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Research Article

Improved Generalized H_2 Filtering for Static Neural Networks with Time-Varying Delay via Free-Matrix-Based Integral Inequality

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This paper focuses on the generalized H_2 filtering of static neural networks with a time-varying delay. The aim of this problem is to design a full-order filter such that the filtering error system is globally asymptotically stable with guaranteed H_2 performance index. By constructing an augmented Lyapunov-Krasovskii functional and applying the free-matrix-based integral inequality to estimate its derivative, an improved delay-dependent condition for the generalized H_2 filtering problem is established in terms of LMIs. Finally, a numerical example is presented to show the effectiveness of the proposed method.

1. Introduction

During the past few decades, neural network (NN), which models the way a biological brain solving problems, has been successfully applied to various engineering fields, such as pattern recognition and vehicle control [1]. Because of its powerful features in learning ability, data processing, function approximation, and adaptiveness, it has gained increasing attention and become a hot topic for scholars [2], while, due to the finite information processing speed, time delays inevitably occur in many NNs and possibly lead to poor performances and complex dynamical behaviors [3]. Thus, extensive researches have been addressed for NNs with time-varying delays.

It is well known that, by choosing the external states of neurons or the internal states of neurons as basic variables, a neural network can be usually classified as a static neural network or a local field neural network [4]. Although these two types of neural network can be equivalent under some assumptions, in many applications, these assumptions cannot always be satisfied. A unified model which combines these two systems together has been constructed recently [5].

Since NNs usually are some highly interconnected networks with a great deal of neurons, it may be very hard or even impossible to acquire all neurons state information completely [6]. However, in some practical applications, to achieve some desired objectives, it is needed to know the neurons state information or estimate it in advance [7]. And the design of filter, which aims to estimate the states of a system via its output measurement, provides a method for the above problems. Consequently, it is of great practical interest to study the filtering problems of NNs with time-varying delays.

For NNs with time-varying delays, the filtering problem was firstly investigated in [8]; both H_∞ and generalized H_2 filter were obtained by solving a set of LMIs. Since then, much effort has been made and many results have been reported on this issue. In [9], the delay-dependent H_2 filter of the Luenberger form was derived for delayed switched Hopfield NNs. Considering that there is only one gain matrix in the Luenberger-type filter to be determined, which may introduce some restrictions to certain extent, [10] further studied the H_2 filtering problem of delayed static NNs based on an Arcak-type filter. Since it contains two gain matrices, the Arcak-type filter is regarded as an extension of the Luenberger-type filter. In addition, another generalized H_2 full-order filter was designed for a class of delayed stochastic NNs in [11], in which three gain matrices were involved.

On the other hand, with the rapid development of the Lyapunov-Krasovskii functional (LKF) theory, plenty of techniques developed for delay systems have been applied to the filter design of delayed NNs, in order to reduce the conservatism of the derived conditions. The condition with less conservatism means it can provide larger feasible regions such that the filtering error system is globally asymptotically stable with an optimal performance index. It has been well recognized that constructing a suitable LKF and effectively estimating its derivative are two key points to reduce the conservatism. In early years, the simple LKFs combined with the Jensen inequality is considered to be the most popular method for the filtering problems of delayed NNs [8]. In order to consider more information about time delays, some triple integral terms were introduced into the LKF in [10]. In [12], the delay-decomposition idea was used to derive the delay-dependent condition such that the filtering error system is globally asymptotically stable with a guaranteed H_∞ performance. Based on the works of [13], less conservative filtering design results were obtained via the free-weighting matrix approach [14]. And by introducing a additional zero equation into the negative-definite conditions of LKFs' derivative, [15] derived a suitable H_∞ filter for the static NNs with mixed time-varying delays. Very recently, [16] achieved the less conservative generalized H_2 performance state estimation via an augmented LKF, Wirtinger integral inequality, and the auxiliary function-based integral inequality.

Although research on the filtering problem of delayed NNs is in progress, due to its importance, approaches or results on this topic are lacking compared with the stability analysis. For instance, some new proposed techniques, which can effectively reduce the conservatism of the derived stability criteria, have not been applied to the filtering problem of delayed NNs, to mention a few, the free-matrix-based integral inequality [17], the relaxed integral inequality [18], and the generalized free-weighting-matrix approach [19].

Due to this consideration, this paper further studies the filtering problem of NNs with a time-varying delay. The main purpose of this paper is to develop a less conservative approach for the generalized H_2 filter design. To tackle this problem, a more general Arcak-type filter is adopted firstly. Then, by employing the LKF theory and using the free-matrix-based integral inequality, some other bounding inequalities and the free-weighting matrix approach, new delay-dependent condition is derived to guarantee the asymptotic stability of the filtering error system of delayed static NNs with guaranteed H_2 performance index. Finally, a numerical example is provided to illustrate the effectiveness of our proposed method.

Notations. Throughout this paper, the notations are standard. $\mathscr{R}^n$ is the set of all $n \times n$ real matrices; $P > 0$ (≥ 0) means that P is a real symmetric and positive-definite (semi-positive-definite) matrix; diag$\{\cdots\}$ denotes a block-diagonal matrix; I and 0 represent the identity matrix and the zero-matrix, respectively; the symmetric term in a symmetric matrix is denoted by $*$; and $\mathrm{Sym}\{X\} = X + X^T$.

