Research Article

The Schwartz Alternating Algorithm for Solving 3D Exterior Helmholtz Problems

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Based on the natural boundary reduction, an overlapping domain decomposition method is discussed for solving exterior Helmholtz problem over a three-dimensional (3D) domain. By introducing two different artificial boundaries, the original unbounded domain is divided into a bounded subdomain and a typical unbounded region, and a Schwartz alternating method is presented. The finite element method and natural boundary element method are alternately applied to solve the problems in the bounded subdomain and the typical unbounded subdomain. Moreover, the convergence of the Schwartz alternating algorithm is studied. Finally, some numerical experiments are presented to show the performance of this method.

1. Introduction

When solving a problem modelled by a linear partial differential equation in the bounded or unbounded domain, many existing methods can be adopted. For the problem in the bounded domain, the finite element method and the finite difference method and so on are very effective. Yet we often find them difficult to be applied to unbounded problem directly. To solve such problems in infinite region numerically, there are a variety of numerical methods (cf. [1–4]) and references therein for more details.

Schwartz alternating algorithm is one of the most efficient techniques for solving problems in the unbounded domain, such as harmonic equation [5, 6], Stokes equation [7], and Helmholtz equation [8]. Over exterior 3D domain, Wu and Yu studied the natural integral equations of Helmholtz problems [9] and overlapping domain decomposition method for harmonic equation [10]. We gave a D-N alternating algorithm for solving 3D exterior Helmholtz problems [11].

In this paper, a Schwartz alternating algorithm based on the natural boundary reduction is devised for the numerical solution of exterior three-dimensional Helmholtz problem. Firstly, an exterior Helmholtz problem is introduced and the corresponding variational form is obtained. Secondly, the Schwartz alternating algorithm is posed. Then, the

convergence rate is discussed and the relationship between contraction factor α and overlapping extent is given. Besides, the error estimate of the algorithm is offered. Finally, some numerical examples are presented to illustrate the feasibility and efficiency of this method.

2. Problem and Its Equivalent Form

We consider the following 3D exterior Helmholtz problem [12]:

$$\begin{aligned} -\Delta u - k^2 u &= 0, \quad \text{in } \Omega^c, \\ u &= g, \quad \text{on } \Gamma_0, \end{aligned} \quad (1)$$

where Ω is a bounded domain in $\mathbb{R}^3$ with regular boundary $\partial\Omega = \Gamma_0$. $u(\mathbf{x})$ is the unknown function and g is a known function, which satisfies appropriate conditions. k denotes the wave number, related to the wavelength λ of the incident wave through $k = 2\pi/\lambda$. In order to assure the existence and uniqueness of the solution of (1), the solution $u(\mathbf{x})$ satisfies the Sommerfeld radiation condition:

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial u}{\partial r} - iku \right) = 0, \quad (2)$$

where (r, θ, φ) denotes the spherical coordinates, $r = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$, $i = \sqrt{-1}$. Condition (2) asserts that the scattered wave is outgoing at infinity.

The corresponding variational form of problems (1)-(2) is

$$\text{Find } u \in H^1(\Omega^c), \quad (3)$$

$$\text{such that } a(u, v) = \langle g, v \rangle, \quad \forall v \in H_0^1(\Omega^c),$$

where

$$\begin{aligned} a(u, v) &= \int_{\Omega^c} (\nabla u \cdot \nabla v - k^2 uv) d\mathbf{x}, \\ \langle g, v \rangle &= \int_{\Gamma_0} g v ds, \end{aligned} \quad (4)$$

$$H_0^1(\Omega^c) = \left\{ v \mid \frac{v}{\sqrt{1+r^2}}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z} \in L^2(\Omega^c) \right\}.$$

Particularly, if Ω is a spherical domain with radius R whose centre is the origin of coordinates, the solution of (1)-(2) is given by the following Poisson integral formula:

$$\begin{aligned} u(r, \theta, \varphi) &= \mathcal{P}u(R, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{h_{\ell}(kr)}{h_{\ell}(kR)} Y_{\ell}^m(\theta, \varphi) \\ &\cdot \int_0^{2\pi} \int_0^{\pi} u(R, \theta', \varphi') \overline{Y_{\ell}^m(\theta', \varphi')} \sin \theta' d\theta' d\varphi', \end{aligned} \quad (5)$$

where $\mathcal{P}$ is called the Poisson integral operator of Helmholtz equation in Ω^c , $h_{\ell}(x) = \sqrt{\pi/2x} H_{\ell+1/2}^{(1)}(x)$ is the first kind of Hankel function, $P_{\ell}^m(\cdot)$ denotes the associated Legendre function of the first kind, $\overline{Y_{\ell}^m(\cdot)}$ denotes the complex conjugate of $Y_{\ell}^m(\cdot)$, and

$$\begin{aligned} Y_{\ell}^m(\theta, \varphi) &= \sqrt{\frac{(\ell-m)!}{(\ell+m)!} \frac{2\ell+1}{4\pi}} P_{\ell}^m(\cos \theta) e^{im\varphi}, \\ P_{\ell}^m(x) &= \begin{cases} \frac{1}{2^{\ell} \ell!} (1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^{\ell}, & m \geq 0, \\ (-1)^{-m} \frac{(\ell+m)!}{(\ell-m)!} P_{\ell}^{-m}(x), & m < 0. \end{cases} \end{aligned} \quad (6)$$

By using (5), we will develop an overlapping domain decomposition method (Schwartz alternating algorithm based on the natural boundary reduction) for problems (1)-(2). Taking the normal derivative of (5), we have the following natural integral equation:

$$\begin{aligned} \frac{\partial u}{\partial n} &= \mathcal{K}u(R, \theta, \varphi) = - \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{k h'_{\ell}(kR)}{h_{\ell}(kR)} Y_{\ell}^m(\theta, \varphi) \\ &\cdot \int_0^{2\pi} \int_0^{\pi} u(R, \theta', \varphi') \overline{Y_{\ell}^m(\theta', \varphi')} \sin \theta' d\theta' d\varphi', \end{aligned} \quad (7)$$

where $\mathcal{K}$ is called the natural integral operator of Helmholtz equation in Ω^c .

