Research Article

Parameter Identification and Hybrid Synchronization in an Array of Coupled Chaotic Systems with Ring Connection: An Adaptive Integral Sliding Mode Approach

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This article presents an adaptive integral sliding mode control (SMC) design method for parameter identification and hybrid synchronization of chaotic systems connected in ring topology. To employ the adaptive integral sliding mode control, the error system is transformed into a special structure containing nominal part and some unknown terms. The unknown terms are computed adaptively. Then the error system is stabilized using integral sliding mode control. The controller of the error system is created that contains both the nominal control and the compensator control. The adapted laws and compensator controller are derived using Lyapunov stability theory. The effectiveness of the proposed technique is validated through numerical examples.

1. Introduction

Ever since the classic effort by Pecora and Carroll [1], the synchronization in chaotic systems became an active research area because of its useful applications [2–5]. Different synchronization schemes for chaotic systems were studied and investigated [6–10]. Among these schemes, hybrid synchronization [11–19] is one in which some of the chaotic systems are synchronized whereas others are antisynchronized. Due to its importance, hybrid synchronization has been the subject of many research works. These works include study of synchronization/antisynchronization for permanent magnet synchronous motors connected in ring topology [11], function projective type synchronization for complex dynamical networks [12], investigation of complete synchronization and antiphase synchronization together [13], hybrid synchronization of networks having heterogeneous systems [14], study of synchronization for fractional-order systems [15], investigation of synchronization for systems with hyperchaotic nature [16, 17], and study of synchronization for complex networks [18, 19]. Study of hybrid synchronization involves multiple chaotic systems like complete, adaptive, global, projective,

and antisynchronization systems [17, 20–27], and synchronization systems in multiple coupled complex networks [28, 29]. Currently, hybrid synchronization of several connected chaotic systems is a hot topic of research and the work includes investigation of complex network synchronization for perturbations and delays [30] and a study of neural networks for synchronization [31].

In this paper, chaotic systems connected in ring topology are considered which can be shown in Figure 1. This arrangement was studied in [32] for the known parameter case. We extend this work and consider that all the parameters of all the systems in the ring connected network are unknown. To reach hybrid synchronization in this system we use adaptive integral SMC.

SMC is a distinct nonlinear control method. The objective of the SMC method is to drive the states of the system to a specific surface, called sliding manifold. When the surface is touched, the dynamic system is required to persist on it afterward. The main disadvantage of SMC is discontinuous control law. In practice, this creeps towards an undesirable occurrence called “chattering.” The closed-loop dynamics of the system in SMC depend only on the design parameters of

FIGURE 1: Chaotic systems connected in ring topology [32].

the switching sliding manifold. Sliding mode control also offers several benefits like simplicity, robustness to external disturbance, and parameter variation and quick response. The integral SMC, a variant of SMC, guarantees the robustness [33] because of eliminating the reaching-phase. Since reaching-phase is eradicated therefore robustness of the dynamic system can be sure all the way through the system response, beginning from initial conditions. The integral sliding mode control combines both the nominal control which steadies the nominal system and the discontinuous control which discards uncertainties.

In this research, we use adaptive integral sliding mode control technique to identify the unknown parameters and to achieve the hybrid synchronization of many chaotic systems connected in the ring topology. By using this technique, the error system is transformed into a special structure containing nominal part and some unknown terms. The unknown terms are computed adaptively. Then the error system is synthesized using integral sliding mode control. The controller of the error system is created that contains both the nominal control and the compensator control. The adaptive laws and compensator controller are derived using Lyapunov stability theory. The effectiveness of the proposed technique is validated through numerical examples.

The remainder of the paper is arranged as follows; Section 2 presents preliminaries and system description. Section 3 presents proposed control strategies for the general case of hybrid synchronization. Section 4 presents application examples. Section 5 presents discussion and simulation results and in the last section, paper is concluded.