2. Preliminaries

Consider the following delayed static NN with external disturbance:

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + f(Wx(t-d(t)) + J) + B_1\omega(t), \\ y(t) &= Cx(t) + Dx(t-d(t)) + B_2\omega(t), \\ z(t) &= Hx(t), \\ x(t) &= \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathscr{R}^n$ is the neuron state vector, $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))]^T$ is the neural activation functions, $\omega(t)$ is the external disturbance belonging to $L_2[0,\infty)$, $y(t) \in \mathscr{R}^p$ is the network output, $z(t)$ is the linear combination of states to be estimated, $J = [J_1, J_2, \ldots, J_n]^T$ is an exogenous input vector, A, W, B_1, B_2, C, D, and H are known real matrices with appropriate dimensions, $\phi(t)$ is the initial function, and $d(t)$ is time-varying delay satisfying:

$$\begin{aligned} 0 \leq d(t) &\leq h, \\ \dot{d}(t) &\leq \mu, \end{aligned} \tag{2}$$

where h and μ are known scalars.

Assumption 1. For each $i = 1, 2, \ldots, n$, there exist real scalars $l_i > 0$ such that the continuous activation function $f_i(\cdot)$ satisfies

$$0 \leq \frac{f_i(a) - f_i(b)}{a - b} \leq l_i, \quad a \neq b \in \mathscr{R}. \tag{3}$$

A full-order Arcak-type filter for the delayed static NN (1) is constructed as

$$\begin{aligned} \dot{\hat{x}}(t) &= -A\hat{x}(t) \\ &\quad + f\left(W\hat{x}(t-d(t)) + J + K_1\left(y(t) - \hat{y}(t)\right)\right) \\ &\quad + K_2\left(y(t) - \hat{y}(t)\right), \\ \hat{y}(t) &= C\hat{x}(t) + D\hat{x}(t-d(t)), \\ \hat{z}(t) &= H\hat{x}(t), \\ \hat{x}(t) &= 0, \quad t \in [-h, 0], \end{aligned} \tag{4}$$

where $\hat{x}(t) = [\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_n(t)]^T \in \mathscr{R}^n$ and matrices K_1, K_2 are the gains of the filter to be determined.

Define the error signals as $e(t) = x(t) - \hat{x}(t)$ and $z_e(t) = z(t) - \hat{z}(t)$. Then, the filtering error system is expressed as follows:

$$\begin{aligned} \dot{e}(t) &= -\left(A + K_2C\right)e(t) - K_2De(t-d(t)) + g(t) \\ &\quad + \left(B_1 - K_2B_2\right)\omega(t), \\ z_e(t) &= He(t), \end{aligned} \tag{5}$$

where $g(t) = f(Wx(t-d(t)) + J) - f(W\hat{x}(t-d(t)) + J + K_1(y(t) - \hat{y}(t)))$.

This paper aims to present a less conservative approach for the H_2 filtering problem of delayed static NN (1). Toward this problem, the following definition is indispensable.

Definition 2. For given $\gamma > 0$, the H_2 filtering problem is said to be solved if a suitable full-order filter (4) can be found such that

(1) the filtering error system (5) with $\omega(t) = 0$ is globally asymptotically stable;

(2) the H_2 performance $\|z_e(t)\|_\infty \le \gamma\|\omega(t)\|_2$ is guaranteed under zero initial conditions for all nonzero $\omega(t) \in \mathscr{L}_2[0,\infty)$, where $\|z_e(t)\|_\infty = \sup_t \sqrt{z_e^T(t) z_e(t)}$ and $\|\omega(t)\|_2 = \sqrt{\int_0^\infty \omega^T(t)\omega(t)dt}$.

Before proceeding, we present the following lemmas, which will be used in the sequel.

Lemma 3 (free-matrix-based integral inequality [17]). *Let ω be a differentiable signal in $[\alpha,\beta] \to \mathscr{R}^n$; for positive definite matrices $R \in \mathscr{R}^{n\times n}$, $X, Z \in \mathscr{R}^{3n\times 3n}$, and any matrices $Y \in \mathscr{R}^{3n\times 3n}$, $M, N \in \mathscr{R}^{3n\times n}$ satisfying*

$$\begin{bmatrix} X & Y & M \\ * & Z & N \\ * & * & R \end{bmatrix} \ge 0 \tag{6}$$

the following inequality holds:

$$-\int_\beta^\alpha \dot{\omega}^T(s) R \dot{\omega}(s)\, ds \le \varpi^T \widehat{\Omega} \varpi, \tag{7}$$

where

$$\begin{aligned} \widehat{\Omega} &= (\alpha-\beta)\left(X + \frac{1}{3}Z\right) + \operatorname{Sym}\{MG_1 + NG_2\}, \\ G_1 &= [I, -I, O], \\ G_2 &= [-I, -I, 2I], \\ \varpi &= \left[\omega^T(\alpha), \omega^T(\beta), \frac{1}{\alpha-\beta}\int_\beta^\alpha \omega^T(s)\, ds\right]^T. \end{aligned} \tag{8}$$

Lemma 4 (Jensen's inequality [20]). *Let ω be a differentiable signal in $[\alpha,\beta] \to \mathscr{R}^n$; for positive definite matric $R \in \mathscr{R}^n$, the following inequalities hold:*

$$\begin{aligned} &\frac{(\alpha-\beta)^2}{2}\int_\beta^\alpha\int_s^\alpha \omega^T(s) R\omega(s)\, ds\, d\theta \\ &\ge \left(\int_\beta^\alpha\int_s^\alpha \omega(s)\, ds\, d\theta\right)^T R \left(\int_\beta^\alpha\int_s^\alpha \omega(s)\, ds\, d\theta\right). \end{aligned} \tag{9}$$

3. Main Result

In this section, by employing an augmented LKF and using the free-matrix-based integral inequality to estimate its derivative, an improved delay-dependent condition is derived such that the filtering error system (5) is globally asymptotically stable with guaranteed generalized H_2 performance indexes.

