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*Research Article*

# Consensus of Fractional-Order Multiagent Systems with Double Integral and Time Delay

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This paper is devoted to the consensus problems for a fractional-order multiagent system (FOMAS) with double integral and time delay, the dynamics of which are double-integrator fractional-order model, where there are two state variables in each agent. The consensus problems are investigated for two types of the double-integrator FOMAS with time delay: the double-integrator FOMAS with time delay whose network topology is undirected topology and the double-integrator FOMAS with time delay whose network topology is directed topology with a spanning tree in this paper. Based on graph theory, Laplace transform, and frequency-domain theory of the fractional-order operator, two maximum tolerable delays are obtained to ensure that the two types of the double-integrator FOMAS with time delay can asymptotically reach consensus. Furthermore, it is proven that the results are also suitable for integer-order dynamical model. Finally, the relationship between the speed of convergence and time delay is revealed, and simulation results are presented as a proof of concept.

## 1. Introduction

In the past decade, an increasing number of scholars have been interested in the consensus problems for multiagent systems with potential applications in biology, control engineering, and physics. Typical applications include flocking [1, 2], swarming [3, 4], formation control [5, 6], sensor networks [7–9], and many other areas. For multiagent systems, consensus means that a group of agents reach an agreement on a common value by exchanging local information with their neighbours. However, due to the limitation of the communication device and the undesirable communication environment, time delays are ubiquitous in the information exchange. The existence of time delays may have a bad impact on the consensus for multiagent systems, so it is meaningful to study the property of multiagent systems with time delays. Up till now, consensus problems involving multiagent systems with time delays have been resolved by many scholars. In [10], the authors discussed the average consensus problem in undirected networks of dynamic agents with fixed and switching topologies as well as multiple time-varying communication delays. In [11], the authors investigated consensus problems of a class of second-order continuous-time multiagent systems with time delay and jointly connected topologies. In [12], the authors investigated the leader-following stationary consensus problem for second-order multiagent systems with time-varying communication delay and switching topology. In [13], the authors proposed a distributed protocol for consensus of second-order multiagent systems with inherent nonlinear dynamics and communication time delay. In [14], the authors solved control problems of agents achieving consensus motions in presence of nonuniform time delays. In [15], the authors investigated robust $H_\infty$ consensus control problems involving input delays for uncertain multiagent systems.

Note that all above multiagent systems concern integer-order dynamics. In fact, many natural phenomena in the complex environment cannot be accurately explained by using the framework of integer-order dynamics, such as electromagnetic waves, viscoelasticity, and the heat flux of the thermal field of the furnace wall [16], while more dynamic properties of natural phenomena can be better revealed by using the framework of fractional-order dynamics, such as food seeking of microbes, collection motion of bacteria in lubrications, and underwater vehicles operating in lentic lakes [17]. Fractional-order derivatives provide an excellent instrument for the description of memories and hereditary effects of various materials and processes which are neglected in classical integer-order dynamics. In addition, it has been stated in [18] that the well-studied integer-order systems were just the special cases of fractional-order ones. Consensus of multiagent systems based on the fractional-order models was early studied by Cao et al. [17]. After that, the research results have been continuously springing up about consensus problems of fractional-order multiagent systems with time delays. In [19], a necessary and sufficient condition was derived to ensure the consensus of fractional-order systems with identical input delays over a directed topology. In [20, 21], the authors successively studied the fractional-order system consensus control with communication delays, where homogeneous dynamics and heterogeneous dynamics were investigated. In [22], the consensus of linear and nonlinear fractional-order multiagent systems with input time delay was studied. In [23], consensus problems were investigated for fractional-order multiagent systems with nonuniform time delays. Recently, a new fractional-order multiagent system with double-integrator model was proposed by authors in [24–26]. Based on the fractional-order stability theory, Mittag-Leffler function, and Laplace transform, the consensus problem of fractional-order multiagent systems with double integral under fixed topology was studied in [24]. By applying Mittag-Leffler function, Laplace transform, and dwell time technique, the consensus of fractional-order multiagent systems with double integral under switching topology was investigated in [25]. The consensus for the fractional-order double-integrator multiagent systems based on the sliding mode estimator was studied in [26]. So far, to the best of our knowledge, there are very few research works done on the consensus problems of fractional-order multiagent systems with double integral and time delay.

Motivated by above analysis, the consensus problems are investigated for the two types of fractional-order multiagent system (FOMAS) with double integral and time delay: the FOMAS with double integral and time delay whose network topology is undirected topology and the FOMAS with double integral and time delay whose network topology is directed topology with a spanning tree in this paper. Firstly, a distributed control protocol based on state feedback of neighbours is designed. By applying the graph theory tools, the closed-loop double-integrator fractional-order dynamics with time delay are established and the consensus problems of the double-integrator FOMAS with time delay are transformed into the problems of the system matrix eigenvalues of the double-integrator FOMAS. By employing Laplace transform of Caputo derivative, the characteristic equation of the system matrix of the double-integrator FOMAS is derived. Then, based on the frequency-domain analysis and the matrix theory tool, the two maximum tolerable delays are obtained to ensure consensus for the double-integrator FOMAS with time delay. The main contribution of this article lies in the research on the double-integrator fractional-order dynamics which can better reveal the essential characteristic or behavior of an object in the complex environment and the consensus of the double-integrator FOMAS with time delay whose network topology $\mathscr{G}$ is, respectively, undirected topology and directed topology.

The remainder of the paper is organized as follows. In Section 2, some basic preliminaries about algebraic graph theory and fractional calculus are shown out. In Sections 3 and 4, problem statement, main results, and some corollaries for the two types of double-integrator FOMAS with time delay whose network topology $\mathscr{G}$ is, respectively, undirected topology and directed topology are proposed. In Section 5, several simulation results are simulated to illustrate the correctness of the proposed theoretical results. Finally, concluding remarks are drawn in Section 6.

## 2. Preliminaries

In this section, basic preliminary knowledge about algebraic graph theory and fractional calculus is introduced for the following analysis.

*2.1. Algebraic Graph Theory.* Algebraic graph theory is a practical framework for analyzing consensus problems. Let $\mathscr{G}(\mathscr{V},\mathscr{E},\mathscr{A})$ be an interaction graph of order $n$, where $\mathscr{V} = \{h_1, h_2, \dots, h_n\}$ is the set of nodes, $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ is the set of edges, and $\mathscr{A} = [a_{ik}] \in \mathbb{R}^{n\times n}$ is a weighted adjacency matrix. The node's indices belong to a finite index set $\mathscr{I} = \{1, 2, \dots, n\}$. If there is a directed edge $e_{ik} \in \mathscr{E}$ which is from node $h_k$ to node $h_i$, then $a_{ik} > 0$; otherwise, $a_{ik} = 0$. Suppose that there are no self-loops; that is, $a_{ii} = 0$. If $a_{ik} = a_{ki} > 0$ for any $i, k \in \mathscr{I}$, $\mathscr{G}$ is an undirected graph; if there exists $a_{ik} \neq a_{ki}$, $\mathscr{G}$ is a directed graph. The set of neighbours of node $v_i$ is denoted by $N_i = \{h_k \in \mathscr{V} : a_{ik} > 0\}$. The Laplacian matrix of the graph $\mathscr{G}$ is defined as $\mathscr{L} = \mathscr{D} - \mathscr{A} \in \mathbb{R}^{n\times n}$, where $\mathscr{D} = \operatorname{diag}\{\deg_{\text{out}}(h_1), \deg_{\text{out}}(h_2), \dots, \deg_{\text{out}}(h_i), \dots, \deg_{\text{out}}(h_n)\}$ is a diagonal matrix with $\deg_{\text{out}}(h_i) = \sum_{k=1}^{n} a_{ik}$. A directed path is a sequence of ordered edges of the form $e_{ii_1}, e_{i_1 i_2}, e_{i_2 i_3}, \dots, e_{i_l k}$, and if there is a path from every node to every other node, the graph is said to be strongly connected. Moreover, a directed graph is said to have a spanning tree, if there exists a node such that every other node has a directed path to this node. It is easy to verify that $\mathscr{L}$ has at least one zero eigenvalue with a corresponding eigenvector $\mathbf{1}$, where $\mathbf{1} = (1, 1, \dots, 1)^T$.

