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Research Article

A Geometric Modeling and Computing Method for Direct Kinematic Analysis of 6-4 Stewart Platforms

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A geometric modeling and solution procedure for direct kinematic analysis of 6-4 Stewart platforms with any link parameters is proposed based on conformal geometric algebra (CGA). Firstly, the positions of the two single spherical joints on the moving platform are formulated by the intersection, dissection, and dual of the basic entities under the frame of CGA. Secondly, a coordinate-invariant equation is derived via CGA operation in the positions of the other two pairwise spherical joints. Thirdly, the other five equations are formulated in terms of geometric constraints. Fourthly, a 32-degree univariate polynomial equation is reduced from a constructed 7 by 7 matrix which is relatively small in size by using a Gröbner-Sylvester hybrid method. Finally, a numerical example is employed to verify the solution procedure. The novelty of the paper lies in that (1) the formulation is concise and coordinate-invariant and has intrinsic geometric intuition due to the use of CGA and (2) the size of the resultant matrix is smaller than those existed.

1. Introduction

The Stewart platform [1] is a fully parallel, six-degree-of-freedom manipulator that generally consists of a base platform, a moving platform, and six limbs connected to each other in parallel. Stewart platforms have been successfully used in a wide variety of fields and industries, ranging from astronomy to flight simulators and are becoming increasingly popular in the machine-tool industry [2]. From the 1980s, Stewart platforms have attracted wide interests from researchers and engineers due to their advantages of simplicity, high stiffness, large load capacity, quick dynamic response, and excellent accuracy.

The direct kinematic analysis of Stewart platforms has been considered a challenging problem, which leads naturally to a system of highly nonlinear algebraic equations with multiple solutions. There are two main approaches to solve these equations: numerical schemes and closed-form solutions. A closed-form solution provides more information about the geometric and kinematic behavior over a numerical solution, and the closed-form univariate polynomial equation has significant theoretical values as it is fundamental to many other kinematic problems. Hence obtaining a closed-form solution to the direct kinematic analysis is clearly preferred in most cases.

In this paper, we will revisit the direct kinematic analysis of 6-4 Stewart platforms, four of which meet the platform pairwise, while the remaining two meet both base and platform singly. Numerous researchers [3–6] have worked on this problem. Hunt (1983) [3] wrongly stated that the maximum number of assembly modes for the problem was 24 by geometrical proof. Innocenti (1995) [4] derived a suitable set of five closure equations and solved the problem by a specifically developed elimination scheme, that is, a constructed 10 by 10 matrix. The number of the solutions is 36 in view of resultant form; however, the numerical result leads to a 32-degree polynomial equation in a single variable. Liao et al. (1995) [5] formulated this problem based on the vector method from equivalent mechanisms and obtained all the 32 solutions by constructed 10 by 10 resultant matrix. The solution procedure is complex due to numerous vector computations. Zhang et al. (2012) [6] also modeled this problem based on the trilateration method and vector method from equivalent mechanisms and the solution procedure is the same as [5]. It is concluded from the above-mentioned literature that the modeling and the solution procedure either are formulated algebraically from the equivalent mechanisms or require resultant elimination. The size of the constructed resultant matrix is all 10 by 10.

Conformal geometric algebra (CGA) [7–10] is a relatively new mathematical tool for geometric representation and computation. Essentially, CGA represents various geometric entities of points, spheres, lines, planes, circles, and point pairs in a systematical hierarchy of multiple grades. More importantly, CGA provides direct algebraic operations on these geometric entities which typically lead to simple, compact, coordinate-invariant formulations and enables complicated symbolic geometric computations. The above-mentioned properties are two superior characteristics of CGA. Hence it is very efficient for geometric modeling and computation for kinematic problem of mechanisms and robotics. In recent decades, CGA has been mostly applied to solve the inverse kinematics problem of the serial mechanisms [11–14] via CGA operation of the geometric entities. In addition, Tanev [15, 16], Kim et al. [17], and Huo et al. [18] employed CGA to study the singularity analysis of PMs. Huo et al. [18] and Li et al. [19] proposed a mobility analysis approach for PMs based on geometric algebra. Zhang et al. [20, 21] and Wei et al. [22] applied CGA to solve the direct kinematics of parallel mechanisms.

In this paper, we will formulate the direct kinematic analysis problem of 6-4 Stewart platform using CGA and then construct a 7 by 7 resultant using Gröbner-Sylvester hybrid method [23, 24] which finally leads to a 32-degree univariate polynomial equation without extraneous roots. The derived coordinate-invariant equation is also applicable to other Stewart platforms or parallel mechanisms whose number of spherical joints on the moving platform is equal to 4.

The rest of the paper is organized as follows: In Section 2, the fundamentals of CGA are introduced. In Section 3, the geometric modeling for the direct kinematic analysis of 6-4 Stewart platforms is formulated based on CGA. Section 4 proposes the elimination procedure and finally reduces a 32-degree univariate polynomial equation from a constructed 7 by 7 matrix by Gröbner-Sylvester hybrid method. In Section 5, a numerical example is provided to verify our solution procedure. Finally, conclusions and future work will be given in Section 6.

2. Fundamentals of Conformal Geometric Algebra

In geometric algebra, the fundamental algebraic operators are the inner product ($\mathbf{A} \cdot \mathbf{B}$), the outer product ($\mathbf{A} \wedge \mathbf{B}$), and the geometric product ($\mathbf{AB} = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B}$).

The 5-dimensional (5D) CGA $\mathbf{G}^{4,1}$ is derived from a 3D Euclidean space $\mathbf{G}^3$ and a 2D Minkowski space $\mathbf{G}^{1,1}$. CGA has five orthonormal basis vectors given by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_+, \mathbf{e}_-\}$ with the following properties:

$$\begin{gathered}
{\mathbf{e}_1}^2 = {\mathbf{e}_2}^2 = {\mathbf{e}_3}^2 = {\mathbf{e}_+}^2 = -{\mathbf{e}_-}^2 = 1, \\
\mathbf{e}_i \cdot \mathbf{e}_j = 0 \quad (i \neq j;\ i, j = 1, 2, 3, +, -), \\
\mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i \quad (i, j = 1, 2, 3, +, -),
\end{gathered} \tag{1}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are the three orthonormal basis vectors in the Euclidean space and $\{\mathbf{e}_+, \mathbf{e}_-\}$ are the two orthogonal basis vectors in Minkowski space.

In addition, two null bases can now be introduced by the vectors

$$\begin{aligned}
\mathbf{e}_0 &= \frac{1}{2}(\mathbf{e}_- - \mathbf{e}_+), \\
\mathbf{e}_\infty &= \mathbf{e}_+ + \mathbf{e}_-,
\end{aligned} \tag{2}$$

with the properties

$$\begin{aligned}
\mathbf{e}_0^2 &= \mathbf{e}_\infty^2 = 0, \\
\mathbf{e}_\infty \cdot \mathbf{e}_0 &= -1,
\end{aligned} \tag{3}$$

where $\mathbf{e}_0$ is the conformal origin and $\mathbf{e}_\infty$ is the conformal infinity.