Theorem 5. *For given scalars h and μ, the H_2 filtering problem of NN (1) with time-varying delay satisfying (2) is solved with $\gamma_{\min}$ if there exist positive definite matrices $P = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} \in \mathscr{R}^{2n}$, $Q_1, Q_2, R, Z \in \mathscr{R}^n$, positive definite diagonal matrices $\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \in \mathscr{R}^n$, symmetric matrices $X_1, X_3, Y_1, Y_3, Z_1, Z_3 \in \mathscr{R}^{3n}$, and any matrices $X_2, Y_2, Z_2 \in \mathscr{R}^{3n}$, $M_1, M_2, N_1, N_2, G_1, G_2 \in \mathscr{R}^{3n\times n}$, $T \in \mathscr{R}^n$, $V_1, V_2 \in \mathscr{R}^{n\times p}$, satisfying the following LMIs:*

$$\begin{bmatrix} X_1 & X_2 & M_1 \\ * & X_3 & M_2 \\ * & * & R \end{bmatrix} \ge 0, \tag{10}$$

$$\begin{bmatrix} Y_1 & Y_2 & N_1 \\ * & Y_3 & N_2 \\ * & * & R \end{bmatrix} \ge 0, \tag{11}$$

$$\begin{bmatrix} Z_1 & Z_2 & G_1 \\ * & Z_3 & G_2 \\ * & * & Z \end{bmatrix} \ge 0, \tag{12}$$

$$\begin{bmatrix} P_1 & P_2 & H^T \\ * & P_3 & O \\ * & * & \gamma^2 I \end{bmatrix} \ge 0, \tag{13}$$

$$\begin{aligned} \Gamma(0) &< 0, \\ \Gamma(h) &< 0, \end{aligned} \tag{14}$$

where

$$\begin{aligned} \Gamma(d(t)) &= \Phi_1(d(t)) + \Phi_2 + \Phi_3(d(t)) + \Phi_4(d(t)) \\ &\quad + \Omega, \\ \Phi_1(d(t)) &= \operatorname{Sym}\{\Pi_1^T P \Pi_2\}, \\ \Phi_2 &= e_1^T(Q_1+Q_2)e_1 - (1-\mu)e_2^T Q_1 e_2 - e_3^T Q_2 e_3, \\ \Phi_3(d(t)) &= he_7^T R e_7 + d(t)\Pi_3^T\left(X_1 + \frac{1}{3}X_3\right)\Pi_3 + (h \\ &\quad - d(t))\Pi_4^T\left(Y_1 + \frac{1}{3}Y_3\right)\Pi_4 + \operatorname{Sym}\{\Pi_3^T M \Pi_5 \\ &\quad + \Pi_4^T N \Pi_6\}, \\ \Phi_4(d(t)) &= \frac{h^2}{2}e_7^T Z e_7 - 2(e_1-e_4)^T Z(e_1-e_4) \\ &\quad - 2(e_2-e_5)^T Z(e_2-e_5) + h(h-d(t))\Pi_3^T\left(Z_1\right. \\ &\quad \left.+ \frac{1}{3}Z_3\right)\Pi_3 + (h-d(t))\operatorname{Sym}\{\Pi_3^T G \Pi_5\}, \end{aligned}$$

$$\Omega = \mathrm{Sym}\left\{e_6^T \Lambda \left(LWe_2 - e_6\right) - e_6^T V_1 \left(Ce_1 + De_2 + B_2 e_8\right) + \left(e_1 + e_7\right)^T \cdot T\left(e_7 + Ae_1 - e_6 - B_1 e_8\right) + \left(e_1 + e_7\right)^T \cdot V_2 \left(Ce_1 + De_2 + B_2 e_8\right)\right\},$$
$$\Pi_1 = \left[e_1^T, d(t) e_4^T + (h - d(t)) e_5^T\right]^T,$$
$$\Pi_2 = \left[e_7^T, e_1^T - e_3^T\right]^T,$$
$$\Pi_3 = \left[e_1^T, e_2^T, e_4^T\right]^T,$$
$$\Pi_4 = \left[e_2^T, e_3^T, e_5^T\right]^T,$$
$$\Pi_5 = \left[e_1^T - e_2^T, -e_1^T - e_2^T + 2e_4^T\right]^T,$$
$$\Pi_6 = \left[e_2^T - e_3^T, -e_2^T - e_3^T + 2e_5^T\right]^T,$$
$$M = \left[M_1, M_2\right],$$
$$N = \left[N_1, N_2\right],$$
$$G = \left[G_1, G_2\right],$$
$$e_i = \left[0_{n\times(i-1)n}, I_n, 0_{n\times(7-i)n}, 0_{n\times p}\right], \quad (i = 1, 2, \ldots, 7),$$
$$e_8 = \left[0_{p\times 7n}, I_p\right],$$
$$L = \mathrm{diag}\left\{l_1, l_2, \ldots, l_n\right\},$$
$$\xi(t) = \left[e^T(t), e^T(t - d(t)), e^T(t - h), \int_{t-d(t)}^{t} \frac{e^T(s)}{d(t)} ds, \int_{t-h}^{t-d(t)} \frac{e^T(s)}{h - d(t)} ds, g^T(t), \dot{e}^T(t), \omega^T(t)\right]^T \tag{15}$$

