Research Article

Application of Dual Hesitant Fuzzy Geometric Bonferroni Mean Operators in Deciding an Energy Policy for the Society

Raja Noshad Jamil and Tabasam Rashid

University of Management and Technology, Lahore 54770, Pakistan

Correspondence should be addressed to Tabasam Rashid; tabasam.rashid@gmail.com

Received 27 February 2018; Revised 11 May 2018; Accepted 27 May 2018; Published 26 June 2018

Academic Editor: Kishin Sadarangani

Copyright © 2018 Raja Noshad Jamil and Tabasam Rashid. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Dual hesitant fuzzy geometric Bonferroni mean is defined for dual hesitant fuzzy sets. Different properties of dual hesitant fuzzy geometric Bonferroni mean are discussed. Some special cases are studied in detail for dual hesitant fuzzy geometric Bonferroni mean. In addition, dual hesitant fuzzy weighted geometric Bonferroni mean and dual hesitant fuzzy Choquet geometric Bonferroni mean are proposed. A multicriteria decision-making method is discussed to find the best alternative among different alternatives by using proposed aggregated operators and an illustrated example is also given to understand our proposal.

1. Introduction

Fuzzy sets (FS) initiated by Zadeh [1] are great invention in the field of sciences. The idea of fuzzy sets received great intension for handling uncertainty and vagueness situations. Often the information taken from human preferences has vagueness. Bellman and Zadeh succeeded to develop a useful technique in decision making for these situations by the help of fuzzy sets theory [2]. Due to use of fuzzy sets as the effective tools to collect and represent arguments in different fields like [3–11], several well-known extensions of fuzzy sets have been developed including interval-valued fuzzy sets (IVFS) [12], intuitionistic fuzzy sets (IFS) [13], interval-valued intuitionistic fuzzy sets (IVIFS) [4], hesitant fuzzy set [14], and dual hesitant fuzzy set (DHFS) [15].

In FS, membership degree of an element a is taken as $m \in [0, 1]$, a single value, but in real life, the degree of nonmembership of a is not equal to $1 - m$ (say m') because there may exist some hesitation degree. Due to that, intuitionistic fuzzy set (IFS) which is a generalization of fuzzy sets theory was suggested by Atanassov [13, 16]. IFS integrates the membership degree of hesitation defined as $m'' = 1 - (m + m')$ where $0 \leq m'' \leq 1$. IFS as the generalization of FS motivates the researchers to apply it in different real life applications. Atanassov [16, 17] showed that the

information and semantic illustration of intuitionistic fuzzy set become more significant, practical, and applicable due to its degree of belongingness, degree of nonbelongingness, and the hesitation boundary. IFS is a useful representation of uncertainty and ambiguity of an entity and hence is able to be used as a great instrument to express data information under a range of different fuzzy situations. Torra [14] extended the concept of fuzzy sets to hesitant fuzzy sets (HFS). In HFS, a decision maker tries to manage those situations where hesitation involves in decision to provide membership value for particular element [18–23]. Zhu et al. initiated to merge both IFS and HFS and introduced the idea of dual hesitant fuzzy sets (DHFS) [15].

Multicriteria decision making is the general observable routine in daily life, which is to select the most suitable selection that possibly takes place from several likely options or to obtain their position by aggregating the performances of each choice under several criteria. Aggregation is a procedure in which different available options are aggregated with different method and return a single value [24, 25]. Dual hesitant fuzzy sets (DHFS) have ability to deal with the circumstances when the assessment of an alternative under each condition is corresponded to several possible values for membership and nonmembership for the same element [15]. In existing literature there is very fewer work existing on

aggregation operator for *DHFS*; particularly no work exists when arguments interrelated with each other. This inadequacy motivated us to develop some aggregative operators for *DHFS* based on geometric Bonferroni mean and Choquet integral. Geometric Bonferroni mean has the properties to capture the interrelationships among arguments [26]. The Choquet integral also is an important tool to consider the correlations among arguments [27]. Both of these operators have ability to aggregate all arguments when they interrelated with each other.

In this research article, we purposed some aggregation operators for *DHFS* as dual hesitant fuzzy geometric Bonferroni mean (*DHFGBM*) and dual hesitant fuzzy weighted geometric Bonferroni mean (*DHFWGBM*). Further in this research, Choquet integral with geometric Bonferroni mean is used to introduce aggregation operators for *DHFS* called dual hesitant fuzzy Choquet geometric Bonferroni mean (*DHFCBGM*).

The rest of the paper is organized as follows: Section 2 briefly reviewed some basic concepts including definition of fuzzy set, its extension, and aggregated operators defined on these generalized fuzzy sets. Section 3 defined a ranking method for dual hesitant fuzzy sets and also defined some operational laws for manipulation of *DHFS*. Section 4 defined *DHFGBM* and discussed different properties of *DHFGBM*. Section 5 is devoted to discussion on Choquet integral (CI) based geometric Bonferroni mean in dual hesitant fuzzy environment. In Section 6, a multicriteria decision-making approach is discussed based on *DHFCBGM* and an illustrated example is given to understand our method. The research article finished in Section 7 with some concluding remarks.

2. Preliminaries

This section is devoted to some significant basic theories which are important to understand the article.

Definition 1 (see [28]). Let $m_1, m_2 > 0$ and $a_i \geq 0$ for all $i = 1, 2, \dots, n$. Then Bonferroni geometric mean (BGM) is defined as

$$\begin{aligned} BGM^{m_1, m_2}(a_1, a_2, \dots, a_n) \\ = \frac{1}{m_1 + m_2} \prod_{i,j=1, i \neq j}^n (m_1 a_i + m_2 a_j)^{1/n(n-1)} \end{aligned} \quad (1)$$

Definition 2 (see [1]). Let X be a set of crisp elements, a fuzzy set A on X is a function $h_A : X \rightarrow [0, 1]$ for any $x \in X$, and the value $h_A(x)$ is said to be the degree of membership of x in A .

Definition 3 (see [13]). Let X be a nonempty set; an intuitionistic fuzzy set A on X is combination of functions $h_A : X \rightarrow [0, 1]$ and $h'_A : X \rightarrow [0, 1]$, where $h_A(x)$ and $h'_A(x)$ represent the degrees of membership and nonmembership for every element $x \in X$ and must satisfy the condition

$$0 \leq h_A(x) + h'_A(x) \leq 1 \quad (2)$$

Let $e = (h, h')$, $e_1 = (h_1, h'_1)$, and $e_2 = (h_2, h'_2)$ be elements of intuitionistic fuzzy set A , then following basic operations introduced by Xu and Yager [29] hold:

- (1) $\alpha e = (1 - (1 - h)^\alpha, h^\alpha), \alpha > 0;$
- (2) $e^\alpha = (h^\alpha, 1 - (1 - h')^\alpha), \alpha > 0;$
- (3) $e_1 \oplus e_2 = (h_1 + h_2 - h_1 h_2, h'_1 h'_2);$
- (4) $e_1 \otimes e_2 = (h_1 h_2, h'_1 + h'_2 - h'_1 h'_2).$

Based on these operation laws intuitionistic fuzzy Bonferroni mean and intuitionistic fuzzy geometric Bonferroni mean are defined as follows.

Definition 4 (see [11]). Let $e_i = (\varepsilon_i, \varepsilon'_i)$ for all $i = 1, 2, \dots, n$ be the elements of intuitionistic fuzzy set I for any $m_1, m_2 > 0$,

$$\begin{aligned} IFB^{m_1, m_2}(e_1, e_2, \dots, e_n) \\ = \left(\frac{1}{n(n-1)} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^n (e_i^{m_1} \otimes e_j^{m_2}) \right) \right)^{1/(m_1+m_2)}, \end{aligned} \quad (3)$$

then IFB^{m_1, m_2} is said to be intuitionistic fuzzy Bonferroni mean (*IFBM*).

Definition 5 (see [30]). Let $e_i = (\varepsilon_i, \varepsilon'_i)$ for all $i = 1, 2, \dots, n$ be the elements of intuitionistic fuzzy set I for any $m_1, m_2 > 0$,

$$\begin{aligned} IFGB^{m_1, m_2}(e_1, e_2, \dots, e_n) = \frac{1}{m_1 + m_2} \\ \cdot \bigotimes_{i,j=1, i \neq j}^n ((m_1 e_i \oplus m_2 e_j) \otimes (m_1 e_j \oplus m_2 e_i))^{2/n(n-1)}, \end{aligned} \quad (4)$$

then $IFGB^{m_1, m_2}$ is said to be intuitionistic fuzzy Bonferroni Geometric mean (*IFGBM*).

Torra [14] proposed *HFS*, which is a more general fuzzy set and permits the membership to include a set of possible values.

Definition 6 (see [14]). Let X be a fixed set; a hesitant fuzzy set A on X is defined in terms of a function $h_A(x)$ that when applied to X it returns to a finite subset of $[0, 1]$.

To be easily understood, Xia and Xu [31] express the *HFS* by a mathematical symbol:

$$E = \{\langle x, h_E(x) \rangle \mid x \in X\}, \quad (5)$$

where $h_E(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set E , and called a hesitant fuzzy element (*HFE*).

Let $e = \{h\}$, $e_1 = \{h_1\}$, and $e_2 = \{h_2\}$ be elements of hesitant fuzzy set A , then following basic operations introduced by Xia et al. [28] hold:

- (1) $e^\alpha = \cup_{h \in e} \{h^\alpha\}, \alpha > 0;$
- (2) $\alpha e = \cup_{h \in e} \{1 - (1 - h)^\alpha\}, \alpha > 0;$

- (3) $h_1 \oplus h_2 = \cup_{h_1 \in e_1, h_2 \in e_2} \{h_1 + h_2 - h_1 h_2\};$
 (4) $h_1 \otimes h_2 = \cup_{h_1 \in e_1, h_2 \in e_2} \{h_1 h_2\}.$

Based on these operational laws hesitant fuzzy geometric Bonferroni mean is defined as follows.

Definition 7 (see [32]). Let $e_i \geq 0$ for all $i = 1, 2, \dots, n$ be a collection of *HFEs*. For any $m_1, m_2 > 0$, then hesitant fuzzy geometric Bonferroni mean (*HFGBM*) is defined as

$$HFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) = \frac{1}{m_1 + m_2} \cdot \left(\bigotimes_{i,j=1, i \neq j}^n ((m_1 e_i \oplus m_2 e_j) \otimes (m_1 e_j \oplus m_2 e_i)) \right)^{2/n(n-1)} \quad (6)$$

In 2012, Zhu, et al. introduced dual hesitant fuzzy set which has both membership and nonmembership degree of an element in hesitant environment [15]. In 2014, Beg and Rashid [5] further discussed dual hesitant fuzzy set with new name intuitionistic fuzzy hesitant set and defined TOPSIS technique.