**Lemma 1** (see [27]). *For an undirected graph, the Laplacian matrix $\mathscr{L}$ has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}$ and other eigenvalues are positive real numbers; that is, $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$, if and only if $\mathscr{G}$ is connected.*

**Lemma 2** (see [27]). *For a directed graph, the Laplacian matrix $\mathscr{L}$ has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}$ and other eigenvalues have a positive real part; that is, $\lambda_1 = 0$, $\lambda_i \neq 0$, $i = 2, 3, \dots, n$, if and only if $\mathscr{G}$ has a directed spanning tree.*

*2.2. Fractional Calculus.* There are many different definitions of fractional-order operators; the Caputo fractional operator and Riemann-Liouville (R-L) fractional operator are the most broadly used to analyze fractional-order dynamical systems. The major advantage of Caputo fractional derivative over R-L fractional derivative is the initial conditions for fractional differential equations can take on the same form as the traditional integer-order differential equations. Therefore, this paper adopts the Caputo fractional operator to study the system dynamics. For an arbitrary real number $a$, the Caputo derivative is defined as

$$ {}_a^C D_t^q f(t) = \frac{1}{\Gamma(p-q)} \int_a^t (t-\theta)^{p-q-1} \left(\frac{d}{d\theta}\right)^p f(\theta)\, d\theta, \quad (1)$$

where $q \in (p-1, p]$ ($p \in Z^+$) denotes the order of the derivative and $\Gamma(\cdot)$ is the Gamma function:

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt. \quad (2)$$

Let $F(s)$ represent the Laplace transform of the function $f(t)$; that is, $F(s) = \mathfrak{L}\{f(t)\} = \int_{0^-}^\infty e^{-st} f(t)dt$, and let $f^{(q)}t$ replace ${}_a^c D_t^q f(t)$; then the Laplace transform is of the Caputo derivative as follows:

$$\mathfrak{L}\{f^q(t)\} = s^q F(s) - \sum_{i=1}^{[q]+1} s^{q-i} f^{(i-1)}(0), \quad (3)$$

where $f(0^-) = \lim_{t\to 0^-} f(t)$ and $f^{(i)}(0^-) = \lim_{t\to 0^-} f^{(i)}(t)$.

## 3. Problem Statement

Consider a double-integrator FOMAS being made up of $n$ agents. Each agent is regarded as a node in the graph $\mathcal{G}$. Each edge $e_{ik} \in \mathcal{E}$ corresponds to an available information channel between agents $i$ and $k$. The dynamical equations of agent $i$ of the double-integrator FOMAS are described as follows:

$$\begin{aligned} x_i^{(\alpha)}(t) &= y_i(t), \\ y_i^{(\alpha)}(t) &= u_i(t), \\ & i \in \mathcal{I}, \end{aligned} \quad (4)$$

where $x_i(t) \in \mathbb{R}$ and $y_i(t) \in \mathbb{R}$, respectively, represent the $i$th agent's two states, $x_i^{(\alpha)}(t)$ and $y_i^{(\alpha)}(t)$, respectively, denote the $\alpha$-order Caputo derivatives of $x_i(t)$ and $y_i(t)$ ($\alpha \in (0, 1]$), and $u_i(t) \in \mathbb{R}$ represents control input.

*Definition 3.* The double-integrator FOMAS (4) reaches consensus, if and only if the states of agents satisfy

$$\begin{aligned} \lim_{t\to+\infty} (x_i(t) - x_k(t)) &= 0, \\ \lim_{t\to+\infty} (y_i(t) - y_k(t)) &= 0, \\ & i, k \in \mathcal{I}. \end{aligned} \quad (5)$$

Assume the following control protocol is given by

$$\begin{aligned} u_i(t) &= \sum_{k\in N_i} a_{ik} \{k_1 [x_k(t) - x_i(t)] + k_2 [y_k(t) - y_i(t)]\}, \end{aligned} \quad (6)$$

where $i, k \in \mathcal{I}$, $k_1, k_2 > 0$ are the scale coefficients, $a_{ik} > 0$ is the $(i,k)$th entry of the adjacency matrix $\mathcal{A}$ in $\mathcal{G}$, and $N_i$ denotes the neighbour set of $i$th agent.

Define the state vector of a single agent as $\zeta_i(t) \triangleq [x_i(t), y_i(t)]^T$; the joint state vector of the double-integrator FOMAS (4) is $\psi(t) \triangleq [\zeta_1(t)^T, \zeta_2(t)^T, \ldots, \zeta_n(t)^T]^T$.

Define two matrixes:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}. \end{aligned} \quad (7)$$

Let $\Phi \triangleq I_n \otimes A - \mathcal{L} \otimes B$; the double-integrator FOMAS (4) without time delay can be described as:

$$\psi^{(\alpha)}(t) = \Phi \psi(t), \quad (8)$$

where $\psi^{(\alpha)}(t)$ denotes the $\alpha$-order Caputo derivative of $\psi(t)$.

Consider that time delay cannot be avoided; the control protocol involving time delay can be given by

$$\begin{aligned} u_i(t) = \sum_{k\in N_i} a_{ik} \{ & k_1 [x_k(t-\tau) - x_i(t-\tau)] \\ & + k_2 [y_k(t-\tau) - y_i(t-\tau)]\}, \end{aligned} \quad (9)$$

where $\tau > 0$ is the time delay.

Now, we adopt protocol (9); the double-integrator FOMAS (4) with time delay can be described as

$$\psi^{(\alpha)}(t) = (I_n \otimes A)\psi(t) - (\mathcal{L} \otimes B)\psi(t-\tau). \quad (10)$$

## 4. Main Results

### *4.1. Consensus of the Double-Integrator FOMAS with Time Delay over Undirected Topology*

**Theorem 4.** *Suppose that a double-integrator FOMAS is composed of n agents whose network topology $\mathcal{G}$ is connected and undirected. By the distributed control protocol (9), the double-integrator FOMAS (10) with time delay can asymptotically reach consensus, if*

$$\begin{aligned} \tau &< \overline{\tau} \\ &= \frac{1}{\overline{\omega}} \left[\pi(1-\alpha) + \arctan \frac{k_2 \overline{\omega}^\alpha \sin(\pi\alpha/2)}{k_1 + k_2 \overline{\omega}^\alpha \cos(\pi\alpha/2)}\right]; \end{aligned} \quad (11)$$

*the value of $\overline{\omega}$ is given by the following equation:*

$$\frac{\overline{\omega}^{2\alpha}}{\sqrt{k_1^2 + k_2^2 \overline{\omega}^{2\alpha} + 2k_1 k_2 \overline{\omega}^\alpha \cos(\pi\alpha/2)}} = \lambda_n, \quad (12)$$

*where $\lambda_n$ is the maximum eigenvalue of the Laplacian matrix $\mathcal{L}$.*

*Proof.* Applying the frequency-domain method to analyze the double-integrator FOMAS (10) and defining $\Psi(s)$ as the Laplace transform of $\psi(t)$, it is obvious that $\Psi(s) = G_\tau^{-1}(s)\psi(0^-)$ and

$$G_\tau(s) = s^\alpha I_{2n} - I_n \otimes A + (\mathscr{L} \otimes B) e^{-\tau s}. \tag{13}$$

For the stability analysis of a fractional-order system, Matignon has pointed out that the stability of the linear fractional-order system can be judged by characteristic roots of the system function and all the poles of the system function have negative real parts if the linear fractional-order system is stable [28, 29]. So we can analyze the root's position of the characteristic polynomial $\det[G_\tau(s)]$ to study consensus of the double-integrator FOMAS (10). Specifically, as $\tau$ increases continuously from zero, the characteristic root will change continuously from the left half plane (LHP) to the right half plane (RHP). Once the characteristic root passes through the imaginary axis to the RHP, the double-integrator FOMAS (10) will be unstable and cannot achieve consensus. Therefore, it is necessary for us to consider a critical condition where the nonzero characteristic root is located just on the imaginary axis and the corresponding time delay is just the maximum tolerable delay of the double-integrator FOMAS (10).