3. Schwartz Alternating Algorithm and Its Convergence

Taking two sphere surfaces $\Gamma_1 = \{(r, \theta, \varphi) \mid r = R_1, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi\}$ and $\Gamma_2 = \{(r, \theta, \varphi) \mid r = R_2, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi\}$, such that $R_1 > R_2$ and Γ_0 is surrounded by Γ_2 , denote

$$\Omega_1 = \{(x, y, z) \mid x^2 + y^2 + z^2 < R_1^2, (x, y, z) \in \Omega^c\},$$

$$\Omega_2 = \{(x, y, z) \mid x^2 + y^2 + z^2 > R_2^2, (x, y, z) \in \Omega^c\},$$

$$\Omega_{11} = \Omega_1 \cap \overline{\Omega_2^c},$$

$$\Omega_{12} = \Omega_1 \cap \Omega_2,$$

$$\Omega_{22} = \Omega_2 \cap \overline{\Omega_1^c},$$

$$V = \left\{ v \mid v \in H_0^1(\Omega^c), \lim_{r \rightarrow \infty} r \left(\frac{\partial v}{\partial r} - ikv \right) = 0 \right\}, \quad (8)$$

$$V_1 = \{v \mid v \in H_0^1(\Omega_1), v|_{\Gamma_1} = 0\},$$

$$\begin{aligned} V_2 &= \left\{ v \mid v \in H_0^1(\Omega_2), v|_{\Gamma_2} = 0, \lim_{r \rightarrow \infty} r \left(\frac{\partial v}{\partial r} - ikv \right) \right. \\ &= 0 \left. \right\}, \end{aligned}$$

$$V_n = \{v \mid v \in H_0^1(\Omega_1), v|_{\Gamma_1} = \lambda^{(n)}\}.$$

Construct the following Schwartz alternating algorithm.

Step 1. Select the initial $u_0 \in V$ and let $\eta^{(0)} = u_0|_{\Gamma_1}$, $n = 0$.

Step 2. Solve the problem in Ω_1 :

$$\begin{aligned} -\Delta u_1^{(2n+1)} - k^2 u_1^{(2n+1)} &= 0, \quad \text{in } \Omega_1, \\ u_1^{(2n+1)} &= \eta^{(n)}, \quad \text{on } \Gamma_1, \\ u_1^{(2n+1)} &= g, \quad \text{on } \Gamma_0. \end{aligned} \quad (9)$$

Step 3. Solve the problem in Ω_2 :

$$\begin{aligned} -\Delta u_2^{(2n+2)} - k^2 u_2^{(2n+2)} &= 0, \quad \text{in } \Omega_2, \\ u_2^{(2n+2)} &= u_1^{(2n+1)}, \quad \text{on } \Gamma_2, \end{aligned} \quad (10)$$

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial u_2^{(2n+2)}}{\partial r} - iku_2^{(2n+2)} \right) = 0.$$

Step 4. Let $\eta^{(n+1)} = u_2^{(2n+2)}|_{\Gamma_1}$.

Step 5. Let $n = n + 1$, and go to Step 2.

Where $u_1^{(2n+1)}$ and $u_2^{(2n+2)}$ are the $2n+1$ th and $2n+2$ th approximate solutions in Ω_1 and Ω_2 , respectively, $\eta^{(0)}$ is an arbitrary function in $H^{1/2}(\Gamma_1)$. Note that, on interface Γ_1 , only $\eta^{(n+1)} = u_2^{(2n+2)}|_{\Gamma_1}$ is needed. So it is unnecessary to solve (10).

Actually we can obtain $\eta^{(n+1)}$ by making use of the following Poisson integral equation:

$$\eta^{(n+1)} = u_2^{(2n+2)}|_{\Gamma_1} = \mathcal{P}(\gamma_2 u_1^{(2n+1)})|_{\Gamma_1}, \quad (11)$$

where γ_2 is the trace operator on Γ_2 . Because of $u_2^{(2n)} \in V_2$ and $u_1^{(2n+1)} \in V_1$, V_1 and V_2 can be regarded as the subspaces of V , if elements of V are extended by zero. Let $u_1^{(2n+1)}|_{\Omega_{22}} = u_2^{(2n)}$ and $u_2^{(2n)}|_{\Omega_{11}} = u_1^{(2n-1)}$; obviously $u^{(\ell)} \in V$, $\ell = 0, 1, 2, \dots$, $u_1^{(2n+1)} - u_2^{(2n)} \in V_1$, and $u_2^{(2n+2)} - u_1^{(2n+1)} \in V_2$. (9) and (10) are, respectively, equivalent to the following variational forms:

$$\begin{aligned} &\text{Find } u_1^{(2n+1)} - u_2^{(2n)} \in V_1, \\ &\text{such that } a(u_1^{(2n+1)} - u, v_1) = 0, \quad \forall v_1 \in V_1. \end{aligned} \quad (12)$$

$$\text{Find } u_2^{(2n)} - u_1^{(2n-1)} \in V_2,$$

$$\text{such that } a(u_2^{(2n)} - u, v_2) = 0, \quad \forall v_2 \in V_2.$$

P_{V_i} denotes V projected to V_i ; change (12) into

$$\begin{aligned} a(u_1^{(2n+1)} - u_2^{(2n)}, v_1) &= a(u - u_2^{(2n)}, v_1), \\ \forall v_1 \in V_1, \quad u_1^{(2n+1)} - u_2^{(2n)} &\in V_1, \\ a(u_2^{(2n)} - u_1^{(2n-1)}, v_2) &= a(u - u_1^{(2n-1)}, v_2), \\ \forall v_2 \in V_2, \quad u_2^{(2n)} - u_1^{(2n-1)} &\in V_2. \end{aligned} \quad (13)$$