Definition 8 (see [15]). Let X be a nonempty set. A dual hesitant fuzzy set A on X is defined as pair of functions $h_A : X \rightarrow [0, 1]$ and $h'_A : X \rightarrow [0, 1]$, which can return to the subsets of $[0, 1]$ and is represented as $A = \{(x, h_A(x), h'_A(x) \mid x \in X\}$. For every element $x \in X$, the condition $0 \leq \max(h_A(x)) + \min(h'_A(x)) \leq 1$ and $0 \leq \min(h_A(x)) + \max(h'_A(x)) \leq 1$ must be satisfied. $e = (h_A(x), h'_A(x))$ is called an dual hesitant fuzzy element (*DHFE*), where $h_A(x)$ is the possible membership values x while $h'_A(x)$ is the possible nonmembership values of x .

3. Manipulation with Dual Hesitant Fuzzy Elements

Ranking is a basic necessity of decision making in any fuzzy environment; therefore, dual hesitant fuzzy numbers must be ranked before any action is taken by a decision maker. To find order between two *DHFE*, we defined accuracy and similarity function as follows.

Definition 9. Let $e = (\varepsilon, \varepsilon')$ be an *DHFE* where $\varepsilon = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ and $\varepsilon' = \{\varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_m\}$, then an accuracy function for *DHFE* is defined as

$$\varphi(e) = \frac{1}{2} \left(\frac{\sum_{i=1}^n \varepsilon_i}{n} + \frac{\sum_{j=1}^m \varepsilon'_j}{m} \right) \quad (7)$$

Definition 10. If $e = (\varepsilon, \varepsilon')$ is *DHFE* where $\varepsilon = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ and $\varepsilon' = \{\varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_m\}$, then a similarity function for *DHFE* is defined as

$$\psi(e) = \frac{\sum_{j=1}^m \varepsilon'_j}{m(2 - \varphi(e))} \quad (8)$$

Order between two *DHFE* $e_1 = (\varepsilon_1, \varepsilon'_1)$ and $e_2 = (\varepsilon_2, \varepsilon'_2)$ based on (7) and (8) is defined as follows.

Definition 11. Let $e_1 = (\varepsilon_1, \varepsilon'_1)$ and $e_2 = (\varepsilon_2, \varepsilon'_2)$ be two *DHFE* where $\varepsilon = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ and $\varepsilon' = \{\varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_m\}$, then order between e_1 and e_2 is as follows:

- (1) if $\psi(e_1) > \psi(e_2)$ then $e_1 > e_2$;
- (2) if $\psi(e_1) = \psi(e_2)$ and $\varphi(e_1) > \varphi(e_2)$ then $e_1 > e_2$;
- (3) if $\psi(e_1) = \psi(e_2)$ and $\varphi(e_1) = \varphi(e_2)$ then $e_1 \approx e_2$.

Definition 12. Let $\lambda > 0$ and $e_k = (\eta_k, \eta'_k)$ ($k = 1, 2, \dots, p$) be collection of *DHFE*, where $\eta_k = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m\}$, $\eta'_k = \{\varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_n\}$ and $\varepsilon_i, \varepsilon'_j \in [0, 1]$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ with condition

$$0 \leq \max(\varepsilon_i) + \min(\varepsilon'_j) \leq 1, \quad (9)$$

and

$$0 \leq \min(\varepsilon_i) + \max(\varepsilon'_j) \leq 1, \quad (10)$$

then the following operations are hold:

(1)

$$\lambda e_1 = \bigcup_{(\eta_1, \eta'_1) \in e_1} \left\{ \left(1 - (1 - \varepsilon_i)^\lambda, (\varepsilon'_j)^\lambda \right) \right\} \quad (11)$$

$$\forall i, j, \text{ also } \varepsilon_i \in \eta_1 \text{ and } \varepsilon'_j \in \eta'_1$$

(2)

$$e_1^\lambda = \bigcup_{(\eta_1, \eta'_1) \in e_1} \left\{ \left((\varepsilon_i)^\lambda, 1 - (1 - \varepsilon'_j)^\lambda \right) \right\} \quad (12)$$

$$\forall i, j \text{ also } \varepsilon_i \in \eta_1 \text{ and } \varepsilon'_j \in \eta'_1$$

(3)

$$\bigoplus_{k=1}^p e_k = \bigcup_{(\eta_k, \eta'_k) \in e_k} \left\{ \left(1 - \prod_{i=1}^m (1 - \varepsilon_{k,i}), \prod_{j=1}^n \varepsilon'_{k,j} \right) \right\} \quad (13)$$

$$\forall i, j \text{ also } \varepsilon_{k,i} \in \eta_k \text{ and } \varepsilon'_{k,j} \in \eta'_k$$

(4)

$$\bigotimes_{k=1}^p e_k = \bigcup_{(\eta_k, \eta'_k) \in e_k} \left\{ \left(\prod_{i=1}^m \varepsilon_{k,i}, 1 - \prod_{j=1}^n (1 - \varepsilon'_{k,j}) \right) \right\} \quad (14)$$

$$\forall i, j \text{ also } \varepsilon_{k,i} \in \eta_k \text{ and } \varepsilon'_{k,j} \in \eta'_k$$

Theorem 13. Let $e_1 = (\varepsilon_1, \varepsilon'_1)$ and $e_2 = (\varepsilon_2, \varepsilon'_2)$ be *DHFEs* and $\lambda > 0$, then

- (i) $\lambda(e_1 \oplus e_2) = \lambda e_1 \oplus \lambda e_2$;
- (ii) $(e_1 \otimes e_2)^\lambda = e_1^\lambda \otimes e_2^\lambda$.

Theorem 14. Let e_{α_i} and e_{β_i} be two collections of *DHFE*, if for any $m_1, m_2 > 0$, one has

$$\zeta_{\alpha_i, \beta_i} = (m_1 e_{\alpha_i} \oplus m_2 e_{\beta_i}) \otimes (m_1 e_{\alpha_i} \oplus m_2 e_{\beta_i}) \quad (15)$$

$$\text{where } (\eta_{\alpha_i}, \eta'_{\alpha_i}) \in e_{\alpha_i} \text{ for } \varepsilon_{\alpha_i} \in \eta_{\alpha_i} \text{ and } \varepsilon'_{\alpha_i} \in \eta'_{\alpha_i}$$

and

$$\zeta_{\beta_i, j, i < j} = (m_1 e_{\beta_i} \oplus m_2 e_{\beta_j}) \otimes (m_1 e_{\beta_j} \oplus m_2 e_{\beta_i}) \quad (16)$$

where $(\eta_{\beta_i}, \eta'_{\beta_i}) \in e_{\beta_i}$ for $\varepsilon_{\beta_i} \in \eta_{\beta_i}$ and $\varepsilon'_{\beta_i} \in \eta'_{\beta_i}$

and if for all $\varepsilon_{\alpha_i} \leq \varepsilon_{\beta_i}$ and for all $\varepsilon'_{\alpha_i} \geq \varepsilon'_{\beta_i}$ then one has

$$\zeta_{\alpha_i, j, i < j} \leq \zeta_{\beta_i, j, i < j} \quad (17)$$

Proof. As $\varepsilon_{\alpha_i} \leq \varepsilon_{\beta_i}$, $\varepsilon_{\alpha_j} \leq \varepsilon_{\beta_j}$ and $\varepsilon'_{\alpha_i} \geq \varepsilon'_{\beta_i}$, $\varepsilon'_{\alpha_j} \geq \varepsilon'_{\beta_j}$ $\forall i, j = 1, 2, \dots, n, i \neq j$, we have

$$1 - (1 - \varepsilon_{\alpha_i})^{m_1} (1 - \varepsilon_{\alpha_j})^{m_2} \leq 1 - (1 - \varepsilon_{\beta_i})^{m_1} (1 - \varepsilon_{\beta_j})^{m_2} \quad (18)$$

$$1 - (1 - \varepsilon_{\alpha_j})^{m_1} (1 - \varepsilon_{\alpha_i})^{m_2} \leq 1 - (1 - \varepsilon_{\beta_j})^{m_1} (1 - \varepsilon_{\beta_i})^{m_2} \quad (19)$$

$$(\varepsilon'_{\alpha_i})^{m_1} (\varepsilon'_{\alpha_j})^{m_2} \geq (\varepsilon'_{\beta_i})^{m_1} (\varepsilon'_{\beta_j})^{m_2} \quad (20)$$

$$(\varepsilon'_{\alpha_j})^{m_1} (\varepsilon'_{\alpha_i})^{m_2} \geq (\varepsilon'_{\beta_j})^{m_1} (\varepsilon'_{\beta_i})^{m_2}; \quad (21)$$

take

$$\begin{aligned} \zeta_{\alpha_i, j, i < j} &= (m_1 e_{\alpha_i} \oplus m_2 e_{\alpha_j}) \otimes (m_1 e_{\alpha_j} \oplus m_2 e_{\alpha_i}) \\ &= \bigcup_{(\varepsilon_{\alpha_i}, \varepsilon'_{\alpha_i}) \in e_{\alpha_i}, (\varepsilon_{\alpha_j}, \varepsilon'_{\alpha_j}) \in e_{\alpha_j}} \left\{ \left(1 - (1 - \varepsilon_{\alpha_i})^{m_1}, (\varepsilon'_{\alpha_i})^{m_1} \right) \right. \\ &\quad \left. \oplus \left(1 - (1 - \varepsilon_{\alpha_j})^{m_2}, (\varepsilon'_{\alpha_j})^{m_2} \right) \right\} \\ &\quad \otimes \bigcup_{(\varepsilon_{\alpha_i}, \varepsilon'_{\alpha_i}) \in e_{\alpha_i}, (\varepsilon_{\alpha_j}, \varepsilon'_{\alpha_j}) \in e_{\alpha_j}} \left\{ \left(1 - (1 - \varepsilon_{\alpha_j})^{m_1}, (\varepsilon'_{\alpha_j})^{m_1} \right) \right. \\ &\quad \left. \oplus \left(1 - (1 - \varepsilon_{\alpha_i})^{m_2}, (\varepsilon'_{\alpha_i})^{m_2} \right) \right\} \\ \zeta_{\alpha_i, j, i < j} &= \bigcup_{(\varepsilon_{\alpha_i}, \varepsilon'_{\alpha_i}) \in e_{\alpha_i}, (\varepsilon_{\alpha_j}, \varepsilon'_{\alpha_j}) \in e_{\alpha_j}} \left\{ \left(1 - (1 - \varepsilon_{\alpha_i})^{m_1} (1 - \varepsilon_{\alpha_j})^{m_2}, (\varepsilon'_{\alpha_i})^{m_1} (\varepsilon'_{\alpha_j})^{m_2} \right) \right\} \\ &\quad \otimes \bigcup_{(\varepsilon_{\alpha_i}, \varepsilon'_{\alpha_i}) \in e_{\alpha_i}, (\varepsilon_{\alpha_j}, \varepsilon'_{\alpha_j}) \in e_{\alpha_j}} \left\{ \left(1 - (1 - \varepsilon_{\alpha_j})^{m_1} (1 - \varepsilon_{\alpha_i})^{m_2}, (\varepsilon'_{\alpha_j})^{m_1} (\varepsilon'_{\alpha_i})^{m_2} \right) \right\}; \end{aligned} \quad (22)$$