Suppose that $s = -j\omega \neq 0$ is a characteristic root of $\det[G_\tau(s)]$, $u = u_1 \otimes [1,0]^T + u_2 \otimes [0,1]^T$ is the corresponding eigenvector, and $\|u\| = 1$, $u_1, u_2 \in \mathbb{C}^n$; then we have the following equation:

$$\left\{(-j\omega)^\alpha I_{2n} - I_n \otimes A + (\mathscr{L} \otimes B) e^{j\omega\tau}\right\} u = 0. \tag{14}$$

Note that all the complex roots of each $\det[G_\tau(s)]$ appear in conjugated pairs; it is just for us to study the situation where $\omega > 0$. Since the elements of the vector obtained by calculating the left part of (14) are equal to zero, it can be easy to obtain that $(-j\omega)^\alpha u_1 = u_2$. Then, on the left side of (14), multiply by $u^H$ (the conjugate transpose of $u$); the following equation can be obtained:

$$\begin{gathered} u^H \left\{(-j\omega)^\alpha I_{2n} - I_n \otimes A + (\mathscr{L} \otimes B) e^{j\omega\tau}\right\} u = 0, \\ \underbrace{u^H \left\{I_n \otimes \begin{bmatrix} (-j\omega)^\alpha & 0 \\ 0 & (-j\omega)^\alpha \end{bmatrix} - I_n \otimes A\right\} u}_{S_1} + \underbrace{u^H \left\{(\mathscr{L} \otimes B) e^{j\omega\tau}\right\} u}_{S_2} = 0, \end{gathered} \tag{15}$$

where

$$\begin{aligned} u^H &= \left\{u_1 \otimes [1,0]^T + u_2 \otimes [0,1]^T\right\}^H \\ &= \left\{u_1^H \otimes [1,0] + u_2^H \otimes [0,1]\right\}. \end{aligned} \tag{16}$$

Because of $(-j\omega)^\alpha u_1 = u_2$, we get

$$\begin{aligned} S_1 &= u_1^H u_1 \otimes (-j\omega)^\alpha + u_1^H u_2 \otimes (-1) + u_2^H u_1 \otimes 0 \\ &\quad + u_2^H u_2 \otimes (-j\omega)^\alpha \\ &= u_1^H u_1 \otimes (-j\omega)^\alpha - u_1^H u_2 + u_2^H u_2 \otimes (-j\omega)^\alpha \\ &= (-j\omega)^\alpha u_1^H u_1 - u_1^H u_2 + (-j\omega)^\alpha u_2^H u_2 \\ &= (-j\omega)^\alpha u_2^H u_2; \end{aligned} \tag{17}$$

because of $u_1 = (-j\omega)^{-\alpha} u_2$, we get

$$\begin{aligned} &S_2 \\ &= \left(u_2^H (\mathscr{L} \otimes I_2) u_1 \otimes k_1 + u_2^H (\mathscr{L} \otimes I_2) u_2 \otimes k_2\right) e^{j\omega\tau} \\ &= \left(k_1 u_2^H (\mathscr{L} \otimes I_2) u_1 + k_2 u_2^H (\mathscr{L} \otimes I_2) u_2\right) e^{j\omega\tau} \\ &= \left[k_1 (-j\omega)^{-\alpha} + k_2\right] u_2^H (\mathscr{L} \otimes I_2) u_2 e^{j\omega\tau}. \end{aligned} \tag{18}$$

Finally, we can get

$$\begin{aligned} &(-j\omega)^\alpha u_2^H u_2 + \left[k_1 (-j\omega)^{-\alpha} + k_2\right] u_2^H (\mathscr{L} \otimes I_2) u_2 e^{j\omega\tau} \\ &\quad = 0, \\ &\left[k_1 (-j\omega)^{-\alpha} + k_2\right] u_2^H (\mathscr{L} \otimes I_2) u_2 e^{j\omega\tau} \\ &\quad = -(-j\omega)^\alpha u_2^H u_2, \end{aligned}$$

$$\begin{aligned} \frac{u_2^H (\mathscr{L} \otimes I_2) u_2 e^{j\omega\tau}}{u_2^H u_2} &= \frac{u^H (\mathscr{L} \otimes I_2) u e^{j\omega\tau}}{u^H u} \\ &= \frac{-(-j\omega)^\alpha}{k_1 (-j\omega)^{-\alpha} + k_2} = \frac{-(-j\omega)^{2\alpha}}{k_1 + k_2 (-j\omega)^\alpha}. \end{aligned} \tag{19}$$

Define

$$a_n = \frac{u^H (\mathscr{L} \otimes I_2) u}{u^H u}; \tag{20}$$

there is

$$a_n e^{j\omega\tau} = \frac{-(-j\omega)^{2\alpha}}{k_1 + k_2 (-j\omega)^\alpha}. \tag{21}$$

Because $j = \cos(\pi/2) + j\sin(\pi/2) = e^{j(\pi/2)}$, $(-j) = \cos(-\pi/2) + j\sin(-\pi/2) = e^{j(-\pi/2)}$, it is easy to know that $-(-j\omega)^{2\alpha} = -\omega^{2\alpha} \cdot (-j)^{2\alpha} = -\omega^{2\alpha} e^{-j\pi\alpha} = \omega^{2\alpha} e^{j\pi(1-\alpha)}$, and $(-j\omega)^\alpha = \omega^\alpha (-j)^\alpha = \omega^\alpha e^{j(-\pi\alpha/2)}$.

So (21) can be simplified to the following equation:

$$\begin{aligned} a_n e^{j\omega\tau} &= \frac{-(-j\omega)^{2\alpha}}{k_1 + k_2 (-j\omega)^\alpha} = \frac{\omega^{2\alpha} e^{j\pi(1-\alpha)}}{k_1 + k_2 \omega^\alpha e^{j(-\pi\alpha/2)}} \\ &= \frac{\omega^{2\alpha} e^{j\pi(1-\alpha)}}{k_1 + k_2 \omega^\alpha \left[\cos(-\pi\alpha/2) + j\sin(-\pi\alpha/2)\right]} \\ &= \frac{\omega^{2\alpha} e^{j\pi(1-\alpha)}}{k_1 + k_2 \omega^\alpha \left(\cos(\pi\alpha/2) - j\sin(\pi\alpha/2)\right)} \\ &= \frac{\omega^{2\alpha} e^{j\pi(1-\alpha)}}{k_1 + k_2 \omega^\alpha \cos(\pi\alpha/2) - j k_2 \omega^\alpha \sin(\pi\alpha/2)} \end{aligned}$$

$$= \frac{\omega^{2\alpha} e^{j[\pi(1-\alpha)+\arctan(k_2\omega^\alpha \sin(\pi\alpha/2)/(k_1+k_2\omega^\alpha \cos(\pi\alpha/2)))]}}{\sqrt{k_1^2 + k_2^2\omega^{2\alpha} + 2k_1k_2\omega^\alpha \cos(\pi\alpha/2)}} \triangleq F(\omega). \tag{22}$$

Take modulus of the both sides of (22), and, according to Lemma 1, we can get the following inequality:

$$M(\omega) \triangleq |F(\omega)| = \left|a_n e^{j\omega\tau}\right| \le a_n = \frac{u^H(\mathscr{L}\otimes I_2)u}{u^H u} \le M(\overline{\omega}) = \lambda_n; \tag{23}$$

it is obvious that $M(\omega)$ is an increasing function for $\omega > 0$, and if $\omega \le \overline{\omega}$, we can get $M(\omega) \le M(\overline{\omega}) = \lambda_n$; that is, inequality (23) is true.

In addition, the principal value of the argument of $F(\omega)$ is as follows:

$$\theta(\omega) \triangleq \arg[F(\omega)] = \pi(1-\alpha) + \arctan\frac{k_2\omega^\alpha \sin(\pi\alpha/2)}{k_1 + k_2\omega^\alpha \cos(\pi\alpha/2)}. \tag{24}$$

Let

$$\begin{aligned}\tau(\omega) &\triangleq \frac{\theta(\omega)}{\omega} \\ &= \frac{\pi(1-\alpha) + \arctan(k_2\omega^\alpha \sin(\pi\alpha/2)/(k_1+k_2\omega^\alpha\cos(\pi\alpha/2)))}{\omega} \\ &= \frac{1}{\omega}\left[\pi(1-\alpha) + \arctan\frac{k_2\omega^\alpha \sin(\pi\alpha/2)}{k_1 + k_2\omega^\alpha\cos(\pi\alpha/2)}\right] \\ &= \underbrace{\frac{\pi}{\omega}(1-\alpha)}_{L_1(\omega)} + \underbrace{\frac{1}{\omega}\arctan\frac{k_2\omega^\alpha \sin(\pi\alpha/2)}{k_1 + k_2\omega^\alpha\cos(\pi\alpha/2)}}_{L_2(\omega)}.\end{aligned} \tag{25}$$