By the definition of projection, we obtain

$$\begin{aligned} u_1^{(2n+1)} - u_2^{(2n)} &= P_{V_1}(u - u_2^{(2n)}), \quad n \geq 0, \\ u_2^{(2n)} - u_1^{(2n-1)} &= P_{V_2}(u - u_1^{(2n-1)}), \quad n \geq 1. \end{aligned} \quad (14)$$

This is equivalent to

$$\begin{aligned} u - u_1^{(2n+1)} &= P_{V_1^\perp}(u - u_2^{(2n)}), \quad n \geq 0, \\ u - u_2^{(2n)} &= P_{V_2^\perp}(u - u_1^{(2n-1)}), \quad n \geq 1, \end{aligned} \quad (15)$$

where $V_i^\perp$ is the orthogonal complementary space of V_i ; that is,

$$P_{V_i^\perp} = I - P_{V_i}, \quad i = 1, 2. \quad (16)$$

Let $e_i^{(j)} = u - u_i^{(j)}$ ($i = 1, 2$) denote the errors. Then (15) can be rewritten as

$$\begin{aligned} e_1^{(2n+1)} &= P_{V_1^\perp} e_2^{(2n)}, \\ e_2^{(2n)} &= P_{V_2^\perp} e_1^{(2n-1)}. \end{aligned} \quad (17)$$

This yields

$$\begin{aligned} e_1^{(2n+1)} &= P_{V_2^\perp} P_{V_1^\perp} e_1^{(2n-1)}, \\ e_2^{(2n+2)} &= P_{V_1^\perp} P_{V_2^\perp} e_2^{(2n)}. \end{aligned} \quad (18)$$

It is easy to know that if $e_i^{(j)}$ ($i = 1, 2$) are convergent, then their limits are in $V_1^\perp \cap V_2^\perp$.

Lemma 1. $V = V_1 + V_2$, $V_1^\perp \cap V_2^\perp = \{0\}$.

Lemma 2. $V = V_1 + V_2$ and for any $v \in V$ there exists a positive constant C_0 such that

$$|v|_1 \leq C_0 \left(|P_{V_1} v|_1^2 + |P_{V_2} v|_1^2 \right)^{1/2}, \quad (19)$$

where $|\cdot|_1 = \sqrt{a(\cdot, \cdot)}$ denotes energy norm.

Proof. If $V = V_1 + V_2$, there exists a positive constant C_0 such that

$$\begin{aligned} \left(|v_1|_1^2 + |v_2|_1^2 \right)^{1/2} &\leq C_0 |v|_1, \\ v &= v_1 + v_2, \\ v_1 &\in V_1, \quad v_2 \in V_2. \end{aligned} \quad (20)$$

Simultaneously,

$$\begin{aligned} |v|_1^2 &= a(v, v_1) + a(v, v_2) = a(P_{V_1} v, v_1) + a(P_{V_2} v, v_2) \\ &\leq \left(|P_{V_1} v|_1^2 + |P_{V_2} v|_1^2 \right)^{1/2} \left(|v_1|_1^2 + |v_2|_1^2 \right)^{1/2} \\ &\leq C_0 |v|_1 \left(|P_{V_1} v|_1^2 + |P_{V_2} v|_1^2 \right)^{1/2}. \end{aligned} \quad (21)$$

□

Theorem 3. Suppose that $V_1^\perp \cap V_2^\perp = \{0\}$; then

$$\begin{aligned} \lim_{n \rightarrow +\infty} |e_1^{(2n+1)}|_1 &= 0, \\ \lim_{n \rightarrow +\infty} |e_2^{(2n)}|_1 &= 0. \end{aligned} \quad (22)$$

Proof. Following (15), we have

$$|e_1^{(2n+1)}|_1 \geq |e_2^{(2n)}|_1 \geq |e_1^{(2n+3)}|_1 \geq |e_2^{(2n+2)}|_1 \geq 0, \quad (23)$$

which means that

$$\lim_{n \rightarrow +\infty} |e_1^{(2n+1)} - e_2^{(2n)}|_1 = 0. \quad (24)$$

Let $\{e_1^{(2n+1)}\}$ and $\{e_2^{(2n)}\}$ be weakly convergent subsequence, so

$$\begin{aligned} \lim_{n \rightarrow +\infty} e_1^{(2n+1)} &= v_1, \\ \lim_{n \rightarrow +\infty} e_2^{(2n)} &= v_2. \end{aligned} \quad (25)$$

Following (24), we can obtain

$$v_1 = v_2 = 0. \quad (26)$$

Following (17), we have

$$\begin{aligned} a(e_1^{(2n+1)}, e_1^{(2n+1)}) &= a(e_1^{(2n-1)}, e_1^{(2n+3)}) = \dots \\ &= a(e_1^{(4n+1)}, e_1^{(1)}) \longrightarrow 0, \\ a(e_2^{(2n)}, e_2^{(2n)}) &= a(e_2^{(2n-2)}, e_2^{(2n+2)}) = \dots \\ &= a(e_2^{(4n)}, e_2^{(0)}) \longrightarrow 0; \end{aligned} \quad (27)$$

namely

$$\begin{aligned}\lim_{n \rightarrow +\infty} |e_1^{(2n+1)}|_1 &= 0, \\ \lim_{n \rightarrow +\infty} |e_2^{(2n)}|_1 &= 0.\end{aligned}\quad (28)$$

□

Theorem 4. *There exists a constant $\alpha \in [0, 1)$, such that*

$$\begin{aligned}|P_{V_1^\perp} P_{V_2^\perp}| &\leq \alpha, \\ |P_{V_2^\perp} P_{V_1^\perp}| &\leq \alpha, \\ |e_1^{(2n+1)}|_1 &\leq \alpha^{2n} |e_1^{(1)}|_1, \\ |e_2^{(2n+2)}|_1 &\leq \alpha^{2n+2} |e_2^{(0)}|_1\end{aligned}\quad (29)$$

hold true.