Blades are the basic computational elements and the basic geometric entities of the geometric algebra. The grade of a blade is simply the number of linearly independent vectors that are "wedged" together. The 5D CGA consists of blades with grades 0, 1, 2, 3, 4, and 5. A linear combination of the k-blades is called a k-vector, and a linear combination of blades with different grades is called a multivector. The blades with the maximum grade in CGA, that is, 5-blades, are called pseudoscalars and denoted by $\mathbf{I}_\mathbf{C}(\mathbf{e}_{\infty 0123}, {\mathbf{I}_\mathbf{C}}^2 = -1)$.

According to (1)–(3), the inner ($\cdot$) and outer ($\wedge$) products of two 1-vectors $\mathbf{u}, \mathbf{v}$ are defined as

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \frac{1}{2}(\mathbf{uv} + \mathbf{vu}), \\
\mathbf{u} \wedge \mathbf{v} &= \frac{1}{2}(\mathbf{uv} - \mathbf{vu}).
\end{aligned} \tag{4}$$

As extension, the inner product of an r-blade $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_r$ with an s-blade $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_s$ can be defined recursively by

$$\begin{aligned}
&(\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_r) \cdot (\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_s) \\
&\quad = \begin{cases}
((\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_r) \cdot \mathbf{v}_1) \cdot (\mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_s) & \text{if } r \geq s \\
(\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{r-1}) \cdot (\mathbf{u}_r \cdot (\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_s)) & \text{if } r < s
\end{cases}
\end{aligned} \tag{5}$$

with

$$\begin{aligned}
(\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_r) \cdot \mathbf{v}_1 &= \sum_{i=1}^{r} (-1)^{r-i} \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{i-1} \\
&\quad \wedge (\mathbf{u}_i \cdot \mathbf{v}_1) \wedge \mathbf{u}_{i+1} \wedge \cdots \wedge \mathbf{u}_r, \\
\mathbf{u}_r \cdot (\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_s) &= \sum_{i=1}^{s} (-1)^{i-1} \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{i-1} \\
&\quad \wedge (\mathbf{u}_r \cdot \mathbf{v}_i) \wedge \mathbf{v}_{i+1} \wedge \cdots \wedge \mathbf{v}_s.
\end{aligned} \tag{6}$$

We define the dual $\mathbf{X}^*$ of a multivector $\mathbf{X}$ by

$$\mathbf{X}^* = \mathbf{X}\mathbf{I}_\mathbf{C}^{-1} = -\mathbf{X}\mathbf{I}_\mathbf{C}, \tag{7}$$

where $\mathbf{I}_\mathbf{C}^{-1}$ is the inverse of $\mathbf{I}_\mathbf{C}$ and is equal to $-\mathbf{I}_\mathbf{C}$.

CGA provides the representation of primitive geometric entities for intuitive expression. The primitive geometric entities in CGA consist of spheres, points, lines, planes, circles, and point pairs. The representation of the geometric entities with respect to the inner product null space (IPNS) and the one with respect to the outer product null space are, respectively, listed in Table 1. These two representations are dual to each other and therefore can be converted by dual operator. In Table 1, the small bold character represents the point or vector in the Euclidean space, while the bold underlined character represents the basic geometric entity in the conformal space. For more information, please refer to [9, 10].

According to (1)–(3), the inner product between two conformal points $\underline{\mathbf{P}}_1, \underline{\mathbf{P}}_2$ is calculated as

$$\begin{aligned}\underline{\mathbf{P}}_1 \cdot \underline{\mathbf{P}}_2 &= \left(\mathbf{p}_1 + \frac{1}{2}\mathbf{p}_1{}^2\mathbf{e}_\infty + \mathbf{e}_0\right) \cdot \left(\mathbf{p}_2 + \frac{1}{2}\mathbf{p}_2{}^2\mathbf{e}_\infty + \mathbf{e}_0\right) \\ &= -\frac{1}{2}\left(\mathbf{p}_1 - \mathbf{p}_2\right)^2 = -\frac{1}{2}d_{12}^2,\end{aligned} \tag{8}$$

where d_{12} denotes the Euclidean distance between the two points.

From (8), we have $\underline{\mathbf{P}} \cdot \underline{\mathbf{P}} = 0$.

In the next section, we will formulate the direct kinematics of 6-4 Stewart platforms via CGA operation and derive the univariate polynomial equation.

3. Geometric Modeling for Direct Kinematics of 6-4 Stewart Platforms Based on CGA

A 6-4 Stewart platform $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4\mathbf{A}_5\mathbf{A}_6$-$\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3\mathbf{B}_4$, shown in Figure 1, has six SPS (S: spherical joint, P: prismatic joint) legs, four of which meet the moving platform pairwise, while the remaining two meet both the base and the moving platform singly. The six limb lengths l_i $(i = 1, 2, \ldots, 6)$ provided by P-joint in every limb are six inputs to control the position and orientation of the moving platform. For the general 6-4 Stewart platforms, the six S-joints on the base and the four S-joints on the moving platform are not restricted to lie in a plane, respectively. Let $\mathbf{a}_i$ $(i = 1, 2, \ldots, 6)$ and $\mathbf{b}_i$ $(i = 1, 2, 3, 4)$ denote the coordinates of the center of the S-joints $\mathbf{A}_i$ and $\mathbf{B}_i$ in the Euclidean space, respectively; let r_i $(i = 1, 2, \ldots, 6)$ denote the distance of the two S-joints $\mathbf{B}_i$ on the moving platform, where the coordinates $\mathbf{b}_i$ $(i = 1, 2, 3, 4)$ are unknown. Next, we will formulate the two single S-joints $\mathbf{B}_3$ and $\mathbf{B}_4$ on the moving platform by the intersection, dissection, and dual of the basic geometric entities under the frame of CGA.

3.1. The CGA Representation of the Positions of Two Spherical Joints $\mathbf{B}_3$ *and* $\mathbf{B}_4$. As seen from Figure 1, in the tetrahedron $\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3\mathbf{A}_5$, the S-joint $\underline{\mathbf{B}}_3$ must be located on a sphere $\underline{\mathbf{S}}_1$ of radius r_2 with its center at point $\underline{\mathbf{B}}_1$, a sphere $\underline{\mathbf{S}}_2$ of the radius r_1 with the center at point $\underline{\mathbf{B}}_2$, and a sphere $\underline{\mathbf{S}}_3$ of the radius l_5 with the center at point $\underline{\mathbf{A}}_5$. Thus, the actual locus of point $\underline{\mathbf{B}}_3$ must be located on the intersection of the three spheres $\underline{\mathbf{S}}_1$, $\underline{\mathbf{S}}_2$, and $\underline{\mathbf{S}}_3$. From the knowledge of geometry, it is known that the locus of this intersection will be a point pair $\underline{\mathbf{B}}_{b3}$. And therefore, according to Table 1, the point pair $\underline{\mathbf{B}}_{b3}$ can be formulated in CGA as

