Research Article

Hybrid B-Spline Collocation Method for Solving the Generalized Burgers-Fisher and Burgers-Huxley Equations

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In this study, we introduce a new numerical technique for solving nonlinear generalized Burgers-Fisher and Burgers-Huxley equations using hybrid B-spline collocation method. This technique is based on usual finite difference scheme and Crank-Nicolson method which are used to discretize the time derivative and spatial derivatives, respectively. Furthermore, hybrid B-spline function is utilized as interpolating functions in spatial dimension. The scheme is verified unconditionally stable using the Von Neumann (Fourier) method. Several test problems are considered to check the accuracy of the proposed scheme. The numerical results are in good agreement with known exact solutions and the existing schemes in literature.

1. Introduction

Nonlinear partial differential equations (PDEs) play a significant role in different fields of science and engineering. Many physical problems are designed in mathematical form using nonlinear PDEs [1–3]. The generalized Burgers-Fisher (GBF) and generalized Burgers-Huxley (GBH) equations describe many physical phenomena. Numerical treatment of these two equations have become a dominant tool due to complexities in finding their solutions.

Consider one-dimensional nonlinear partial differential equation of the following form:

$$u_t + \alpha u^\delta u_x - \epsilon u_{xx} = \beta f(u), \quad a \leq x \leq b, \quad 0 \leq t \leq T \quad (1)$$

subject to the initial condition,

$$u(x, 0) = g(x), \quad a \leq x \leq b, \quad (2)$$

and the boundary conditions,

$$\begin{aligned} u(a, t) &= g_1(t), \\ u(b, t) &= g_2(t), \end{aligned} \quad (3)$$

$$0 \leq t \leq T,$$

where $u = u(x, t)$ and $f(u)$ are some nonlinear expressions in terms of $u(x, t)$ provided that $f(u) = u(1 - u^\delta)$ for GBF equation and $f(u) = u(1 - u^\delta)(u^\delta - \gamma)$ for GBH equation while $\alpha, \beta, \gamma, \delta, \epsilon$ are constants such that $\alpha, \beta \geq 0, \delta > 0, \gamma \in (0, 1), \epsilon \in (0, 1]$. The nonlinear diffusion models generated from (1) have a significant role in nonlinear physics and of great practical interest.

1.1. Model I. The GBF equation has a lot of applications in the fields such as fluid mechanics [4], gas dynamics, plasma physics [4], number theory, elasticity [5], and heat conduction [6]. This equation becomes the classical Fisher equation when $\alpha = 0, \delta = 1$ which is one of the significant structures in population biology and is given by

$$u_t - \epsilon u_{xx} = \beta u(1 - u). \quad (4)$$

Kolmogorov et al. [7] wrote down the same equation for the description of dynamic spread of a combustion front. It arises in several phenomena involving perturbation spreads in excitable mediums, spreading of bacterial colonies [8], spread of reaction fronts in chemically bistable systems [9], and switching in nonlinear optics [10].

The exact solution of GBF is

$$u(x, t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} \tanh(d_1(x - d_2 t)) \right)^{1/\delta}, \quad (5)$$

subject to the initial condition,

$$u(x, 0) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} \tanh(d_1 x) \right)^{1/\delta}, \quad (6)$$

and the boundary conditions,

$$u(a, t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} \tanh(d_1(a - d_2 t)) \right)^{1/\delta} \quad (7)$$

$$u(b, t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} \tanh(d_1(b - d_2 t)) \right)^{1/\delta},$$

where

$$d_1 = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta(1+\delta)}}{4(1+\delta)} \gamma \delta, \quad (8)$$

$$d_2 = \frac{\gamma \alpha}{(1+\delta)} - \frac{(1+\delta-\gamma)(-\alpha + \sqrt{\alpha^2 + 4\beta(1+\delta)})}{2(1+\delta)}.$$

Wang et al. [11] investigated the exact solution of GBH equation with the help of nonlinear transformations.

1.2. Model II. Satsuma et al. [12] investigated the GBH equation in 1987. This equation reduces to the Huxley equation [11] when $\alpha = 0$, $\delta = 1$, $\epsilon = 1$ which describes nerve pulse propagation in nerve fibres and wall motion in liquid crystals [13, 14]. It can be expressed as follows:

$$u_t - u_{xx} = \beta u(1-u)(u-\gamma). \quad (9)$$

By considering a well-known experiment in liquid crystals, a similarity between the motion of a wall in liquid crystals and nerve propagation was discussed in [14]. These models have been studied widely in the last decades due to their importance in neurobiology. Hodgkin and Huxley [15] suggested their famous Hodgkin-Huxley model for nerve propagation in 1952. It takes the form of Burgers equation by considering $\beta = 0$, $\delta = 1$, $\epsilon = 1$. In nonlinear dissipative systems [16], it describes the far field of wave propagation and can be expressed as follows:

$$u_t + \alpha u u_x - u_{xx} = 0. \quad (10)$$

It becomes a FitzHugh-Nagumo (FN) equation when $\alpha = 0$, $\beta = 1$, $\delta = 1$, $\epsilon = 1$ are chosen. Basically, it is reaction diffusion equation utilized in circuit theory and biology [17] and its mathematical form is

$$u_t - u_{xx} = u(1-u)(u-\gamma). \quad (11)$$

When $\alpha \neq 0$, $\beta \neq 0$, $\delta = 1$, this equation also reduces to prototype model named as Burgers-Huxley equation. It describes the interaction between diffusion transports, convection, and reaction mechanisms [18] and is given by

$$u_t + \alpha u u_x - \epsilon u_{xx} = \beta u(1-u)(u-\gamma). \quad (12)$$

The exact solution of GBH equation can be written as follows:

$$u(x, t) = (0.5 + 0.5 \tanh(\sigma_1(x - \sigma_2 t)))^{1/\delta}, \quad (13)$$

subject to the initial condition,

$$u(x, 0) = (0.5 + 0.5 \tanh(\sigma_1 x))^{1/\delta}, \quad (14)$$

and the boundary conditions,

$$u(a, t) = (0.5 + 0.5 \tanh(\sigma_1(a - \sigma_2 t)))^{1/\delta}, \quad (15)$$

$$u(b, t) = (0.5 + 0.5 \tanh(\sigma_1(b - \sigma_2 t)))^{1/\delta},$$

where

$$\sigma_1 = \frac{-\alpha \delta}{2(1+\delta)}, \quad (16)$$

$$\sigma_2 = \frac{\alpha}{(1+\delta)} + \frac{\beta(1+\delta)}{\alpha}.$$

This solution was investigated by Xinyi and Yuekai [19] which is the generalization of the preceding results.

Several numerical techniques have been developed to find the numerical solution of GBF and GBH equations. Javidi presented the numerical solution of GBH equation using spectral collocation method [20] and pseudospectral and preconditioning [21] and Chebyshev polynomials to develop a new domain decomposition algorithm [22]. Golbabai and Javidi [23] applied a spectral domain decomposition technique for the numerical solution of GBF equation. Darvishi et al. [24] investigated the numerical solution of GBH equation by adopting a spectral collocation method and Darvishi et al.'s preconditioning. Sari et al. [25] presented the numerical solution of GBF equation by applying a compact finite difference scheme. Hammad and El-Azab [26] computed the numerical solution of two types of equations, namely, GBF and GBH, using $2N$ order compact finite difference scheme. A computational meshless method was developed by Khattak [27] for solving the GBH equation.

Sari and Gurarslan [28] obtained the numerical solution of the GBH equation using a polynomial differential quadrature method. Malik et al. [29] developed a heuristic scheme for the numerical solution of the GBF equation based on the hybridization of Exp-function method with nature inspired algorithm. The problem was converted into a nonlinear ordinary differential equation (ODE) by substitution. The travelling wave solution was approximated by the Exp-function method with unknown parameters. Dehghan et al. [30] developed two numerical methods based on the interpolating scaling functions and mixed collocation finite difference schemes for the numerical solution of the GBH equation.