2. System Description and Preliminaries

The structure of chaotic systems connected in ring topology is described as

$$\begin{aligned}
 \dot{x}_1 &= f_1(x_1) + F_1(x_1)\theta_1 + D_1(x_M - x_1) \\
 \dot{x}_2 &= f_2(x_2) + F_2(x_2)\theta_2 + D_2(x_1 - x_2) \\
 &\vdots \\
 \dot{x}_M &= f_M(x_M) + F_M(x_M)\theta_M + D_M(x_{M-1} - x_M),
 \end{aligned} \tag{1}$$

where $x_1, x_2, \dots, x_M$ are vectors and $x_o = (x_{o1}, x_{o2}, \dots, x_{om})^T$, $f_o : \mathfrak{R}^m \rightarrow \mathfrak{R}^m$, $o = 1, 2, \dots, M$ are the nonlinear continuous function, and $\theta_o \in \mathfrak{R}^g$, $o = 1, 2, 3, 4, \dots, M$ are vectors of unknown parameters. $F_o(x_o) \in \mathfrak{R}^{m \times g}$, $o = 1, 2, 3, 4, \dots, M$ are matrices, $D_o = \text{diag}\{d_{o1}, d_{o2}, \dots, d_{om}\}$, $o = 1, 2, 3, 4, \dots, M$ are m dimensional diagonal matrices, and

$d_{op} \geq 0$ represent diagonal matrix parameters. When $f_o(\cdot) \neq f_p(\cdot)$, $o, p = 1, 2, 3, 4, \dots, M$, then the above arrangement has nonidentical systems.

Remark 1. In (1), all the chaotic systems are connected in ring topology [17], because the first system is connected with M th and second is connected with first; and so the final chaotic system is connected with $M - 1$ system.

So the hybrid synchronization for the above system (1) may be expressed as follows:

$$\begin{aligned}
 \dot{x}_1 &= f_1(x_1) + F_1(x_1)\theta_1 + D_1(x_M - x_1) \\
 \dot{x}_2 &= f_2(x_2) + F_2(x_2)\theta_2 + D_2(x_1 - x_2) + u_1 \\
 &\vdots \\
 \dot{x}_M &= f_M(x_M) + F_M(x_M)\theta_M + D_M(x_{M-1} - x_M) \\
 &\quad + u_{M-1}.
 \end{aligned} \tag{2}$$

Definition 2. For system (2), we define that it exhibits hybrid synchronization if controllers u_o , $o = 1, 2, 3, 4, \dots, M - 1$ exist such that all trajectories $x_1(t), x_2(t), x_3(t), x_4(t), \dots, x_M(t)$ for any initial condition $(x_1(0), x_2(0), x_3(0), x_4(0), \dots, x_M(0))$ satisfy the following:

(1) For the antisynchronization errors $e_{ro} = (e_{ro1}, e_{ro2}, \dots, e_{rom})^T$, $o = 1, 2, 3, 4, \dots, M - 1$, we have that

$$\lim_{t \rightarrow \infty} \|e_{ro}\| = \lim_{t \rightarrow \infty} \|x_o(t) + x_{o+1}(t)\| = 0, \tag{3}$$

$o = 1, 2, 3, 4, \dots, M - 1.$

(2) The Synchronization errors $e'_{rp} = (e'_{rp1}, e'_{rp2}, \dots, e'_{rpm})^T$ and $e'_{rq} = (e'_{rq1}, e'_{rq2}, \dots, e'_{rqm})^T$ if M ($M \geq 3$) is odd; then e'_{rp} and e'_{rq} satisfy

$$\lim_{t \rightarrow \infty} \|e'_{rp}\| = \lim_{t \rightarrow \infty} \|x_{p+2}(t) - x_p(t)\| = 0, \tag{4}$$

$p = 1, 3, 5, 7, 9, \dots, M - 2$

$$\lim_{t \rightarrow \infty} \|e'_{rq}\| = \lim_{t \rightarrow \infty} \|x_{q+2}(t) - x_q(t)\| = 0,$$