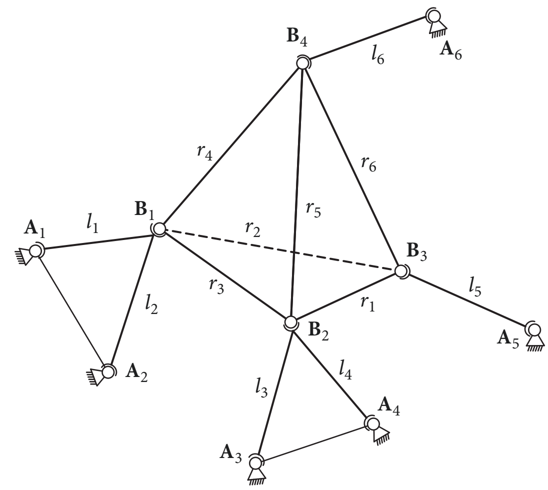

Figure 1: The geometric model of a 6-4 Stewart platform.

$$\underline{\mathbf{B}}_{b3} = \underline{\mathbf{S}}_1 \wedge \underline{\mathbf{S}}_2 \wedge \underline{\mathbf{S}}_3, \tag{9}$$

where the three spheres $\underline{\mathbf{S}}_1$, $\underline{\mathbf{S}}_2$, and $\underline{\mathbf{S}}_3$ can be represented in CGA as

$$\begin{aligned}\underline{\mathbf{S}}_1 &= \underline{\mathbf{S}}_{B_1B_3} = \underline{\mathbf{B}}_1 - \frac{1}{2}r_2{}^2\mathbf{e}_\infty, \\ \underline{\mathbf{S}}_2 &= \underline{\mathbf{S}}_{B_2B_3} = \underline{\mathbf{B}}_2 - \frac{1}{2}r_1{}^2\mathbf{e}_\infty, \\ \underline{\mathbf{S}}_3 &= \underline{\mathbf{S}}_{A_5B_3} = \underline{\mathbf{A}}_5 - \frac{1}{2}l_5{}^2\mathbf{e}_\infty.\end{aligned} \tag{10}$$

According to Table 1, the three centers of S-joints $\underline{\mathbf{B}}_1$, $\underline{\mathbf{B}}_2$, and $\underline{\mathbf{A}}_5$ can be represented in CGA as

$$\begin{aligned}\underline{\mathbf{B}}_1 &= \mathbf{b}_1 + \frac{1}{2}\mathbf{b}_1{}^2\mathbf{e}_\infty + \mathbf{e}_0, \\ \underline{\mathbf{B}}_2 &= \mathbf{b}_2 + \frac{1}{2}\mathbf{b}_2{}^2\mathbf{e}_\infty + \mathbf{e}_0, \\ \underline{\mathbf{A}}_5 &= \mathbf{a}_5 + \frac{1}{2}\mathbf{a}_5{}^2\mathbf{e}_\infty + \mathbf{e}_0.\end{aligned} \tag{11}$$

According to (7), the dual $\underline{\mathbf{B}}_{b3}^*$ of the point pair $\underline{\mathbf{B}}_{b3}$ is represented as

$$\underline{\mathbf{B}}_{b3}^* = \underline{\mathbf{B}}_{\mathbf{b}3}\mathbf{I}_C{}^{-1} = -\underline{\mathbf{B}}_{\mathbf{b}3}\mathbf{I}_C. \tag{12}$$

Point $\underline{\mathbf{B}}_3$ is dissected from the dual of the point pair $\underline{\mathbf{B}}_{b3}^*$ in the conformal space as [21, 22]

$$\underline{\mathbf{B}}_3 = \frac{\mathbf{T}_{b2}}{A_{\mathrm{var}}} \pm \frac{\sqrt{B_{\mathrm{var}}}}{A_{\mathrm{var}}}\mathbf{T}_{b1}, \tag{13}$$

Table 1: List of conformal geometric entities.

Entity	Representation	Grade	Dual representation	Grade
Point	$\underline{\mathbf{P}} = \mathbf{p} + \frac{1}{2}\mathbf{p}^2\mathbf{e}_\infty + \mathbf{e}_0$	1	$\underline{\mathbf{P}}^* = \underline{\mathbf{S}}_1 \wedge \underline{\mathbf{S}}_2 \wedge \underline{\mathbf{S}}_3 \wedge \underline{\mathbf{S}}_4$	4
Sphere	$\underline{\mathbf{S}} = \mathbf{p} + \frac{1}{2}(\mathbf{p}^2 - \rho^2)\mathbf{e}_\infty + \mathbf{e}_0$	1	$\underline{\mathbf{S}}^* = \underline{\mathbf{P}}_1 \wedge \underline{\mathbf{P}}_2 \wedge \underline{\mathbf{P}}_3 \wedge \underline{\mathbf{P}}_4$	4
Plane	$\underline{\boldsymbol{\pi}} = \mathbf{n} + d\mathbf{e}_\infty$	1	$\underline{\boldsymbol{\pi}}^* = \mathbf{e}_\infty \wedge \underline{\mathbf{P}}_1 \wedge \underline{\mathbf{P}}_2 \wedge \underline{\mathbf{P}}_3$	4
Line	$\underline{\mathbf{L}} = \underline{\boldsymbol{\pi}}_1 \wedge \underline{\boldsymbol{\pi}}_2$	2	$\underline{\mathbf{L}}^* = \mathbf{e}_\infty \wedge \underline{\mathbf{P}}_1 \wedge \underline{\mathbf{P}}_2$	3
Circle	$\underline{\mathbf{C}} = \underline{\mathbf{S}}_1 \wedge \underline{\mathbf{S}}_2$	2	$\underline{\mathbf{C}}^* = \underline{\mathbf{P}}_1 \wedge \underline{\mathbf{P}}_2 \wedge \underline{\mathbf{P}}_3$	3
Point pair	$\underline{\mathbf{P}}_\mathbf{p} = \underline{\mathbf{S}}_1 \wedge \underline{\mathbf{S}}_2 \wedge \underline{\mathbf{S}}_3$	3	$\underline{\mathbf{P}}_\mathbf{p}^* = \underline{\mathbf{P}}_1 \wedge \underline{\mathbf{P}}_2$	2

where $\mathbf{T}_{b1} = \mathbf{e}_\infty \cdot \underline{\mathbf{B}}_{b3}^*$, $\mathbf{T}_{b2} = \mathbf{T}_{b1} \cdot \underline{\mathbf{B}}_{b3}^*$, $A_{\text{var}} = \mathbf{T}_{b1} \cdot \mathbf{T}_{b1}$, and $B_{\text{var}} = \underline{\mathbf{B}}_{b3}^* \cdot \underline{\mathbf{B}}_{b3}^*$. Please note that the expression of point $\underline{\mathbf{B}}_3$ in (13) is in its standard and normalized form; that is, the magnitude is equal to 1.