similarly,

$$\begin{aligned} \zeta_{\beta_i, j, i < j} &= \bigcup_{(\varepsilon_{\beta_i}, \varepsilon'_{\beta_i}) \in e_{\beta_i}, (\varepsilon_{\beta_j}, \varepsilon'_{\beta_j}) \in e_{\beta_j}} \left\{ \left(1 - (1 - \varepsilon_{\beta_i})^{m_1} (1 - \varepsilon_{\beta_j})^{m_2}, (\varepsilon'_{\beta_i})^{m_1} (\varepsilon'_{\beta_j})^{m_2} \right) \right\} \\ &\quad \otimes \bigcup_{(\varepsilon_{\beta_i}, \varepsilon'_{\beta_i}) \in e_{\beta_i}, (\varepsilon_{\beta_j}, \varepsilon'_{\beta_j}) \in e_{\beta_j}} \left\{ \left(1 - (1 - \varepsilon_{\beta_j})^{m_1} (1 - \varepsilon_{\beta_i})^{m_2}, (\varepsilon'_{\beta_j})^{m_1} (\varepsilon'_{\beta_i})^{m_2} \right) \right\} \end{aligned} \quad (23)$$

Let $\tau_{\alpha_i, j, i < j} \in \zeta_{\alpha_i, j, i < j}$ and $\tau_{\beta_i, j, i < j} \in \zeta_{\beta_i, j, i < j}$ $\forall i, j = 1, 2, 3, \dots, n; i < j$ then by Definition 12 and (18)–(23), we have $\tau_{\alpha_i, j, i < j} \leq \tau_{\beta_i, j, i < j}$ $\forall i, j = 1, 2, 3, \dots, n$ with $i < j$ and hence

$$\zeta_{\alpha_i, j, i < j} \leq \zeta_{\beta_i, j, i < j} \quad (24)$$

is the required result. $\square$

Theorem 15. Let e_i be a collection of DHFE and for any $m_1, m_2 > 0$, one has

$$\begin{aligned} \zeta_{i, j, i < j} &= (m_1 e_i \oplus m_2 e_j) \otimes (m_1 e_j \oplus m_2 e_i), \\ e_i^- &= \bigcup_{(\varepsilon_i, \varepsilon'_i) \in e_i} \left\{ (\min \{\varepsilon_i\}, \max \{\varepsilon'_i\}) \right\} \end{aligned} \quad (25)$$

and

$$e_i^+ = \bigcup_{(\varepsilon_i, \varepsilon'_i) \in e_i} \left\{ (\max \{\varepsilon_i\}, \min \{\varepsilon'_i\}) \right\}; \quad (26)$$

if $(\varepsilon^-, \varepsilon'^+) \in e_i^-$ and $(\varepsilon^+, \varepsilon'^-) \in e_i^+$ for $i, j = 1, 2, \dots, n; i < j$ with $\varepsilon^- \leq \varepsilon^+$ and $\varepsilon'^+ \geq \varepsilon'^-$ then

$$\begin{aligned} &\bigcup_{(\varepsilon^-, \varepsilon'^+) \in e_i^-} \left\{ \left((1 - (1 - \varepsilon^-)^{m_1 + m_2})^2, 1 - (1 - (\varepsilon'^+)^{m_1 + m_2})^2 \right) \right\} \leq \zeta_{\alpha_i, j, i < j} \\ &\leq \bigcup_{(\varepsilon^+, \varepsilon'^-) \in e_i^+} \left\{ \left((1 - (1 - \varepsilon^+)^{m_1 + m_2})^2, 1 - (1 - (\varepsilon'^-)^{m_1 + m_2})^2 \right) \right\} \end{aligned} \quad (27)$$

Proof. Since $\varepsilon^- \leq \varepsilon_i \leq \varepsilon^+$ and $\varepsilon'^- \leq \varepsilon'_i \leq \varepsilon'^+$ ($i = 1, 2, \dots, n$) then

$$\begin{aligned} 1 - (1 - \varepsilon^-)^{m_1 + m_2} &\leq 1 - (1 - \varepsilon_i)^{m_1} (1 - \varepsilon_j)^{m_2} \\ &\leq 1 - (1 - \varepsilon^+)^{m_1 + m_2}, \end{aligned} \quad (28)$$

$$1 - (1 - \varepsilon^-)^{m_1+m_2} \leq 1 - (1 - \varepsilon_j)^{m_1} (1 - \varepsilon_i)^{m_2} \leq 1 - (1 - \varepsilon^+)^{m_1+m_2}, \quad (29)$$

$$(\varepsilon'^-)^{m_1+m_2} \leq \varepsilon_i'^{m_1} \varepsilon_j'^{m_2} \leq (\varepsilon'^+)^{m_1+m_2}, \quad (30)$$

and

$$(\varepsilon'^-)^{m_1+m_2} \leq \varepsilon_j'^{m_1} \varepsilon_i'^{m_2} \leq (\varepsilon'^+)^{m_1+m_2} \quad (31)$$

as

$$\begin{aligned} \zeta_{i,j,i < j} &= (m_1 e_i \oplus m_2 e_j) \otimes (m_1 e_j \oplus m_2 e_i) \\ &= \bigcup_{(\varepsilon_i, \varepsilon_i') \in e_i, (\varepsilon_j, \varepsilon_j') \in e_j} \left\{ \left((1 - (1 - \varepsilon_i)^{m_1} (1 - \varepsilon_j)^{m_2}), \right. \right. \\ &\quad \left. \left. (\varepsilon_i'^{m_1} \varepsilon_j'^{m_2}) \right) \right\} \\ &\otimes \bigcup_{(\varepsilon_i, \varepsilon_i') \in e_i, (\varepsilon_j, \varepsilon_j') \in e_j} \left\{ \left((1 - (1 - \varepsilon_j)^{m_1} (1 - \varepsilon_i)^{m_2}), \right. \right. \\ &\quad \left. \left. (\varepsilon_j'^{m_1} \varepsilon_i'^{m_2}) \right) \right\}; \end{aligned} \quad (32)$$

according to Definition 12 and (28)–(32) we have

$$\begin{aligned} &\bigcup_{(\varepsilon^-, \varepsilon'^+) \in e_i^-} \left\{ \left((1 - (1 - \varepsilon^-)^{m_1+m_2})^2, 1 \right. \right. \\ &\quad \left. \left. - (1 - (\varepsilon'^+)^{m_1+m_2})^2 \right) \right\} \leq \zeta_{\alpha_{i,j,i < j}} \\ &\leq \bigcup_{(\varepsilon^+, \varepsilon'^-) \in e_i^+} \left\{ \left((1 - (1 - \varepsilon^+)^{m_1+m_2})^2, 1 \right. \right. \\ &\quad \left. \left. - (1 - (\varepsilon'^-)^{m_1+m_2})^2 \right) \right\} \end{aligned} \quad (33)$$

which is required. $\square$

Remark 16. Let e_i be a collection of DHFE then for any $m_1, m_2 > 0$ we have

$$\begin{aligned} &(m_1 e_i \oplus m_2 e_j) \otimes (m_1 e_j \oplus m_2 e_i) \\ &= (m_2 e_i \oplus m_1 e_j) \otimes (m_2 e_j \oplus m_1 e_i) \end{aligned} \quad (34)$$

4. Dual Hesitant Fuzzy Geometric Bonferroni Operators

In multicriteria decision making, the qualifications of an option under a certain condition may be characterized by many potential options. To aggregate all the potential options of an alternative under some criteria, we presented a generalization of the GBM in dual hesitant fuzzy environments, which is defined as follows.

Definition 17. Let $m_1, m_2 > 0$, and e_i ($i = 1, 2, \dots, n$) be the collection of DHFE, then dual hesitant fuzzy geometric Bonferroni mean (DHFGBM) is defined as

$$\begin{aligned} DHFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) &= \frac{1}{m_1 + m_2} \\ &\cdot \bigotimes_{\substack{i,j=1 \\ i \neq j}}^n \left((m_1 e_i \oplus m_2 e_j) \otimes (m_1 e_j \oplus m_2 e_i) \right)^{2/n(n-1)}; \end{aligned} \quad (35)$$

with the help of operational laws (11)–(14), defined on DHFE, we further derive the following results.