Zhang et al. [31] developed a new kind of exact finite difference scheme for solving Burgers equation and Burgers-Fisher equation using the solitary wave solution. Biazar and Mohammadi [32] solved GBH equation using differential transform method (DTM). Bratsos [33, 34] solved GBH equation using a modified predictor-corrector method based

on a second- and fourth-order time finite difference scheme. Zarebnia and Aliniya [35] used a mesh-free collocation method based on sinc functions for solving the Burgers-Huxley equation. Batiha et al. [36] applied He's variational iteration method (VIM) without any discretization to solve the GBH equation. Morufu [37] developed an improved algorithm for solving GBF equation based on a Maple code. Hashim et al. [38] applied Adomian decomposition method (ADM) to get rapidly convergent analytical series solution of the GBH equation.

Zhao et al. [39] approximated the GBF equation using the pseudospectral method based on Crank-Nicolson/leapfrog scheme. The approximate solutions were obtained for the GBH and GBF equations using the Adomian and discrete Adomian decomposition methods [40, 41]. Inan and Bahadir [42] obtained a numerical solution of the GBH equation using implicit exponential finite difference method. Celik [43] proposed a Chebyshev wavelet collocation method based on truncated Chebyshev wavelet series for the solution of GBH equation. Moreover, the numerical solution of proposed GBH equation was obtained using several numerical methods named Galerkin method [44], implicit and fully implicit exponential finite difference methods [45], Haar wavelet method [46], conditionally bounded and symmetry-preserving method [47], linearly implicit compact scheme [48], positive and bounded finite element method [49], explicit solution scheme [50], exponential time differencing scheme [51], and higher order finite difference schemes [52].

The B-spline collocation scheme is a well-known interpolating or approximating scheme which provides a good approximation rate, computationally fast, numerically consistent, and has ability to reproduce the shape of the data with second order of continuity as compared to polynomials. Recently, several numerical schemes based on different types of B-spline functions were applied to find the numerical solutions of the differential equations. Mittal and Tripathi [53] proposed a numerical scheme based on modified cubic B-spline functions to get the approximate solutions of GBF and GBH equations. Mittal and Jain [54] obtained a numerical solution of nonlinear Burgers equation using a modified cubic B-spline collocation method. Singh et al. [55] developed a numerical scheme for solving the GBH equation using modified cubic B-spline differential quadrature method (MCB-DQM) and numerical results can be obtained using SSP-RK43 scheme. Reza [56] implemented the cubic B-spline collocation scheme based on the finite difference scheme for solving the GBH equation. Reza [57] developed a numerical method based on exponential B-spline with finite difference approximations to solve the GBF equation. Recently, Bukhari et al. [58] applied local radial basis functions differential collocation (LRBDQ) method to compute the numerical solution of GBH equation.

1.3. Motivation of the Study. The finite difference scheme is not the only tool for computing approximations to the solution of boundary value problems. There are various approximation techniques which have been examined by many researchers. Spline interpolation method is one of the most effective approximation methods on account of its

FIGURE 1: Hybrid B-spline functions with parameter $\mu = 2, 1.5, 1, 0$ and -1 .

simplicity and practicality. The main advantage of using this method is that it is able to approximate the analytical curve up to certain smoothness. Therefore, the spline method has the flexibility to get the approximation at any point in the domain with more accurate results compared to the usual finite difference method. This, thus, provides the motivation for this study on examining the accuracy of hybrid B-spline on solving nonlinear partial differential equation. However, one of the limitations of classical B-spline interpolation is that it does not possess any free parameter for the curve modification. Therefore, the shape of the curve is incapable of being altered once the control points are determined. On the other hand, spline interpolation is a global interpolation; any changes of the data point will require solving all the linear systems again. The advantage of using hybrid B-spline is that it possesses a free parameter μ to control the global shape of curve. An appropriate choice of the parameter rises the order of accuracy of the scheme. Hybrid B-spline basis function reduces to cubic trigonometric B-spline and cubic B-spline function when $\mu = 0$ and 1 , respectively. This research focuses on the value of $\mu > 1$. Figure 1 depicts the graph of cubic trigonometric B-spline when $\mu = 0$, cubic B-spline function when $\mu = 1$, and the effect of parameter μ for proposed hybrid B-spline function. Therefore, the superiority of this spline interpolation method on proposed problem is to be examined.

Although a finite difference scheme is only able to give the approximations at selected points, this method is relatively simple and very much easy to implement. Hence, an idea of combining finite difference approach with hybrid B-spline interpolation method for solving proposed problem also naturally arose. Here, hybrid B-spline is used to model the solution curve at each level of time. Thus, it is applied to interpolate the solutions at time t while finite difference scheme is used to discretize the time derivative. The obtained results are more accurate than some available methods in the literature. Stability analysis of the proposed method is presented and shown to be unconditionally stable without any restriction on the choice of step sizes h and Δt . An advantage of the proposed hybrid B-spline collocation method (HBSCM) outlined in this study is that it produces a spline function on each new time line which can be used to obtain the solution at any intermediate point in the spatial direction

whereas the finite difference approach yields the solution only at the selected points.

This article is organized as follows: In Section 2, hybrid B-spline collocation method, a combination of cubic B-spline function and cubic trigonometric B-spline function with one free parameter μ , is constructed and applied to obtain the numerical solutions of the proposed equations. In Section 3, the method is proved unconditionally stable by Von Neumann approach. In Section 4, several numerical cases of GBF and GBH equations are considered to show the feasibility of the proposed method. Finally, the conclusion of this study is provided.

2. Materials and Methods

This section introduces the hybrid B-spline basis function and derivation of proposed HBSCM for solving the GBF and GBH equations.

2.1. Hybrid B-Spline Basis Function. For the discretization of the grid region $[a, b] \times [0, T]$ an equally divided mesh Ω with grid points (x_j, t_k) is considered. Here $x_j = a + jh$, $t_k = k\Delta t$ where $j = 0, 1, \dots, n$ and $k = 0, 1, \dots, N$ while h and Δt are spatial size and time step, respectively. Hybrid B-spline collocation basis function can be written as follows:

$$H_j^4(x) = \begin{cases} \frac{\mu}{6h^3}(x-x_j)^3 + \frac{(1-\mu)}{\zeta}r^3(x_j) & x \in [x_j, x_{j+1}] \\ \frac{\mu}{6h^3}(h^3 + 3h^2(x-x_{j+1}) + 3h(x-x_{j+1})^2 - 3(x-x_{j+1})^3) + \frac{(1-\mu)}{\zeta}(r(x_j)(r(x_j)s(x_{j+2}) + s(x_{j+3})r(x_{j+1})) + s(x_{j+4})r^2(x_{j+1}))) & x \in [x_{j+1}, x_{j+2}] \\ \frac{\mu}{6h^3}(h^3 + 3h^2(x_{j+3}-x) + 3h(x_{j+3}-x)^2 - 3(x_{j+3}-x)^3) + \frac{(1-\mu)}{\zeta}(s(x_{j+4})(r(x_{j+1})s(x_{j+3}) + s(x_{j+4})r(x_{j+2})) + r(x_j)s^2(x_{j+3}))) & x \in [x_{j+2}, x_{j+3}] \\ \frac{\mu}{6h^3}(x_{j+4}-x)^3 + \frac{(1-\mu)}{\zeta}s^3(x_{j+4}) & x \in [x_{j+3}, x_{j+4}] \\ 0 & \text{otherwise,} \end{cases} \quad (17)$$

where $r(x_j) = \sin((x-x_j)/2)$, $s(x_j) = \sin((x_j-x)/2)$, $\zeta = \sin(h/2)\sin(h)\sin(3h/2)$ and $\mu \in R$.

The approximate solution $u(x_j, t_k)$ to the exact solution $u_{\text{exc}}(x, t)$ can be expressed as follows [59–64]:

$$u_j^k(x, t) = \sum_{m=j-1}^{j+1} D_m^k(t) H_m^4(x), \quad (18)$$

where $D_m^k(t)$ are time-dependent unknowns to be determined.