For the S-joint $\underline{\mathbf{B}}_4$ on the moving platform, it can be seen from Figure 1 that, in the pentahedron $\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3\mathbf{B}_3\mathbf{A}_6$, point $\underline{\mathbf{B}}_4$ is the intersection point of four spheres $\underline{\mathbf{S}}_4$, $\underline{\mathbf{S}}_5$, $\underline{\mathbf{S}}_6$, and $\underline{\mathbf{S}}_7$, that is, the sphere $\underline{\mathbf{S}}_4$ of the radius r_4 with its center at point $\underline{\mathbf{B}}_1$, the sphere $\underline{\mathbf{S}}_5$ of the radius r_5 with the center at point $\underline{\mathbf{B}}_2$, the sphere $\underline{\mathbf{S}}_6$ of the radius l_6 with the center at point $\underline{\mathbf{A}}_6$, and the sphere $\underline{\mathbf{S}}_7$ of the radius r_6 with the center at point $\underline{\mathbf{B}}_3$. Therefore according to Table 1, the dual of point $\underline{\mathbf{B}}_4$ is represented in CGA as

$$\underline{\mathbf{B}}_4^* = \underline{\mathbf{S}}_4 \wedge \underline{\mathbf{S}}_5 \wedge \underline{\mathbf{S}}_6 \wedge \underline{\mathbf{S}}_7, \tag{14}$$

where the four spheres $\underline{\mathbf{S}}_4$, $\underline{\mathbf{S}}_5$, $\underline{\mathbf{S}}_6$, and $\underline{\mathbf{S}}_7$ can be represented in CGA as

$$\begin{aligned} \underline{\mathbf{S}}_4 &= \underline{\mathbf{S}}_{B_1B_4} = \underline{\mathbf{B}}_1 - \frac{1}{2}{r_4}^2\mathbf{e}_\infty, \\ \underline{\mathbf{S}}_5 &= \underline{\mathbf{S}}_{B_2B_4} = \underline{\mathbf{B}}_2 - \frac{1}{2}{r_5}^2\mathbf{e}_\infty, \\ \underline{\mathbf{S}}_6 &= \underline{\mathbf{S}}_{A_6B_4} = \underline{\mathbf{A}}_6 - \frac{1}{2}{l_6}^2\mathbf{e}_\infty, \\ \underline{\mathbf{S}}_7 &= \underline{\mathbf{S}}_{B_3B_4} = \underline{\mathbf{B}}_3 - \frac{1}{2}{r_6}^2\mathbf{e}_\infty. \end{aligned} \tag{15}$$

According to Table 1, the centers of S-joint $\underline{\mathbf{A}}_6$ can be represented in CGA as $\underline{\mathbf{A}}_6 = \mathbf{a}_6 + (1/2){\mathbf{a}_6}^2\mathbf{e}_\infty + \mathbf{e}_0$.

Point $\underline{\mathbf{B}}_4$ is reduced from (7) and (14) as

$$\underline{\mathbf{B}}_4 = \underline{\mathbf{B}}_4^*\mathbf{I}_C^{-1} = -\underline{\mathbf{B}}_4^*\mathbf{I}_C = -\underline{\mathbf{S}}_7 \cdot \underline{\mathbf{B}}_{b4}^*, \tag{16}$$

where $\underline{\mathbf{B}}_{b4}^* = -(\underline{\mathbf{S}}_4 \wedge \underline{\mathbf{S}}_5 \wedge \underline{\mathbf{S}}_6)\mathbf{I}_C$ and is dual to the point pair $\underline{\mathbf{B}}_{b4}$, which is generated by the intersection of three spheres $\underline{\mathbf{S}}_4$, $\underline{\mathbf{S}}_5$, and $\underline{\mathbf{S}}_6$.

Please note the expression of point $\underline{\mathbf{B}}_4$ is not in its standard and normalized form; that is, the magnitude is not equal to 1 and we can obtain its standard and normalized form by dividing (16) using its magnitude $(-\mathbf{e}_\infty \cdot \underline{\mathbf{B}}_4)$.

3.2. The Derivation of the Coordinate-Invariant Polynomial Equation. According to (8), we can readily obtain

$$\begin{gathered} \underline{\mathbf{B}}_4 \cdot \underline{\mathbf{B}}_4 = 0 \Longleftrightarrow \\ \left(-\underline{\mathbf{S}}_7 \cdot \underline{\mathbf{B}}_{b4}^*\right) \cdot \left(-\underline{\mathbf{S}}_7 \cdot \underline{\mathbf{B}}_{b4}^*\right) = 0. \end{gathered} \tag{17}$$

However, due to the volume sign of the tetrahedron $\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3\mathbf{B}_4$, it will lead to the symmetric extraneous roots with respect to the plane $\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3$ if we obtain the coordinate-invariant equation only from (17). In order to avoid the extraneous roots, we will use its volume of the tetrahedron $\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3\mathbf{B}_4$ to geometrically model the direct kinematics of 6-4 Stewart platforms.

The volume of the tetrahedron $\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3\mathbf{B}_4$ is expressed in CGA as [7]

$$V = -\left(\mathbf{e}_\infty \wedge \underline{\mathbf{B}}_1 \wedge \underline{\mathbf{B}}_2 \wedge \underline{\mathbf{B}}_3 \wedge \underline{\mathbf{B}}_4\right)\mathbf{I}_C, \tag{18}$$

where V is a known scalar and in fact V is six times the volume of the tetrahedron $\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3\mathbf{B}_4$. When four points $\mathbf{B}_i$ $(i = 1, 2, 3, 4)$ on the moving platform lie in the same plane, $V = 0$; if not, V may be positive or negative depending on point $\mathbf{B}_4$ locating above or below the plane $\mathbf{B}_1\mathbf{B}_2\mathbf{B}_3$.

Multiplying both sides of (18) with a scalar $V_0 = -(\mathbf{e}_\infty \wedge \underline{\mathbf{A}}_6 \wedge \underline{\mathbf{B}}_1 \wedge \underline{\mathbf{B}}_2 \wedge \underline{\mathbf{B}}_3)\mathbf{I}_C$ and then expanding it according to (5), we obtain

$$V * V_0 = -\left(\underline{\mathbf{B}}_3 \cdot \underline{\mathbf{A}}_6\right)S_0 + C_{\text{var}}, \tag{19}$$

where

$$\begin{aligned} V_0 &= -\left(\mathbf{e}_\infty \wedge \underline{\mathbf{A}}_6 \wedge \underline{\mathbf{B}}_1 \wedge \underline{\mathbf{B}}_2 \wedge \underline{\mathbf{B}}_3\right)\mathbf{I}_C = -\underline{\mathbf{B}}_3 \cdot \left(\mathbf{e}_\infty \right. \\ &\quad \left. \wedge \underline{\mathbf{A}}_6 \wedge \underline{\mathbf{B}}_1 \wedge \underline{\mathbf{B}}_2\right)\mathbf{I}_C, \\ C_{\text{var}} &= -S_1 - \left(\underline{\mathbf{B}}_1 \cdot \underline{\mathbf{A}}_6\right)S_2 + \left(\underline{\mathbf{B}}_2 \cdot \underline{\mathbf{A}}_6\right)S_3 - \frac{1}{2}{l_6}^2S_4, \\ V^2 &= \frac{1}{4}\left(-{r_1}^2{r_2}^2{r_3}^2 - {r_1}^4{r_4}^2 + {r_1}^2{r_2}^2{r_4}^2 + {r_1}^2{r_3}^2{r_4}^2 \right. \\ &\quad - {r_1}^2{r_4}^4 + {r_1}^2{r_2}^2{r_5}^2 - {r_2}^4{r_5}^2 + {r_2}^2{r_3}^2{r_5}^2 \\ &\quad + {r_1}^2{r_4}^2{r_5}^2 + {r_2}^2{r_4}^2{r_5}^2 - {r_3}^2{r_4}^2{r_5}^2 - {r_2}^2{r_5}^4 \\ &\quad + {r_1}^2{r_3}^2{r_6}^2 + {r_2}^2{r_3}^2{r_6}^2 - {r_3}^4{r_6}^2 - {r_3}^2{r_6}^4 \\ &\quad + {r_1}^2{r_4}^2{r_6}^2 - {r_2}^2{r_4}^2{r_6}^2 + {r_3}^2{r_4}^2{r_6}^2 - {r_1}^2{r_5}^2{r_6}^2 \\ &\quad \left. + {r_2}^2{r_5}^2{r_6}^2 + {r_3}^2{r_5}^2{r_6}^2\right), \\ S_0 &= \frac{1}{4}\left(-{r_1}^2{r_3}^2 - {r_2}^2{r_3}^2 + {r_3}^4 - {r_1}^2{r_4}^2 + {r_2}^2{r_4}^2 \right. \\ &\quad \left. - {r_3}^2{r_4}^2 + {r_1}^2{r_5}^2 - {r_2}^2{r_5}^2 - {r_3}^2{r_5}^2 + 2{r_3}^2{r_6}^2\right), \end{aligned}$$