Proof. Substituting $P_{V_1^\perp} v$ into (19), we have

$$\begin{aligned}|P_{V_1^\perp} v|_1^2 &= |P_{V_2^\perp} P_{V_1^\perp} v|_1^2 + |P_{V_2} P_{V_1^\perp} v|_1^2 \\ &\geq |P_{V_2^\perp} P_{V_1^\perp} v|_1^2 + \frac{1}{C_0^2} |P_{V_1^\perp} v|_1^2.\end{aligned}\quad (30)$$

Equivalently

$$|P_{V_2^\perp} P_{V_1^\perp} v|_1^2 \leq \left(1 - \frac{1}{C_0^2}\right) |P_{V_1^\perp} v|_1^2 \leq \alpha |v|_1^2, \quad \forall v \in V, \quad (31)$$

where

$$\alpha = \sqrt{1 - \frac{1}{C_0^2}} \in [0, 1); \quad (32)$$

namely

$$|P_{V_2^\perp} P_{V_1^\perp}| \leq \alpha. \quad (33)$$

Similarly

$$|P_{V_1^\perp} P_{V_2^\perp}| \leq \alpha. \quad (34)$$

Following (17) and (18), it comes that

$$\begin{aligned}|e_1^{(2n+1)}|_1 &\leq \alpha |P_{V_1^\perp} e^{(2n-1)}|_1 = \alpha |P_{V_1^\perp} P_{V_2^\perp} e^{(2n-2)}|_1 \\ &\leq \alpha^2 |P_{V_2^\perp} e^{(2n-2)}|_1 = \alpha^2 |e^{(2n-1)}|_1.\end{aligned}\quad (35)$$

By recursion we obtain

$$|e_1^{(2n+1)}|_1 \leq \alpha^{2n} |e_1^{(1)}|_1. \quad (36)$$

Similarly, we obtain

$$|e_2^{(2n+2)}|_1 \leq \alpha^{2n+2} |e_2^{(0)}|_1. \quad (37)$$

□

4. Analysis of Convergence Rate

Absolutely, the convergence rate of the above Schwartz alternating algorithm is closely related to the overlapping degree of Ω_1 and Ω_2 . Although it can be deduced intuitively that the larger the overlapping part is, the faster the convergence rate will be; yet we find it difficult to analyse the convergence rate for general unbounded domain Ω^c . However, under certain assumptions, we can find out the relationship between contraction factor α and overlapping degree of Ω_1 and Ω_2 .

Consider the following boundary value problem in domain Ω_1 :

$$\begin{aligned}-\Delta u - k^2 u &= 0, \quad \text{in } \Omega_1, \\ u &= f, \quad \text{on } \Gamma_1, \\ u &= g, \quad \text{on } \Gamma_0.\end{aligned}\quad (38)$$

Lemma 5. *If $f \in H^{1/2}(\Gamma_1)$ and $g \in H^{1/2}(\Gamma_0)$, then u is the solution of (38).*

$$\begin{aligned}u &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{(R_1^{2\ell+1} - R^{2\ell+1}) r^{\ell+1}} \\ &\cdot \left[(R_1^{2\ell+1} - r^{2\ell+1}) R^{\ell+1} \frac{h_\ell(kr)}{h_\ell(kR)} G_\ell^m \right. \\ &\left. + (r^{2\ell+1} - R^{2\ell+1}) R_1^{\ell+1} \frac{h_\ell(kr)}{h_\ell(kR_1)} F_\ell^m \right] Y_\ell^m(\theta, \varphi),\end{aligned}\quad (39)$$

where

$$\begin{aligned}F_\ell^m &= \int_0^{2\pi} \int_0^\pi f(R_1, \theta, \varphi) \overline{Y_\ell^m(\theta, \varphi)} \sin \theta d\theta d\varphi, \\ G_\ell^m &= \int_0^{2\pi} \int_0^\pi g(R, \theta, \varphi) \overline{Y_\ell^m(\theta, \varphi)} \sin \theta d\theta d\varphi.\end{aligned}\quad (40)$$

Theorem 6. *If we apply the Schwartz alternating algorithm given in Section 3 to problems (1)-(2) and $u_0 \in H^{1/2}(\Gamma_0)$, then*

$$\|u - u_1^{(2n)}\|_{H^1(\Omega_1)} \leq C_1 \alpha^n, \quad (41)$$

$$\|u - u_2^{(2n+1)}\|_{W_0^1(\Omega_2)} \leq C_2 \alpha^{n+1}$$

hold true, where constants C_1 and C_2 are independent of n , while

$$\alpha = \frac{R_2 - R}{R_1 - R}. \quad (42)$$

Proof. On one hand, for $u_0 \in H^{1/2}(\Gamma_0)$, we have

$$\begin{aligned}u_0(R, \theta, \varphi) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{h_\ell(kR)}{h_\ell(kR)} U_\ell^m Y_\ell^m(\theta, \varphi) \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} U_\ell^m Y_\ell^m(\theta, \varphi), \\ \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (1 + \ell^2)^{1/2} |U_\ell^m|^2 &< +\infty,\end{aligned}\quad (43)$$

where

$$U_\ell^m = \int_0^{2\pi} \int_0^\pi u_0(R, \theta, \varphi) \overline{Y_\ell^m(\theta, \varphi)} \sin \theta d\theta d\varphi. \quad (44)$$