and the gain matrices K_1, K_2 of the filter of (4) can be designed as

$$K_1 = (\Lambda L)^{-1} V_1, \qquad K_2 = T^{-1} V_2. \tag{16}$$

Proof. Let us construct a new augmented LKF candidate as

$$V(e_t) = \sum_{i=1}^{4} V_i(e_t), \tag{17}$$

where

$$V_1(e_t) = \begin{bmatrix} x(t) \\ \int_{t-h}^{t} e(s)\, ds \end{bmatrix}^T P \begin{bmatrix} x(t) \\ \int_{t-h}^{t} e(s)\, ds \end{bmatrix},$$
$$V_2(e_t) = \int_{t-d(t)}^{t} e^T(s) Q_1 e(s)\, ds + \int_{t-h}^{t} e^T(s) Q_2 e(s)\, ds,$$
$$V_3(e_t) = \int_{-h}^{0} \int_{t+\theta}^{t} \dot{e}^T(s) R \dot{e}(s)\, ds\, d\theta,$$
$$V_4(e_t) = \int_{-h}^{0} \int_{\theta}^{t} \int_{t+u}^{t} \dot{e}^T(s) Z \dot{e}(s)\, ds\, du\, d\theta. \tag{18}$$

Taking the derivative of $V_1(e_t)$ along the filtering error system (5) yields

$$\dot{V}_1(e_t) = 2 \begin{bmatrix} x(t) \\ \int_{t-h}^{t} e(s)\, ds \end{bmatrix}^T P \begin{bmatrix} \dot{x}(t) \\ e(t) - e(t-h) \end{bmatrix} = \xi^T(t) \Phi_1(d(t)) \xi(t). \tag{19}$$

By the time derivative of $V_2(e_t)$, we get

$$\dot{V}_2(e_t) = e^T(t)(Q_1 + Q_2) e(t) - \left(1 - \dot{d}(t)\right) e^T(t - d(t)) Q_1 e(t - d(t)) - e^T(t-h) Q_2 e(t-h) = \xi^T(t) \Phi_2 \xi(t). \tag{20}$$

Calculating the derivative of $V_3(e_t)$, we have

$$\dot{V}_3(e_t) = h\dot{e}^T(t) R \dot{e}(t) - \int_{t-d(t)}^{t} \dot{e}^T(s) R \dot{e}(s)\, ds - \int_{t-h}^{t-d(t)} \dot{e}^T(s) R \dot{e}(s)\, ds. \tag{21}$$

Then by applying Lemma 3 to estimate the above R-dependent integral term, if LMIs (10) and (11) hold, one has

$$-\int_{t-d(t)}^{t} \dot{e}^T(s) R \dot{e}(s)\, ds \le d(t) \chi_1^T \left(X_1 + \frac{1}{3} X_3\right) \chi_1 + \mathrm{Sym}\left\{\chi_1^T M_1 \chi_2 + \chi_1^T M_2 \chi_3\right\}, \tag{22}$$

$$-\int_{t-h}^{t-d(t)} \dot{e}^T(s) R \dot{e}(s)\, ds \le (h - d(t)) \chi_4^T \left(Y_1 + \frac{1}{3} Y_3\right) \chi_4 + \mathrm{Sym}\left\{\chi_4^T N_1 \chi_5 + \chi_4^T N_2 \chi_6\right\}, \tag{23}$$

where

$$\chi_1 = \left[e^T(t), e^T(t - d(t)), \frac{1}{d(t)} \int_{t-d(t)}^{t} e^T(s)\, ds\right]^T,$$
$$\chi_2 = e(t) - e(t - d(t)),$$

$$\chi_3 = -e(t) - e(t-d(t)) + \frac{2}{d(t)} \int_{t-d(t)}^{t} e(s)\,ds,$$

$$\chi_4 = \left[e^T(t-d(t)), e^T(t-h), \frac{1}{h-d(t)} \cdot \int_{t-h}^{t-d(t)} e^T(s)\,ds\right]^T,$$

$$\chi_5 = e(t-d(t)) - e(t-h),$$

$$\chi_6 = -e(t-d(t)) - e(t-h) + \frac{2}{h-d(t)} \cdot \int_{t-h}^{t-d(t)} e(s)\,ds. \tag{24}$$

Combining (21) with (23), it is clear that

$$\dot{V}_3(e_t) \le \xi^T(t)\,\Phi_3(d(t))\,\xi(t). \tag{25}$$

Finally, the derivative of $V_4(e_t)$ can be obtained as

$$\begin{aligned}\dot{V}_4(e_t) &= \frac{h^2}{2}\dot{e}^T(t) Z \dot{e}(t) - \int_{t-h}^{t}\int_{\theta}^{t} \dot{e}^T(s) Z \dot{e}(s)\,ds\,d\theta \\ &= \frac{h^2}{2}\dot{e}^T(t) Z \dot{e}(t) \\ &\quad - \int_{t-d(t)}^{t}\int_{\theta}^{t} \dot{e}^T(s) Z \dot{e}(s)\,ds\,d\theta \\ &\quad - \int_{t-h}^{t-d(t)}\int_{\theta}^{t-d(t)} \dot{e}^T(s) Z \dot{e}(s)\,ds\,d\theta \\ &\quad - (h-d(t)) \int_{t-d(t)}^{t} \dot{e}^T(s) Z \dot{e}(s)\,ds.\end{aligned} \tag{26}$$