Define $p = \sin(\pi\alpha/2)$, $q = \cos(\pi\alpha/2)$. Obviously, $p^2 + q^2 = 1$. Then the first derivative of is $\tau(\omega)$ as follows:

$$\Gamma(\omega) \triangleq \frac{d\tau(\omega)}{d\omega} = L_1'(\omega) + L_2'(\omega), \tag{26}$$

where

$$\begin{aligned}L_1'(\omega) &= -\frac{\pi}{\omega^2}(1-\alpha) \le 0, \\ L_2'(\omega) &= \frac{\left[\alpha k_2 p\omega^{\alpha-1}(k_1+k_2q\omega^\alpha) - \alpha k_2^2 pq\omega^{2\alpha-1}\right]}{\left\{\omega\left[(k_1+k_2q\omega^\alpha)^2 + k_2^2p^2\omega^{2\alpha}\right]\right\}} - \frac{\{\arctan[k_2p\omega^\alpha/(k_1+k_2q\omega^\alpha)]\}}{\omega^2} \\ &= \frac{\left[\alpha k_2 p\omega^{\alpha}(k_1+k_2q\omega^\alpha) - \alpha k_2^2 pq\omega^{2\alpha}\right]}{\left\{\omega^2\left[(k_1+k_2q\omega^\alpha)^2 + k_2^2p^2\omega^{2\alpha}\right]\right\}} - \frac{\{\arctan[k_2p\omega^\alpha/(k_1+k_2q\omega^\alpha)]\}}{\omega^2} \\ &= \frac{\alpha k_1 k_2 p\omega^\alpha}{\left\{\omega^2\left[(k_1+k_2q\omega^\alpha)^2 + k_2^2p^2\omega^{2\alpha}\right]\right\}} - \frac{\{\arctan[k_2p\omega^\alpha/(k_1+k_2\omega^\alpha q)]\}}{\omega^2} \\ &= \frac{\left\{\alpha k_1k_2p\omega^\alpha/\left[(k_1+k_2q\omega^\alpha)^2 + k_2^2p^2\omega^{2\alpha}\right] - \arctan[k_2p\omega^\alpha/(k_1+k_2q\omega^\alpha)]\right\}}{\omega^2}.\end{aligned} \tag{27}$$

Construct the following function:

$$\mathscr{H}(\omega) = \arctan\left[\frac{k_2p\omega^\alpha}{(k_1+k_2q\omega^\alpha)}\right] - \frac{\alpha k_1k_2p\omega^\alpha}{\left[(k_1+k_2q\omega^\alpha)^2 + k_2^2p^2\omega^{2\alpha}\right]}; \tag{28}$$

then the first derivative of $\mathscr{H}$ is as follows:

$$\begin{aligned}\mathscr{H}'(\omega) &= \frac{\left[\alpha k_2 p\omega^{\alpha-1}(k_1+k_2q\omega^\alpha) - \alpha k_2^2 pq\omega^{2\alpha-1}\right]}{\left[(k_1+k_2q\omega^\alpha)^2 + k_2^2p^2\omega^{2\alpha}\right]} \\ &\quad - \frac{\left\{\alpha^2k_1k_2p\omega^{\alpha-1}\left[(k_1+k_2q\omega^\alpha)^2 + k_2^2p^2\omega^{2\alpha}\right] - \left[2(k_1+k_2q\omega^\alpha)\alpha k_2q\omega^{\alpha-1} + 2\alpha k_2^2p^2\omega^{2\alpha-1}\right]\alpha k_1k_2p\omega^\alpha\right\}}{\left[(k_1+k_2q\omega^\alpha)^2 + k_2^2p^2\omega^{2\alpha}\right]^2} \\ &= \frac{\left\{\alpha k_1k_2p\omega^{\alpha-1}\left[(k_1+k_2q\omega^\alpha)^2 + k_2^2p^2\omega^{2\alpha}\right] - \alpha^2k_1k_2p\omega^{\alpha-1}\left[(k_1+k_2q\omega^\alpha)^2 + k_2^2p^2\omega^{2\alpha}\right] + \left[2(k_1+k_2q\omega^\alpha)\alpha k_2q\omega^{\alpha-1} + 2\alpha k_2^2p^2\omega^{2\alpha-1}\right]\alpha k_1k_2p\omega^\alpha\right\}}{\left[(k_1+k_2q\omega^\alpha)^2 + k_2^2p^2\omega^{2\alpha}\right]^2}.\end{aligned} \tag{29}$$

Note that the denominator of $\mathscr{H}'(\omega)$ is greater than 0; we just need to consider the molecule of $\mathscr{H}'(\omega)$. $\xi(\omega)$ denotes the molecule of $\mathscr{H}'(\omega)$; we have

$$\begin{aligned}\xi(\omega) &= \alpha k_1 k_2 p\omega^{\alpha-1}\left[(k_1+k_2q\omega^{\alpha})^2+k_2^2p^2\omega^{2\alpha}\right] \\ &\quad -\alpha^2 k_1k_2p\omega^{\alpha-1}\left[(k_1+k_2q\omega^{\alpha})^2+k_2^2p^2\omega^{2\alpha}\right] \\ &\quad +\left[2(k_1+k_2q\omega^{\alpha})\alpha k_2q\omega^{\alpha-1}+2\alpha k_2^2p^2\omega^{2\alpha-1}\right] \\ &\quad \cdot\alpha k_1k_2p\omega^{\alpha} = \alpha(1-\alpha)k_1k_2p\omega^{\alpha-1}\left[k_1^2\right. \\ &\quad \left.+2k_1k_2q\omega^{\alpha}+k_2^2\omega^{2\alpha}\right]+\left[2\alpha k_1k_2q\omega^{\alpha-1}\right. \\ &\quad \left.+2\alpha k_2^2\omega^{2\alpha-1}\right]\alpha k_1k_2p\omega^{\alpha} \\ &= \alpha k_1k_2p\omega^{\alpha-1}\left[(1-\alpha)k_1^2+2(1-\alpha)k_1k_2q\omega^{\alpha}\right. \\ &\quad \left.+(1-\alpha)k_2^2\omega^{2\alpha}+2\alpha k_1k_2q\omega^{\alpha}+2\alpha k_2^2\omega^{2\alpha}\right] \\ &= \alpha k_1k_2p\omega^{\alpha-1}\left[(1-\alpha)k_1^2+2k_1k_2q\omega^{\alpha}\right. \\ &\quad \left.+(1+\alpha)k_2^2\omega^{2\alpha}\right]>0.\end{aligned} \tag{30}$$

Due to $\xi(\omega)>0$ and $\mathscr{H}'(\omega)>0$, one has $\mathscr{H}(\omega)>\mathscr{H}(0)=0$ for $\omega>0$. Moreover, $\mathscr{H}(\omega)>0$; thus, $L_2'(\omega)<0$, which means that $\Gamma(\omega)<0$. Therefore, $\tau(\omega)$ is a decreasing function for $\omega>0$; when $\omega\le\overline{\omega}$, one has

$$\overline{\tau}=\tau(\overline{\omega})\le\tau(\omega). \tag{31}$$

Note that the above inequality (31) is based on the assumption that $\det[G_\tau(s)]$'s root of the double-integrator FOMAS (10) exists on the imaginary axis. If we let $\tau<\overline{\tau}$, we can obtain the following inequality:

$$\tau(\omega)=\frac{\theta(\omega)}{\omega}=\frac{\arg\left(a_ne^{j\omega\tau}\right)}{\omega}\le\frac{\max\{\omega\tau\}}{\omega}<\frac{\omega\overline{\tau}}{\omega}=\overline{\tau}. \tag{32}$$

Inequality (32) contradicts inequality (31). That is, as long as $\tau<\overline{\tau}$, we can ensure that all the characteristic roots of $\det[G_\tau(s)]$ are located in the LHP of the imaginary axis; the double-integrator FOMAS (10) can achieve consensus. On the other hand, when $\tau=\overline{\tau}$, $-j\overline{\omega}$ is a characteristic root of $\det[G_\tau(s)]$, and the corresponding vector $u(\overline{\omega})$ makes $|a_n|=\lambda_n$ hold. So $\overline{\tau}$ is called the maximum tolerable delay. In addition, when $\tau>\overline{\tau}$, the double-integrator FOMAS (10) must have characteristic roots of $\det[G_\tau(s)]$ which are located in the RHP of the imaginary axis. According to the principle of stability, the double-integrator FOMAS (10) cannot achieve consensus.