The values of $H_j^4(x)$ and its derivatives at node $x = x_j$ are given by

$$H_m^4(x_j) = \begin{cases} a_1 = \frac{\mu}{6} + (1-\mu)\sin^2\left(\frac{h}{2}\right)\text{cosec}(h)\text{cosec}\left(\frac{3h}{2}\right) & m = j \pm 1 \\ a_2 = \frac{2\mu}{3} + (1-\mu)\frac{2}{1+2\cos(h)} & m = j \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{d}{dx}H_m^4(x_j) = \begin{cases} a_3 = -\left(\frac{\mu}{2h} + (1-\mu)\frac{3}{4}\text{cosec}\left(\frac{3h}{2}\right)\right) & m = j-1 \\ a_4 = \left(\frac{\mu}{2h} + (1-\mu)\frac{3}{4}\text{cosec}\left(\frac{3h}{2}\right)\right) & m = j+1 \\ 0 & \text{otherwise,} \end{cases} \quad (19)$$

$$\frac{d^2}{dx^2}H_m^4(x_j) = \begin{cases} a_5 = \frac{\mu}{h^2} + (1-\mu)\frac{3(1+3\cos(h))\text{cosec}^2(h/2)}{16(2\cos(h/2)+\cos(3h/2))} & m = j \pm 1 \\ a_6 = -\left(\frac{2\mu}{h^2} + (1-\mu)\frac{3\cos^2(h/2)\text{cosec}^2(h/2)}{(2+4\cos(h))}\right) & m = j \\ 0 & \text{otherwise.} \end{cases}$$

From (17)–(19), the values of $u_j^k(x, t)$ and their derivatives at the knots are calculated in terms of time parameters $D_j^k(t)$ as follows:

$$\begin{aligned} u_j^k &= a_1 D_{j-1}^k + a_2 D_j^k + a_1 D_{j+1}^k, \\ (u_x)_j^k &= a_3 D_{j-1}^k + 0 D_j^k + a_4 D_{j+1}^k, \\ (u_{xx})_j^k &= a_5 D_{j-1}^k + a_6 D_j^k + a_5 D_{j+1}^k. \end{aligned} \quad (20)$$

Equation (18) and boundary conditions given in (3) are used to obtain the approximate solution at end points of the mesh as

$$\begin{aligned} u(x_0, t_{k+1}) &= a_1 D_{-1}^{k+1} + a_2 D_0^{k+1} + a_1 D_1^{k+1} = g_1(t_{k+1}), \\ u(x_n, t_{k+1}) &= a_1 D_{n-1}^{k+1} + a_2 D_n^{k+1} + a_1 D_{n+1}^{k+1} = g_2(t_{k+1}). \end{aligned} \quad (21)$$

2.2. Numerical Solution of the Generalized Burgers-Fisher and Generalized Burgers-Huxley Equations. By utilizing temporal discretization and Crank-Nicolson approach, (1) can be written as

$$\begin{aligned} \frac{u_j^{k+1} - u_j^k}{\Delta t} + \frac{\left(\alpha (u^\delta u_x)_j^{k+1} - \epsilon (u_{xx})_j^{k+1} \right)}{2} \\ + \frac{\left(\alpha (u^\delta u_x)_j^k - \epsilon (u_{xx})_j^k \right)}{2} \\ = \beta \left(\frac{f(u_j^k) + f(u_j^{k+1})}{2} \right), \end{aligned} \quad (22)$$

where k and $k + 1$ describe successive time positions and $f(u_j^k) = u_j^k(1 - (u^\delta)_j^k)$ for GBF and $f(u_j^k) = u_j^k(1 - (u_j^k)^\delta)((u_j^k)^\delta - \gamma)$ for GBH equations. After simplification, (22) for GBF can be written as follows:

$$\begin{aligned} u_j^{k+1} + 0.5\alpha\Delta t (u^\delta u_x)_j^{k+1} - 0.5\epsilon\Delta t (u_{xx})_j^{k+1} \\ - 0.5\Delta t\beta u_j^{k+1} \left(1 - (u^\delta)_j^{k+1} \right) \\ = u_j^k - 0.5\alpha\Delta t (u^\delta u_x)_j^k + 0.5\epsilon\Delta t (u_{xx})_j^k \\ + 0.5\Delta t\beta u_j^k \left(1 - (u^\delta)_j^k \right) \end{aligned} \quad (23)$$

and (22) for GBH equation can be written as follows:

$$\begin{aligned} u_j^{k+1} + 0.5\alpha\Delta t (u^\delta u_x)_j^{k+1} - 0.5\epsilon\Delta t (u_{xx})_j^{k+1} \\ - 0.5\Delta t\beta u_j^{k+1} \left(1 - (u^\delta)_j^{k+1} \right) \left((u_j^{k+1})^\delta - \gamma \right) \\ = u_j^k - 0.5\alpha\Delta t (u^\delta u_x)_j^k + 0.5\epsilon\Delta t (u_{xx})_j^k \\ + \Delta t\beta u_j^k \left(1 - (u^\delta)_j^k \right) \left((u_j^k)^\delta - \gamma \right). \end{aligned} \quad (24)$$

The system, thus obtained, on simplifying (23) for GBF and (24) for GBH problem using (20) consists of $n + 1$ nonlinear equations in $n + 3$ unknowns $D^{k+1} = (D_{-1}^{k+1}, D_0^{k+1}, D_1^{k+1}, \dots, D_{n+1}^{k+1})$ at the time level t_{k+1} . Further two equations are included in the resulting system to obtain a unique solution of the problem using the boundary conditions given in (21). The initial vector D^0 can be obtained from initial condition given in (2) [59–64]. Thus, the resulting system becomes a matrix system of dimension $(n+3) \times (n+3)$ which is a tridiagonal system that can be solved by Thomas algorithm [65–67].

3. Stability

This section discusses the Von Neumann criteria to investigate the stability of GBF and GBH equations. In the product term, consider $u^\delta = \tau$ where τ is taken [68] as locally constant, the GBF equation is described as follows:

$$u_t + \alpha\tau u_x - \epsilon u_{xx} = (1 - \tau)\beta u \quad (25)$$

and applying same procedure for nonlinear term as in [64], the GBH equation can be converted to the following:

$$u_t + \alpha\tau u_x - \epsilon u_{xx} = -\gamma\beta u. \quad (26)$$

Utilizing same procedure as stated in (22) and setting $1 - \tau = \lambda$, the above two GBF and GBH equations take the following forms:

$$\begin{aligned} b_1 u_j^{k+1} + b_2 (u_x)_j^{k+1} - b_3 (u_{xx})_j^{k+1} \\ = b_4 u_j^k - b_2 (u_x)_j^k + b_3 (u_{xx})_j^k, \\ a_{11} u_j^{k+1} + b_2 (u_x)_j^{k+1} - b_3 (u_{xx})_j^{k+1} \\ = a_{22} u_j^k - b_2 (u_x)_j^k + b_3 (u_{xx})_j^k, \end{aligned} \quad (27)$$

where $a_{11} = 2 + \beta\gamma\Delta t$, $a_{22} = 2 - \beta\gamma\Delta t$, $b_1 = 2 - \beta\lambda\Delta t$, $b_2 = \alpha\tau\Delta t$, $b_3 = \epsilon\Delta t$, $b_4 = 2 + \beta\lambda\Delta t$

Substitute (23) into (27) and simplifying yield

$$\begin{aligned} q_1 D_{j-1}^{k+1} + q_2 D_j^{k+1} + q_3 D_{j+1}^{k+1} \\ = q_4 D_{j-1}^k + q_5 D_j^k + q_6 D_{j+1}^k, \\ p_1 D_{j-1}^{k+1} + p_2 D_j^{k+1} + p_3 D_{j+1}^{k+1} \\ = p_4 D_{j-1}^k + p_5 D_j^k + p_6 D_{j+1}^k, \end{aligned} \quad (28)$$

where

$$\begin{aligned} p_1 &= a_{11}a_1 + b_2a_3 - b_3a_5, \\ p_2 &= a_{11}a_2 - b_3a_6, \\ p_3 &= a_{11}a_1 + b_2a_4 - b_3a_5, \\ p_4 &= a_{22}a_1 - b_2a_3 + b_3a_5, \\ p_5 &= a_{22}a_2 + b_3a_6, \end{aligned}$$

$$\begin{aligned}
p_6 &= a_{22}a_1 - b_2a_4 + b_3a_5, \\
q_1 &= b_1a_1 + b_2a_3 - b_3a_5, \\
q_2 &= b_1a_2 - b_3a_6, \\
q_3 &= b_1a_1 + b_2a_4 - b_3a_5, \\
q_4 &= b_4a_1 - b_2a_3 + b_3a_5, \\
q_5 &= b_4a_2 + b_3a_6, \\
q_6 &= b_4a_1 - b_2a_4 + b_3a_5.
\end{aligned}$$