$$S_1 = \frac{1}{8}\left(-2{r_1}^2{r_2}^2{r_3}^2 - {r_1}^4{r_4}^2 + {r_1}^2{r_2}^2{r_4}^2 + {r_1}^2{r_3}^2{r_4}^2 + {r_1}^2{r_2}^2{r_5}^2 - {r_2}^4{r_5}^2 + {r_2}^2{r_3}^2{r_5}^2 + {r_1}^2{r_3}^2{r_6}^2 + {r_2}^2{r_3}^2{r_6}^2 - {r_3}^4{r_6}^2\right),$$

$$S_2 = \frac{1}{4}\left({r_1}^4 - {r_1}^2{r_2}^2 - {r_1}^2{r_3}^2 + 2{r_1}^2{r_4}^2 - {r_1}^2{r_5}^2 - {r_2}^2{r_5}^2 + {r_3}^2{r_5}^2 - {r_1}^2{r_6}^2 + {r_2}^2{r_6}^2 - {r_3}^2{r_6}^2\right),$$

$$S_3 = \frac{1}{4}\left({r_1}^2{r_2}^2 - {r_2}^4 + {r_2}^2{r_3}^2 + {r_1}^2{r_4}^2 + {r_2}^2{r_4}^2 - {r_3}^2{r_4}^2 - 2{r_2}^2{r_5}^2 - {r_1}^2{r_6}^2 + {r_2}^2{r_6}^2 + {r_3}^2{r_6}^2\right),$$

$$S_4 = \frac{1}{4}\left({r_1}^4 - 2{r_1}^2{r_2}^2 + {r_2}^4 - 2{r_1}^2{r_3}^2 - 2{r_2}^2{r_3}^2 + {r_3}^4\right). \tag{20}$$

By transposing and combining the terms in (19), we have

$$\underline{\mathbf{B}}_3 \cdot \mathbf{N} = C_{\text{var}}, \tag{21}$$

where $\mathbf{N} = -V * (e \wedge \underline{\mathbf{A}}_6 \wedge \underline{\mathbf{B}}_1 \wedge \underline{\mathbf{B}}_2)\mathbf{I}_C + S_0 * \underline{\mathbf{A}}_6$.

Substituting (13) into (21) and then expanding (21), we have

$$\pm\frac{\sqrt{B_{\text{var}}}\left(\mathbf{T}_{b1} \cdot \mathbf{N}\right)}{A_{\text{var}}} = C_{\text{var}} - \frac{\left(\mathbf{T}_{b2} \cdot \mathbf{N}\right)}{A_{\text{var}}}. \tag{22}$$

Taking the square of both sides of (22) and combining terms, we obtain

$$\frac{\left(\left(\mathbf{T}_{b2} \cdot \mathbf{N}\right)^2 - B_{\text{var}}\left(\mathbf{T}_{b1} \cdot \mathbf{N}\right)^2\right)}{A_{\text{var}}^2} - \frac{2C_{\text{var}}\left(\mathbf{T}_{b2} \cdot \mathbf{N}\right)}{A_{\text{var}}} + C_{\text{var}}^2 = 0. \tag{23}$$

Simplifying (23) by using (1)–(5) and taking only the numerator, we have

$$E_{\text{var}} - B_{\text{var}}D_{\text{var}} - 2C_{\text{var}}\left(\mathbf{T}_{b2} \cdot \mathbf{N}\right) + A_{\text{var}}C_{\text{var}}^2 = 0, \tag{24}$$

where $(\mathbf{T}_{b2} \cdot \mathbf{N})^2 - B_{\text{var}}(\mathbf{T}_{b1} \cdot \mathbf{N})^2 = A_{\text{var}}(E_{\text{var}} - B_{\text{var}}D_{\text{var}})$, $D_{\text{var}} = \mathbf{N} \cdot \mathbf{N}$, and $E_{\text{var}} = (\underline{\mathbf{B}}_{b3}^* \wedge \mathbf{N}) \cdot (\underline{\mathbf{B}}_{b3}^* \wedge \mathbf{N})$. The detailed expansion of the term $(\mathbf{T}_{b2} \cdot \mathbf{N})^2 - B_{\text{var}}(\mathbf{T}_{b1} \cdot \mathbf{N})^2$ is given in the appendix.

The derivation of (24) is coordinate-invariant due to the use of CGA, and (24) depends on only the design parameters, the inputs, and the positions of points $\mathbf{B}_1$ and $\mathbf{B}_2$.

3.3. The Derivation of the Other Five Constraint Equations. Equation (24) is the first constraint equation in the positions of two pairwise S-joints $\mathbf{B}_1$ and $\mathbf{B}_2$ on the moving platform. According to the limb length and (8), the other five constraint equations can be formulated as

$$\underline{\mathbf{B}}_1 \cdot \underline{\mathbf{A}}_1 = -\frac{1}{2}{l_1}^2, \tag{25}$$

$$\underline{\mathbf{B}}_1 \cdot \underline{\mathbf{A}}_2 = -\frac{1}{2}{l_2}^2, \tag{26}$$

$$\underline{\mathbf{B}}_2 \cdot \underline{\mathbf{A}}_3 = -\frac{1}{2}{l_3}^2, \tag{27}$$

$$\underline{\mathbf{B}}_2 \cdot \underline{\mathbf{A}}_4 = -\frac{1}{2}{l_4}^2, \tag{28}$$

$$\underline{\mathbf{B}}_1 \cdot \underline{\mathbf{B}}_2 = -\frac{1}{2}{r_3}^2. \tag{29}$$

Equations (24)–(29) are the six constraint equations for direct kinematics of 6-4 Stewart platforms. In next section, we will derive a univariate high-degree polynomial equation for this problem by using Gröbner-Sylvester hybrid method [23, 24].