This completes the proof of Theorem 4. □

**Corollary 5.** *Suppose that a double-integrator FOMAS is composed of $n$ agents whose network topology $\mathscr{G}$ is connected and undirected. When the fractional order satisfies $\alpha=1$, by the distributed control protocol (9), the double-integrator FOMAS (10) with time delay can asymptotically reach consensus, if*

$$\tau<\overline{\tau}=\frac{1}{\overline{\omega}}\left[\arctan\left(\frac{k_2}{k_1}\overline{\omega}\right)\right]; \tag{33}$$

*the value of $\overline{\omega}$ is given by the following equation:*

$$\frac{\overline{\omega}^2}{\sqrt{k_1^2+(k_2\overline{\omega})^2}}=\lambda_n, \tag{34}$$

*where $\lambda_n$ is the maximum eigenvalue of the Laplacian matrix $\mathscr{L}$.*

### *4.2. Consensus of the Double-Integrator FOMAS with Time Delay over Directed Topology*

**Theorem 6.** *Suppose that a double-integrator FOMAS is composed of $n$ agents whose network topology $\mathscr{G}$ is directed and has a spanning tree. By the distributed control protocol (9), the double-integrator FOMAS (10) with time delay can asymptotically reach consensus, if*

$$\begin{aligned}\tau<\overline{\tau}=\min_{|\lambda_i|\ne0}&\left\{\frac{1}{\overline{\omega}_i}\left[\pi(1-\alpha)\right.\right. \\ &\left.\left.+\arctan\frac{k_2\overline{\omega}_i^{\alpha}\sin(\pi\alpha/2)}{k_1+k_2\overline{\omega}_i^{\alpha}\cos(\pi\alpha/2)}-\arg(\lambda_i)\right]\right\},\end{aligned} \tag{35}$$

*where $\arg(\lambda_i)\in(-\pi/2,\pi/2)$ and $\lambda_i$ is the $i$th eigenvalue of the Laplacian matrix $\mathscr{L}$. The value of $\overline{\omega}_i$ is given by the following equation:*

$$\frac{\overline{\omega}_i^{2\alpha}}{\sqrt{k_1^2+k_2^2\overline{\omega}_i^{2\alpha}+2k_1k_2\overline{\omega}_i^{\alpha}\cos(\pi\alpha/2)}}=|\lambda_i|. \tag{36}$$

*Proof.* Similarly to the proof method of Theorem 4, we suppose that $s=-j\omega\ne0$ is the characteristic root of the system matrix on the imaginary axis, $u$ is the corresponding eigenvector, and $u=u_1\otimes[1,0]^T+u_2\otimes[0,1]^T$ and $\|u\|=1$, $u_1,u_2\in\mathbb{C}^n$. According to Lemma 2, we can get the following equation:

$$\begin{aligned}B_a&\triangleq a_ne^{j\omega\tau}=\frac{-(-j\omega)^{2\alpha}}{k_1+k_2(-j\omega)^{\alpha}} \\ &=\frac{\omega^{2\alpha}e^{j\pi(1-\alpha)}}{k_1+k_2\omega^{\alpha}\cos(\pi\alpha/2)-jk_2\omega^{\alpha}\sin(\pi\alpha/2)}.\end{aligned} \tag{37}$$

Taking modulus of the both sides of (37), we get

$$\begin{aligned}|B_a|&=\left|\frac{\omega^{2\alpha}e^{j\pi(1-\alpha)}}{k_1+k_2\omega^{\alpha}\cos(\pi\alpha/2)-jk_2\omega^{\alpha}\sin(\pi\alpha/2)}\right| \\ &=\frac{\omega^{2\alpha}}{\sqrt{k_1^2+k_2^2\omega^{2\alpha}+2k_1k_2\omega^{\alpha}\cos(\pi\alpha/2)}},\end{aligned} \tag{38}$$

where $|B_a|$ is an increasing function for $\omega>0$; thus $\omega(|B_a|)$ is also an increasing function for $|B_a|$.

Calculate the principal value of the argument of (37); we get

$$\arg(B_a)=\pi(1-\alpha)+\arctan\frac{k_2\omega^{\alpha}\sin(\pi\alpha/2)}{k_1+k_2\omega^{\alpha}\cos(\pi\alpha/2)}. \tag{39}$$

According to the definition of $B_a$ in (37), we get

$$\arg(B_a)\le\arg(a_n)+\max(\omega\tau), \tag{40}$$

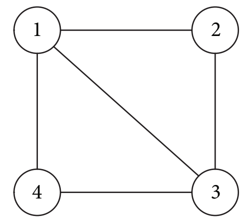


Figure 1: The connected interaction topology in Example 1.

so one has

$$\max(\omega\tau) \geq \pi(1-\alpha) + \arctan\frac{k_2\omega^\alpha \sin(\pi\alpha/2)}{k_1 + k_2\omega^\alpha \cos(\pi\alpha/2)} - \arg(a_n). \tag{41}$$

Due to $a_n = u^H(\mathscr{L} \otimes \mathrm{I}_2)u/u^H u$, the possible values of $a_n$ must be nonzero eigenvalues of the Laplacian matrix $\mathscr{L}$ of the graph $\mathscr{G}$; that is, $a_n = \lambda_i$ $(\lambda_i \neq 0)$, which makes the maximum tolerable delay $\overline{\tau}$ minimized. So when $|B_a| \leq |\lambda_i|$, $\omega(|B_a|) \leq \omega(|\lambda_i|) = \overline{\omega}_i$. If we let $\tau < \overline{\tau}$, one has

$$\begin{aligned}
\max(\omega\tau) < \overline{\omega}_i\overline{\tau} &= \min_{|\lambda_i|\neq 0}\left\{\frac{\left[\pi(1-\alpha) + \arctan\left(k_2\overline{\omega}_i^\alpha \sin(\pi\alpha/2)/\left(k_1 + k_2\overline{\omega}_i^\alpha \cos(\pi\alpha/2)\right)\right) - \arg(\lambda_i)\right]}{\overline{\omega}_i}\right\}\overline{\omega}_i \\
&= \min_{|\lambda_i|\neq 0}\left\{\frac{\left[\pi(1-\alpha) + \arctan\left(k_2\overline{\omega}_i^\alpha \sin(\pi\alpha/2)/\left(k_1 + k_2\overline{\omega}_i^\alpha \cos(\pi\alpha/2)\right)\right) - \arg(a_n)\right]}{\overline{\omega}_i}\right\}\overline{\omega}_i \\
&\leq \pi(1-\alpha) + \arctan\frac{k_2\omega^\alpha \sin(\pi\alpha/2)}{k_1 + k_2\omega^\alpha \cos(\pi\alpha/2)} - \arg(a_n).
\end{aligned} \tag{42}$$

Inequality (42) contradicts inequality (41). That is, when $\tau < \overline{\tau}$, the eigenvalues of the system matrix of the double-integrator FOMAS (10) cannot reach or cross the imaginary axis; then the double-integrator FOMAS (10) will remain stable and consensus of the double-integrator FOMAS (10) can be achieved. On the other hand, when $\tau > \overline{\tau}$, there must exist at least one eigenvalue of the system matrix of the double-integrator FOMAS (10) in the RHP; then the states of the double-integrator FOMAS (10) are no longer convergent; the double-integrator FOMAS (10) cannot reach consensus.

This completes the proof of Theorem 6. □

**Corollary 7.** *Suppose that a double-integrator FOMAS is composed of n agents whose network topology $\mathscr{G}$ is directed and has a spanning tree. When the fractional order satisfies $\alpha = 1$, by the distributed control protocol (9), the double-integrator FOMAS (10) with time delay can asymptotically reach consensus, if $k_1 \in (0, \rho k_2^2)$ [30] and*

$$\tau < \overline{\tau} = \min_{|\lambda_i|\neq 0}\left\{\frac{1}{\overline{\omega}_i}\left[\arctan\left(\frac{k_2}{k_1}\overline{\omega}_i\right) - \arg(\lambda_i)\right]\right\}, \tag{43}$$

*where $\rho = \min_{|\lambda_i|\neq 0}\{|\lambda_i|^2 \operatorname{Re}(\lambda_i)/\operatorname{Im}(\lambda_i)^2\}$, $\arg(\lambda_i) \in (-\pi/2, \pi/2)$, and $\lambda_i$ is the ith eigenvalue of the Laplacian matrix $\mathscr{L}$. The value of $\overline{\omega}_i$ is given by the following equation:*

$$\frac{\overline{\omega}_i^2}{\sqrt{k_1^2 + (k_2\overline{\omega}_i)^2}} = |\lambda_i|. \tag{44}$$

## 5. Simulation Results

To illustrate the correctness of the theoretical results, numerical simulations will be given in this section.

