a_1$ and $a_3$ are negative while $a_2$ is positive, a stable periodic orbit is obtained. The stable equilibrium point at origin becomes unstable and this scenario arises for the values of parameters conditions in (59). A stable periodic orbit is shown in Figure 10 when $a_1 = -0.7$, $a_2 = 0.4$, and $a_3 = -0.5$. Periodic orbits also occur when parameters $a_1$, $a_2$, and $a_3$ satisfy conditions in (62). Finally, a 2D torus appears in system dynamics when parameters $a_1$ and $a_2$ are positive while $a_3$ is negative. This is clarified in Figure 11 where $a_1 = 0.45$, $a_2 = 0.3$, and $a_3 = -0.2$. The values of parameters $a_1$, $a_2$, and $a_3$ in this example satisfy the conditions in (64). The associated Lyapunov exponents spectrum is plotted and shown in Figure 12.
5.3. Effects of Time-Delayed Coupling. In this part, the influences of time-delayed coupling on the aforementioned numerical examples are numerically investigated. For case 1 of ring model the mathematical model can be rewritten in the following form:

\[ \ddot{x}_j - \epsilon \dot{x}_j (a_j - \alpha_j x_j^3) + \beta_j x_j - b_j \epsilon \left( \sum_{k=1, k \neq j}^3 x_k^\tau \right) = 0, \quad j = 1, 2, 3, \]  

(65)

\[ x_k^\tau = x_k(t - \tau) \] and \( \tau \) denotes constant time delay value. It is observed that for the values of parameters used in Figure 13 the equilibrium point loses its stability when time delay value increases over a certain value (approximately 0.4) and a periodic orbit is induced.

However, the effects of time delay coupling on other dynamics of system are almost negligible as shown in Figure 14.

For case 2 of ring model the rate equations can be reformulated as follows:

\[ \dot{x}_1 - \epsilon \dot{x}_1 (a_1 - \alpha_1 x_1^3) + \beta_1 x_1 - b_1 \epsilon (x_2^\tau + x_3^\tau) = 0, \]
\[ \dot{x}_2 - \epsilon \dot{x}_2 (a_2 - \alpha_3 x_2^3 - \alpha_2 x_2^3) + \beta_2 x_2 - b_2 \epsilon (x_1^\tau + x_3^\tau) = 0, \]
\[ \dot{x}_3 - \epsilon \dot{x}_3 (a_3 - \alpha_4 x_3^3) + \beta_3 x_3 - b_3 \epsilon (x_1^\tau + x_2^\tau) = 0. \]

Although the effect of time-delayed coupling on the stable equilibrium point at the origin and torus is measly as shown in Figure 15, it is observed that it has an obvious impact on periodic solution that exists when conditions are satisfied as shown in Figure 16.

6. Conclusion

In this work, we have investigated new analytical results concerning the dynamical behaviors of three coupled memristor-based oscillators with identical and different nonlinearities.
in ring configuration. The stability regions in parameters space are attained for each type of induced dynamics. The analytical perturbation techniques were employed to achieve this goal analytically. Furthermore, numerical simulations are performed and they agreed with analytical results obtained. It was demonstrated that the proposed models exhibit complex dynamics including periodic orbits, 2D tori, and 3D tori in ring coupling configuration. A detailed analytical investigations of the effects of time-delayed couplings, with different time delays, can be considered in future work. Also, the mathematical modeling of coupled memristor-based oscillators using fractional order derivatives, in order to accurately incorporate memory effects, can be examined.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


Figure 15: Phase portraits of case 2 of ring model with time-delayed coupling obtained for (a) $\tau = 2$, $\epsilon = 0.1$, $a_1 = -1$, $a_2 = -1$, $a_3 = -1$, $\beta_1 = 3$, $\beta_2 = 1$, $\beta_3 = 2$ and $b_1 = b_2 = b_3 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ and (b) $\tau = 0.5$, $\epsilon = 0.1$, $a_1 = 0.45$, $a_2 = 0.3$, $a_3 = -0.2$, $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 1$ and $b_1 = b_2 = b_3 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$.

Figure 16: Phase portraits of case 2 of ring model with time-delayed coupling obtained for two values of $\tau$, $\epsilon = 0.1$, $a_1 = -1$, $a_2 = -0.5$, $a_3 = -1$, $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 2$ and $b_1 = b_2 = b_3 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ in system (66). Time delay values are for (a) $\tau = 0.7$ and (b) $\tau = 0.1$. 