4. Solution Procedure

4.1. An Univariate Equation by Using the Gröbner-Sylvester Hybrid Method. In this section, the main aim is to obtain the univariate high-degree polynomial equation, from which the solutions of direct kinematics of 6-4 Stewart platforms can be obtained. In order to get the position of the four S-joints on the moving platform, first of all, we attach a reference coordinate frame $\mathbf{O}$-XYZ with its origin $\mathbf{O}$ anywhere, and let $\mathbf{a}_i$ $(a_{xi}, a_{yi}, a_{zi})^{\mathrm{T}}$ $(i = 1, \ldots, 6)$ denote the coordinate of the S-joint $\mathbf{A}_i$ and $\mathbf{b}_i$ $(b_{xi}, b_{yi}, b_{zi})^{\mathrm{T}}$ $(i = 1, \ldots, 4)$ denote the coordinate of the S-joint $\mathbf{B}_i$ in the reference frame. And therefore, (24)–(29) are six constraint equations in variables b_{x1}, b_{y1}, b_{z1}, b_{x2}, b_{y2}, b_{z2}.

For (24)–(29), changing the exponent of the three variables b_{x1}, b_{y1}, and b_{z1} four times the original one and using the Gröbner basis theory under the degree reverse lexicographic term ordering with $b_{x1}^4 > b_{y1}^4 > b_{z1}^4 > b_{x2} > b_{y2} > b_{z2}$ yield a Gröbner basis with 15 polynomials with the suppressed variable b_{z2} as follows:

$$gb_1 = g_1\left(1, \underline{b_{x2}}, b_{y2}\right),$$

$$gb_2 = g_2\left(1, b_{y2}, \underline{b_{y2}^2}\right),$$

$$gb_3 = g_3\left(1, \underline{b_{x1}}, b_{z1}\right),$$

$$gb_4 = g_4\left(1, b_{z1}, b_{y2}, b_{y1}b_{y2}, \underline{b_{y2}b_{z1}}\right),$$

$$gb_6 = g_6\left(1, \underline{b_{y1}^2}, b_{z1}, b_{z1}^2\right),$$

$$gb_{11} = g_{11}\left(1, b_{y1}, b_{z1}, b_{y2}, b_{y1}b_{z1}, b_{y2}b_{z1}, b_{z1}^2, \underline{b_{y1}b_{z1}^2}\right),$$

$$gb_{14} = g_{14}\left(1, b_{y1}, b_{z1}, b_{y2}, b_{y1}b_{z1}, b_{y2}b_{z1}, b_{z1}^2, \underline{b_{z1}^3}\right),$$

Table 2: Input data.

O-X-Y-Z	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{A}_5$	$\mathbf{A}_6$
a_{ix}	0	5	−2	−3	6	−3
a_{iy}	0	0	4	−1	−2	5
a_{iz}	0	1	−1	1	2	−1
The distances r_i between the S-joints $\mathbf{B}_i$	$r_1 = 2\sqrt{6},\ r_2 = 1,\ r_3 = \sqrt{21},\ r_4 = \sqrt{69},\ r_5 = 4,\ r_6 = 6\sqrt{2}$					
The lengths of six limbs l_i	$l_1 = 5.74,\ l_2 = 3.32,\ l_3 = 4.58,\ l_4 = 5.39,\ l_5 = 4.69,\ l_6 = 4.58$					

$$gb_{15} = g_{15}\left(1, b_{y1}, b_{z1}, b_{y2}, b_{y1}b_{z1}, b_{y2}b_{z1}, b_{z1}^2, \underline{b_{y2}b_{z1}^2}\right),$$
$$gb_i = g_i\left(1, b_{y1}, b_{z1}, b_{y2}, b_{y1}b_{z1}, b_{y2}b_{z1}, b_{z1}^2\right)$$
$$(i = 5, 7\text{–}10, 12, 13), \tag{30}$$

where $g_i(1, b_{y1}, \ldots)$ $(i = 1, 2, \ldots, 15)$ means the polynomial of the product terms in the bracket, of which the coefficients are comprised of products of power in the variable b_{z2} and real constants depending on the link parameters and inputs only.

By analyzing the bases in (30), we will notice that in gb_i $(i = 1\text{–}4, 6, 11, 14, 15)$, the product terms underlined only exist in their own bases, and their coefficients are real constants depending on the link parameters and inputs only. And therefore when we set up the matrix form equation, we only need other 7 polynomials to construct the following equation due to no effect on the solution procedure after deleting gb_i $(i = 1\text{–}4, 6, 11, 14, 15)$:

$$\mathbf{G}\left(1, b_{y1}, b_{z1}, b_{y2}, b_{y1}b_{z1}, b_{y2}b_{z1}, b_{z1}^2\right)^{\mathrm{T}} = \mathbf{0}, \tag{31}$$

where $\mathbf{G}$ is a 7 by 7 matrix, of which the elements are polynomials in b_{z2}.

The vanishing of the determinant of the coefficient matrix $\mathbf{G}$ gives necessary condition for polynomials (30) to have common solutions; that is, (24)–(29) have common solutions, so we can get an equation in b_{z2}:

$$|\mathbf{G}| = 0, \tag{32}$$

where $|\mathbf{G}|$ is the determinant of the matrix $\mathbf{G}$.

By expanding each element of the matrix $\mathbf{G}$, we can get the degrees in b_{z2} as shown below:

$$\begin{bmatrix} 2 & 1 & 2 & 2 & \times & 1 & \times \\ 9 & 8 & 4 & 5 & 0 & 0 & 0 \\ 9 & 8 & 5 & 4 & 0 & 0 & 0 \\ 9 & 8 & 4 & 4 & 1 & 0 & 0 \\ 9 & 8 & 4 & 4 & 0 & 0 & 1 \\ 10 & 8 & 4 & 4 & 0 & 0 & 0 \\ 9 & 9 & 4 & 4 & 0 & 0 & 0 \end{bmatrix}. \tag{33}$$

Therefore expanding (32), we can obtain an equation of the highest degree $1 + 5 + 5 + 1 + 1 + 10 + 9 = 32$ in b_{z2} as follows:

$$\sum_{i=0}^{32} c_i b_{z2}^i = 0, \tag{34}$$

where c_i are real constants depending on the link parameters and inputs only.

4.2. Back Substitution. Solving (34), all the 32 solutions for b_{z2} can be gotten. In (31), we choose any 6 of 7 equations and substitute the 32 solutions of b_{z2} into the linear system; using the Cramer's rule, we can obtain the corresponding values of b_{y1}, b_{z1}, and b_{y2} by solving the system linearly. And substitute the 32 solutions of b_{z2} into gb_1 and gb_3, we can obtain the values of b_{x1} and b_{x2}, respectively. Now the coordinates of two S-joints $\mathbf{B}_1$ and $\mathbf{B}_2$ are all obtained.