$q = 2, 4, 6, 8, 10, \dots, M - 3$

and if M ($M \geq 4$) is even, then e'_{rp} and e'_{rq} satisfy

$$\lim_{t \rightarrow \infty} \|e'_{rp}\| = \lim_{t \rightarrow \infty} \|x_{p+2}(t) - x_p(t)\| = 0,$$

$p = 1, 3, 5, 7, 9, \dots, M - 3$ \tag{5}

$$\lim_{t \rightarrow \infty} \|e'_{rq}\| = \lim_{t \rightarrow \infty} \|x_{q+2}(t) - x_q(t)\| = 0,$$

$q = 2, 4, 6, 8, 10, \dots, M - 2.$

We notice from this definition that the error systems e_{ro} , e'_{rp} , and e'_{rq} are globally and asymptotically stable. This shows that, to ensure hybrid synchronization, we have to design controllers $u_o(t)$ to make e_{ro} , e'_{rp} , and e'_{rq} converge to zero.

3. Parameter Identification and Hybrid Synchronization for the General Case

For antisynchronization, the error vectors are

$$\begin{aligned} e_{r1} &= x_2 + x_1, \\ e_{r2} &= x_3 + x_2, \dots, e_{rM-1} = x_M + x_{M-1}. \end{aligned} \quad (6)$$

Let $\hat{\theta}_o$ be estimates of θ_o , $o = 1, 2, 3, 4, \dots, M$, respectively, and let $\tilde{\theta}_o = \theta_o - \hat{\theta}_o$, $o = 1, 2, 3, 4, \dots, M$ be errors while estimating the parameters θ_o , $o = 1, 2, \dots, M$, respectively. Derivative of (6) leads to the following:

$$\begin{aligned} \dot{e}_{r1} &= \dot{x}_2 + \dot{x}_1 = f_2(x_2) + F_2(x_2)\hat{\theta}_2 + F_2(x_2)\tilde{\theta}_2 \\ &\quad + D_2(x_1 - x_2) + u_1 + f_1(x_1) + F_1(x_1)\hat{\theta}_1 \\ &\quad + F_1(x_1)\tilde{\theta}_1 + D_1(x_M - x_1) \\ \dot{e}_{r2} &= \dot{x}_3 + \dot{x}_2 = f_3(x_3) + F_3(x_3)\hat{\theta}_3 + F_3(x_3)\tilde{\theta}_3 \\ &\quad + D_3(x_2 - x_3) + u_2 + f_2(x_2) + F_2(x_2)\hat{\theta}_2 \end{aligned}$$

$$\begin{aligned} &+ F_2(x_2)\tilde{\theta}_2 + D_2(x_1 - x_2) + u_1 \\ \dot{e}_{r3} &= \dot{x}_4 + \dot{x}_3 = f_4(x_4) + F_4(x_4)\hat{\theta}_4 + F_4(x_4)\tilde{\theta}_4 \\ &\quad + D_4(x_3 - x_4) + u_3 + f_3(x_3) + F_3(x_3)\hat{\theta}_3 \\ &\quad + F_3(x_3)\tilde{\theta}_3 + D_3(x_2 - x_3) + u_2 \\ &\quad \vdots \\ \dot{e}_{rM-1} &= \dot{x}_M + q\dot{x}_{M-1} = f_M(x_M) + F_M(x_M)\hat{\theta}_M \\ &\quad + F_M(x_M)\tilde{\theta}_M + D_M(x_{M-1} - x_M) + u_{M-1} \\ &\quad + f_{M-1}(x_{M-1}) + F_{M-1}(x_{M-1})\hat{\theta}_{M-1} \\ &\quad + F_{M-1}(x_{M-1})\tilde{\theta}_{M-1} + D_{M-1}(x_{M-2} - x_{M-1}) \\ &\quad + u_{M-2} \end{aligned} \quad (7)$$