For the first double integral term in $\dot{V}_4(e_t)$, we can do the following treatment based on Lemma 4:

$$\begin{aligned}&- \int_{t-d(t)}^{t}\int_{\theta}^{t} \dot{e}^T(s) Z \dot{e}(s)\,ds\,d\theta \\ &\le -\frac{2}{d^2(t)} \left(\int_{t-d(t)}^{t}\int_{\theta}^{t} \dot{e}(s)\,ds\,d\theta\right)^T \\ &\quad \cdot Z \left(\int_{t-d(t)}^{t}\int_{\theta}^{t} \dot{e}(s)\,ds\,d\theta\right) \\ &= -\frac{2}{d^2(t)} \left(d(t)e(t) - \int_{t-d(t)}^{t} e(s)\,ds\right)^T \\ &\quad \cdot Z \left(d(t)e(t) - \int_{t-d(t)}^{t} e(s)\,ds\right) \\ &= -2\left(e(t) - \int_{t-d(t)}^{t} \frac{e(s)}{d(t)}ds\right)^T \\ &\quad \cdot Z \left(e(t) - \int_{t-d(t)}^{t} \frac{e(s)}{d(t)}ds\right).\end{aligned} \tag{27}$$

Similarly, the estimation of the second integral term can be done as

$$\begin{aligned}&- \int_{t-h}^{t-d(t)}\int_{\theta}^{t-d(t)} \dot{e}^T(s) Z \dot{e}(s)\,ds\,d\theta \\ &\le -\frac{2}{(h-d(t))^2} \left(\int_{t-h}^{t-d(t)}\int_{\theta}^{t-d(t)} \dot{e}(s)\,ds\,d\theta\right)^T \\ &\quad \cdot Z \left(\int_{t-h}^{t-d(t)}\int_{\theta}^{t-d(t)} \dot{e}(s)\,ds\,d\theta\right) \\ &= -2\left(e(t-d(t)) - \int_{t-h}^{t-d(t)} \frac{e(s)}{h-d(t)}ds\right)^T \\ &\quad \cdot Z \left(e(t-d(t)) - \int_{t-h}^{t-d(t)} \frac{e(s)}{h-d(t)}ds\right).\end{aligned} \tag{28}$$

And by utilizing Lemma 3, if LMI (12) holds, we obtain

$$\begin{aligned}&- (h-d(t)) \int_{t-d(t)}^{t} \dot{e}^T(s) Z \dot{e}(s)\,ds \le (h-d(t)) \\ &\quad \cdot \Big(h\chi_1^T \left(Z_1 + \frac{1}{3}Z_3\right)\chi_1 \\ &\quad + \mathrm{Sym}\left\{\chi_1^T G_1 \chi_2 + \chi_1^T G_2 \chi_3\right\}\Big).\end{aligned} \tag{29}$$

From (26) to (29), we get

$$\dot{V}_4(e_t) \le \xi^T(t)\,\Phi_4(d(t))\,\xi(t). \tag{30}$$

Then, taking assumption of the activation function (3) into account yields

$$\begin{aligned}0 \le 2g^T(t)\Lambda\,[&L(We(t-d(t)) - K_1Ce(t) \\ &- K_1De(t-d(t)) - K_1B_2\omega(t)) - g(t)],\end{aligned} \tag{31}$$

where

$$\begin{aligned}V_1 &= \Lambda L K_1 \\ &\Longrightarrow 0 \\ &\le 2e_6^T\Lambda(LWe_2 - e_6) - 2e_6^T V_1 (Ce_1 + De_2 + B_2e_8).\end{aligned} \tag{32}$$

Based on the filtering error system (5), for any appropriately dimensioned matrix T, the following equation holds:

$$\begin{aligned}0 = 2(e(t) + \dot{e}(t))^T T\,[&\dot{e}(t) + (A + K_2C)e(t) \\ &+ K_2De(t-d(t)) - g(t) - (B_1 - K_2B_2)\omega(t)],\end{aligned} \tag{33}$$

where

$$\begin{aligned}V_2 &= TK_2 \\ &\Longrightarrow 0 \\ &= 2(e_1 + e_7)^T T(e_7 + Ae_1 - e_6 - B_1e_8) \\ &\quad + 2(e_1 + e_7)^T V_2 (Ce_1 + De_2 + B_2e_8).\end{aligned} \tag{34}$$

Combining (19), (20), (21), (22), (23), (25), (26), (27), (28), (29), (30), (32), and (34), one can easily yield

$$\dot{V}(e_t) \leq \xi^T(t)\,\Gamma(d(t))\,\xi(t). \tag{35}$$

If $\Gamma(d(t)) < 0$, then $\dot{V}(e_t) < 0$. Thus, if $\Gamma(0) < 0$ and $\Gamma(h) < 0$ are satisfied, then $\dot{V}(e_t) \leq -\epsilon\|e(t)\|^2$ for a sufficient small scalar $\epsilon > 0$, which implies the filtering error system (5) with $\omega(t) = 0$, is asymptotically stable.