Theorem 18. Let $m_1, m_2 > 0$ and e_i ($i = 1, 2, \dots, n$) be the collection of DHFE. Then DHFGBM is an DHFE and

$$\begin{aligned} DHFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) &= \bigcup_{(\varepsilon_{i,j}, \varepsilon'_{i,j}) \in \zeta_{i,j}} \left\{ \left(1 \right. \right. \\ &\quad \left. \left. - \left(1 - \left(\prod_{i,j=1, i < j}^n \varepsilon_{i,j} \right)^{2/n(n-1)} \right)^{1/(m_1+m_2)} \right. \right. \\ &\quad \left. \left. \left(1 - \prod_{i,j=1, i < j}^n (1 - \varepsilon'_{i,j})^{2/n(n-1)} \right)^{1/(m_1+m_2)} \right) \right\} \end{aligned} \quad (36)$$

where

$$\zeta_{i,j,i < j} = (m_1 e_i \oplus m_2 e_j) \otimes (m_1 e_j \oplus m_2 e_i) \quad (37)$$

Proof. From (11)–(14) and Theorem 13 we see that $\zeta_{i,j}$ is also an DHFE. By (35) and (37) we have

$$\begin{aligned} DHFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) &= \frac{1}{m_1 + m_2} \left(\bigotimes_{i,j=1}^n (\zeta_{i,j,i < j})^{2/n(n-1)} \right) \end{aligned} \quad (38)$$

Theorem 13 applies on (38); we obtain

$$\begin{aligned} DHFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) &= \frac{1}{m_1 + m_2} \left(\bigotimes_{i,j=1}^n \zeta_{i,j,i < j} \right)^{2/n(n-1)}; \end{aligned} \quad (39)$$

by (14)

$$= \frac{1}{m_1 + m_2} \bigcup_{(\varepsilon_{i,j}, \varepsilon'_{i,j}) \in \zeta_{i,j;i < j}} \left\{ \left(\prod_{i,j=1,i < j}^n \varepsilon_{i,j}, 1 - \prod_{i,j=1,i < j}^n (1 - \varepsilon'_{i,j}) \right)^{2/n(n-1)} \right\}; \quad (40)$$

from (12)

$$= \frac{1}{m_1 + m_2} \bigcup_{(\varepsilon_{i,j}, \varepsilon'_{i,j}) \in \zeta_{i,j;i < j}} \left\{ \left(\left(\prod_{i,j=1,i < j}^n \varepsilon_{i,j} \right)^{2/n(n-1)}, 1 - \left(\prod_{i,j=1,i < j}^n (1 - \varepsilon'_{i,j}) \right)^{2/n(n-1)} \right) \right\} \quad (41)$$

using (11), we have

$$DHFGMB^{m_1, m_2}(e_1, e_2, \dots, e_n) = \bigcup_{(\varepsilon_{i,j}, \varepsilon'_{i,j}) \in \zeta_{i,j;i < j}} \left\{ \left(1 - \left(1 - \left(\prod_{i,j=1,i < j}^n \varepsilon_{i,j} \right)^{2/n(n-1)} \right)^{1/(m_1 + m_2)}, \left(1 - \prod_{i,j=1,i < j}^n (1 - \varepsilon'_{i,j})^{2/n(n-1)} \right)^{1/(m_1 + m_2)} \right) \right\}; \quad (42)$$

the proof is completed. $\square$

Example 19. Let us consider three DHFE, $e_1 = (\{0.1\}, \{0.3\})$, $e_2 = (\{0.2, 0.4\}, \{0.1\})$, and $e_3 = (\{0.3\}, \{0.1, 0.2\})$. If $m_1 = m_2 = 1$, then

$$DHGBM^{1,1}(e_1, e_2, e_3) = \{(0.0660, 0.0726, 0.0784, 0.0799, 0.0863, 0.0932, 0.0950, 0.1027, 0.1132), (0.2150, 0.2223, 0.2294, 0.2369, 0.2436, 0.2500, 0.2569, 0.2629, 0.2688)\} \quad (43)$$

Theorem 20. Let $e_i = (\varepsilon_i, \varepsilon'_i)$, $(i = 1, 2, \dots, n)$ be the collection of DHFEs, then by theorem one has the following results:

- (1) If $e_i = (\varepsilon_i, \varepsilon'_i) = (\{0\}, \{0\}) \forall i = 1, 2, \dots, n$ then $DHFGMB = (\{0\}, \{0\})$
- (2) If $e_i = (\varepsilon_i, \varepsilon'_i) = (\{1\}, \{0\}) \forall i = 1, 2, \dots, n$ then $DHFGMB = (\{1\}, \{0\})$

- (3) If $e_i = (\varepsilon_i, \varepsilon'_i) = (\{0\}, \{1\}) \forall i = 1, 2, \dots, n$ then $DHFGMB = (\{0\}, \{1\})$

Next we discussed some special cases of DHFGMB by replacing the parameters m_1 and m_2 as follows.

Case 1. If $m_2 \rightarrow 0$ then by (35) we have

$$\begin{aligned} DHFGMB^{m_1, m_2}(e_1, e_2, \dots, e_n) &= \frac{1}{m_1 + m_2} \bigotimes_{i,j=1,i < j}^n ((m_1 e_i \oplus m_2 e_j) \otimes (m_1 e_j \oplus m_2 e_i))^{2/n(n-1)} = \frac{1}{m_1} \left(\bigotimes_{i,j=1,i < j}^n (m_1 e_i \otimes m_1 e_j)^{2/n(n-1)} \right) = \frac{1}{m_1} \\ &\cdot \bigcup_{(\varepsilon_i, \varepsilon'_i) \in e_i, (\varepsilon_j, \varepsilon'_j) \in e_j} \left\{ \left(\prod_{i,j=1,i < j}^n \left(\frac{(1 - (1 - \varepsilon_i)^{m_1})^{2/n(n-1)}}{(1 - (1 - \varepsilon_j)^{m_1})^{2/n(n-1)}} \right), 1 - \prod_{i,j=1,i < j}^n \left(1 - \left(\frac{(1 - \varepsilon_i^{m_1})^{2/n(n-1)}}{(1 - \varepsilon_j^{m_1})^{2/n(n-1)}} \right) \right) \right) \right\} \\ &= \bigcup_{(\varepsilon_i, \varepsilon'_i) \in e_i, (\varepsilon_j, \varepsilon'_j) \in e_j} \left\{ \left(1 - \left(1 - \prod_{i,j=1,i < j}^n \left(\frac{(1 - (1 - \varepsilon_i)^{m_1})^{2/n(n-1)}}{(1 - (1 - \varepsilon_j)^{m_1})^{2/n(n-1)}} \right) \right)^{1/m_1}, \left(1 - \prod_{i,j=1,i < j}^n \left(1 - \left(\frac{(1 - \varepsilon_i^{m_1})^{2/n(n-1)}}{(1 - \varepsilon_j^{m_1})^{2/n(n-1)}} \right) \right) \right)^{1/m_1} \right) \right\} \\ &= DHFGMB^{m_1, 0}(e_1, e_2, \dots, e_n) \end{aligned} \quad (44)$$

which is called generalized dual hesitant fuzzy geometric Bonferroni mean (GDHFGBM).

Case 2. If $m_1 = 1$ and $m_2 \rightarrow 0$ then (35) becomes

$$\begin{aligned} DHFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) &= \bigotimes_{i,j=1}^n (e_i \otimes e_j)^{2/n(n-1)} \\ &= \bigcup_{(\varepsilon_i, \varepsilon'_i) \in e_i, (\varepsilon_j, \varepsilon'_j) \in e_j} \left\{ \left(1 - \left(1 - \prod_{i,j=1, i < j}^n ((1 - (1 - \varepsilon_i))^2)^{2/n(n-1)} \right) \right. \right. \\ &\quad \left. \left(1 - \prod_{i,j=1, i < j}^n \left(1 - \left(1 - ((1 - \varepsilon'_i)^2)^{2/n(n-1)} \right) \right) \right) \right\} \\ &= DHFGBM^{1,0}(e_i) \end{aligned} \quad (45)$$

which is called dual hesitant fuzzy unit geometric Bonferroni mean.

Case 3. If $m_1 = 2$ and $m_2 \rightarrow 0$,

$$\begin{aligned} DHFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) &= \frac{1}{2} \bigotimes_{i,j=1}^n (2e_i \otimes 2e_j)^{2/n(n-1)} \\ &= \bigcup_{(\varepsilon_i, \varepsilon'_i) \in e_i, (\varepsilon_j, \varepsilon'_j) \in e_j} \left\{ \left(1 - \left(1 - \prod_{i,j=1, i < j}^n \left(\left(1 - (1 - \varepsilon_i)^2 \right)^{2/n(n-1)} \right)^{1/2} \right. \right. \right. \\ &\quad \left. \left(1 - \prod_{i,j=1, i < j}^n \left(1 - \left(1 - \left(\frac{1 - \varepsilon_i'^2}{1 - \varepsilon_j'^2} \right)^{2/n(n-1)} \right) \right) \right)^{1/2} \right) \right\} \\ &= DHFGBM^{2,0}(e_1, e_2, \dots, e_n) \end{aligned} \quad (46)$$

$DHFGBM^{2,0}(e_1, e_2, \dots, e_n)$ is called dual hesitant fuzzy square geometric Bonferroni mean.

Case 4. If $m_1 = 1$ and $m_2 = 1$, let

$$\begin{aligned} \zeta_{i,j,i < j}^{1,1} &= (e_i \oplus e_j) \otimes (e_j \oplus e_i) \\ &= \bigcup_{(\varepsilon_{i,j}^{1,1}, \varepsilon_{i,j}^{'1,1}) \in \zeta_{i,j}^{1,1}, i < j} \{(\varepsilon_{i,j}^{1,1}, \varepsilon_{i,j}^{'1,1})\}, \end{aligned} \quad (47)$$

then (35) reduces to the following:

$$\begin{aligned} DHFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) &= \frac{1}{2} \bigotimes_{i,j=1, i < j}^n ((e_i \oplus e_j) \otimes (e_j \\ &\quad \oplus e_i))^{2/n(n-1)} = \frac{1}{2} \bigcup_{(\varepsilon_{i,j}^{1,1}, \varepsilon_{i,j}^{'1,1}) \in \zeta_{i,j}^{1,1}, i < j} \left\{ \left(\prod_{i,j=1, i < j}^n \varepsilon_{i,j}^{1,1}, 1 - \prod_{i,j=1, i < j}^n (1 - \varepsilon_{i,j}^{'1,1}) \right) \right\}^{2/n(n-1)} \\ &= \frac{1}{2} \bigcup_{(\varepsilon_{i,j}^{1,1}, \varepsilon_{i,j}^{'1,1}) \in \zeta_{i,j}^{1,1}, i < j} \left\{ \left(\left(\prod_{i,j=1, i < j}^n \varepsilon_{i,j}^{1,1} \right)^{2/n(n-1)}, 1 - \left(1 - \left(1 - \prod_{i,j=1, i < j}^n (1 - \varepsilon_{i,j}^{'1,1}) \right) \right)^{2/n(n-1)} \right) \right\} \\ &= \bigcup_{(\varepsilon_{i,j}^{1,1}, \varepsilon_{i,j}^{'1,1}) \in \zeta_{i,j}^{1,1}, i < j} \left\{ \left(1 - \left(1 - \left(\prod_{i,j=1, i < j}^n \varepsilon_{i,j}^{1,1} \right)^{2/n(n-1)} \right)^{1/2}, \right. \right. \\ &\quad \left. \left(1 - \left(1 - \left(1 - \prod_{i,j=1, i < j}^n (1 - \varepsilon_{i,j}^{'1,1}) \right) \right)^{2/n(n-1)} \right)^{1/2} \right) \right\} \end{aligned} \quad (48)$$

which is called an dual hesitant fuzzy interrelated square geometric Bonferroni mean.