Now substituting $D_j^k = \rho^k e^{i\omega jh}$ into (28). After simplification dividing both sides by $\rho^k e^{i\omega jh}$, we obtain the following expressions given in (29) and (30) for GBF and GBH equations, respectively:

$$\rho = \frac{c_1 + id_1}{c_2 + id_2}, \quad i = \sqrt{-1}, \quad (29)$$

$$\rho = \frac{l_1 + im_1}{l_2 + im_2}, \quad i = \sqrt{-1}, \quad (30)$$

where

$$\begin{aligned}
c_1 &= \cos(\phi)(q_6 + q_4) + q_5, \\
d_1 &= \sin(\phi)(q_6 - q_4), \\
c_2 &= \cos(\phi)(q_3 + q_1) + q_2, \\
d_2 &= \sin(\phi)(q_3 - q_1), \\
l_1 &= \cos(\phi)(p_6 + p_4) + p_5, \\
m_1 &= \sin(\phi)(p_6 - p_4), \\
l_2 &= \cos(\phi)(p_3 + p_1) + p_2, \\
m_2 &= \sin(\phi)(p_3 - p_1).
\end{aligned}$$

Since the wave number $\omega = 2\pi/\lambda$ where λ is the wave length so $\phi = \omega h = 2\pi/N$ for $0 < \phi < \pi$. The amplification factor ρ is a complex number; therefore the stability condition $|\rho| \leq 1$ yields the following relation by adopting same procedure as in [60].

For GBF equation,

$$|\rho| = \sqrt{\left(\frac{c_1c_2 + d_1d_2}{c_2^2 + d_2^2}\right)^2 + \left(\frac{d_1c_2 - c_1d_2}{c_2^2 + d_2^2}\right)^2} \leq 1 \quad (31)$$

and, for GBH equation,

$$|\rho| = \sqrt{\left(\frac{l_1l_2 + m_1m_2}{l_2^2 + m_2^2}\right)^2 + \left(\frac{m_1l_2 - l_1m_2}{l_2^2 + m_2^2}\right)^2} \leq 1. \quad (32)$$

Substituting the values into (31) and (32), we obtain the following expressions for GBF equation:

$$|\rho| = \sqrt{\frac{q_4^2 + q_5^2 + q_6^2 + 2q_5(q_4 + q_6)\cos(\phi) + 2q_4q_6\cos(2\phi)}{q_1^2 + q_2^2 + q_3^2 + 2q_2(q_1 + q_3)\cos(\phi) + 2q_1q_3\cos(2\phi)}} \quad (33)$$

and, for GBH equation,

$$|\rho| = \sqrt{\frac{p_4^2 + p_5^2 + p_6^2 + 2p_5(p_4 + p_6)\cos(\phi) + 2p_4p_6\cos(2\phi)}{p_1^2 + p_2^2 + p_3^2 + 2p_2(p_1 + p_3)\cos(\phi) + 2p_1p_3\cos(2\phi)}}. \quad (34)$$

Taking extreme value $\phi = \pi$, (33) and (34) become

$$\begin{aligned}
|\rho| &= \sqrt{\frac{(q_4 - q_5 + q_6)^2}{(q_1 - q_2 + q_3)^2}}, \\
|\rho| &= \sqrt{\frac{(p_4 - p_5 + p_6)^2}{(p_1 - p_2 + p_3)^2}}.
\end{aligned} \quad (35)$$

By substituting the values from (28) into (35) amplification factor ρ for GBF equation is

$$|\rho| = \frac{(\xi_2 + \xi_5 + \xi_9) - (\xi_1 + \xi_3 + \xi_4 + \xi_6 + \xi_7 + \xi_8 + \xi_{10})}{(\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 + \xi_7 + \xi_{10}) - (\xi_8 + \xi_9)} \quad (36)$$

and for GBH equation

$$|\rho| = \frac{(\delta_5 + \delta_2) - (\delta_1 + \delta_3 + \delta_4 + \delta_6 + \delta_7 + 3h^2(1 - \mu)(8 + (3 - 4\beta\gamma)\Delta t) + (-8 + (9 + 4\beta\gamma)\Delta t)\cos(h))\csc^2(h/2)\sec(h/2)}{\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + \delta_7 + 6h^2(1 - \mu)((-8 - (9 + 4\beta\gamma)\Delta t) + 6\Delta t\csc^2(h/2))\sec(h/2)}, \quad (37)$$

where

$$\begin{aligned}
\xi_1 &= 96\Delta t\mu\epsilon, \\
\xi_2 &= 16h^2(6 - 5\mu), \\
\xi_3 &= 8h^2\beta\Delta t(6 - 5\mu)\lambda, \\
\xi_4 &= 192\Delta t\epsilon\mu\cos(h),
\end{aligned}$$

$$\begin{aligned}
\xi_5 &= 32h^2\mu\cos(h), \\
\xi_6 &= 16h^2\beta\Delta t\mu\lambda\cos(h), \\
\xi_7 &= 36h^2\epsilon\Delta t(1 - \mu)\cot^2(h/2), \\
\xi_8 &= 48h^2(1 - \mu)\sec(h/2), \\
\xi_9 &= 24h^2(1 - \mu)\beta\Delta t\lambda\sec(h/2),
\end{aligned}$$

TABLE 1: Absolute errors of GBH equation at different values of time and parameter μ taking $\Delta t = 0.0001$ for case 1.

x	t	$\mu = 1.1$	$\mu = 1.5$	$\mu = 1.9$
0.1	0.05	7.722E-09	7.703E-09	7.684E-09
	0.1	1.129E-08	1.126E-08	1.123E-08
	1.0	1.685E-08	1.681E-08	1.677E-08
	5.0	1.685E-08	1.681E-08	1.677E-08
	10	1.685E-08	1.681E-08	1.677E-08
0.5	0.05	1.734E-08	1.730E-08	1.726E-08
	0.1	2.881E-08	2.874E-08	2.867E-08
	1.0	4.682E-08	4.670E-08	4.658E-08
	5.0	4.682E-08	4.670E-08	4.658E-08
	10	4.682E-08	4.670E-08	4.658E-08
0.9	0.05	7.722E-09	7.703E-09	7.684E-09
	0.1	1.129E-08	1.126E-08	1.123E-08
	1.0	1.685E-08	1.681E-08	1.677E-08
	5.0	1.686E-08	1.681E-08	1.677E-08
	10	1.686E-08	1.681E-08	1.677E-08

$$\xi_{10} = 18h^2(1-\mu)\epsilon\Delta t(1+3\cos(h))/(1-\cos(h))\sec(h/2),$$

$$\delta_1 = 96\Delta t\mu,$$

$$\delta_2 = 16h^2(6-5\mu),$$

$$\delta_3 = 8h^2\beta\gamma\Delta t(6-5\mu),$$

$$\delta_4 = 192\Delta t\mu\cos(h),$$

$$\delta_5 = 32h^2\mu\cos(h),$$

$$\delta_6 = 16h^2\beta\gamma\Delta t\mu\cos(h),$$

$$\delta_7 = 36h^2\Delta t(1-\mu)\cot^2(h/2).$$

Since the numerator is less than the denominator in (36) and (37), so amplification factor $|\rho| \leq 1$ for both GBF and GBH equations which clearly demonstrate that the proposed scheme is unconditionally stable.