Considering the definition of H_2 performance, we define

$$J(t) = V(e_t) - \int_0^t \omega^T(s)\,\omega(s)\,ds. \tag{36}$$

Then, under the zero-initial conditions $V(e_t)|_{t=0} = 0$ and $V(e_t) > 0$ for $t > 0$, the following inequality is given:

$$\begin{aligned} J(t) &= V(e_t) - V(e_t)\big|_{t=0} - \int_0^t \omega^T(s)\,\omega(s)\,ds \\ &= \int_0^t \left[\dot{V}(e(s)) - \omega^T(s)\,\omega(s)\right] ds < 0. \end{aligned} \tag{37}$$

That is,

$$V(e_t) < \int_0^t \omega^T(s)\,\omega(s)\,ds. \tag{38}$$

In addition, it follows that LMI (13) and Schur complement that

$$\begin{bmatrix} H^T H & O \\ * & O \end{bmatrix} \leq \gamma^2 \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix}. \tag{39}$$

Then, the following holds:

$$\begin{aligned} z_e^T(t)\,z_e(t) &= e^T(t)\,H^T H e(t) \\ &= \begin{bmatrix} x(t) \\ \int_{t-h}^t e(s)\,ds \end{bmatrix}^T \begin{bmatrix} H^T H & O \\ * & O \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-h}^t e(s)\,ds \end{bmatrix} \\ &\leq \gamma^2 \begin{bmatrix} x(t) \\ \int_{t-h}^t e(s)\,ds \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-h}^t e(s)\,ds \end{bmatrix} \\ &= \gamma^2 V_1(e_t) \leq \gamma^2 V(e_t) \leq \gamma^2 \int_0^t \omega^T(s)\,\omega(s)\,ds \\ &\leq \gamma^2 \int_0^\infty \omega^T(s)\,\omega(s)\,ds. \end{aligned} \tag{40}$$

Taking the supremum over $t > 0$, one has $\|z_e(t)\|_\infty^2 \leq \gamma\|\omega(t)\|_2^2$, which means $\|z_e(t)\|_\infty \leq \gamma\|\omega(t)\|_2$. Therefore, when (10)–(14) hold, the generalized H_2 filtering problem of NN (1) with time-varying delay satisfying (2) is solved with $\gamma_{\min}$. This completes the proof. □

Remark 6. The generalized H_2 design of a type of static NN with a time-varying delay is solved in Theorem 5. Different from [8–10, 12–15], we solve the filtering problem based on an Arcak-type filter, in which there are two gain matrices to be determined. Compared with the widely used Luenberger-type filter, it contains an additional gain matrix K_1 included in the activation function. So, it can be expected to lead a better generalized H_2 performance. And if we set $K_1 = 0$, the Arcak-type filter is then reduced to the Luenberger-type filter.

Remark 7. The triple integral term is introduced into the LKF in this paper. When bounding its derivative, the derived double integral term $-\int_{t-h}^t \int_\theta^t \dot{e}^T(s) Z \dot{e}(s)\,ds\,d\theta$ is divided into three parts, two double integral terms and a single integral term. By using the double Jensen inequality to estimate the double integral terms and using the free-matrix-based integral inequality to estimate the single integral term, the derivation of the triple term is accomplished without ignoring any terms. It should be pointed out that this technique is more effective than the ones used in [10], in which the derived double integral term $-\int_{t-h}^t \int_\theta^t \dot{e}^T(s) Z \dot{e}(s)\,ds\,d\theta$ is estimated in its entirety.

Remark 8. To reduce the conservatism of the derived conditions, Wirtinger integral inequality was used to estimate single integral terms in [16], while in the presented study, the free-matrix-based integral inequality is employed. The use of the free-matrix-based integral inequality brings several advantages for Theorem 5. On the one hand, by introducing some free matrices, it gives more degrees of freedom to the derived condition. On the other hand, it has been proved in [17] that the free-matrix-based integral inequality contains Wirtinger integral inequality as a special case. That means, it can provide a tighter bound than Wirtinger integral inequality. And if the same LKF is chosen, our designed filter will be better than the one designed in [16].

Remark 9. There is some room for further improvement of our proposed method:

(i) Notice that when the free-matrix-based integral inequality is employed to improve the proposed condition, the number of decision variables is increased, which further increases the computation complexity of the condition. It has been pointed out in [3] that the calculation complexity is also an important consideration during the application of the proposed LMI-based condition to physical systems. Thus, the results considering both the conservatism and the computation complexity need more investigation.

(ii) Recently, some new approaches have been developed for the stability analysis of delayed neural networks, which can be applied into the generalized H_2 filtering problem of neural networks, such as the delay-decomposition approach [21] and the free-matrix-based double integral inequality [22].

Notice that, in many applications, the derivative bound of the time delay is unknown. By setting $Q_1 = 0$, we can easily derive the delay-dependent but delay derivative-independent condition of the generalized H_2 filtering problem of neural network (1) based on Theorem 5.

Theorem 10. *For given scalars h, the H_2 filtering problem of NN (1) with time-varying delay satisfying $0 \leq d(t) \leq h$ is solved with $\gamma_{\min}$ if there exist positive definite matrices $P = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} \in \mathscr{R}^{2n}$, $Q_2, R, Z \in \mathscr{R}^n$, positive definite diagonal matrices $\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \in \mathscr{R}^n$, symmetric matrices $X_1, X_3, Y_1, Y_3, Z_1, Z_3 \in \mathscr{R}^{3n}$, and any matrices $X_2, Y_2, Z_2 \in \mathscr{R}^{3n}$, $M_1, M_2, N_1, N_2, G_1, G_2 \in \mathscr{R}^{3n\times n}$, $T \in \mathscr{R}^n$, $V_1, V_2 \in \mathscr{R}^{n\times p}$, satisfying LMIs (10)–(14).*

4. Numerical Examples

In this section, we provide a well-used numerical example to demonstrate the effectiveness of the proposed approach.