*Example 1* (simulations for Theorem 4). Consider a double-integrator FOMAS with four agents whose dynamics are described by (10) with $\alpha = 0.8$, $k_1 = 1$, and $k_2 = 1$. The connected interaction graph $\mathscr{G}$ of the double-integrator FOMAS (10) is undirected and is shown in Figure 1, and the Laplacian matrix of which is as follows:

$$\mathscr{L} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}. \tag{45}$$

It is easy to obtain that the four eigenvalues of the Laplacian matrix $\mathscr{L}$ are 0, 2, 4, and 4, respectively; then $\lambda_n = 4$. According to Theorem 4, the maximum tolerable delay $\overline{\tau}$ of the double-integrator FOMAS (10) is 0.266567 s. Assume that the initial states of the double-integrator FOMAS (10) are taken as $x_1(t=0) = -3$, $y_1(t=0) = -3$, $x_2(t=0) = -1.5$, $y_2(t=0) = -1.5$, $x_3(t=0) = 1.5$, $y_3(t=0) = 1.5$, and $x_4(t=0) = 3$, $y_4(t=0) = 3$. Then three different time delays are used for simulation: (1) $\tau = 0.26$ s, (2) $\tau = 0.25$ s, and (3) $\tau = 0.24$ s.

Figures 2, 3, and 4 show the trajectories of all the agents' states $x_i(t)$. Figures 5, 6, and 7 show the trajectories of all

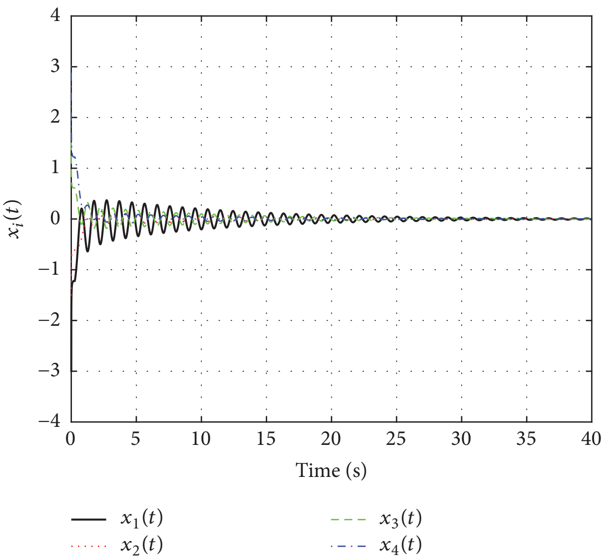


FIGURE 2: The trajectories of $x_i(t)$ when (1)$\tau < \overline{\tau}$ in Example 1.

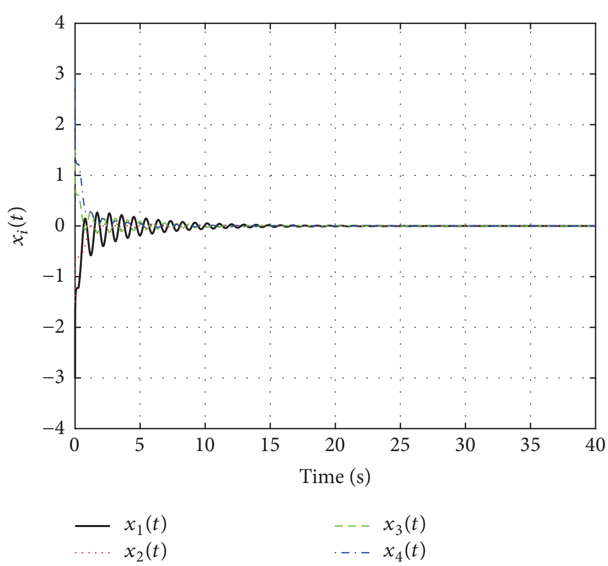


FIGURE 3: The trajectories of $x_i(t)$ when (2)$\tau < \overline{\tau}$ in Example 1.

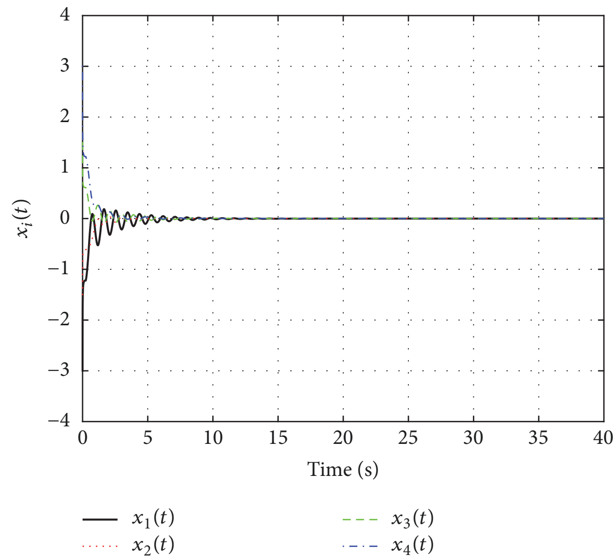


FIGURE 4: The trajectories of $x_i(t)$ when (3)$\tau < \overline{\tau}$ in Example 1.

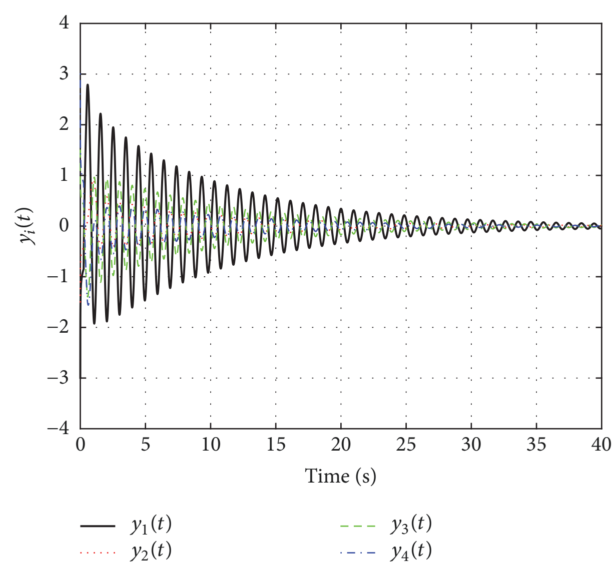


FIGURE 5: The trajectories of $y_i(t)$ when (1)$\tau < \overline{\tau}$ in Example 1.

the agents' states $y_i(t)$. It is clear that the double-integrator FOMAS (10) can reach consensus.

By comparing the simulation results of Figures 2, 3, and 4 and Figures 5, 6, and 7, we can find that as the time delay stays away from the maximum tolerable delay $\overline{\tau}$ when the communication topology and the fractional-order $\alpha$ of the double-integrator FOMAS (10) stay the same, the convergence speed will become faster, whereas the convergence speed will become slower as the time delay is close to the maximum tolerable delay $\overline{\tau}$.

In order to make a comparison, under the same conditions, suppose that $\tau = 0.27$ s. Figures 8 and 9 show the trajectories of all the agents' states $x_i(t)$ and $y_i(t)$, respectively. It is obvious that the double-integrator FOMAS (10) cannot reach consensus.

The above simulation results are consistent with Theorem 4. So the correctness of Theorem 4 is validated.

*Example 2* (simulations for Theorem 6). Consider a double-integrator FOMAS with four agents whose dynamics are described by (10) with $\alpha = 0.8$, $k_1 = 1$, and $k_2 = 1$. The directed graph $\mathscr{G}$ of the double-integrator FOMAS (10) has a spanning tree and is shown in Figure 10, the Laplacian matrix of which is as follows:

$$\mathscr{L} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 2 \end{bmatrix}. \qquad (46)$$

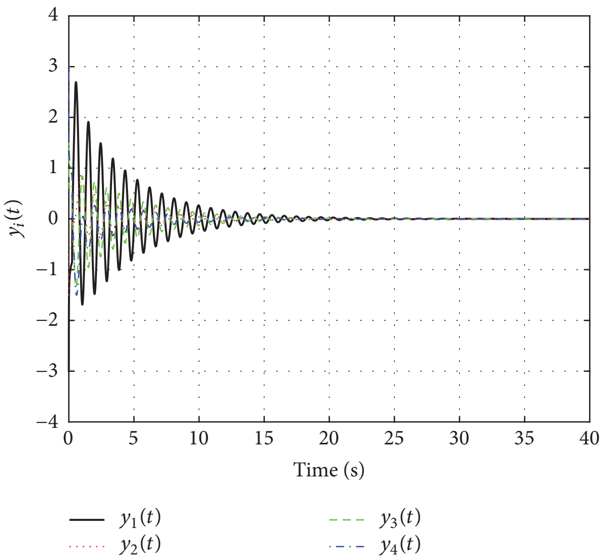


FIGURE 6: The trajectories of $y_i(t)$ when (2)$\tau < \overline{\tau}$ in Example 1.