Following [9], we obtain

$$u(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{h_\ell(kr)}{h_\ell(kR)} U_\ell^m Y_\ell^m(\theta, \varphi), \quad r > R. \quad (45)$$

On the other hand, for $\tilde{u} \in W_0^1(\Omega^c)$, we have $\tilde{u}|_{\Gamma_1} \in H^{1/2}(\Gamma_1)$, while

$$\tilde{u}|_{\Gamma_1} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \tilde{U}_\ell^m Y_\ell^m(\theta, \varphi), \quad (46)$$

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (1 + \ell^2)^{-1/2} |\tilde{U}_\ell^m|^2 < +\infty,$$

where

$$\tilde{U}_\ell^m = \int_0^{2\pi} \int_0^\pi \tilde{u}(R_1, \theta, \varphi) \overline{Y_\ell^m(\theta, \varphi)} \sin \theta d\theta d\varphi. \quad (47)$$

According to Lemma 5, it comes that

$$u_1^{(2n)} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{R^{\ell+1} U_\ell^m}{r^{\ell+1}} - \alpha_\ell^n \frac{r^{2\ell+1} - R^{2\ell+1}}{R_1^{2\ell+1} - R^{2\ell+1}} \right. \\ \left. \cdot \frac{R^{\ell+1} U_\ell^m - R_1^{\ell+1} \tilde{U}_\ell^m}{r^{\ell+1}} \right) \frac{h_\ell(kr)}{h_\ell(kR)} Y_\ell^m(\theta, \varphi), \quad (48)$$

where

$$\alpha_\ell = \frac{R_2^{2\ell+1} - R^{2\ell+1}}{R_1^{2\ell+1} - R^{2\ell+1}}. \quad (49)$$

Suppose that $n \in \overline{Z^-}$; we have

$$u_1^{(2n)}|_{\Gamma_2} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{R^{\ell+1} U_\ell^m}{R_2^{\ell+1}} - \alpha_\ell^{n+1} \frac{R^{\ell+1} U_\ell^m - R_1^{\ell+1} \tilde{U}_\ell^m}{R_2^{\ell+1}} \right) \frac{h_\ell(kR_2)}{h_\ell(kR)} Y_\ell^m(\theta, \varphi), \\ u_2^{(2n+1)} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{R^{\ell+1} U_\ell^m}{R_2^{\ell+1}} - \alpha_\ell^{n+1} \right. \\ \left. \cdot \frac{R^{\ell+1} U_\ell^m - R_1^{\ell+1} \tilde{U}_\ell^m}{R_2^{\ell+1}} \right) \frac{R_2^{\ell+1}}{r^{\ell+1}} \frac{h_\ell(kr)}{h_\ell(kR)} Y_\ell^m(\theta, \varphi). \quad (50)$$

Following trace theorem, we have

$$u_2^{(2n+1)}|_{\Gamma_1} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{R^{\ell+1} U_\ell^m}{R_1^{\ell+1}} - \alpha_\ell^{n+1} \frac{R^{\ell+1} U_\ell^m - R_1^{\ell+1} \tilde{U}_\ell^m}{R_1^{\ell+1}} \right) \\ \cdot \frac{h_\ell(kR_1)}{h_\ell(kR)} Y_\ell^m(\theta, \varphi). \quad (51)$$

By Lemma 5 again, we obtain

$$u_1^{(2n+2)} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{R^{\ell+1} U_\ell^m}{r^{\ell+1}} - \alpha_\ell^{n+1} \frac{r^{2\ell+1} - R^{2\ell+1}}{R_1^{2\ell+1} - R^{2\ell+1}} \right. \\ \left. \cdot \frac{R^{\ell+1} U_\ell^m - R_1^{\ell+1} \tilde{U}_\ell^m}{r^{\ell+1}} \right) \frac{R_2^{\ell+1}}{r^{\ell+1}} \frac{h_\ell(kr)}{h_\ell(kR)} Y_\ell^m(\theta, \varphi). \quad (52)$$

By induction, it comes that

$$u|_{\Omega_1} - u_1^{(2n)} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \alpha_\ell^n \frac{r^{2\ell+1} - R^{2\ell+1}}{R_1^{2\ell+1} - R^{2\ell+1}} \\ \cdot \frac{R^{\ell+1} U_\ell^m - R_1^{\ell+1} \tilde{U}_\ell^m}{r^{\ell+1}} \frac{R_2^{\ell+1}}{r^{\ell+1}} \frac{h_\ell(kr)}{h_\ell(kR_1)} Y_\ell^m(\theta, \varphi). \quad (53)$$

Denoting

$$e_N^{(2n)} = \sum_{\ell=0}^N \sum_{m=-\ell}^{\ell} \alpha_\ell^n \\ \cdot \frac{r^{2\ell+1} - R^{2\ell+1}}{R_1^{2\ell+1} - R^{2\ell+1}} \frac{R^{\ell+1} U_\ell^m - R_1^{\ell+1} \tilde{U}_\ell^m}{r^{\ell+1}} \frac{R_2^{\ell+1}}{r^{\ell+1}} \frac{h_\ell(kr)}{h_\ell(kR_1)} Y_\ell^m(\theta, \varphi), \quad (54)$$