4. Numerical Results and Discussion

This section presents the numerical results of GBF and GBH equations with initial (2) and boundary conditions (3) by HBSCM. To test the accuracy of present method, several numerical tests for different values of $\alpha, \beta, \gamma, \delta$ are provided where L_∞, L_2 , and order of convergence p are calculated by

$$\begin{aligned} L_\infty &= \max |u_j - u_{\text{excj}}|, \\ L_2 &= \frac{\sqrt{\sum_{j=1}^n |u_j - u_{\text{excj}}|^2}}{\sqrt{\sum_{j=1}^n |u_{\text{excj}}|^2}}, \\ p &= \frac{\log(L_\infty(n)/L_\infty(2n))}{\log(2)}. \end{aligned} \quad (38)$$

Numerical results obtained by HBSCM are compared with given exact solutions and the approximate methods existing

in the literature. The programming of the proposed problem is carried out in Matlab R2015b. Numerical results are computed at different time levels with smaller storage which are tabulated in different Tables.

Consider the following numerical test cases for GBH and GBF equations to show the accuracy of proposed method.

4.1. Numerical Test Cases for Model I

- (1) When $\alpha = 1, \beta = 1, \gamma = 0.001, \epsilon = 1$, GBH equation can be described as follows:

$$u_t + u^\delta u_x - u_{xx} = u(1 - u^\delta)(u^\delta - 0.001). \quad (39)$$

The obtained errors at $\delta = 1$ corresponding to different values of parameter μ are listed in Table 1. Table 2 shows a comparison of the absolute errors at $\mu = 3.9, \delta = 1, 2, 3$ obtained by proposed method HBSCM with the methods existing in the literature named compact finite difference scheme (CFDS) [26], a fourth-order improved numerical scheme (FONS) [33], variational iteration method (VIM) [36], Adomian decomposition method (ADM) [38, 40], implicit exponential finite difference method (IEFM) [42], and modified cubic B-spline (MCBS) [55]. The obtained results are compared with Haar wavelet method (HWM) [46] at $t = 0.8$ in Table 3 while a comparison between HBSCM and a new domain decomposition method (NDDA) [22] can be observed in Table 4. The results of the proposed method in terms of errors comparative to Local Radial Basis Function Differential Collocation method (LRBFDQ) [58] is provided in Table 5. It can be observed that increase in Δt did not disturb the accuracy of HBSCM and our method still approximates the exact solution quite adequately due to hybrid parameter. The graphical representations of error terms at different time levels have been carried out in Figures 2 and 3. A prominent difference in the

TABLE 2: Comparison of absolute errors calculated by HBSCM with the existing methods at $\delta = 1, \mu = 3.9$ for case 1.

x	$t = 0.05$			$t = 0.1$			$t = 1.0$		
	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\delta = 1$									
HBSCM	$7.587E-09$	$1.704E-08$	$7.587E-09$	$1.109E-08$	$2.830E-08$	$1.109E-08$	$1.656E-08$	$4.599E-08$	$1.656E-08$
CFDS [26]	$7.700E-09$	$1.728E-08$	$7.700E-09$	$1.126E-08$	$2.873E-08$	$1.126E-08$	$1.686E-08$	$4.684E-08$	$1.686E-08$
FONS [33]	$1.264E-09$	$1.977E-08$	$4.602E-08$	$6.395E-09$	$3.996E-08$	$7.663E-08$	$3.292E-07$	$3.792E-07$	$4.292E-07$
VIM [36]	$1.874E-08$	$1.874E-08$	$1.874E-08$	$3.748E-08$	$1.374E-08$	$3.748E-08$	$3.748E-07$	$3.748E-07$	$3.748E-07$
ADM [38]	$1.874E-08$	$1.874E-08$	$1.874E-08$	$3.748E-08$	$3.748E-08$	$3.748E-08$	$3.748E-07$	$3.748E-07$	$3.748E-07$
ADM [40]	$1.937E-07$	$1.937E-07$	$1.937E-07$	$3.874E-07$	$3.874E-07$	$3.874E-07$	$3.875E-06$	$3.875E-06$	$3.875E-06$
IEFM [42]	$1.544E-08$	$3.469E-08$	$1.544E-08$	$2.258E-08$	$5.764E-08$	$2.259E-08$	$3.372E-08$	$9.369E-08$	$3.373E-08$
MCBS [55]	—	—	—	$1.111E-08$	$2.870E-08$	$1.111E-08$	$1.668E-08$	$4.665E-08$	$1.668E-08$
$\delta = 2$									
HBSCM	$3.545E-07$	$7.960E-07$	$3.544E-07$	$5.182E-07$	$1.322E-06$	$5.182E-07$	$7.734E-07$	$2.148E-06$	$7.734E-07$
VIM [36]	—	—	—	$5.515E-05$	$5.510E-05$	—	—	—	—
ADM [38]	—	—	—	$5.515E-05$	$5.511E-05$	—	—	—	—
IEFM [42]	$1.402E-06$	$3.150E-06$	$1.402E-06$	$2.051E-06$	$5.233E-06$	$2.051E-06$	$3.056E-06$	$8.490E-06$	$3.056E-06$
$\delta = 3$									
HBSCM	$1.294E-06$	$2.906E-06$	$1.293E-06$	$1.892E-06$	$4.828E-06$	$1.892E-06$	$2.823E-06$	$7.841E-06$	$2.823E-06$
IEFM [42]	$8.789E-06$	$1.973E-05$	$8.788E-06$	$1.284E-05$	$3.278E-05$	$1.284E-05$	$1.902E-05$	$5.285E-05$	$1.902E-05$

TABLE 3: Comparison of absolute errors of GBH equation at $\mu = 3.9, \Delta t = 0.0001$ taking different values of δ for case 1.

$t = 0.8$	$\delta = 1$			$\delta = 8$		
x	0.1	0.5	0.9	0.1	0.5	0.9
HBSCM	$5.870E-09$	$1.342E-08$	$5.870E-09$	$6.805E-06$	$1.555E-05$	$6.800E-06$
HWM [46]	$1.586E-08$	$4.662E-08$	$1.586E-08$	$1.404E-05$	$4.123E-05$	$1.404E-05$

TABLE 4: Comparison of maximum errors of GBH equation at $\mu = 3.9, \Delta t = 0.00005$ taking different values of δ for case 1.

$t = 1.0$	$\delta = 1$	$\delta = 4$	$\delta = 8$
HBSCM	$1.342E-08$	$5.325E-06$	$1.555E-05$
NDDA [22]	$4.685E-08$	$1.532E-05$	$4.141E-05$

TABLE 5: Comparison of absolute errors calculated by HBSCM with the existing method at $\Delta t = 0.01, \mu = 2.6$ for case 1.

x	$t = 0.05$			$t = 0.1$			$t = 1.0$		
	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\delta = 1$									
HBSCM	$8.48E-11$	$1.91E-10$	$8.58E-11$	$1.23E-10$	$3.18E-10$	$1.25E-10$	$1.87E-10$	$5.22E-10$	$1.88E-10$
LRBDQ [58]	$7.97E-10$	$1.87E-08$	$3.83E-08$	$1.59E-09$	$3.75E-08$	$7.66E-08$	$1.60E-08$	$3.75E-07$	$7.65E-07$
$\delta = 2$									
HBSCM	$1.14E-08$	$2.57E-08$	$1.14E-08$	$1.67E-08$	$4.26E-08$	$1.66E-08$	$2.44E-08$	$6.77E-08$	$2.43E-08$
LRBDQ [58]	$1.15E-09$	$8.70E-07$	$1.75E-06$	$2.26E-09$	$1.75E-06$	$3.50E-06$	$1.37E-08$	$1.75E-05$	$3.49E-05$
$\delta = 3$									
HBSCM	$7.69E-08$	$1.72E-07$	$7.65E-08$	$1.11E-07$	$2.85E-07$	$1.11E-07$	$1.63E-07$	$4.54E-07$	$1.63E-07$
LRBDQ [58]	$9.40E-07$	$3.19E-06$	$7.30E-06$	$1.89E-06$	$6.83E-06$	$1.47E-05$	$1.89E-05$	$6.38E-05$	$1.46E-04$

TABLE 6: Comparison of error norms at $\alpha = 0.001$, $\beta = 0.001$, $\gamma = 0.001$, $\Delta t = 0.0001$, $\epsilon = 1$, $\delta = 2$ for case 2.

t	HBSCM		LRBF [58]	
	L_∞	L_2	L_∞	L_2
0.0005	$5.9337E-13$	$8.8011E-14$	$2.2152E-11$	$1.3610E-11$
0.001	$1.1867E-12$	$2.3285E-13$	$4.4301E-11$	$2.7153E-11$
0.5	$1.4722E-10$	$8.7302E-10$	$2.2152E-08$	$1.3565E-08$
1	$1.4831E-10$	$1.3634E-09$	$4.4304E-08$	$2.7130E-08$
5	$1.4832E-10$	$3.2646E-09$	$2.2152E-07$	$1.3536E-07$

FIGURE 2: Absolute errors at different time levels when $\Delta t = 0.0001$, $\mu = 3.9$, $\delta = 1$ for case 1. FIGURE 3: Absolute errors at different time levels when $\Delta t = 0.01$, $\mu = 2.6$, $\delta = 1$ for case 1.

absolute errors can be visualized in the figures and tables due to the hybrid parameter. It is pertinent to claim that the proposed method provides accurate and improves results as compared to others.