Example 1. Consider the delayed static NN (1) with the following parameters:

$$A = \begin{bmatrix} 0.96 & 0 & 0 \\ 0 & 1.22 & 0 \\ 0 & 0 & 0.78 \end{bmatrix},$$
$$W = \begin{bmatrix} -0.32 & 0.75 & -1.42 \\ 1.21 & 0.41 & -0.50 \\ 0.42 & 0.82 & -1.06 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix},$$
$$H = \begin{bmatrix} 0.8 & 0 & 0.5 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \tag{41}$$
$$C = \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & -0.5 & 0.6 \end{bmatrix},$$
$$D = \begin{bmatrix} 0 & -1.2 & 0.2 \\ 0 & 0 & 0.5 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0.4 \\ -0.3 \end{bmatrix}.$$

This example has been studied in [10, 16], and we take these literatures for comparison study. To verify the advantages of our proposed approach, we calculate the optimal H_2 performance bounds index $\gamma_{\min}$ for various h, μ, and L such that filtering error system (5) is globally asymptotically stable with generalized H_2 performance.

Firstly, let $L = \operatorname{diag}\{l_1, l_2, \ldots, l_n\} = I$; for given h and μ, the optimal generalized H_2 performance index $\gamma_{\min}$ derived by Theorem 5 and the ones reported in [10, 16] are given in Table 1, where (α, β) represents $h = \alpha$, $\mu = \beta$ and "infeasible" means that no feasible solution can be found in the corresponding criteria.

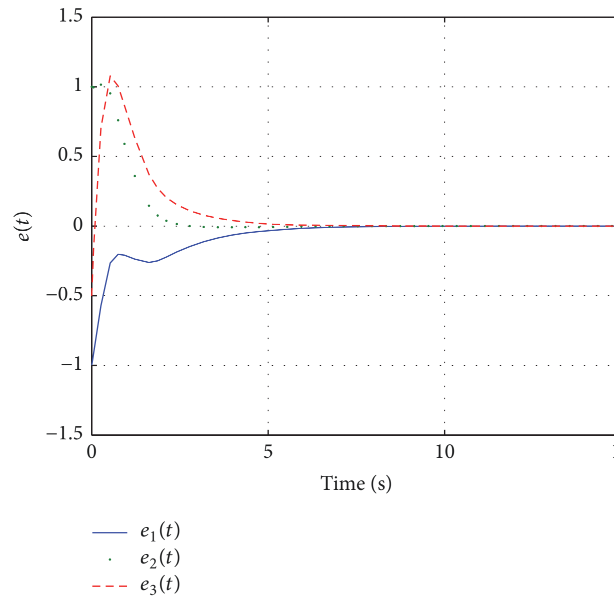

Figure 1: State response of the error $e(t)$.

It is clearly shown from Table 1 that, compared with [10, 16], much better generalized H_2 performance is achieved by Theorem 5 proposed in this paper. It should be noticed that, in this paper, we choose the same filter as [10, 16]; thus, the better H_2 performance effectively illustrates the superiority of our proposed approaches, which is mainly based on the free-matrix-based integral inequality.

Then let $f(x) = 1.57\tanh(x)$, $d(t) = 0.3\sin(1.5t) + 1.2$, and $w(t) = \sin(t)e^{-5t}$, for $h = 1.5$, $\mu = 0.3$; by solving the LMIs in Theorem 5, the gain matrices of our designed filter (4) are obtained as

$$K_1 = \begin{bmatrix} 1.0591 & 1.2990 \\ 0.1519 & 0.2687 \\ 2.4379 & 3.3466 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} 0.0944 & -0.6005 \\ 0.3315 & -0.5100 \\ 2.1756 & 1.2563 \end{bmatrix} \tag{42}$$

with the optimal generalized H_2 performance index $\gamma = 0.0505$. Figure 1 shows the state responses of error system (5) under initial condition $e(0) = [-1, 1, -0.5]^T$. The resulting responses obviously demonstrate the asymptotic stability of the simulated error system.

In addition, the optimal generalized H_2 performance index $\gamma_{\min}$ for $h = 0.5$, $\mu = 0.7$ and different L is listed in Table 2, from which one can see that Theorem 5 in this paper outperforms the ones in [10, 16].

Moreover, the optimal generalized H_2 performance index $\gamma_{\min}$ derived by Theorem 10 for $h = 0.8$, unknown μ, and different L is listed in Table 3.

Table 1: Optimal generalized H_2 performance index $\gamma_{\min}$ for $L = I$ and different (h, μ).

(h, μ)	$(1, 0.4)$	$(1.5, 0.3)$	$(3, 0.8)$	$(5, 1.2)$	$(6, 1.5)$
Theorem 1 [10]	0.7981	Infeasible	Infeasible	Infeasible	Infeasible
Theorem 2 [16]	1.34×10^{-4}	1.2763	Infeasible	Infeasible	Infeasible
Theorem 5	1.33×10^{-4}	0.0505	0.6639	0.9137	1.9523

Table 2: Optimal generalized H_2 performance index $\gamma_{\min}$ for $h = 0.5$, $\mu = 0.7$, and different L.

L	$1.6I$	$1.8I$	$2.5I$	$3.4I$
Theorem 1 [10]	3.3728	Infeasible	Infeasible	Infeasible
Theorem 2 [16]	0.5983	1.5563	Infeasible	Infeasible
Theorem 5	0.2765	0.4015	0.6896	4.7676

Table 3: Optimal generalized H_2 performance index $\gamma_{\min}$ for $h = 0.5$, unknown μ, and different L.