which can be written as

$$\begin{aligned} &\begin{bmatrix} \dot{e}_{r1} \\ \dot{e}_{r2} \\ \vdots \\ \dot{e}_{rM-1} \end{bmatrix} \\ &= \begin{bmatrix} f_2(x_2) + f_1(x_1) + F_2(x_2)\hat{\theta}_2 + F_1(x_1)\hat{\theta}_1 + D_2(x_1 - x_2) + D_1(x_M - x_1) \\ f_3(x_3) + f_2(x_2) + F_3(x_3)\hat{\theta}_3 + F_2(x_2)\hat{\theta}_2 + D_3(x_2 - x_3) + D_2(x_1 - x_2) \\ \vdots \\ f_M(x_M) + f_{M-1}(x_{M-1}) + F_M(x_M)\hat{\theta}_M + F_{M-1}(x_{M-1})\hat{\theta}_{M-1} + D_M(x_{M-1} - x_M) + D_{M-1}(x_{M-2} - x_{M-1}) \end{bmatrix} \\ &\quad + \begin{bmatrix} F_2(x_2)\tilde{\theta}_2 + F_1(x_1)\tilde{\theta}_1 \\ F_3(x_3)\tilde{\theta}_3 + F_2(x_2)\tilde{\theta}_2 \\ \vdots \\ F_M(x_M)\tilde{\theta}_M + F_{M-1}(x_{M-1})\tilde{\theta}_{M-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{M-1} \end{bmatrix}. \end{aligned} \quad (8)$$

By choosing

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{M-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} e_{r2} \\ e_{r3} \\ \vdots \\ e_{rM-1} \\ v \end{bmatrix} - A \right\}, \quad (9)$$

where v is the new input vector and

$$A = \begin{bmatrix} f_2(x_2) + f_1(x_1) + F_2(x_2)\hat{\theta}_2 + F_1(x_1)\hat{\theta}_1 + D_2(x_1 - x_2) + D_1(x_M - x_1) \\ f_3(x_3) + f_2(x_2) + F_3(x_3)\hat{\theta}_3 + F_2(x_2)\hat{\theta}_2 + D_3(x_2 - x_3) + D_2(x_1 - x_2) \\ \vdots \\ f_M(x_M) + f_{M-1}(x_{M-1}) + F_M(x_M)\hat{\theta}_M + F_{M-1}(x_{M-1})\hat{\theta}_{M-1} + D_M(x_{M-1} - x_M) + D_{M-1}(x_{M-2} - x_{M-1}) \end{bmatrix} \quad (10)$$

then system (7) becomes

$$\begin{aligned} \dot{e}_{r1} &= e_{r2} + F_2(x_2)\tilde{\theta}_2 + F_1(x_1)\tilde{\theta}_1 \\ \dot{e}_{r2} &= e_{r3} + F_3(x_3)\tilde{\theta}_3 + F_2(x_2)\tilde{\theta}_2 \\ \dot{e}_{r3} &= e_{r4} + F_4(x_4)\tilde{\theta}_4 + F_3(x_3)\tilde{\theta}_3 \\ &\vdots \\ \dot{e}_{rM-2} &= e_{rM-1} + F_{M-1}(x_{M-1})\tilde{\theta}_{M-1} \\ &\quad + F_{M-2}(x_{M-2})\tilde{\theta}_{M-2} \\ \dot{e}_{rM-1} &= v + F_M(x_M)\tilde{\theta}_M + F_{M-1}(x_{M-1})\tilde{\theta}_{M-1}. \end{aligned} \quad (11)$$

To employ the integral sliding mode control, choose the nominal system for (11) as

$$\begin{aligned} \dot{e}_{r1} &= e_{r2} \\ \dot{e}_{r2} &= e_{r3} \\ \dot{e}_{r3} &= e_{r4} \\ &\vdots \\ \dot{e}_{rM-2} &= e_{rM-1} \\ \dot{e}_{rM-1} &= v_0. \end{aligned} \quad (12)$$

To stabilize the system (11), we choose the Hurwitz sliding surface for system (12) as $\sigma_0 = (1 + d/dt)^{M-2} e_{r1} = e_{r1} + c_1 e_{r2} + \dots + c_{M-3} e_{rM-2} + e_{rM-1}$, where the coefficients c_o are chosen in such a way that σ_0 becomes Hurwitz polynomial. Then $\dot{\sigma}_0 = e_{r2} + c_1 e_{r3} + c_2 e_{r4} + \dots + c_{M-3} e_{rM-1} + v_0$.