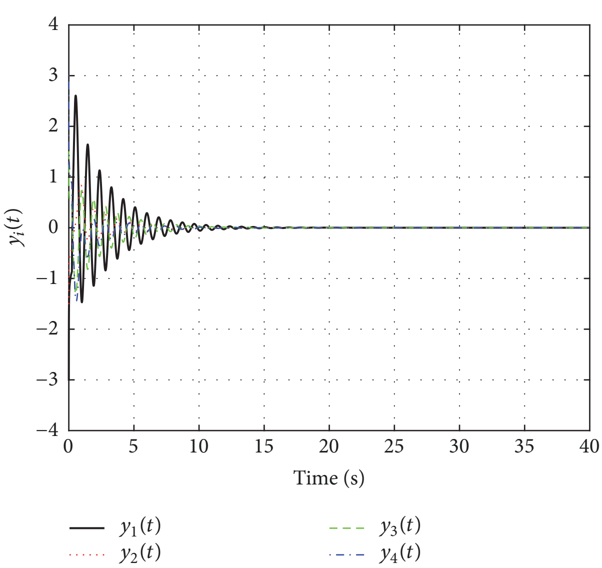


FIGURE 7: The trajectories of $y_i(t)$ when (3)$\tau < \overline{\tau}$ in Example 1.

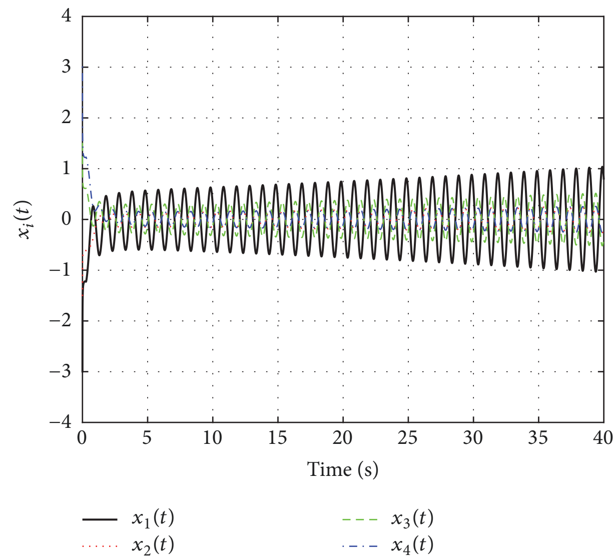


FIGURE 8: The trajectories of $x_i(t)$ when $\tau > \overline{\tau}$ in Example 1.

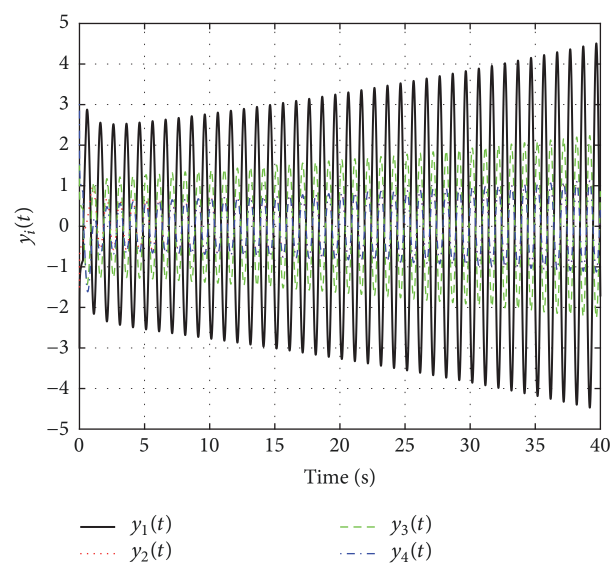


FIGURE 9: The trajectories of $y_i(t)$ when $\tau > \overline{\tau}$ in Example 1.

It is easy to obtain that the four eigenvalues of the Laplacian matrix $\mathscr{L}$ are $2, 0, 1.5000 + 0.866i$ and $1.5000 - 0.866i$, respectively. According to Theorem 6, the maximum tolerable delay $\overline{\tau}$ of the double-integrator FOMAS (10) is 0.3847 s. Assume that the initial states of the double-integrator FOMAS (10) are taken as $x_1(t = 0) = -3$, $y_1(t = 0) = -3$, $x_2(t = 0) = -1.5$, $y_2(t = 0) = -1.5$, $x_3(t = 0) = 1.5$, $y_3(t = 0) = 1.5$, and $x_4(t = 0) = 3$, $y_4(t = 0) = 3$. Then three different time delays are used simulation: (1) $\tau = 0.37$ s, (2) $\tau = 0.36$ s, and (3) $\tau = 0.35$ s.

Figures 11, 12, and 13 show the trajectories of all the agents' states $x_i(t)$. Figures 14, 15, and 16 show the trajectories of all the agents' states $y_i(t)$. It is clear that the double-integrator FOMAS (10) can reach consensus.

By comparing the simulation results of Figures 11, 12, and 13 and Figures 14, 15, and 16, we can find that as the time delay stays away from the maximum tolerable delay $\overline{\tau}$ when the communication topology and the fractional-order $\alpha$ of the double-integrator FOMAS (10) stay the same, the convergence speed will become faster, whereas the convergence speed will become slower as the time delay is close to the maximum tolerable delay $\overline{\tau}$.

In order to make a comparison, under the same conditions, suppose that $\tau = 0.39$ s. Figures 17 and 18 show the trajectories of all the agents' states $x_i(t)$ and $y_i(t)$, respectively. It is obvious that the double-integrator FOMAS (10) cannot reach consensus.

The above simulation results are consistent with Theorem 6. So the correctness of Theorem 6 is validated.

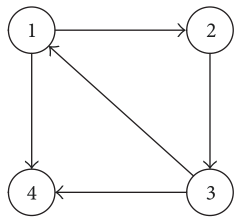


Figure 10: The connected interaction topology in Example 2.

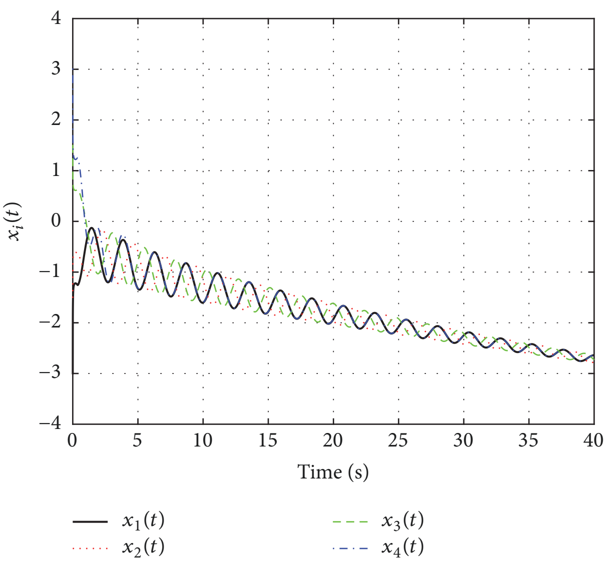


Figure 11: The trajectories of $x_i(t)$ when (1)$\tau < \overline{\tau}$ in Example 2.

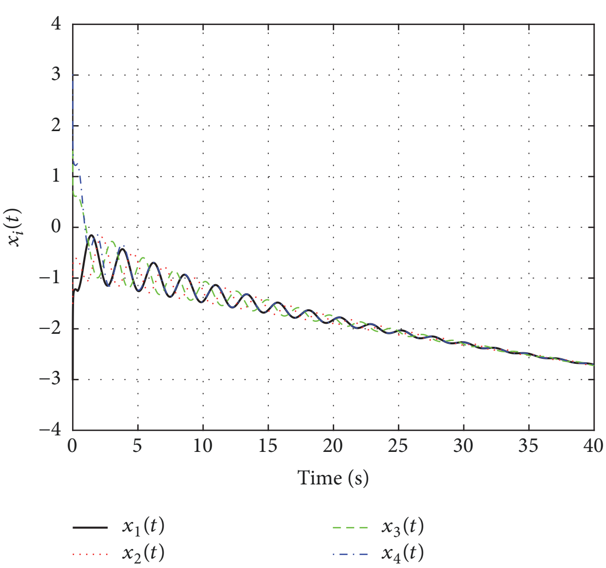


Figure 12: The trajectories of $x_i(t)$ when (2)$\tau < \overline{\tau}$ in Example 2.