(53) is changed into

$$\lim_{N \rightarrow +\infty} |e_N^{(2n)} - (u - u_1^{(2n)})|_{H^1(\Omega_1)} = 0. \quad (55)$$

Clearly

$$\Delta e_N^{(2n)} = 0, \quad (56)$$

noting that

$$0 < \alpha_\ell = \frac{R_2^{2\ell+1} - R^{2\ell+1}}{R_1^{2\ell+1} - R^{2\ell+1}} \leq \frac{R_2 - R}{R_1 - R} = \alpha, \quad \ell = 0, 1, \dots \quad (57)$$

Following (56) and (57), there exists C_1 independent of n , such that

$$\|e_N^{(2n)}\|_{H^1(\Omega_1)} \leq C_1 \alpha^n. \quad (58)$$

Similarly, we obtain

$$\|e_N^{(2n)}\|_{W_0^1(\Omega_2)} \leq C_2 \alpha^{n+1}. \quad (59)$$

It can be seen from Theorem 6 that the larger the overlapping part of Ω_1 and Ω_2 is, the smaller the contraction factor α is and, consequently, the faster the Schwartz alternating algorithm converges. $\square$

5. Error Estimates of Algorithms

Subdivide Ω_1 into hexahedrons. Let $S_h(\Omega_1)$ denote the linear finite element space over Ω_1 . Putting

$$\dot{S}_h(\Omega_1) = \{v_h \mid v_h \in S_h(\Omega_1), v_h|_{\Gamma_0 \cup \Gamma_1} = 0\}, \quad (60)$$

$\dot{S}_h(\Omega_1)$ can be regarded as the subspace of V if its elements are extended by zero. We first establish the following discrete Schwartz alternating algorithm:

$$\begin{aligned} \text{Find } w_{1h}^{(2n+1)} &= u_{1h}^{(2n+1)} - u_h^{(2n)} \in \dot{S}_h(\Omega_1), \\ \text{such that } a(w_{1h}^{(2n+1)}, v_h) &= -a(u_h^{(2n)}, v_h), \quad (61) \\ &\forall v_h \in \dot{S}_h(\Omega_1), \end{aligned}$$

$$\begin{aligned} \text{Find } w_{2h}^{(2n+2)} &= u_{2h}^{(2n+2)} - u_h^{(2n+1)} \in V_2, \\ \text{such that } a(w_{2h}^{(2n+2)}, v) &= -a(u_h^{(2n+1)}, v), \quad (62) \\ &\forall v_h \in V_2, \end{aligned}$$

with

$$\begin{aligned} u_h^{(2n+1)} &= \tilde{u} + \begin{cases} \sum_{i=0}^n w_{1h}^{(2i+1)}, & \text{on } \overline{\Omega} \setminus \Omega_2, \\ \sum_{i=0}^n w_{1h}^{(2i+1)} + \sum_{j=0}^{n-1} [\mathcal{P}\gamma w_{1h}^{(2j+1)} - w_{1h}^{(2j+1)}] + \varepsilon_n(\mathcal{P}\gamma \tilde{u} - \tilde{u}), & \text{in } \Omega_1 \setminus (\overline{\Omega} \setminus \Omega_2), \\ \sum_{j=0}^{n-1} \mathcal{P}\gamma w_{1h}^{(2j+1)} + \varepsilon_n(\mathcal{P}\gamma \tilde{u} - \tilde{u}), & \text{on } \Omega \setminus \Omega_1, \end{cases} \\ u_h^{(2n+2)} &= \tilde{u} + \begin{cases} \sum_{i=0}^n w_{1h}^{(2i+1)}, & \text{on } \overline{\Omega} \setminus \Omega_2, \\ \sum_{i=0}^n w_{1h}^{(2i+1)} + \sum_{j=0}^{n-1} [\mathcal{P}\gamma w_{1h}^{(2j+1)} - w_{1h}^{(2j+1)}] + (\mathcal{P}\gamma \tilde{u} - \tilde{u}), & \text{in } \Omega_1 \setminus (\overline{\Omega} \setminus \Omega_2), \\ \sum_{j=0}^{n-1} \mathcal{P}\gamma w_{1h}^{(2j+1)} + (\mathcal{P}\gamma \tilde{u} - \tilde{u}), & \text{on } \Omega \setminus \Omega_1, \end{cases} \quad (65) \\ \varepsilon_n &= \begin{cases} 0, & n = 0, \\ 1, & n > 0, \end{cases} \end{aligned}$$

and the term corresponding to $\sum_{j=0}^{n-1}$ vanishes while $n = 0$. Define

$$\begin{aligned} A_h(\Omega_2) &= \left\{ \mathcal{P}\gamma(v_h + \beta_1 \tilde{u} + \beta_2 w) \right. \\ &\quad \left. - (v_h + \beta_1 \tilde{u} + \beta_2 w)|_{\overline{\Omega}_2} \mid v_h \in \dot{S}_h(\Omega_1), \beta_1, \beta_2 \right. \\ &\quad \left. \in R, w = u - \tilde{u} \right\}. \end{aligned} \quad (66)$$

Then, extending the elements of $A_h(\Omega_2)$ by zero, we have

$$A_h(\Omega_2) \subset V_2 \subset V. \quad (67)$$

$$\begin{aligned} u_h^{(2n+1)} &= \begin{cases} u_{1h}^{(2n+1)}, & \text{in } \Omega_1, \\ u_h^{(2n)}, & \text{in } \Omega \setminus \Omega_1, \end{cases} \\ u_h^{(2n+2)} &= \begin{cases} u_{1h}^{(2n+1)}, & \text{in } \Omega \setminus \Omega_2, \\ u_{2h}^{(2n+2)}, & \text{in } \Omega_2, \end{cases} \quad (63) \\ u_h^0 &= \tilde{u}. \end{aligned}$$