By choosing $v_0 = -e_{r2} - c_1 e_{r3} - c_2 e_{r4} - \dots - c_{M-3} e_{rM-1} - k\sigma_0 - k \text{sign}(\sigma_0)$, $k > 0$, we have $\dot{\sigma}_0 = -k \text{sign}(\sigma_0) - k\sigma_0$; therefore $\sigma_0 \rightarrow 0$, which gives $e_{r1}, e_{r2}, \dots, e_{rM-1} \rightarrow 0$. Therefore, system (12) becomes asymptotically stable.

The sliding surface for system (9) is $\sigma = \sigma_0 + z$. The term z in the sliding surface is an integral term computed later. To avoid the reaching-phase, choose $z(0)$ such that $\sigma(0) = 0$. By choosing $v = v_0 + v_s$ where v_0 is the nominal input vector

and v_s is compensator terms computed later, then the time derivative of the sliding surface becomes as

$$\begin{aligned} \dot{\sigma} &= \dot{\sigma}_0 + \dot{z} \\ &= \dot{e}_{r1} + c_1 \dot{e}_{r2} + c_2 \dot{e}_{r3} + c_3 \dot{e}_{r4} + \dots + c_{M-3} \dot{e}_{rM-2} \\ &\quad + \dot{e}_{rM-1} + \dot{z} \\ &= e_{r2} + c_1 e_{r3} + c_2 e_{r4} + \dots + c_{M-3} e_{rM-1} + v_0 + v_s + \dot{z} \\ &\quad + F_1(x_1)\tilde{\theta}_1 + (1 + c_1)F_2(x_2)\tilde{\theta}_2 \\ &\quad + (c_1 + c_2)F_3(x_3)\tilde{\theta}_3 + (c_2 + c_3)F_4(x_4)\tilde{\theta}_4 + \dots \\ &\quad + (c_{M-4} + c_{M-3})F_{M-2}(x_{M-2})\tilde{\theta}_{M-2} \\ &\quad + (c_{M-3} + 1)F_{M-1}(x_{M-1})\tilde{\theta}_{M-1} + F_M(x_M)\tilde{\theta}_M. \end{aligned} \quad (13)$$

By choosing a Lyapunov function: $V = (1/2)\{\sigma^T \sigma + \tilde{\theta}_1^T \tilde{\theta}_1 + \tilde{\theta}_2^T \tilde{\theta}_2 + \sum_{o=1}^{M-4} \tilde{\theta}_{o+2}^T \tilde{\theta}_{o+2} + \tilde{\theta}_{M-1}^T \tilde{\theta}_{M-1} + \tilde{\theta}_M^T \tilde{\theta}_M\}$, designing the adaptive laws for $\tilde{\theta}_o, \hat{\theta}_o$, $o = 1, 2, \dots, M$ and computing v_s such that $\dot{V} < 0$.

Theorem 3. Consider a Lyapunov function $V = (1/2)\{\sigma^T \sigma + \tilde{\theta}_1^T \tilde{\theta}_1 + \tilde{\theta}_2^T \tilde{\theta}_2 + \sum_{o=1}^{M-4} \tilde{\theta}_{o+2}^T \tilde{\theta}_{o+2} + \tilde{\theta}_{M-1}^T \tilde{\theta}_{M-1} + \tilde{\theta}_M^T \tilde{\theta}_M\}$. Then $\dot{V} < 0$ if the adaptive laws for $\tilde{\theta}_o, \hat{\theta}_o$, $o = 1, 2, 3, 4, \dots, M$ and the value of v_s are chosen as