Corollary 21. If $e_i = (\varepsilon_i, \varepsilon'_i) = (\varepsilon, \varepsilon') = e$, such that $\varepsilon, \varepsilon' \in [0, 1]$ for all $i = 1, 2, \dots, n$, then for any $m_1, m_2 > 0$ is

$$\begin{aligned} DHFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) &= \frac{1}{m_1 + m_2} \\ &\cdot \bigotimes_{i,j=1, i < j}^n ((m_1 e_i \oplus m_2 e_j) \otimes (m_1 e_j \oplus m_2 e_i))^{2/n(n-1)} \\ &= \bigcup_{(\eta, \eta') \in \zeta} \left\{ \left(1 - (1 - \eta)^{1/(m_1 + m_2)}, (\eta')^{1/(m_1 + m_2)} \right) \right\} \end{aligned} \quad (49)$$

where $(\eta, \eta') \in \zeta$ and

$$\zeta = (m_1 e \oplus m_2 e) \otimes (m_1 e \oplus m_2 e) \quad (50)$$

Proof. From (11)-(14) and Theorem 13 we see that $\zeta_{i,j}$ is also an DHFE

$$\begin{aligned} \zeta_{i,j,i < j} &= (m_1 e_i \oplus m_2 e_j) \otimes (m_1 e_j \oplus m_2 e_i) \\ &= (m_1 e \oplus m_2 e) \otimes (m_1 e \oplus m_2 e) = \zeta \end{aligned} \quad (51)$$

Let n be total number of DHFEs and Y be the collection of all $\zeta_{i,j,i < j}$, then the total number of elements in Y is

$$O(Y) = \frac{n(n-1)}{2}; \quad (52)$$

take

$$\begin{aligned}
 DHFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) \\
 &= \frac{1}{m_1 + m_2} \left(\bigotimes_{\substack{i,j=1 \\ i < j}}^n (\zeta_{i,j})^{2/n(n-1)} \right) \\
 &= \frac{1}{m_1 + m_2} \left(\bigotimes_{\substack{i,j=1 \\ i < j}}^n (\zeta)^{2/n(n-1)} \right) \quad (53)
 \end{aligned}$$

and if $(\eta, \eta') \in \zeta$,

then

$$\begin{aligned}
 &= \frac{1}{m_1 + m_2} \bigcup_{(\eta, \eta') \in \zeta} \left\{ \left(\prod_{i,j=1}^n \eta, 1 \right. \right. \\
 &\quad \left. \left. - \prod_{i,j=1}^n (1 - \eta') \right)^{2/n(n-1)} \right\} = \frac{1}{m_1 + m_2} \\
 &\quad \cdot \bigcup_{(\eta, \eta') \in \zeta} \left\{ \left(\left(\prod_{i,j=1}^n \eta \right)^{2/n(n-1)}, 1 \right. \right. \\
 &\quad \left. \left. - \left(\prod_{i,j=1}^n (1 - \eta') \right)^{2/n(n-1)} \right) \right\} = \bigcup_{(\eta, \eta') \in \zeta} \left\{ \left(1 \right. \right. \\
 &\quad \left. \left. - \left(1 - \left(\prod_{i,j=1}^n \eta \right)^{2/n(n-1)} \right)^{1/(m_1 + m_2)} \right. \right. \\
 &\quad \left. \left. \left(1 - \left(\prod_{i,j=1}^n (1 - \eta') \right)^{2/n(n-1)} \right)^{1/(m_1 + m_2)} \right) \right\} \\
 &= \bigcup_{(\eta, \eta') \in \zeta} \left\{ \left(1 - \left(1 - \left(\prod_{i,j=1}^n \eta \right)^{2/n(n-1)} \right)^{1/(m_1 + m_2)} \right. \right. \\
 &\quad \left. \left. \left(1 - \left(\prod_{i,j=1}^n (1 - \eta') \right)^{2/n(n-1)} \right)^{1/(m_1 + m_2)} \right) \right\}
 \end{aligned} \quad (54)$$

by equation (52),

$$\prod_{i,j=1}^n \eta = \eta^{n(n-1)/2},$$

$$\prod_{i,j=1}^n (1 - \eta) = (1 - \eta)^{n(n-1)/2}$$

therefore, $DHFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n)$

$$= \bigcup_{(\eta, \eta') \in \zeta} \left\{ \left(1 - \left(1 - \eta \right)^{1/(m_1 + m_2)}, \left(\eta' \right)^{1/(m_1 + m_2)} \right) \right\} \quad (55)$$

□

Corollary 22 (monotonicity). Let $(\eta_{\alpha_i}, \eta'_{\alpha_i}) \in e_{\alpha_i}$ and $(\eta_{\beta_i}, \eta'_{\beta_i}) \in e_{\beta_i}$ be two collections of DHFE; also $\varepsilon_{\alpha_i} \in \eta_{\alpha_i}$, $\varepsilon_{\beta_i} \in \eta_{\beta_i}$, $\varepsilon'_{\alpha_i} \in \eta'_{\alpha_i}$, and $\varepsilon'_{\beta_i} \in \eta'_{\beta_i}$ with the conditions $\varepsilon_{\alpha_i} \leq \varepsilon_{\beta_i}$ and $\varepsilon'_{\alpha_i} \geq \varepsilon'_{\beta_i}$ $\forall i, j = 1, 2, \dots, n, i \neq j$, then $DHFGBM^{m_1, m_2}(e_{\alpha_i}) \leq DHFGBM^{m_1, m_2}(e_{\beta_i})$.

Proof. By (24) and (42) we have

$$\begin{aligned}
 &1 - \left(1 - \left(\prod_{i,j=1, i < j}^n \varepsilon_{\alpha_{i,j}} \right)^{2/n(n-1)} \right)^{1/(m_1 + m_2)} \\
 &\leq 1 - \left(1 - \left(\prod_{i,j=1, i < j}^n \varepsilon_{\beta_{i,j}} \right)^{2/n(n-1)} \right)^{1/(m_1 + m_2)}
 \end{aligned} \quad (56)$$

$$\begin{aligned}
 &\left(1 - \prod_{i,j=1, i < j}^n (1 - \varepsilon'_{\alpha_{i,j}})^{2/n(n-1)} \right)^{1/(m_1 + m_2)} \\
 &\geq \left(1 - \prod_{i,j=1, i < j}^n (1 - \varepsilon'_{\beta_{i,j}})^{2/n(n-1)} \right)^{1/(m_1 + m_2)};
 \end{aligned} \quad (57)$$

by Definition 12 and (56) and (57) we have

$$\begin{aligned}
 &\frac{1}{m_1 + m_2} \bigotimes_{i,j=1}^n (\zeta_{\alpha_{i,j}})^{2/n(n-1)} = \bigcup_{(\varepsilon_{i,j}, \varepsilon'_{i,j}) \in \zeta_{i,j, i < j}} \left\{ \left(1 \right. \right. \\
 &\quad \left. \left. - \left(1 - \left(\prod_{i,j=1, i < j}^n \varepsilon_{\alpha_{i,j}} \right)^{2/n(n-1)} \right)^{1/(m_1 + m_2)} \right. \right. \\
 &\quad \left. \left. \left(1 - \prod_{i,j=1, i < j}^n (1 - \varepsilon'_{\alpha_{i,j}})^{2/n(n-1)} \right)^{1/(m_1 + m_2)} \right) \right\} \\
 &\leq \bigcup_{(\varepsilon_{i,j}, \varepsilon'_{i,j}) \in \zeta_{i,j, i < j}} \left\{ \left(1 \right. \right. \\
 &\quad \left. \left. - \left(1 - \left(\prod_{i,j=1, i < j}^n \varepsilon_{\beta_{i,j}} \right)^{2/n(n-1)} \right)^{1/(m_1 + m_2)} \right. \right. \\
 &\quad \left. \left. \left(1 - \prod_{i,j=1, i < j}^n (1 - \varepsilon'_{\beta_{i,j}})^{2/n(n-1)} \right)^{1/(m_1 + m_2)} \right) \right\}
 \end{aligned}$$

$$\left(1 - \prod_{i,j=1, i < j}^n (1 - \varepsilon'_{\beta_{i,j}})^{2/n(n-1)}\right)^{1/(m_1+m_2)} \Bigg\} \\ \leq \frac{1}{m_1 + m_2} \bigotimes_{i,j=1}^n (\zeta_{\beta_{i,j}})^{2/n(n-1)}; \quad (58)$$

hence

$$\frac{1}{m_1 + m_2} \bigotimes_{i,j=1}^n (\zeta_{\alpha_{i,j}})^{2/n(n-1)} \\ \leq \frac{1}{m_1 + m_2} \bigotimes_{i,j=1}^n (\zeta_{\beta_{i,j}})^{2/n(n-1)}; \quad (59)$$

the proof is completed. $\square$

Corollary 23 (bounded). Let $(\varepsilon_{\alpha_i}, \varepsilon'_{\alpha_i}) \in e_{\alpha_i}$ and $(\varepsilon_{\beta_i}, \varepsilon'_{\beta_i}) \in e_{\beta_i}$ be two collections of DHFEs, $e_i^- = \cup_{(\varepsilon_i, \varepsilon'_i) \in e_i} \{(\min\{\varepsilon_i\}, \max\{\varepsilon'_i\})\}$ and $e_i^+ = \cup_{(\varepsilon_i, \varepsilon'_i) \in e_i} \{(\max\{\varepsilon_i\}, \min\{\varepsilon'_i\})\}$; if $(\varepsilon^-, \varepsilon'^+) \in e_i^-$ and $(\varepsilon^+, \varepsilon'^-) \in e_i^+$, $i, j = 1, 2, \dots, n$; $i \neq j$ with the condition $\varepsilon^- \leq \varepsilon^+$ and $\varepsilon'^+ \geq \varepsilon'^-$, then for any $m_1, m_2 > 0$, one has

$$\bigcup_{(\varepsilon^-, \varepsilon'^+) \in e_i^-} \left\{ \left(\left(1 - \left(1 - \left(1 - \varepsilon^- \right)^{m_1+m_2} \right)^2 \right)^{1/(m_1+m_2)} \right), \right. \\ \left. \left(1 - \left(1 - \left(1 - \left(\varepsilon'^+ \right)^{m_1+m_2} \right)^2 \right)^{1/(m_1+m_2)} \right) \right\} \\ \leq DHFGBM^{m_1, m_2} \quad (60) \\ \leq \bigcup_{(\varepsilon^+, \varepsilon'^-) \in e_i^+} \left\{ \left(\left(1 - \left(1 - \left(1 - \varepsilon^+ \right)^{m_1+m_2} \right)^2 \right)^{1/(m_1+m_2)} \right), \right. \\ \left. \left(1 - \left(1 - \left(1 - \left(\varepsilon'^- \right)^{m_1+m_2} \right)^2 \right)^{1/(m_1+m_2)} \right) \right\}$$