For the coordinates of S-joint $\mathbf{B}_3$, it cannot be determined only in terms of (13) after substituting the coordinates of two S-joint $\mathbf{B}_1$ and $\mathbf{B}_2$. The choice of the positive or negative sign can be obtained from the ratio of two terms $C_{\text{var}} - (\mathbf{T}_{b2} \cdot \mathbf{N})/A_{\text{var}}$ and $\sqrt{B_{\text{var}}}(\mathbf{T}_{b1} \cdot \mathbf{N})/A_{\text{var}}$. If the ratio is equal to 1, we choose the positive sign and vice versa. If both the two terms are equal to 0, we choose the sign depending on whether the term B_{var} equals 0. If the term B_{var} equals 0, we can choose any sign and it will reduce to the same results. If the term B_{var} is not equal to 0, we determine the sign by making sure that the value of $(-e_\infty \cdot \underline{\mathbf{B}}_4)$ is not equal to 0. After we choose the right sign in (13), we can obtain the coordinates of point $\mathbf{B}_3$. The coordinates of the S-joint $\mathbf{B}_4$ can be obtained from (16) by dividing its magnitude $(-\mathbf{e}_\infty \cdot \underline{\mathbf{B}}_4)$ after substituting the coordinates of the joints $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$.

5. Numerical Example

In order to validate the solution procedure, the link parameters and inputs of the numerical example are given in Table 2 and are the same as [6]. Due to the space limitation, the real solutions are only given in Table 3. The computation time of the closed-form solution is just about 0.3 s in Mathematica 11.0 running on a PC with Intel Core i7-6700 CPU @ 3.4 GHz and 8 GB RAM. The approach is very efficient.

6. Conclusion

The paper has proposed a CGA-based formulation and solution procedure for the direct kinematic analysis of 6-4

Table 3: Ten real solutions.

i		$\mathbf{B}_1$	$\mathbf{B}_2$	$\mathbf{B}_3$	$\mathbf{B}_4$
	X	3.9672	1.6694	3.2853	−1.2546
1	Y	0.4233	1.6113	0.5543	0.9734
	Z	4.1267	0.3440	4.8464	−2.3100
	X	3.9844	0.0656	4.0347	−3.9222
2	Y	0.8633	2.9776	0.9909	3.2402
	Z	4.0407	2.9579	5.0312	3.1266
	X	3.9990	−0.0514	3.9757	−4.0473
3	Y	1.1023	3.0297	1.0191	2.8821
	Z	3.9674	3.0296	4.9636	2.9236
	X	4.0646	−0.2895	4.5438	−3.1386
4	Y	1.7833	3.1288	2.3513	5.9173
	Z	3.6396	3.1583	2.9704	3.4851
	X	4.1190	1.6609	4.9908	−0.2122
5	Y	2.1540	1.3125	2.4709	2.5765
	Z	3.3678	−0.1071	2.9944	−3.7077
	X	4.6080	0.4144	4.8572	−3.4566
6	Y	3.2959	2.8064	2.3558	1.8759
	Z	0.9226	2.7044	0.6897	2.3178
	X	5.2094	1.6165	5.3389	−2.2620
7	Y	1.2109	1.7665	0.2194	1.2649
	Z	−2.0842	0.7055	−2.0784	1.5457
	X	5.2173	1.6477	5.8529	0.0286
8	Y	−1.1033	1.2742	−0.3823	3.7367
	Z	−2.1238	−0.5095	−2.3997	2.1950
	X	5.2341	0.9164	5.1481	−2.3352
9	Y	−0.8230	0.7105	−1.7370	1.8885
	Z	−2.2078	−2.2846	−2.6045	−4.2944
	X	5.2535	0.7010	5.5092	−2.9281
10	Y	0.1857	0.64471	0.9000	0.4219
	Z	−2.3051	−2.5567	−1.6533	−0.8893

Stewart platforms with any link parameters. Thanks to the intuitiveness of CGA, the representations of the positions of two single spherical joints have explicit geometric meaning. A coordinate-invariant polynomial equation was derived via CGA operation and it is feasible to other Stewart platforms or parallel mechanisms whose number of spherical joints on the moving platform is equal to 4. The univariate polynomial equation has been derived by constructing a 7 by 7 resultant matrix which is more compact and smaller than those published in the literature. Compared with the previous methods in the literature, the main contribution of the paper lies in that the formulation has geometric meaning due to the intuitiveness of CGA and, in addition, the size of the matrix is smaller than those existed. In future, we will extend this approach to the direct kinematics of other Stewart platforms or complicated parallel mechanisms.

Appendix

Derivation of the Term $(\mathbf{T}_{b2} \cdot \mathbf{N})^2 - B_{\text{var}}(\mathbf{T}_{b1} \cdot \mathbf{N})^2$

The expansion of $(\mathbf{T}_{b2} \cdot \mathbf{N})^2 - B_{\text{var}}(\mathbf{T}_{b1} \cdot \mathbf{N})^2$ is expressed as

$$\begin{aligned}
&(\mathbf{T}_{b2} \cdot \mathbf{N})^2 - B_{\text{var}} (\mathbf{T}_{b1} \cdot \mathbf{N})^2 \\
&\quad = (\mathbf{T}_{b2} \wedge \mathbf{N}) \cdot (\mathbf{T}_{b2} \wedge \mathbf{N}) + (\mathbf{T}_{b2} \cdot \mathbf{T}_{b2}) (\mathbf{N} \cdot \mathbf{N}) \\
&\qquad - B_{\text{var}} (\mathbf{T}_{b1} \cdot \mathbf{N})^2 \\
&\quad = ((\mathbf{T}_{b1} \cdot \underline{\mathbf{B}}_{b3}^*) \wedge \mathbf{N}) \cdot ((\mathbf{T}_{b1} \cdot \underline{\mathbf{B}}_{b3}^*) \wedge \mathbf{N}) \\
&\qquad - B_{\text{var}} (\mathbf{T}_{b1} \cdot \mathbf{N})^2 + (\mathbf{T}_{b2} \cdot \mathbf{T}_{b2}) (\mathbf{N} \cdot \mathbf{N}) \\
&\quad = (\mathbf{T}_{b1} \cdot (\underline{\mathbf{B}}_{b3}^* \wedge \mathbf{N}) - (\mathbf{T}_{b1} \cdot \mathbf{N}) \underline{\mathbf{B}}_{b3}^*) \\
&\qquad \cdot (\mathbf{T}_{b1} \cdot (\underline{\mathbf{B}}_{b3}^* \wedge \mathbf{N}) - (\mathbf{T}_{b1} \cdot \mathbf{N}) \underline{\mathbf{B}}_{b3}^*) \\
&\qquad - B_{\text{var}} (\mathbf{T}_{b1} \cdot \mathbf{N})^2 - A_{\text{var}} B_{\text{var}} (\mathbf{N} \cdot \mathbf{N}) \\
&\quad = (\mathbf{T}_{b1} \cdot (\underline{\mathbf{B}}_{b3}^* \wedge \mathbf{N})) \cdot (\mathbf{T}_{b1} \cdot (\underline{\mathbf{B}}_{b3}^* \wedge \mathbf{N})) \\
&\qquad + (\mathbf{T}_{b1} \cdot \mathbf{N})^2 (\underline{\mathbf{B}}_{b3}^* \cdot \underline{\mathbf{B}}_{b3}^*) \\
&\qquad - 2 (\mathbf{T}_{b1} \cdot \mathbf{N}) (\mathbf{T}_{b1} \cdot (\underline{\mathbf{B}}_{b3}^* \wedge \mathbf{N})) \cdot \underline{\mathbf{B}}_{b3}^* \\
&\qquad - B_{\text{var}} (\mathbf{T}_{b1} \cdot \mathbf{N})^2 - A_{\text{var}} B_{\text{var}} (\mathbf{N} \cdot \mathbf{N}) \\
&\quad = (\mathbf{T}_{b1} \cdot \mathbf{T}_{b1}) (\underline{\mathbf{B}}_{b3}^* \wedge \mathbf{N}) \cdot (\underline{\mathbf{B}}_{b3}^* \wedge \mathbf{N}) \\
&\qquad + B_{\text{var}} (\mathbf{T}_{b1} \cdot \mathbf{N})^2 - B_{\text{var}} (\mathbf{T}_{b1} \cdot \mathbf{N})^2 \\
&\qquad - A_{\text{var}} B_{\text{var}} (\mathbf{N} \cdot \mathbf{N})
\end{aligned}$$