By the Poisson integral formula, the solution of (62) can be given as

$$u_{2h}^{(2n+2)} = \mathcal{P}\gamma_2 u_h^{(2n+1)}, \quad (64)$$

where $\gamma_2 : H^1(\Omega_1) \rightarrow H^{1/2}(\Gamma_2)$ denotes the Dirichlet trace operator and $\mathcal{P} : H^{1/2}(\Gamma_2) \rightarrow H_0^1(\Omega_2)$ denotes the Poisson integral operator. It is easy to verify that

Now we introduce the following variational problem:

$$\begin{aligned} \text{Find } v_h^* &\in \dot{S}_h(\Omega_1) + A_h(\Omega_2), \\ \text{such that } a(v_h^*, v_h) &= -a(\tilde{u}, v_h), \quad (68) \\ &\forall v_h \in \dot{S}_h(\Omega_1) + A_h(\Omega_2). \end{aligned}$$

Obviously, (68) exists as a unique solution. Assume that

$$u_h^* = v_h^* + \tilde{u}. \quad (69)$$

Similarly in [10], we have the following error estimates.

TABLE 1: $(a, R_2, m) = (1, 1.75, 2)$.

N_1	N_2	s	n	0	1	2	3	4
4	8	2	$e(n)$	2.5231×10^{-1}	7.2245×10^{-2}	6.1215×10^{-2}	4.2233×10^{-2}	4.0159×10^{-2}
			$e_h(n)$		9.8566×10^{-2}	2.6452×10^{-2}	6.1205×10^{-3}	1.7354×10^{-3}
			$q_h(n)$			3.7270	3.6219	3.5267
8	16	4	$e(n)$	2.1628×10^{-1}	6.8132×10^{-2}	4.2354×10^{-2}	2.1532×10^{-2}	2.1533×10^{-2}
			$e_h(n)$		1.5982×10^{-1}	4.5357×10^{-2}	1.2986×10^{-2}	3.7281×10^{-3}
			$q_h(n)$			3.5236	3.4926	3.4432
16	32	8	$e(n)$	1.8437×10^{-1}	3.5468×10^{-2}	2.3459×10^{-2}	2.0245×10^{-2}	1.7342×10^{-2}
			$e_h(n)$		1.2183×10^{-1}	3.7647×10^{-2}	1.1743×10^{-2}	3.6502×10^{-3}
			$q_h(n)$			3.2361	3.2058	3.2172

TABLE 2: $(a, R_2, m) = (1, 1.75, 2)$.

N_1	N_2	s	n	5	6	7	8	9
4	8	2	$e(n)$	4.0029×10^{-2}	3.9021×10^{-2}	3.9002×10^{-2}	3.8876×10^{-2}	3.8651×10^{-2}
			$e_h(n)$	5.2202×10^{-4}	1.7305×10^{-4}	5.7476×10^{-5}	1.9094×10^{-5}	6.2964×10^{-6}
			$q_h(n)$	3.3245	3.0166	3.0108	3.0102	3.0326
8	16	4	$e(n)$	2.0522×10^{-2}	2.0614×10^{-2}	1.8631×10^{-2}	1.7532×10^{-2}	1.7092×10^{-2}
			$e_h(n)$	1.1646×10^{-3}	3.7540×10^{-4}	1.2175×10^{-4}	3.9789×10^{-5}	1.3206×10^{-5}
			$q_h(n)$	3.2012	3.1823	3.0832	3.0601	3.0129
16	32	8	$e(n)$	6.9625×10^{-3}	3.8703×10^{-3}	2.7213×10^{-3}	2.8173×10^{-3}	2.6785×10^{-3}
			$e_h(n)$	1.1663×10^{-3}	3.7108×10^{-4}	1.2291×10^{-4}	4.0792×10^{-5}	1.3538×10^{-5}
			$q_h(n)$	3.1295	3.1432	3.0190	3.0132	3.0130

Theorem 7. For the discrete Schwartz alternating algorithms (61)-(62) and the constant α in Theorem 4, there hold the following error estimates

$$\begin{aligned} |u - u_h^{(2n+1)}|_1 &\leq Ch + \alpha^{2n} |u_h^* - u_h^{(1)}|_1, \\ |u - u_h^{(2n+2)}|_1 &\leq Ch + \alpha^{2n+2} |u_h^* - u_h^{(0)}|_1. \end{aligned} \quad (70)$$

Since $u - u_h^{(n)} \in V$, (70) also hold true if the norm in their left hand sides is replaced by the norm of $H_0^1(\Omega)$.

6. Numerical Examples

To test the effectiveness of the method in this paper, we give some numerical examples, using the discrete Schwartz alternating algorithm in Section 5. In all examples, the exact solutions are known. The purpose of these examples is to check the convergence in terms of iteration and mesh size h .

Suppose that $\Omega^c = \{(r, \theta, \varphi) \mid |r| > a, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi\}$ is the exterior domain of sphere $x^2 + y^2 + z^2 = a^2$. The artificial boundary is $\Gamma_1 = \{(r, \theta, \varphi) \mid |r| = ma, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi\}$ ($m > 1, m \in \mathbb{Z}^+$), $\Gamma_2 = \{(r, \theta, \varphi) \mid |r| = R_2, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi\}$. Split the bounded domain Ω_1 into finite element mesh as follows. We subdivide the bounded domain Ω_1 using the following family of planes $\theta = (\pi/N_1)j$ ($j = 0, 1, \dots, N_1$), $\varphi_\ell = (2\pi/N_2)\ell$ ($\ell = 0, 1, \dots, N_2$), and $r = ia/s$ ($i = s, s+1, \dots, ms$), where s is a positive integer. Now we obtain a uniform hexahedrons partition of Ω_1 with mesh size of $h = a/s$. Let $S_h(\Omega_1)$ be the piecewise trilinear finite element space on Ω_1 . Substitute

$$\tilde{u}^I = \sum_{P_i \in \Sigma_0} g_0(P_i) L_i(\mathbf{x}) \quad (71)$$

for $\tilde{u}$ (where $L_i(\mathbf{x})$ denotes the interpolation basis function corresponding to node P_i).