Proof. By (27) and (42) we have

$$\left\{ \left(\left(1 - \left(1 - \left(1 - \varepsilon^- \right)^{m_1+m_2} \right)^2 \right)^{1/(m_1+m_2)} \right), \left(1 - \left(1 - \left(1 - \left(\varepsilon'^+ \right)^{m_1+m_2} \right)^2 \right)^{1/(m_1+m_2)} \right) \right\} \leq \left\{ \left(1 - \left(1 - \left(\prod_{i,j=1, i < j}^n \varepsilon_{\beta_{i,j}} \right)^{2/n(n-1)} \right)^{1/(m_1+m_2)} \right), \left(1 - \prod_{i,j=1, i < j}^n \left(1 - \varepsilon'_{\beta_{i,j}} \right)^{2/n(n-1)} \right)^{1/(m_1+m_2)} \right\} \leq \left\{ \left(1 - \left(1 - \left(1 - \left(1 - \varepsilon^+ \right)^{m_1+m_2} \right)^2 \right)^{1/(m_1+m_2)} \right), \left(1 - \left(1 - \left(1 - \left(\varepsilon'^- \right)^{m_1+m_2} \right)^2 \right)^{1/(m_1+m_2)} \right) \right\}$$

$$\left(1 - \left(1 - \left(1 - \left(1 - \varepsilon^+ \right)^{m_1+m_2} \right)^2 \right)^{1/(m_1+m_2)} \right), \left(1 - \left(1 - \left(1 - \left(\varepsilon'^- \right)^{m_1+m_2} \right)^2 \right)^{1/(m_1+m_2)} \right) \right\} \quad (61)$$

By (27) and (61) we have the required result. $\square$

Corollary 24. Let $e_i = (\eta_i, \eta'_i)$ ($i = 1, 2, \dots, n$) be collection of DHFEs. Let $\varepsilon_i = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p) \in \eta_i$, $\varepsilon'_i = (\varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_q) \in \eta'_i$, $m_1, m_2 \in \mathbb{N}$, $\varepsilon_i = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p) \in \eta_i$, and $\varepsilon'_i = (\varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_q) \in \eta'_i$, where “.” represents any permutations of ε_i and ε'_i , respectively, then

$$DHFGBM^{m_1, m_2}(e_i) = DHFGBM^{m_1, m_2}(e'_i) \quad (62)$$

where $\zeta_{i,j;i < j} = (m_1 e_i \oplus m_2 e_j) \otimes (m_1 e_j \oplus m_2 e_i)$ and $\zeta_{i,j;i < j} = (m_1 e'_i \oplus m_2 e'_j) \otimes (m_1 e'_j \oplus m_2 e'_i)$ $i, j = 1, 2, \dots, n, i \neq j$.

Proof. Given that $e_i = (\eta_i, \eta'_i)$ and $\varepsilon_i \in \eta_i$, $\varepsilon'_i \in \eta'_i$, $m_1, m_2 \in \mathbb{N}$, $\varepsilon_i \in \eta_i$, and $\varepsilon'_i \in \eta'_i$, then

$$DHFGBM^{m_1, m_2}(\eta_i, \eta'_i) \\ = \frac{1}{m_1 + m_2} \left(\bigotimes_{i,j=1}^n (\zeta_{i,j;i < j})^{2/n(n-1)} \right) \\ = \frac{1}{m_1 + m_2} \left(\bigotimes_{i,j=1}^n (\zeta_{i,j;i < j})^{2/n(n-1)} \right) \\ = DHFGBM^{m_1, m_2}(\eta_i, \eta'_i) \quad (63)$$

$\square$

Corollary 25. Let $e_i = (\eta_i, \eta'_i)$ ($i = 1, 2, \dots, n$) be collection of DHFEs, where $\varepsilon_i \in \eta_i$, $\varepsilon'_i \in \eta'_i$ with $\varepsilon_i = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p)$ and $\varepsilon'_i = (\varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_q)$, $m_1, m_2 \in \mathbb{N}$, then

$$DHFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) \\ = DHFGBM^{m_2, m_1}(e_1, e_2, \dots, e_n) \quad (64)$$

Proof. One has

$$DHFGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) \\ = \frac{1}{m_1 + m_2} \left(\bigotimes_{i,j=1, i < j}^n ((m_1 e_i \oplus m_2 e_j) \otimes (m_1 e_j \oplus m_2 e_i))^{2/n(n-1)} \right)$$

$$= \frac{1}{m_2 + m_1} \left(\bigotimes_{i,j=1;i < j}^n ((m_2 e_i \oplus m_1 e_j)) \right. \\ \left. \otimes (m_2 e_j \oplus m_1 e_i) \right)^{2/n(n-1)} = DHFGMB^{m_2, m_1}(e_1, \\ e_2, \dots, e_n) \quad (65)$$

which is required proof. $\square$

In practical applications often, we have to deal with complicated situations, considering not only the importance of individual arguments but also the relations among them. In order to consider the connections among criteria in multicriteria decision-making problems, we developed a dual hesitant fuzzy weighted geometric Bonferroni mean (DHFWGBM).

Definition 26. Let e_i ($i = 1, 2, \dots, n$) be a collection of DHFE and $w = (w_i)$ be the weight vector of e_i , where w_i indicates the importance degree of e_i , satisfying $w_i > 0$ ($i = 1, 2, \dots, n$) and $\sum_{i=1}^n w_i = 1$. Then dual hesitant fuzzy weighted geometric Bonferroni mean (DHFWGBM) for $m_1, m_2 > 0$ is

$$A_w^{\sim i, j: i < j} = \left\{ \zeta_{\sigma_{i,j}(1)}^w, \zeta_{\sigma_{i,j}(2)}^w, \dots, \zeta_{\sigma_{i,j}(n-1)}^w \right\} \quad \text{such that } \zeta_{i,j:i < j}^w \notin A^{\sim i, j: i < j} \text{ and } \zeta_{\sigma_{i,j}(l-1)}^w \geq \zeta_{\sigma_{i,j}(l)}^w, \quad \forall l = 2, 3, \dots, n-1 \quad (67)$$

and

$$B_{\sigma_{i,j}(k)}^w = \emptyset \quad \text{when } k = 1 \text{ and} \\ B_{\sigma_{i,j}(k)}^w = \zeta_{\sigma_{i,j}(l): l \leq k}^w \quad \text{when } k \geq 2 \quad (68)$$

If $E = \{e_1, e_2, \dots, e_n\}$ is the set of criteria and $E^{\sim i, j} = E - \{e_i, e_j\}$ is the set of all criteria except e_i and e_j , then a fuzzy measure $\omega_{i,j:i < j}$ on $E^{\sim i, j}$ is

$$\omega_{i,j:i < j}(E^{\sim i, j}) \longrightarrow [0, 1] \quad (69)$$

which satisfied the following conditions:

- (1) $\omega_{i,j:i < j}(\emptyset) = 0$
- (2) $\omega_{i,j:i < j}(E^{\sim i, j}) = 1$
- (3) $\omega_{i,j:i < j}(G) \leq \omega_{i,j:i < j}(H)$

If $\omega_{i,j:i < j}$ is a fuzzy measure on $E^{\sim i, j}$ defined in (69) and if we take

$$a_w^{\sim i, j: i < j} = (m_1 e_i^{w_i} \oplus m_2 e_j^{w_j}) \otimes (m_1 e_j^{w_j} \oplus m_2 e_i^{w_i}) \quad (70)$$

$$DHFWGBM^{m_1, m_2}(e_1, e_2, \dots, e_n) \\ = \frac{1}{m_1 + m_2} \left(\bigotimes_{i,j=1;i < j}^n ((m_1 e_i^{w_i} \oplus m_2 e_j^{w_j})) \right. \\ \left. \otimes (m_1 e_j^{w_j} \oplus m_2 e_i^{w_i}) \right)^{2/n(n-1)} \quad (66)$$

5. Choquet Integral and Geometric Bonferroni Mean

As Bonferroni mean has the subjective characteristics to capture interrelationships among arguments, Choquet integral has ability to represent objective characteristics of arguments. Sometimes in multicriteria decision-making problems, it should be required to take the choice of the decision makers; the Choquet integral can deal with these situations; i.e., it is possible to take not only the importance of each criteria but also the correlations of criteria. Therefore, both techniques are often used in practical situations that why in this research paper we merged the DHFGMB with the Choquet integral as follows.

Let

then

$$\mu_{i,j:i < j}^w = \bigoplus_{i,j=1,i \neq j}^n a_w^{\sim i, j: i < j} \\ \cdot \left(\omega_{i,j:i < j} \left(B_{\sigma_{i,j:i < j}(k)}^w \right) - \omega_{i,j:i < j} \left(B_{\sigma_{i,j:i < j}(k-1)}^w \right) \right) \quad (71)$$

where $a_w^{\sim i, j: i < j} \in A^{\sim i, j: i < j}$ and $k = 2, 3, \dots, n$.