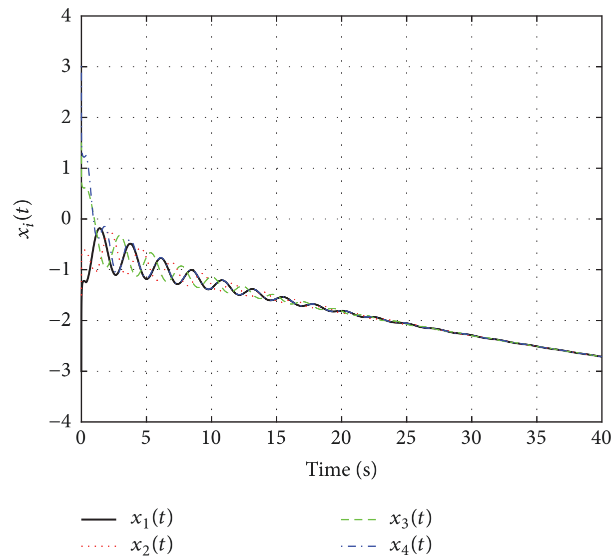


Figure 13: The trajectories of $x_i(t)$ when (3)$\tau < \overline{\tau}$ in Example 2.

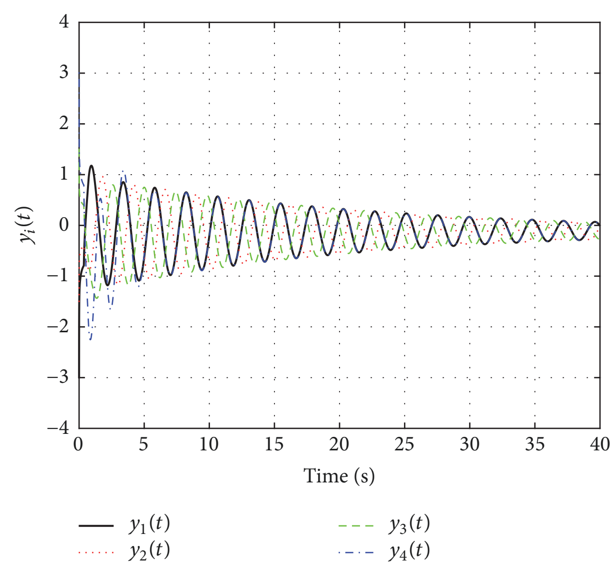


Figure 14: The trajectories of $y_i(t)$ when (1)$\tau < \overline{\tau}$ in Example 2.

## 6. Conclusion

In this paper, the consensus control problems of the double-integrator FOMAS with time delay are studied. First of all, the consensus problem is investigated for the double-integrator FOMAS with time delay over undirected topology, and a maximum tolerable delay is obtained to ensure the consensus. Then, the consensus problem is investigated for the double-integrator FOMAS with time delay over directed topology, and a maximum tolerable delay is obtained to ensure the consensus. Moreover, extending above results, some corollaries are obtained for the corresponding integer-order multiagent systems, which are the same as traditional integer-order systems. Finally, the relationship between the speed of convergence and time delay is revealed, and the correctness of our theoretical results is validated by the simulations.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

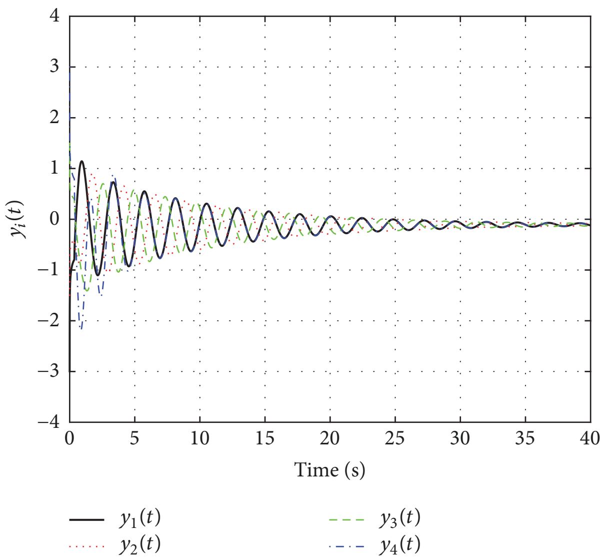


Figure 15: The trajectories of $y_i(t)$ when (2)$\tau < \overline{\tau}$ in Example 2.

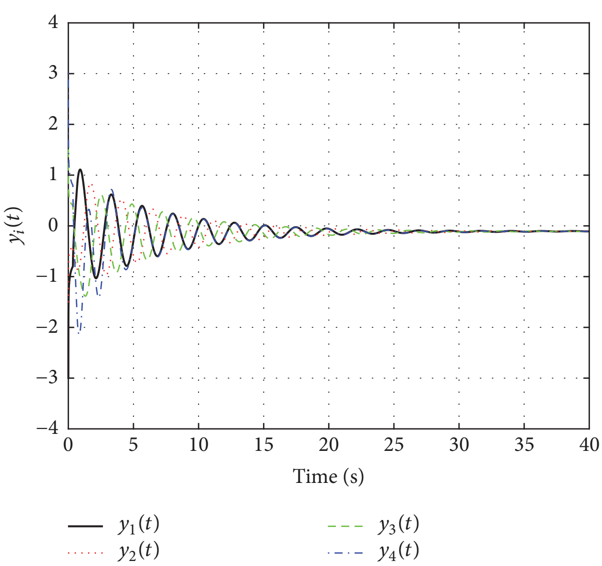


Figure 16: The trajectories of $y_i(t)$ when (3)$\tau < \overline{\tau}$ in Example 2.

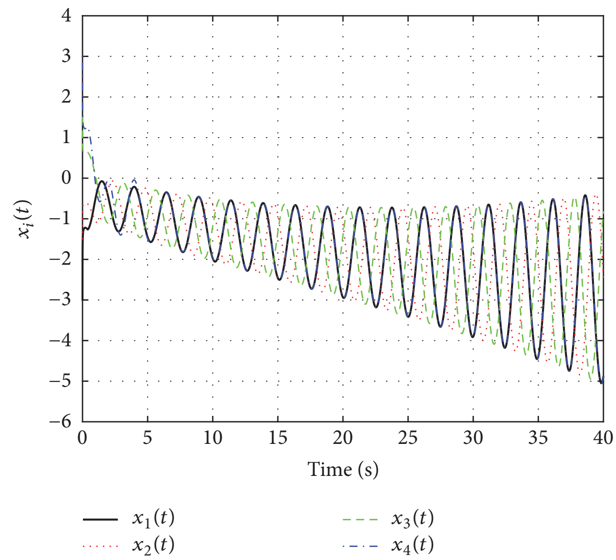


Figure 17: The trajectories of $x_i(t)$ when $\tau > \overline{\tau}$ in Example 2.

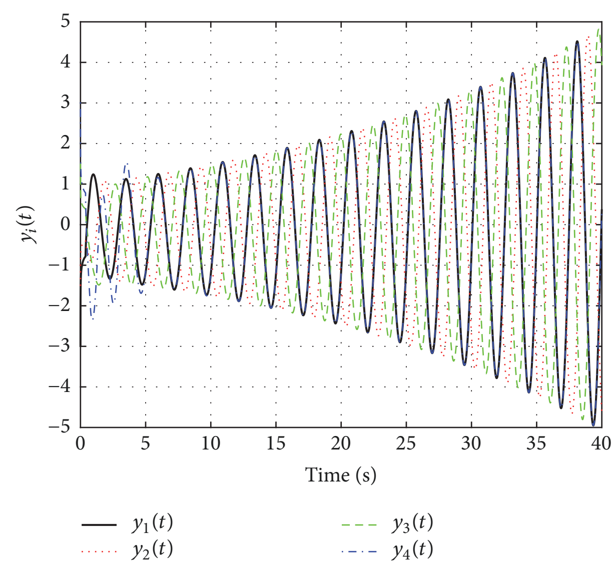


Figure 18: The trajectories of $y_i(t)$ when $\tau > \overline{\tau}$ in Example 2.

## Acknowledgments

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