$$
\begin{aligned}
&= A_{\text{var}}\left(\left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right) \cdot \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right) - B_{\text{var}}\left(\mathbf{N} \cdot \mathbf{N}\right)\right) \\
&= A_{\text{var}}\left(E_{\text{var}} - B_{\text{var}} D_{\text{var}}\right),
\end{aligned} \tag{A.1}
$$

where $D_{\text{var}} = \mathbf{N} \cdot \mathbf{N}$, and $E_{\text{var}} = (\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}) \cdot (\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N})$.

During the process of expansion, we use the following expressions:

$$
\begin{aligned}
&\left(\mathbf{T}_{b2} \wedge \mathbf{N}\right) \cdot \left(\mathbf{T}_{b2} \wedge \mathbf{N}\right) \\
&\quad = -\left(\mathbf{T}_{b2} \cdot \mathbf{T}_{b2}\right)\left(\mathbf{N} \cdot \mathbf{N}\right) + \left(\mathbf{T}_{b2} \cdot \mathbf{N}\right)^{2}, \\
&\mathbf{T}_{b1} \cdot \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right) = \left(\mathbf{T}_{b1} \cdot \underline{\mathbf{B}}_{b3}^{*}\right) \wedge \mathbf{N} + \left(\mathbf{T}_{b1} \cdot \mathbf{N}\right) \underline{\mathbf{B}}_{b3}^{*}, \\
&\mathbf{T}_{b2} \cdot \mathbf{T}_{b2} = \left(\mathbf{T}_{b1} \cdot \underline{\mathbf{B}}_{b3}^{*}\right) \cdot \left(\mathbf{T}_{b1} \cdot \underline{\mathbf{B}}_{b3}^{*}\right) \\
&\quad = -\left(\mathbf{T}_{b1} \cdot \mathbf{T}_{b1}\right)\left(\underline{\mathbf{B}}_{b3}^{*} \cdot \underline{\mathbf{B}}_{b3}^{*}\right) = -A_{\text{var}}\left(\underline{\mathbf{B}}_{b3}^{*} \cdot \underline{\mathbf{B}}_{b3}^{*}\right) \\
&\quad = -A_{\text{var}} B_{\text{var}}, \\
&\left(\mathbf{T}_{b1} \wedge \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right)\right) \cdot \left(\mathbf{T}_{b1} \wedge \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right)\right) \\
&\quad = -\left(\mathbf{T}_{b1} \cdot \mathbf{T}_{b1}\right)\left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right) \cdot \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right) \\
&\qquad + \left(\mathbf{T}_{b1} \cdot \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right)\right) \cdot \left(\mathbf{T}_{b1} \cdot \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right)\right)
\end{aligned} \tag{A.2}
$$

and according to $\mathbf{T}_{b1} \wedge \underline{\mathbf{B}}_{b3}^{*} = 0$, we obtain

$$
\begin{aligned}
&\left(\mathbf{T}_{b1} \cdot \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right)\right) \cdot \left(\mathbf{T}_{b1} \cdot \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right)\right) = \left(\mathbf{T}_{b1} \cdot \mathbf{T}_{b1}\right) \\
&\quad \cdot \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right) \cdot \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right), \\
&\left(\mathbf{T}_{b1} \cdot \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right)\right) \cdot \underline{\mathbf{B}}_{b3}^{*} = 0, \\
&\because \mathbf{T}_{b1} = \left(\underline{\mathbf{B}}_{31} - \underline{\mathbf{B}}_{32}\right), \\
&\underline{\mathbf{B}}_{b3}^{*} = \left(\underline{\mathbf{B}}_{31} \wedge \underline{\mathbf{B}}_{32}\right), \\
&\therefore \left(\mathbf{T}_{b1} \cdot \left(\underline{\mathbf{B}}_{b3}^{*} \wedge \mathbf{N}\right)\right) \cdot \underline{\mathbf{B}}_{b3}^{*} = \left(\left(\underline{\mathbf{B}}_{31} - \underline{\mathbf{B}}_{32}\right)\right. \\
&\quad \left. \cdot \left(\underline{\mathbf{B}}_{31} \wedge \underline{\mathbf{B}}_{32} \wedge \mathbf{N}\right)\right) \cdot \left(\underline{\mathbf{B}}_{31} \wedge \underline{\mathbf{B}}_{32}\right) \\
&\quad = \left(-\left(\underline{\mathbf{B}}_{31} \cdot \underline{\mathbf{B}}_{32}\right)\left(\underline{\mathbf{B}}_{31} \wedge \mathbf{N}\right)\right. \\
&\quad + \left(\underline{\mathbf{B}}_{31} \cdot \mathbf{N}\right)\left(\underline{\mathbf{B}}_{31} \wedge \underline{\mathbf{B}}_{32}\right) - \left(\underline{\mathbf{B}}_{31} \cdot \underline{\mathbf{B}}_{32}\right)\left(\underline{\mathbf{B}}_{32} \wedge \mathbf{N}\right) \\
&\quad \left. - \left(\underline{\mathbf{B}}_{32} \cdot \mathbf{N}\right)\left(\underline{\mathbf{B}}_{31} \wedge \underline{\mathbf{B}}_{32}\right)\right) \cdot \left(\underline{\mathbf{B}}_{31} \wedge \underline{\mathbf{B}}_{32}\right) \\
&\quad = \left(-\left(\underline{\mathbf{B}}_{31} \cdot \underline{\mathbf{B}}_{32}\right)^{2}\left(\underline{\mathbf{B}}_{31} \cdot \mathbf{N}\right)\right. \\
&\quad + \left(\underline{\mathbf{B}}_{31} \cdot \mathbf{N}\right)\left(\underline{\mathbf{B}}_{31} \cdot \underline{\mathbf{B}}_{32}\right)^{2} + \left(\underline{\mathbf{B}}_{31} \cdot \underline{\mathbf{B}}_{32}\right)^{2}\left(\underline{\mathbf{B}}_{32} \cdot \mathbf{N}\right) \\
&\quad \left. - \left(\underline{\mathbf{B}}_{32} \cdot \mathbf{N}\right)\left(\underline{\mathbf{B}}_{31} \cdot \underline{\mathbf{B}}_{32}\right)^{2}\right) = 0.
\end{aligned} \tag{A.3}
$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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