$$\dot{z} = -e_{r2} - \sum_{o=3}^{M-1} c_{o-2} e_{ro} - v_0,$$

$$v_s = -k\sigma$$

$$\dot{\tilde{\theta}}_1 = -F_1^T(x_1)\sigma - k_1 \tilde{\theta}_1,$$

$$\dot{\tilde{\theta}}_1 = -\dot{\tilde{\theta}}_1$$

$$\dot{\tilde{\theta}}_2 = -(1 + c_1)F_2^T(x_2)\sigma - k_2 \tilde{\theta}_2,$$

$$\dot{\tilde{\theta}}_2 = -\dot{\tilde{\theta}}_2$$

$$\dot{\tilde{\theta}}_{o+2} = -(c_o + c_{o+1})F_{o+2}^T(x_{o+2})\sigma - k_{o+2} \tilde{\theta}_{o+2},$$

$$\dot{\tilde{\theta}}_{o+2} = -\dot{\tilde{\theta}}_{o+2},$$

$$o = 1, \dots, M-4$$

$$\begin{aligned}\dot{\tilde{\theta}}_{M-1} &= -(c_{M-3} + 1) F_{M-1}^T(x_{M-1}) \sigma - k_{M-1} \tilde{\theta}_{M-1}, \\ \dot{\tilde{\theta}}_{M-1} &= -\dot{\tilde{\theta}}_{M-1} \\ \dot{\tilde{\theta}}_M &= -F_M^T(x_M) \sigma - k_M \tilde{\theta}_M, \\ \dot{\tilde{\theta}}_M &= -\dot{\tilde{\theta}}_M.\end{aligned}\tag{14}$$

Proof. Since

$$\begin{aligned}\dot{V} &= \sigma^T \dot{\sigma} + \tilde{\theta}_1^T \dot{\tilde{\theta}}_1 + \tilde{\theta}_2^T \dot{\tilde{\theta}}_2 + \sum_{o=1}^{M-4} \tilde{\theta}_{o+2}^T \dot{\tilde{\theta}}_{o+2} + \tilde{\theta}_{M-1}^T \dot{\tilde{\theta}}_{M-1} \\ &+ \tilde{\theta}_M^T \dot{\tilde{\theta}}_M = \sigma^T \left\{ e_{r2} + \sum_{o=3}^{M-1} c_{o-2} e_{ro} + v_0 + v_s + \dot{z} \right\} \\ &+ \tilde{\theta}_1^T \left\{ \dot{\tilde{\theta}}_1 + F_1^T(x_1) \sigma \right\} + \tilde{\theta}_2^T \left\{ \dot{\tilde{\theta}}_2 \right. \\ &+ (1 + c_1) F_2^T(x_2) \sigma \left. \right\} \\ &+ \sum_{o=1}^{M-4} \tilde{\theta}_{o+2}^T \left\{ \dot{\tilde{\theta}}_{o+2} + (c_o + c_{o+1}) F_{o+2}^T(x_{o+2}) \sigma \right\} \\ &+ \tilde{\theta}_{M-1}^T \left\{ \dot{\tilde{\theta}}_{M-1} + (c_{M-3} + 1) F_{M-1}^T(x_{M-1}) \sigma \right\} \\ &+ \tilde{\theta}_M^T \left\{ \dot{\tilde{\theta}}_M + F_M^T(x_M) \sigma \right\}\end{aligned}\tag{16}$$

by using (14) we have

$$\dot{V} = -k\sigma^2 - \sum_{o=1}^M k_o \tilde{\theta}_o^T \tilde{\theta}_o.\tag{17}$$

From this we conclude that $\sigma, \tilde{\theta}_o \rightarrow 0$. Since $\sigma \rightarrow 0$, therefore $e_o \rightarrow 0$, $o = 1, 2, 3, \dots, M-1$. Thus the antisynchronization is achieved.