- (2) When $\alpha = 0.001$, $\beta = 0.001$, $\gamma = 0.001$, $\Delta t = 0.0001$, $\epsilon = 1$, $\delta = 2$, GBH equation can be [58] described as follows:

$$u_t + 0.001u^2u_x - u_{xx} = 0.001u(1 - u^2)(u^2 - 0.001). \quad (40)$$

In Table 6, the error norms are calculated at different time levels for $\Delta t = 0.0001$ and compared with LRBF [58]. Figures 4 and 5 depict the error norms at different time levels. It can be concluded that the proposed method is more accurate than LRBF [58].

- (3) When $\alpha = 0.1$, $\beta = 0.001$, $\gamma = 0.0001$, $\epsilon = 1$, GBH equation is expressed as follows:

$$u_t + 0.1u^\delta u_x - u_{xx} = 0.001u(1 - u^\delta)(u^\delta - 0.0001). \quad (41)$$

A comparison of the absolute errors calculated by HBSCM at $\delta = 1, 2, 8$ is presented in Table 7 with the

existing methods named spectral collocation method (SCM) [20] and IEFM [42]. The order of convergence calculated numerically by HBSCM is compared with IEFM [42] in Table 8. It can be concluded that present method has a rapid convergence. Figure 6 depicts the absolute errors which clearly demonstrate that increase in time does not disturb the accuracy of the obtained results

- (4) When $\alpha = 0$, $\beta = 1$, $\gamma = 0.001$, $\mu = 1.65$, $\Delta t = 0.001$, $\epsilon = 1$, GBH can be represented as follows:

$$u_t - u_{xx} = u(1 - u^\delta)(u^\delta - 0.001). \quad (42)$$

In Table 9, we tabulate a comparison between HBSCM and the existing methods CFDS [26], FONS [33], ADM [38], Galerkin method (GM) [44], higher order finite difference method (HFDM) [52], MCBS [55], cubic B-spline algorithm (CBSA) [56], and optimal homotopy asymptotic method (OHAM) [69] at $\delta = 1, 3$. Computations of the absolute errors and two different types of error norms have been mentioned in Tables 10 and 11, respectively. Figure 7 exhibits the computed results at different time levels

TABLE 7: Comparison of absolute errors at different values of δ and parameter $\mu = 1.9$ for case 3.

x	$t = 0.2$			$t = 0.5$			$t = 0.8$		
	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\delta = 1$									
HBSCM	$1.315E-13$	$3.569E-13$	$1.315E-13$	$1.490E-13$	$4.135E-13$	$1.490E-13$	$1.499E-13$	$4.165E-13$	$1.499E-13$
SCM [20]	—	—	—	$8.381E-12$	$4.879E-11$	$2.859E-11$	—	—	—
IEFM [42]	$1.561E-13$	$4.238E-13$	$1.561E-13$	$1.770E-13$	$4.911E-13$	$1.770E-13$	$1.780E-13$	$4.946E-13$	$1.780E-13$
$\delta = 2$									
HBSCM	$1.928E-11$	$5.232E-11$	$1.928E-11$	$2.1847E-11$	$6.062E-11$	$2.184E-11$	$2.198E-11$	$6.105E-11$	$2.198E-11$
SCM [20]	—	—	—	$5.554E-10$	$3.815E-09$	$2.236E-09$	—	—	—
IEFM [42]	$3.838E-11$	$1.041E-10$	$3.838E-11$	$4.349E-11$	$1.206E-10$	$4.349E-11$	$4.375E-11$	$1.215E-10$	$4.375E-11$
$\delta = 8$									
HBSCM	$8.877E-10$	$2.408E-09$	$8.877E-10$	$1.005E-09$	$2.790E-09$	$1.005E-09$	$1.011E-09$	$2.810E-09$	$1.011E-09$
SCM [20]	—	—	—	$3.395E-08$	$1.976E-07$	$1.158E-07$	—	—	—
IEFM [42]	$1.743E-08$	$4.732E-08$	$1.743E-08$	$1.976E-08$	$5.483E-08$	$1.976E-08$	$1.988E-08$	$5.522E-08$	$1.988E-08$

TABLE 8: Order of convergence (p) at $t = 1$ for case 3.

n	2	4	8	16	32	64
L_∞	$3.054E-06$	$7.008E-07$	$1.761E-07$	$4.874E-08$	$1.710E-08$	$9.212E-09$
HBSCM (p)	—	2.123	1.992	1.853	1.510	0.892
IEFM [42]	—	0.175608438	0.078691220	0.041403087	0.021266315	0.011014546

FIGURE 4: L_∞ errors at $\delta = 2$, $t = 0.0005$ for case 2. FIGURE 5: L_2 errors at $\delta = 2$, $t = 0.0005$ for case 2.

which enable us to claim that the proposed scheme is superior to the existing method in the terms of accuracy.

- (5) When $\alpha = 5$, $\delta = 2$, $\beta = 10$, $\mu = 1.6$, $\Delta t = 0.0001$, $\epsilon = 1$, the following GBH can be obtained:

$$u_t + 5u^2 u_x - u_{xx} = 10u(1 - u^2)(u^2 - \gamma). \quad (43)$$

The absolute errors are recorded in Table 12 at different values of $\gamma = 10^{-3}, 10^{-4}, 10^{-5}$, calculated by HBSCM and compared with IEFM [42]. Figure 8 explains these errors graphically at $\gamma = 10^{-3}$. A prominent difference in the accuracy can be seen in the figure and table due to the hybrid parameter.

TABLE 9: Comparison of absolute errors computed by HBSCM with other methods at $\alpha = 0$, $\beta = 1$, $\gamma = 0.001$ for case 4.

x	$t = 0.05$			$t = 0.1$			$t = 1.0$		
	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\delta = 1$									
HBSCM	1.030E-08	2.313E-08	1.030E-08	1.506E-08	3.844E-08	1.506E-08	2.248E-08	6.246E-08	2.248E-08
CFDS [26]	1.026E-08	2.304E-08	1.026E-08	1.502E-08	3.832E-08	1.502E-08	2.248E-08	6.246E-08	2.248E-08
FONS [33]	2.498E-08	2.498E-08	2.498E-08	4.997E-08	4.997E-08	4.997E-08	4.997E-07	4.997E-07	4.997E-07
ADM [40]	1.874E-07	1.874E-07	1.875E-07	3.749E-07	3.749E-07	3.750E-07	3.750E-07	3.750E-07	3.750E-07
GM [44]	1.069E-08	9.259E-09	7.892E-09	2.318E-08	2.174E-08	2.038E-08	2.487E-07	2.472E-07	2.459E-07
HFDM [52]	1.030E-08	2.313E-08	1.030E-08	1.506E-08	3.844E-08	1.506E-08	2.248E-08	6.246E-08	2.248E-08
MCBS [55]	1.004E-08	2.304E-08	1.004E-08	1.479E-08	3.825E-08	1.479E-08	2.220E-08	6.216E-08	2.220E-08
CBSA [56]	1.029E-08	2.313E-08	1.029E-08	1.502E-08	3.843E-08	1.502E-08	2.248E-08	6.246E-08	2.248E-08
OHAM [69]	2.498E-08	2.498E-08	2.498E-08	4.997E-08	4.997E-08	4.997E-08	4.997E-07	4.997E-07	4.997E-07
$\delta = 3$									
HBSCM	1.635E-06	3.673E-06	1.635E-06	2.391E-06	6.102E-06	2.390E-06	3.567E-06	9.907E-06	3.566E-06
CFDS [26]	1.630E-06	3.658E-06	1.629E-06	2.385E-06	6.083E-06	2.385E-06	3.567E-06	9.907E-06	3.566E-06
FONS [33]	3.967E-06	3.966E-06	3.965E-06	7.934E-06	7.933E-06	7.931E-06	7.934E-06	7.933E-06	7.931E-06
ADM [40]	1.984E-06	1.983E-06	1.983E-06	3.968E-06	3.967E-06	3.966E-06	3.966E-06	3.965E-06	3.964E-06
HFDM [52]	1.635E-06	3.672E-06	1.635E-06	2.391E-06	6.101E-06	2.390E-06	3.567E-06	9.907E-06	3.566E-06
MCBS [55]	1.594E-06	3.658E-06	1.594E-06	2.347E-06	6.072E-06	2.347E-06	3.522E-06	9.861E-06	3.521E-06
CBSA [56]	1.634E-06	3.672E-06	1.634E-06	2.390E-06	6.101E-06	2.390E-06	3.567E-06	9.907E-06	3.566E-06