Denote e the maximum node-error on $\bar{\Omega}_i$

$$e(n) = \sup_{P_i \in \bar{\Omega}_i} |u(P_i) - u_{1h}^{(2n)}(P_i)|; \quad (72)$$

e_h denotes the maximum node-error of the adjacent two-steps on nodes

$$e_h(n) = \sup_{P_i \in \bar{\Omega}_i} |u_{1h}^{(2n+2)}(P_i) - u_{1h}^{(2n)}(P_i)|; \quad (73)$$

q_h is the convergence rate $q_h(n) = e_h(n-1)/e_h(n)$. We substitute $\sum_{\ell=0}^M$ for $\sum_{\ell=0}^\infty$ in the computing of the entries of stiffness matrix.

Example 1. We take that the exact solution of problem (1) is

$$u(r, \theta, \varphi) = -\sqrt{\frac{3}{4\pi}} \frac{kr + i}{k^2 r^2} e^{ikr} \cos \theta, \quad (74)$$

where

$$g = -\sqrt{\frac{3}{4\pi}} \frac{k + i}{k^2} e^{ik} \cos \theta. \quad (75)$$

Let $k = 2.0$, $a = 1$, $m = 2$, $R_2 = 1.75$, and $M = 4$; the numerical results are shown in Tables 1 and 2.

Example 2. We take that the exact solution of problem (1) is

$$\begin{aligned} u(r, \theta, \varphi) \\ = -\sqrt{\frac{15}{32\pi}} \left(\frac{3i}{k^3 r^3} + \frac{3}{k^2 r^2} - \frac{i}{kr} \right) e^{i(kr+\varphi)} \sin 2\theta, \end{aligned} \quad (76)$$

TABLE 3: $(a, R_2, m) = (1, 1.85, 2)$.

N_1	N_2	s	n	0	1	2	3	4
4	8	2	$e(n)$	2.5231×10^{-1}	5.2541×10^{-2}	5.0562×10^{-2}	4.5623×10^{-2}	3.8128×10^{-2}
			$e_h(n)$		7.5434×10^{-1}	2.1411×10^{-2}	6.6464×10^{-3}	2.1131×10^{-3}
			$q_h(n)$			3.5231	3.2215	3.1452
8	16	4	$e(n)$	2.1628×10^{-1}	6.2315×10^{-2}	5.1532×10^{-2}	3.8321×10^{-2}	2.2532×10^{-2}
			$e_h(n)$		1.4526×10^{-1}	4.4753×10^{-2}	1.4856×10^{-2}	5.0306×10^{-3}
			$q_h(n)$			3.2458	3.0124	2.9532
16	32	8	$e(n)$	1.8437×10^{-1}	2.4623×10^{-2}	2.3245×10^{-2}	2.1244×10^{-2}	2.0132×10^{-2}
			$e_h(n)$		1.0157×10^{-1}	3.6381×10^{-2}	1.4196×10^{-2}	5.7773×10^{-3}
			$q_h(n)$			2.7918	2.5628	2.4572

TABLE 4: $(a, R_2, m) = (1, 1.85, 2)$.

N_1	N_2	s	n	5	6	7	8	9
4	8	2	$e(n)$	3.7170×10^{-2}	3.5319×10^{-2}	3.4321×10^{-2}	3.2112×10^{-2}	2.9518×10^{-2}
			$e_h(n)$	2.3888×10^{-4}	9.6127×10^{-5}	4.1760×10^{-5}	1.8519×10^{-5}	8.2215×10^{-6}
			$q_h(n)$	2.5621	2.4851	2.3019	2.2549	2.2526
8	16	4	$e(n)$	2.0213×10^{-2}	1.9526×10^{-2}	1.9432×10^{-2}	1.9542×10^{-2}	1.8926×10^{-2}
			$e_h(n)$	1.3314×10^{-3}	4.8891×10^{-4}	2.1574×10^{-4}	9.5621×10^{-5}	4.2462×10^{-5}
			$q_h(n)$	2.8001	2.7232	2.2662	2.2572	2.2519
16	32	8	$e(n)$	1.9524×10^{-2}	9.5324×10^{-3}	8.4526×10^{-3}	8.0219×10^{-3}	7.8243×10^{-3}
			$e_h(n)$	1.4861×10^{-3}	6.4006×10^{-4}	2.8198×10^{-4}	1.2506×10^{-4}	5.5502×10^{-5}
			$q_h(n)$	2.4562	2.3218	2.2699	2.2548	2.2532

where

$$g = -\sqrt{\frac{15}{32\pi}} \frac{3i + 3k - ik^2}{k^3} e^{i(k+\varphi)} \sin 2\theta. \quad (77)$$

Let $k = 2.0$, $a = 1$, $m = 2$, $R_2 = 1.85$, and $M = 4$; the numerical results are shown in Tables 3 and 4.

As can be seen from Tables 3 and 4, the discrete Schwartz alternating algorithm is geometrically convergent and $q_h(n)$ is nearly not affected by mesh parameter h . With the increase of R_2 , $q_h(n)$ is smaller. So the larger the overlapping part is, the faster the convergence rate will be. All these are in accord with the theoretical analyses.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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