The controllers designed for the antisynchronization are used for the complete synchronization. For this we consider two cases:

Case 1. For odd number of systems and M ($M \geq 3$), synchronization error is expressed like

$$\begin{aligned}e'_{rp}(t) &= x_{p+2}(t) - x_p(t), \\ p &= 1, 3, 5, 7, 9, \dots, M-2 \\ e'_{rq}(t) &= x_{q+2}(t) - x_q(t), \\ q &= 2, 4, 6, 8, 10, \dots, M-3.\end{aligned}\tag{18}$$

Then

$$\begin{aligned}\lim_{t \rightarrow \infty} \|e'_{rp}\| &= \lim_{t \rightarrow \infty} \|x_{p+2}(t) - x_p(t)\| \\ \lim_{t \rightarrow \infty} \|x_{p+2}(t) + x_{p+1}(t) - x_{p+1}(t) - x_p(t)\| \\ &\leq \lim_{t \rightarrow \infty} \|x_{p+2}(t) + x_{p+1}(t)\| \\ &+ \lim_{t \rightarrow \infty} \|x_{p+1}(t) + x_p(t)\| = 0\end{aligned}\tag{19}$$

due to antisynchronization.

Thus

$$\lim_{t \rightarrow \infty} \|e'_{rp}\| = 0, \quad p = 1, 3, 5, 7, 9, \dots, M-2.\tag{20}$$

Similarly

$$\begin{aligned}\lim_{t \rightarrow \infty} \|e'_{rp}\| &= \lim_{t \rightarrow \infty} \|x_{q+2}(t) - x_q(t)\| \\ &= \lim_{t \rightarrow \infty} \|x_{q+2}(t) + x_{q+1}(t) - x_{q+1}(t) - x_q(t)\| \\ &\leq \lim_{t \rightarrow \infty} \|x_{q+2}(t) + x_{q+1}(t)\| \\ &+ \lim_{t \rightarrow \infty} \|x_{q+1}(t) + x_q(t)\| = 0\end{aligned}\tag{21}$$

$$\lim_{t \rightarrow \infty} \|e'_{rp}\| = 0, \quad q = 2, 4, 6, 8, 10, \dots, M-3.$$

Case 2. For even number of systems and M ($M \geq 4$), synchronization error is expressed like

$$\begin{aligned}e'_{rp} &= x_{p+2}(t) - x_p(t), \\ e'_{rq} &= x_{q+2}(t) - x_q(t) \\ \lim_{t \rightarrow \infty} \|e'_{rp}\| &= \lim_{t \rightarrow \infty} \|x_{p+2}(t) - x_p(t)\| \\ &= \lim_{t \rightarrow \infty} \|x_{p+2}(t) + x_{p+1}(t) - x_{p+1}(t) - x_p(t)\| \\ &\leq \lim_{t \rightarrow \infty} \|x_{p+2}(t) + x_{p+1}(t)\| \\ &+ \lim_{t \rightarrow \infty} \|x_{p+1}(t) + x_p(t)\| = 0 \\ \lim_{t \rightarrow \infty} \|e'_{rp}\| &= 0, \quad p = 1, 3, 5, 7, 9, \dots, M-3.\end{aligned}\tag{22}$$

Similarly

$$\begin{aligned}\lim_{t \rightarrow \infty} \|e'_{rq}\| &= \lim_{t \rightarrow \infty} \|x_{q+2}(t) - x_q(t)\| = 0, \\ q &= 2, 4, 6, 8, 10, \dots, M-2.\end{aligned}\tag{23}$$

So complete synchronization is too achieved. $\square$

4. Application Examples

To illustrate the control design procedure for reaching the hybrid synchronization behavior, we present two examples with $M = 3$ and $M = 4$.