Then the Choquet integral of $a_w^{\sim i, j: i < j}$ with respect to fuzzy measure $\omega_{i,j:i < j}$ is defined as follows:

$$C_{\omega_{i,j:i < j}}^w(a_w^{\sim i, j: i < j}) = \frac{\sum_{k_2=1}^{n_2} \varepsilon_{i,j:i < j}^{t_{w_2}}}{n_2(2 - \varphi(a_w^{\sim i, j: i < j}))} \\ \text{for } i, j = 1, 2, \dots, n \quad (72)$$

$$\text{where, } \varphi(a_w^{\sim i, j: i < j}) = \frac{1}{2} \left(\frac{\sum_{k_1=1}^{n_1} \varepsilon_{i,j:i < j}^{t_{w_1}}}{n_1} + \frac{\sum_{k_2=1}^{n_2} \varepsilon_{i,j:i < j}^{t_{w_2}}}{n_2} \right)$$

where $(\varepsilon_{i,j:i < j}^{t_{w_1}}, \varepsilon_{i,j:i < j}^{t_{w_2}}) \in \mu_{i,j:i < j}^w$

Definition 27. Let e_i ($i = 1, 2, \dots, n$) be a collection of DHFE and $w = (w_i)$ be the weight vector of e_i , where w_i indicates the importance degree of e_i , satisfying $w_i > 0$ ($i = 1, 2, \dots, n$) and $\sum_{i=1}^n w_i = 1$. Then dual hesitant fuzzy Choquet geometric Bonferroni mean (DHFCGBM) is as follows for any $m_1, m_2 > 0$:

$$DHFCGBM_w^{m_1, m_2}(e_1, e_2, \dots, e_n) = \frac{1}{m_1 + m_2} \left(\bigotimes_{i,j=1:i < j}^n \left(((m_1 e_i^{w_i} \oplus m_2 e_j^{w_j}) \otimes (m_1 e_j^{w_j} \oplus m_2 e_i^{w_i}))^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)} \right) \quad (73)$$

where $C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})$ be the Choquet integral defined in (72)

Theorem 28. Let $m_1, m_2 > 0$, $e_i = (\varepsilon_i, \varepsilon_i')$ for $i = 1, 2, \dots, n$ be the collection of DHFE, then aggregated value of DHFCGBM is also an DHFE and equal to

$$\begin{aligned} & DHFCGBM_w^{m_1, m_2}(e_1, e_2, \dots, e_n) \\ &= \frac{1}{m_1 + m_2} \bigcup_{\substack{(w_{i,j}, l_{w_{i,j}}) \in \zeta_{i,j;i < j}^{w_{i,j}} \\ (\varepsilon_{i,j}, \varepsilon_{i,j}') \in \zeta_{i,j;i < j}^{w_{i,j}}}} \left\{ \bigotimes_{i,j=1:i < j}^n \left(((\varepsilon_{i,j}^{w_{i,j}})^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)}, 1 - \left((1 - \varepsilon_{i,j}')^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)} \right) \right\} \end{aligned} \quad (74)$$

Proof. It is obvious from (11)-(14) and Theorem 13 that

$$\zeta_{i,j;i < j}^{w_{i,j}} = (m_1 e_i^{w_i} \oplus m_2 e_j^{w_j}) \otimes (m_1 e_j^{w_j} \oplus m_2 e_i^{w_i}), \quad (75)$$

an DHFE. Therefore from (73) and (75) we have

$$\begin{aligned} & DHFCGBM_w^{m_1, m_2}(e_1, e_2, \dots, e_n) = \frac{1}{m_1 + m_2} \bigotimes_{i,j=1:i < j}^n \left(((m_1 e_i^{w_i} \oplus m_2 e_j^{w_j}) \otimes (m_1 e_j^{w_j} \oplus m_2 e_i^{w_i}))^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)} \\ &= \frac{1}{m_1 + m_2} \bigotimes_{i,j=1:i < j}^n \left((\zeta_{i,j;i < j}^{w_{i,j}})^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)} = \frac{1}{m_1 + m_2} \\ &\cdot \bigcup_{\substack{(w_{i,j}, l_{w_{i,j}}) \in \zeta_{i,j;i < j}^{w_{i,j}} \\ (\varepsilon_{i,j}, \varepsilon_{i,j}') \in \zeta_{i,j;i < j}^{w_{i,j}}}} \left\{ \bigotimes_{i,j=1:i < j}^n \left(((\varepsilon_{i,j}^{w_{i,j}})^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)}, 1 - \left((1 - \varepsilon_{i,j}')^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)} \right) \right\} = \frac{1}{m_1 + m_2} \\ &\cdot \bigcup_{\substack{(w_{i,j}, l_{w_{i,j}}) \in \zeta_{i,j;i < j}^{w_{i,j}} \\ (\varepsilon_{i,j}, \varepsilon_{i,j}') \in \zeta_{i,j;i < j}^{w_{i,j}}}} \left\{ \bigotimes_{i,j=1:i < j}^n \left(((\varepsilon_{i,j}^{w_{i,j}})^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)}, 1 - \left((1 - \varepsilon_{i,j}')^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)} \right) \right\} = \frac{1}{m_1 + m_2} \\ &\cdot \bigcup_{\substack{(w_{i,j}, l_{w_{i,j}}) \in \zeta_{i,j;i < j}^{w_{i,j}} \\ (\varepsilon_{i,j}, \varepsilon_{i,j}') \in \zeta_{i,j;i < j}^{w_{i,j}}}} \left\{ \left(\prod_{i,j=1:i < j}^n ((\varepsilon_{i,j}^{w_{i,j}})^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)}, 1 - \prod_{i,j=1:i < j}^n \left((1 - \varepsilon_{i,j}')^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)} \right) \right\} \\ &= \bigcup_{\substack{(w_{i,j}, l_{w_{i,j}}) \in \zeta_{i,j;i < j}^{w_{i,j}} \\ (\varepsilon_{i,j}, \varepsilon_{i,j}') \in \zeta_{i,j;i < j}^{w_{i,j}}}} \left\{ \left(1 - \prod_{i,j=1:i < j}^n \left((\varepsilon_{i,j}^{w_{i,j}})^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)} \right)^{1/(m_1+m_2)}, \left(1 - \prod_{i,j=1:i < j}^n \left((1 - \varepsilon_{i,j}')^{C_{w_{i,j;i < j}}^w(a_w^{i,j;i < j})} \right)^{2/n(n-1)} \right)^{1/(m_1+m_2)} \right) \right\} \end{aligned} \quad (76)$$

□

6. An Approach to Multicriteria Decision Making with CI Based GBM

In this section, we developed a method by applying *DHFWGBM* and *DHFCGBM* operators for multicriteria decision making. There are m decision makers (*DMs*) who want to select the best alternative from the collection of P_i ($i = 1, 2, \dots, n$) options; $W = \{w_1, w_2, \dots, w_k\}$ are the weights for C_k ($k = 1, 2, \dots, p$) criteria to be considered by *DMs*. *DMs* used *DHFS* for their opinion for each alternative; i.e., *DMs* predilections are in the form of optimistic felling with hesitant situations (membership value) and denial felling also with hesitant situations (nonmembership value). We now introduced the following methodology to deal with the above discussed situation. The following steps are required to find the best choice among the participated alternatives.

Step 1. Firstly represent each *DMs* opinion in matrix form, called dual hesitant fuzzy matrix (*DHFM*) $D_j = [(\{\epsilon_{r_1}\}, \{\epsilon_{r_2}\})]_{n \times p}$ $r_1, r_2 \in \mathbb{N}$, n and p represented the total numbers of alternatives and criteria, respectively. Let w_m be the weight vector for *DMs* and w_c be weight vector for criteria.

Step 2. Find a matrix $D_{n \times p}$ by aggregating each corresponding element of D_j ($j = 1, 2, \dots, m$) matrices by aggregation operator *DHFWGBM* and also taking aggregation weights for resultant element by calculating simple arithmetic mean on the weights w_c suggested by each *DMs* for every criteria.

Step 3. Aggregate all values of each alternative in the matrix $D_{n \times p}$ by *DHFCGBM* and then find $\psi_i(P_i) \forall i = 1, 2, \dots, n$.

Step 4. The best alternative is $\max(\psi_i)$.

6.1. Illustrative Example. The following example illustrated the suggested method.

Example 29. Energy is an essential part for the growth of social-economic development of any society. Thus it is very important to adopt proper energy policy for economic development and environment. Let us assume that there are five alternatives (energy projects) P_i ($i = 1, 2, 3, 4, 5$) to be invested and three criteria to be considered: C_1 —technological; C_2 —environmental; C_3 —economic. Two *DMs* are requested to evaluate the performances of the five alternatives. To deal with this energy policy problem, consider weights according to preference of *DMs* for each alternative in group decision making.

Case 1.

Step 1. The *DMs* provide their preferences over all the alternatives P_i ($i = 1, 2, 3, 4, 5$) w.r.t. the criteria C_k ($k = 1, 2, 3$) based on *DHFS*; let weight vectors for the criteria be $w_{c_1} = \{0.4, 0.3, 0.3\}$ and $w_{c_2} = \{0.3, 0.4, 0.3\}$, with respect to *DMs*, respectively. The following D_j ($j = 1, 2$) represented the suggestion of *DMs*, in *DHFE*:

$$D_1 = \begin{bmatrix} (\{0.3, 0.6\}, \{0.3\}) & (\{0.6\}, \{0.3\}) & (\{0.7\}, \{0.2\}) \\ (\{0.6\}, \{0.3\}) & (\{0.3, 0.5\}, \{0.4\}) & (\{0.7\}, \{0.1, 0.2\}) \\ (\{0.4\}, \{0.3, 0.5\}) & (\{0.6\}, \{0.2, 0.3\}) & (\{0.5\}, \{0.5\}) \\ (\{0.6, 0.7\}, \{0.1, 0.2\}) & (\{0.4\}, \{0.5\}) & (\{0.6\}, \{0.3\}) \\ (\{0.6\}, \{0.3\}) & (\{0.5, 0.6\}, \{0.3\}) & (\{0.8\}, \{0.1\}) \end{bmatrix} \quad (77)$$

$$D_2 = \begin{bmatrix} (\{0.4\}, \{0.4\}) & (\{0.6, 0.7\}, \{0.1, 0.2\}) & (\{0.4\}, \{0.5\}) \\ (\{0.6, 0.7\}, \{0.1, 0.2\}) & (\{0.6\}, \{0.3\}) & (\{0.3, 0.4\}, \{0.6\}) \\ (\{0.5, 0.6\}, \{0.3\}) & (\{0.4, 0.5\}, \{0.3\}) & (\{0.5, 0.7\}, \{0.2, 0.3\}) \\ (\{0.4\}, \{0.5\}) & (\{0.4\}, \{0.2\}) & (\{0.3, 0.4\}, \{0.5\}) \\ (\{0.6, 0.7\}, \{0.1, 0.2\}) & (\{0.4\}, \{0.4, 0.5\}) & (\{0.3, 0.4\}, \{0.4\}) \end{bmatrix} \quad (78)$$

Step 2. Apply aggregation operator *DHFWGBM* as taking $w_{p_1} = 0.5 = w_{p_2}$ on D_j ($j = 1, 2$). The following $D_{5 \times 3}$ matrix

represented aggregation values of each alternative P_i ($i = 1, 2, \dots, 5$):

$$D_{5 \times 3} = \begin{bmatrix} (\{0.6621, 0.7075, 0.7633\}, \{0.1571\}) & (\{0.8151, 0.8284, 0.8428\}, \{0.0748, 0.0926, 0.1074\}) & (\{0.7989\}, \{0.1440\}) \\ (\{0.8152, 0.8284, 0.8484\}, \{0.0748, 0.0925, 0.1074\}) & (\{0.7370, 0.7614, 0.7890\}, \{0.1572\}) & (\{0.7767, 0.7874, 0.7989\}, \{0.1123, 0.1391, 0.1614\}) \\ (\{0.7296, 0.7458, 0.7633\}, \{0.1337, 0.1582, 0.1793\}) & (\{0.7633, 0.7757, 0.7890\}, \{0.1074, 0.1212, 0.1337\}) & (\{0.7591, 0.7874, 0.8201\}, \{0.1439, 0.1626, 0.1793\}) \\ (\{0.7633, 0.7803, 0.7989\}, \{0.1002, 0.1241, 0.1440\}) & (\{0.6963\}, \{0.1074\}) & (\{0.7367, 0.7498, 0.7633\}, \{0.1793\}) \\ (\{0.8152, 0.8284, 0.8428\}, \{0.0748, 0.0925, 0.1074\}) & (\{0.7296, 0.7458, 0.7633\}, \{0.1571, 0.1686, 0.1793\}) & (\{0.8209, 0.8295, 0.8386\}, \{0.1572\}) \end{bmatrix} \quad (78)$$

FIGURE 1: Graphical representation of case 1.