TABLE 10: Comparison of absolute errors calculated by HBSCM with the existing method at $\Delta t = 0.01$, $\mu = 2.6$ for case 4.

x	$t = 0.05$			$t = 0.1$			$t = 1.0$		
	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\delta = 1$									
HBSCM	2.50E-09	5.63E-09	2.50E-09	3.65E-09	9.33E-09	3.65E-09	5.41E-09	1.50E-08	5.41E-09
LRBDQ [58]	1.37E-09	2.50E-08	1.37E-09	2.70E-09	5.00E-08	2.70E-09	2.70E-08	5.00E-07	2.70E-08
$\delta = 2$									
HBSCM	1.1E-07	2.5E-07	1.1E-07	1.6E-07	4.1E-07	1.6E-07	2.4E-07	6.7E-07	2.4E-07
LRBDQ [58]	6.1E-08	1.1E-06	6.1E-08	1.2E-07	2.2E-06	1.2E-07	1.1E-06	2.2E-05	1.2E-06
$\delta = 3$									
HBSCM	3.98E-07	8.93E-07	3.97E-07	5.79E-07	1.48E-06	5.79E-07	8.56E-07	2.37E-06	8.55E-07
LRBDQ [58]	2.17E-07	3.97E-06	2.19E-07	4.33E-07	7.93E-06	4.37E-07	4.32E-06	7.93E-05	4.36E-06

TABLE 11: Comparison of error norms for $\alpha = 0$, $\beta = 1$, $\gamma = 0.001$, $\Delta t = 0.01$, $\epsilon = 1$, $\delta = 2$ of case 4.

t	HBSCM		LRBF [58]	
	L_∞	L_2	L_∞	L_2
0.05	2.5183E-07	1.2105E-07	5.1347E-08	3.0711E-08
0.1	4.1746E-07	2.8221E-07	1.0269E-07	6.1418E-08
1	6.7151E-07	1.9621E-06	1.0262E-06	6.1359E-07
5	1.3669E-06	4.6487E-06	5.1155E-06	3.0649E-06
50	2.0079E-05	1.3669E-05	4.9458E-05	3.0125E-05

- (6) When $\alpha = 1$, $\delta = 1$, $\gamma = 0.0001$, $\mu = 1.35$, $\Delta t = 0.0001$, $\epsilon = 1$, GBH equation is

$$u_t + uu_x - u_{xx} = \beta u(1-u)(u-0.0001). \quad (44)$$

The computed results by HBSCM at $\beta = 1, 10, 100$ are recorded in Table 13 and compared with the existing method IEFM [42]. Figure 9 illustrates the behaviour of these errors graphically at $\beta = 1$.

TABLE 12: Comparison of absolute errors obtained by HBSCM with existing methods at different values of γ for case 5.

x	$t = 0.2$			$t = 0.5$			$t = 0.8$		
	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\gamma = 10^{-3}$									
HBSCM	$6.230E-06$	$1.690E-06$	$6.231E-06$	$7.056E-06$	$1.958E-05$	$7.057E-06$	$7.094E-06$	$1.970E-05$	$7.095E-06$
IEFM [42]	$2.075E-05$	$5.633E-05$	$2.076E-05$	$2.339E-05$	$6.495E-05$	$2.340E-05$	$2.340E-05$	$6.503E-05$	$2.341E-05$
$\gamma = 10^{-4}$									
HBSCM	$1.971E-07$	$5.350E-07$	$1.971E-07$	$2.233E-07$	$6.198E-07$	$2.233E-07$	$2.247E-07$	$6.242E-07$	$2.247E-07$
IEFM [42]	$6.580E-07$	$1.785E-06$	$6.580E-07$	$7.453E-07$	$2.068E-06$	$7.453E-07$	$7.494E-07$	$2.081E-06$	$7.495E-07$
$\gamma = 10^{-5}$									
HBSCM	$6.235E-09$	$1.692E-08$	$6.235E-09$	$7.065E-09$	$1.960E-08$	$7.065E-09$	$7.108E-09$	$1.974E-08$	$7.108E-09$
IEFM [42]	$2.081E-08$	$5.648E-08$	$2.081E-08$	$2.358E-08$	$6.545E-08$	$2.358E-08$	$2.372E-08$	$6.591E-08$	$2.373E-08$

TABLE 13: Comparison of absolute errors by HBSCM with other methods at $\alpha = 1$, $\gamma = 10^{-4}$, $\delta = 1$ for case 6.

x	$t = 0.2$			$t = 0.5$			$t = 0.8$		
	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\beta = 1$									
HBSCM	$1.479E-10$	$4.015E-10$	$1.479E-10$	$1.676E-10$	$4.653E-10$	$1.676E-10$	$1.687E-10$	$4.686E-10$	$1.687E-10$
IEFM [42]	$2.959E-10$	$8.029E-10$	$2.959E-10$	$3.353E-10$	$9.304E-10$	$3.353E-10$	$3.373E-10$	$9.370E-10$	$3.373E-10$
$\beta = 10$									
HBSCM	$1.775E-09$	$4.818E-09$	$1.775E-09$	$2.012E-09$	$5.583E-09$	$2.012E-09$	$2.024E-09$	$5.622E-09$	$2.024E-09$
IEFM [42]	$4.142E-09$	$1.124E-08$	$4.142E-09$	$4.694E-09$	$1.302E-08$	$4.694E-09$	$4.723E-09$	$1.311E-08$	$4.723E-09$
$\beta = 100$									
HBSCM	$1.905E-08$	$5.171E-08$	$1.905E-08$	$2.159E-08$	$5.991E-08$	$2.159E-08$	$2.172E-08$	$6.034E-08$	$2.172E-08$
IEFM [42]	$4.662E-08$	$1.265E-07$	$4.662E-08$	$5.283E-08$	$1.466E-07$	$5.283E-08$	$5.314E-08$	$1.476E-07$	$5.314E-08$

FIGURE 6: Absolute errors at different time levels when $\delta = 1$ for case 3. FIGURE 7: Absolute errors at $\alpha = 0$, $\beta = 1$, $\gamma = 0.001$, $\delta = 1$ for case 4.