L	$1.6I$	$1.8I$	$2.5I$	$2.91I$
Theorem 10	0.5156	0.5868	1.0891	5.1122

5. Conclusion

In this paper, the generalized H_2 filtering problem of static NNs with a time-varying delay has been investigated. First, an Arcak-type filter has been constructed. Then, based on an augmented LKF, the free-matrix-based integral inequality, the free-weighting matrix approach, and Jensen inequality, a suitable delay-dependent condition expressed in terms of LMIs has been derived such that the filtering error system of the considered static NNs is globally asymptotically stable with guaranteed H_2 performance index. Moreover, by solving some coupled LMIs, the optimal performance index and the gain matrices of the designed H_2 filter have been obtained. The effectiveness and advantages of the proposed method have been demonstrated by an illustrative example finally.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding this paper.

References

[1] L. O. Chua and L. Yang, "Cellular Neural Networks: Applications," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 35, no. 10, pp. 1273–1290, 1988.

[2] X.-M. Zhang, Q.-L. Han, and X. Yu, "Survey on Recent Advances in Networked Control Systems," *IEEE Transactions on Industrial Informatics*, vol. 12, no. 5, pp. 1740–1752, 2016.

[3] C.-K. Zhang, Y. He, L. Jiang, and M. Wu, "Stability analysis for delayed neural networks considering both conservativeness and complexity," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 27, no. 7, pp. 1486–1501, 2016.

[4] H.-B. Zeng, Y. He, P. Shi, M. Wu, and S.-P. Xiao, "Dissipativity analysis of neural networks with time-varying delays," *Neurocomputing*, vol. 168, pp. 741–746, 2015.

[5] C.-K. Zhang, Y. He, L. Jiang, Q. H. Wu, and M. Wu, "Delay-dependent stability criteria for generalized neural networks with two delay components," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 25, no. 7, pp. 1263–1276, 2014.

[6] Y. He, Q. G. Wang, and C. Lin, "An improved H8 filter design for systems with time-varying interval delay," *IEEE Trans. Circuits Syst. II*, vol. 53, p. 1235, 2006.

[7] Y. He, G. P. Liu, D. Rees, and M. Wu, "Improved H∞ filtering for systems with a time-varying delay," *Circuits, Systems and Signal Processing*, vol. 29, no. 3, pp. 377–389, 2010.

[8] H. Huang and G. Feng, "Delay-dependent H∞ and generalized H2 filtering for delayed neural networks," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 56, no. 4, pp. 846–857, 2009.

[9] C. K. Ahn and M. K. Song, "L2-L∞ filtering for time-delayed switched hopfield neural networks," *International Journal of Innovative Computing, Information and Control*, vol. 7, no. 4, pp. 1831–1844, 2011.

[10] D. Hu, H. Huang, and T. Huang, "Design of Arcak-type generalized H2 filter for delayed static neural networks," *Circuits, Systems and Signal Processing*, vol. 33, no. 11, pp. 3635–3648, 2014.

[11] M. Hua, H. Tan, and J. Chen, "Delay-dependent H^{∞} and generalized H^2 filtering for stochastic neural networks with time-varying delay and noise disturbance," *Neural Computing and Applications*, vol. 25, no. 3-4, pp. 613–624, 2014.

[12] Y. J. Li and Y. N. Liang, "Delay-dependent H∞ filtering for neural networks with time delay," *Applied Mechanics and Materials*, vol. 511-512, pp. 875–879, 2014.

[13] Y. He, G. P. Liu, D. Rees, and M. Wu, "Stability analysis for neural networks with time-varying interval delay," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 18, no. 6, pp. 1850–1854, 2007.

[14] Y. Li, J. Li, and M. Hua, "New results of H8 filtering for neural network with time-varying delay," *Int. J. Innov. Comput. Inf. Control*, vol. 10, pp. 2309–2323, 2014.

[15] G. Liu, S. Zhou, X. Luo, and K. Zhang, "New filter design for static neural networks with mixed time-varying delays," *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics): Preface*, vol. 9772, pp. 117–129, 2016.

[16] Y. Shu, X.-G. Liu, Y. Liu, and J. H. Park, "Improved Results on Guaranteed Generalized H2 Performance State Estimation for Delayed Static Neural Networks," *Circuits, Systems and Signal Processing*, vol. 36, no. 8, pp. 3114–3142, 2017.

[17] H.-B. Zeng, Y. He, M. Wu, and J. She, "Free-matrix-based integral inequality for stability analysis of systems with time-varying delay," *Institute of Electrical and Electronics Engineers Transactions on Automatic Control*, vol. 60, no. 10, pp. 2768–2772, 2015.

[18] C.-K. Zhang, Y. He, L. Jiang, M. Wu, and H.-B. Zeng, "Stability analysis of systems with time-varying delay via relaxed integral inequalities," *Systems & Control Letters*, vol. 92, pp. 52–61, 2016.

[19] C.-K. Zhang, Y. He, L. Jiang, W.-J. Lin, and M. Wu, "Delay-dependent stability analysis of neural networks with time-varying delay: a generalized free-weighting-matrix approach," *Applied Mathematics and Computation*, vol. 294, pp. 102–120, 2017.

[20] J. Sun, G. P. Liu, J. Chen, and D. Rees, "Improved delay-range-dependent stability criteria for linear systems with time-varying delays," *Automatica*, vol. 46, no. 2, pp. 466–470, 2010.

[21] C. Ge, C. Hua, and X. Guan, "New delay-dependent stability criteria for neural networks with time-varying delay using delay-decomposition approach," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 25, no. 7, pp. 1378–1383, 2014.

[22] C. Hua, S. Wu, X. Yang, and X. Guan, "Stability analysis of time-delay systems via free-matrix-based double integral inequality," *International Journal of Systems Science*, vol. 48, no. 2, pp. 257–263, 2017.