Step 3. Aggregate all values of each alternative, by applying DHFCGBM on $D_{5 \times 3}$ and find $\psi(P_1) = 0.002201$, $\psi(P_2) = 0.001646$, $\psi(P_3) = 0.002166$, $\psi(P_4) = 0.002195$, and $\psi(P_5) = 0.002020$.

Step 4. $\psi(P_1) > \psi(P_4) > \psi(P_3) > \psi(P_5) > \psi(P_2) \Rightarrow P_1 > P_4 > P_3 > P_5 > P_2$.

Hence project P_1 is the best choice and P_2 is the worst choice in the given scenario.

Case 2.

Step 1. Repeat Step 1 as in Case 1.

Step 2. Now apply aggregation operator DHFWGBM as taking $w_{p_1} = 0.3$ and $w_{p_2} = 0.7$ on D_j ($j = 1, 2$). The following $D_{5 \times 3}$ matrix represented aggregation values of each alternative P_i ($i = 1, 2, \dots, 5$):

$D_{5 \times 3}$

$$= \begin{bmatrix} (\{0.6867, 0.7314, 0.7870\}, \{0.1430\}) & (\{0.8306, 0.8432, 0.8570\}, \{0.0677, 0.0841, 0.09778\}) & (\{0.8203\}, \{0.1290\}) \\ (\{0.8306, 0.8472, 0.8570\}, \{0.0677, 0.0841, 0.0978\}) & (\{0.7515, 0.7765, 0.8050\}, \{0.1450\}) & (\{0.8087, 0.8113, 0.8203\}, \{0.0992, 0.1233, 0.1433\}) \\ (\{0.7498, 0.7640, 0.7792\}, \{0.1225, 0.1464, 0.1669\}) & (\{0.7870, 0.7973, 0.8083\}, \{0.0978, 0.1108, 0.1225\}) & (\{0.7792, 0.8042, 0.8381\}, \{0.1348, 0.1517, 0.1669\}) \\ (\{0.7870, 0.8029, 0.8203\}, \{0.0893, 0.1111, 0.1291\}) & (\{0.7218\}, \{0.09903\}) & (\{0.7661, 0.7763, 0.7870\}, \{0.1618\}) \\ (\{0.8306, 0.8423, 0.8549\}, \{0.0694, 0.0855, 0.0990\}) & (\{0.7547, 0.7702, 0.7870\}, \{0.1430, 0.1527, 0.1618\}) & (\{0.8427, 0.8495, 0.8567\}, \{0.1430\}) \end{bmatrix} \quad (79)$$

Step 3. Aggregate all values of each alternative, by applying DHFCGBM on updated $D_{5 \times 3}$ and calculated new values $\psi(P_1) = 0.00159$, $\psi(P_2) = 0.00139$, $\psi(P_3) = 0.00180$, $\psi(P_4) = 0.00177$, and $\psi(P_5) = 0.00167$.

Step 4. $\psi(P_3) > \psi(P_4) > \psi(P_5) > \psi(P_1) > \psi(P_2) \Rightarrow P_3 > P_4 > P_5 > P_1 > P_2$.

Hence project P_3 is the best choice and P_2 is the worst choice, when D_2 has some priority with D_1 .

7. Conclusion

In this paper, we defined a ranking method for DHFE. DHFE is a generalization of both intuitionistic and hesitant fuzzy elements so it inherited properties of both intuitionistic and hesitant fuzzy sets. In the article dual hesitant fuzzy geometric Bonferroni mean developed and proved the fact that DHFGBM is also DHFE. Several particular cases and properties of DHFGBM have been examined, defined weighted dual hesitant fuzzy geometric Bonferroni mean. In

the article we discussed a fuzzy measure for dual hesitant fuzzy environment and then defined Choquet integral for DHFE, Choquet integral combined with DHFGBM and defined dual hesitant fuzzy Choquet geometric Bonferroni mean. A multicriteria decision-making technique developed based on the aggregation operators is defined in the article. An illustrated example is also discussed to understand the developed multicriteria decision-making method. We also observed and saw in Figures 1 and 2 that when a decision maker dominates others opinion, then overall results change accordingly.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

FIGURE 2: Graphical representation of case 2.

References

- [1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
- [2] R. E. Bellman and L. A. Zadeh, "Decision-making in a fuzzy environment," *Management Science*, vol. 17, pp. B141–B164, 1970/71.
- [3] Z. Bashir, T. Rashid, and M. Iqbal, "Hesitant Probabilistic Fuzzy PReference Relations in Decision Making," *Mathematical Problems in Engineering*, Art. ID 5383487, 24 pages, 2018.
- [4] G. Beliakov, H. Bustince, D. P. Goswami, U. K. Mukherjee, and N. R. Pal, "On averaging operators for Atanassov's intuitionistic fuzzy sets," *Information Sciences*, vol. 181, no. 6, pp. 1116–1124, 2011.
- [5] I. Beg and T. Rashid, "Group Decision Making Using Intuitionistic Hesitant Fuzzy Sets," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 14, no. 3, pp. 181–187, 2014.
- [6] I. Beg and T. Rashid, "Hesitant 2-tuple linguistic information in multiple attributes group decision making," *Journal of Intelligent and Fuzzy Systems*, vol. 30, no. 1, Article ID 151737, pp. 109–116, 2015.
- [7] T.-Y. Chen, "Bivariate models of optimism and pessimism in multi-criteria decision-making based on intuitionistic fuzzy sets," *Information Sciences*, vol. 181, no. 11, pp. 2139–2165, 2011.
- [8] D. Dubois, "The role of fuzzy sets in decision sciences: old techniques and new directions," *Fuzzy Sets and Systems*, vol. 184, pp. 3–28, 2011.
- [9] T. Rashid, I. Beg, and S. M. Husnine, "Robot selection by using generalized interval-valued fuzzy numbers with TOPSIS," *Applied Soft Computing*, vol. 21, pp. 462–468, 2014.
- [10] C. Wei, P. Wang, and Y. Zhang, "Entropy, similarity measure of interval-valued intuitionistic fuzzy sets and their applications," *Information Sciences*, vol. 181, no. 19, pp. 4273–4286, 2011.
- [11] Z. S. Xu, "Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators," *Knowledge-Based Systems*, vol. 24, no. 6, pp. 749–760, 2011.
- [12] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, NY, USA, 1980.
- [13] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.
- [14] V. Torra, "Hesitant fuzzy sets," *International Journal of Intelligent Systems*, vol. 25, no. 6, pp. 529–539, 2010.
- [15] B. Zhu, Z. Xu, and M. Xia, "Dual hesitant fuzzy sets," *Journal of Applied Mathematics*, Art. ID 879629, 13 pages, 2012.
- [16] K. Atanassov and G. Gargov, "Interval valued intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 31, no. 3, pp. 343–349, 1989.
- [17] K. T. Atanassov, *Intuitionistic Fuzzy Sets*, Physica-Verlag, New York, NY, USA, 1999.
- [18] J. Liu and M. Sun, "Generalized power average operator of hesitant fuzzy numbers and its application in multiple attribute decision making," *Journal of Computational Information Systems*, vol. 9, no. 8, pp. 3051–3058, 2013.
- [19] T. Rashid and S. M. Husnine, "Multicriteria group decision making by using trapezoidal valued hesitant fuzzy sets," *The Scientific World Journal*, vol. 2014, Article ID 304834, 8 pages, 2014.
- [20] M. Xia, Z. Xu, and N. Chen, "Some hesitant fuzzy aggregation operators with their application in group decision making," *Group Decision and Negotiation*, vol. 22, no. 2, pp. 259–279, 2013.
- [21] Z. Xu and X. Zhang, "Hesitant fuzzy multi-attribute decision making based on TOPSIS with incomplete weight information," *Knowledge-Based Systems*, vol. 52, pp. 53–64, 2013.
- [22] D. Yu, Y. Wu, and W. Zhou, "Multi-criteria decision making based on Choquet integral under hesitant fuzzy environment," *Journal of Computational Information Systems*, vol. 7, pp. 4506–4513, 2011.
- [23] Z. Zhang, "Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making," *Information Sciences*, vol. 234, pp. 150–181, 2013.
- [24] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap, *Aggregation Functions*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2009.
- [25] E. Pap, *Null-Additive Set Functions*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1995.
- [26] R. R. Yager, "On generalized Bonferroni mean operators for multi-criteria aggregation," *International Journal of Approximate Reasoning*, vol. 50, no. 8, pp. 1279–1286, 2009.
- [27] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning—I," *Information Sciences*, vol. 8, pp. 199–249, 1975.
- [28] M. Xia, Z. Xu, and B. Zhu, "Geometric Bonferroni means with their application in multi-criteria decision making," *Knowledge-Based Systems*, vol. 40, pp. 88–100, 2013.
- [29] Z. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *International Journal of General Systems*, vol. 35, no. 4, pp. 417–433, 2006.
- [30] W. Zhou and J.-M. He, "Intuitionistic fuzzy geometric bonferroni means and their application in multicriteria decision making," *International Journal of Intelligent Systems*, vol. 27, no. 12, pp. 995–1019, 2012.
- [31] M. Xia and Z. Xu, "Hesitant fuzzy information aggregation in decision making," *International Journal of Approximate Reasoning*, vol. 52, no. 3, pp. 395–407, 2011.
- [32] B. Zhu, Z. Xu, and M. Xia, "Hesitant fuzzy geometric Bonferroni means," *Information Sciences*, vol. 205, pp. 72–85, 2012.

Submit your manuscripts at
www.hindawi.com