4.2. Numerical Test Cases for Model II

- (7) When $\alpha = 1$, $\beta = 1$, $\Delta t = 0.00001$, $\epsilon = 1$, $\delta = 1$, GBF equation in the domain $[-1, 1]$ can be written as follows:

$$u_t + uu_x - u_{xx} = u(1 - u). \quad (45)$$

In Table 14, we record the absolute errors obtained by HBSCM for three different values of hybrid parameter μ and compare them with the cubic B-spline method (CBSM) [53]. Table 15 establishes a comparison of error norms L_2 and L_∞ calculated by HBSCM at

TABLE 14: Comparison of absolute errors with [53] $\alpha = 1, \beta = 1, \delta = 1$ for case 7.

x	t	$\mu = 1.1$	$\mu = 1.5$	$\mu = 1.9$	CBSM [53]
-0.5	0.5	$3.142E-06$	$1.463E-06$	$4.226E-07$	$6.108E-06$
	1.0	$3.408E-06$	$1.177E-06$	$4.825E-07$	$5.496E-06$
	2.0	$4.505E-06$	$1.766E-06$	$5.298E-07$	$3.758E-06$
	4.0	$4.149E-06$	$3.644E-06$	$7.470E-07$	$6.071E-07$
0.0	0.5	$4.329E-06$	$2.196E-06$	$5.588E-07$	$3.072E-06$
	1.0	$4.495E-06$	$1.621E-06$	$6.447E-07$	$4.118E-06$
	2.0	$5.650E-06$	$2.109E-06$	$6.744E-07$	$4.559E-06$
	4.0	$6.247E-06$	$4.725E-06$	$9.711E-07$	$3.117E-07$
0.5	0.5	$3.514E-06$	$1.893E-06$	$1.339E-07$	$1.159E-06$
	1.0	$3.437E-06$	$1.279E-06$	$4.973E-07$	$3.680E-06$
	2.0	$4.175E-06$	$1.499E-06$	$5.035E-07$	$5.724E-06$
	4.0	$6.525E-06$	$3.595E-06$	$7.400E-07$	$1.672E-07$
0.9	0.5	$9.634E-07$	$5.331E-07$	$0.456E-07$	$1.315E-06$
	1.0	$8.952E-07$	$3.360E-07$	$1.303E-07$	$4.758E-06$
	2.0	$1.081E-06$	$3.814E-07$	$1.308E-07$	$7.098E-06$
	4.0	$9.281E-06$	$9.499E-07$	$1.950E-07$	$2.364E-07$

TABLE 15: Comparison of error norms L_∞ and L_2 calculated by HBSCM at $\mu = 1.9$ with [53] for case 7.

T	HBSCM		CBSM [53]	
	L_∞	L_2	L_∞	L_2
0.5	$1.343E-07$	$1.750E-07$	$9.172E-06$	$6.953E-06$
1.0	$1.347E-07$	$1.688E-07$	$6.852E-06$	$6.131E-06$
2.0	$1.345E-07$	$1.059E-07$	$7.304E-06$	$7.228E-06$
4.0	$1.593E-07$	$9.761E-08$	$6.443E-07$	$6.702E-07$

FIGURE 8: Absolute errors at $\alpha = 5, \beta = 10, \gamma = 0.001, \delta = 2$ for case 6. FIGURE 9: Absolute errors at $\alpha = 1, \beta = 1, \gamma = 0.0001, \delta = 1$ for case 7.

$\mu = 1.5$ with CBSM [53]. The behaviour of the absolute errors is shown in Figure 10. The exact and approximate solutions are presented graphically in Figure 11 at different time levels. Figures 12 and 13 represent the pictorial view of L_2 and L_∞ errors,

respectively. Figures 14 and 15 indicate that the proposed HBSCM approximates the exact solution quite nicely. It can be concluded from these figures and tables that the proposed method is more accurate than existing methods.

TABLE 16: Comparison of absolute errors with existing methods at $\alpha = 0.001$, $\beta = 0.001$, $\delta = 1$ for case 8.

x	$t = 0.001$			$t = 0.01$		
	0.1	0.5	0.9	0.1	0.5	0.9
HBSCM	$3.056E - 10$	$2.658E - 10$	$3.055E - 10$	$1.962E - 09$	$2.659E - 09$	$1.962E - 09$
EFM[29]	$1.970E - 08$	$3.580E - 09$	$1.800E - 08$	$1.970E - 08$	$3.880E - 09$	$1.740E - 08$
IA [37]	$2.524E - 05$	$2.524E - 05$	$2.524E - 05$	$5.999E - 10$	$5.999E - 10$	$5.999E - 10$
ADM [40]	$1.937E - 06$	$1.937E - 06$	$1.937E - 06$	$1.937E - 05$	$1.937E - 05$	$1.937E - 05$
OHAM [69]	$2.250E - 08$	$4.580E - 08$	$4.580E - 08$	$2.250E - 07$	$4.580E - 07$	$4.580E - 07$
NAT [70]	$3.466E - 04$	$9.870E - 03$	$9.424E - 05$	$1.178E - 05$	$3.740E - 05$	$6.378E - 08$

TABLE 17: The absolute errors calculated by HBSCM and other methods at $\alpha = 1$, $\beta = 0$, $\delta = 3$ for case 9.

x	$t = 0.0001$			$t = 0.001$		
	0.1	0.5	0.9	0.1	0.5	0.9
HBSCM	$1.260E - 09$	$5.543E - 10$	$4.512E - 11$	$1.147E - 08$	$5.530E - 09$	$5.148E - 10$
CMM [27]	—	—	—	$1.900E - 05$	$1.600E - 05$	$1.500E - 05$
EFM [29]	$4.550E - 07$	$5.660E - 07$	$7.000E - 07$	$4.600E - 07$	$5.610E - 07$	$6.950E - 07$
IA [37]	$8.780E - 08$	$4.484E - 07$	$4.175E - 07$	$2.220E - 07$	$2.635E - 07$	$3.645E - 07$
ADM [40]	$4.463E - 04$	$1.860E - 03$	$9.318E - 04$	$4.439E - 04$	$1.847E - 03$	$9.042E - 04$
NAT [70]	$4.780E - 04$	$1.436E - 04$	$7.659E - 05$	$7.865E - 03$	$5.323E - 04$	$7.115E - 04$

FIGURE 10: Absolute errors at $\alpha = 1$, $\beta = 1$, $\delta = 1$ for case 7. FIGURE 11: Exact and approximate solutions at $\alpha = 1$, $\beta = 1$, $\delta = 1$ for case 7.

- (8) When $\alpha = 0.001$, $\beta = 0.001$, $\mu = 1.3$, $\epsilon = 1$, $\delta = 1$, GBF equation in the domain $[0, 1]$ can be described as

$$u_t + 0.001uu_x - u_{xx} = 0.001u(1 - u). \quad (46)$$

The absolute errors calculated at different time levels by proposed HBSCM are compared with the existing methods Exp-function method (EFM) [29],

improved algorithm (IA) [37], ADM [40], OHAM [69], and Nonlinear Analytic Technique (NAT) [70] in Table 16. It is concluded that the proposed method is more accurate than these methods.

- (9) When $\alpha = 1$, $\beta = 0$, $\Delta t = 0.00001$, $\mu = 1.4$, $\epsilon = 1$, $\delta = 3$, GBF in the domain $[0, 1]$ is represented as follows:

$$u_t + u^3u_x - u_{xx} = 0. \quad (47)$$

FIGURE 12: L_2 error norm at $\alpha = 1$, $\beta = 1$, $\delta = 1$ for case 7. FIGURE 13: L_∞ error norm at $\alpha = 1$, $\beta = 1$, $\delta = 1$ for case 7. FIGURE 14: Exact solution in space at $\alpha = 1$, $\beta = 1$, $\delta = 1$, $t = 0.9$ for case 7. FIGURE 15: Approximate solution in space at $\alpha = 1$, $\beta = 1$, $\delta = 1$, $t = 0.9$ for case 7.

The absolute errors are calculated by proposed HBSCM and compared with the existing methods: computational meshless method (CMM) [27], EFM [29], IA [37], ADM [40], and NAT [70] at two different time levels which are tabulated in Table 17. A remarkable difference in accuracy can be noted from this table.

5. Concluding Remarks

In this paper, a hybrid B-spline collocation method is developed for solving one-dimensional GBH and GBF equations with known initial and boundary conditions. A finite difference scheme and hybrid B-spline function are used to discretize the time and spatial derivatives, respectively. The hybrid B-spline method considered in this study is simple and straight forward by the application point of view. The obtained results are presented in Tables 1–17 and graphically shown in Figures 1–15. It is found that the numerical results are in excellent agreement with the analytical solutions. The proposed scheme is not only accurate but also quite different from the schemes [53–57] due to presence of parameter. The parameter $\mu > 1$ provides the better approximation to the exact solution as compared to classical cubic B-spline function. The stability analysis using Von Neumann approach has also been presented. It is shown that the method is unconditionally stable for any step length.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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