Example 1 ($M = 3$). Chen chaotic system, Lorenz chaotic system, and Lü chaotic system, in that order, are selected for this numerical example and are expressed below:

$$\begin{aligned}
 \dot{x}_{11} &= -35x_{11} + 35x_{12} + d_{11}(x_{31} - x_{11}) \\
 \dot{x}_{12} &= -7x_{11} + 28x_{12} + x_{11}x_{13} + d_{12}(x_{32} - x_{12}) \\
 \dot{x}_{13} &= -3x_{13} + x_{11}x_{12} + d_{13}(x_{33} - x_{13}) \\
 \dot{x}_{21} &= -36x_{21} + 36x_{22} + d_{21}(x_{11} - x_{21}) + u_{11} \\
 \dot{x}_{22} &= 20x_{22} - x_{21}x_{23} + d_{22}(x_{12} - x_{22}) + u_{12} \\
 \dot{x}_{23} &= -3x_{23} + x_{21}x_{22} + d_{23}(x_{13} - x_{23}) + u_{13} \\
 \dot{x}_{31} &= -10x_{31} + 10x_{32} + d_{31}(x_{21} - x_{31}) + u_{21} \\
 \dot{x}_{32} &= 28x_{31} - x_{32} - x_{31}x_{33} + d_{32}(x_{22} - x_{32}) + u_{22} \\
 \dot{x}_{33} &= -\frac{8}{3}x_{33} + x_{31}x_{32} + d_{33}(x_{23} - x_{33}) + u_{23}.
 \end{aligned} \tag{24}$$

By assuming that the systems parameters are unknown, we write systems (24) as

$$\begin{aligned}
 \dot{x}_{11} &= a_1x_{11} + b_1x_{12} + d_{11}(x_{31} - x_{11}) \\
 \dot{x}_{12} &= c_1x_{11} + d_{12}x_{12} - x_{11}x_{13} + d_{12}(x_{32} - x_{12}) \\
 \dot{x}_{13} &= e_1x_{13} + x_{11}x_{12} + d_{13}(x_{33} - x_{13}) \\
 \dot{x}_{21} &= a_2x_{21} + b_2x_{22} + d_{21}(x_{11} - x_{21}) + u_{11} \\
 \dot{x}_{22} &= c_2x_{22} - x_{21}x_{23} + d_{22}(x_{12} - x_{22}) + u_{12} \\
 \dot{x}_{23} &= e_2x_{23} + x_{21}x_{22} + d_{23}(x_{13} - x_{23}) + u_{13} \\
 \dot{x}_{31} &= a_3x_{31} + b_3x_{32} + d_{31}(x_{21} - x_{31}) + u_{21} \\
 \dot{x}_{32} &= c_3x_{31} - x_{32} - x_{31}x_{33} + d_{32}(x_{22} - x_{32}) + u_{22} \\
 \dot{x}_{33} &= e_3x_{33} + x_{31}x_{32} + d_{33}(x_{23} - x_{33}) + u_{23}.
 \end{aligned} \tag{25}$$

Let $\hat{a}_o, \hat{b}_o, \hat{c}_o, \hat{e}_{ro}, \hat{d}_1$, $o = 1, 2, 3$ be estimates of $a_o, b_o, c_o, e_{ro}, d_1$, $o = 1, 2, 3$, respectively, and let $\tilde{a}_o = a_o - \hat{a}_o$, $\tilde{b}_o = b_o - \hat{b}_o$, $\tilde{c}_o = c_o - \hat{c}_o$, $\tilde{e}_{ro} = e_{ro} - \hat{e}_{ro}$, $\tilde{d}_1 = d_1 - \hat{d}_1$, $o = 1, 2, 3$ be errors in estimation of a_o, b_o, c_o, e_o, d_1 , $o = 1, 2, 3$, respectively.

Defining

$$\begin{aligned}
 x_1 &= \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}, \\
 x_2 &= \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}, \\
 x_3 &= \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix},
 \end{aligned}$$

$$u_1 = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \end{bmatrix},$$

$$u_2 = \begin{bmatrix} u_{21} \\ u_{22} \\ u_{23} \end{bmatrix},$$

$$\hat{\theta} = \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \\ \hat{d} \end{bmatrix},$$

$$\tilde{\theta} = \begin{bmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{c} \\ \tilde{d} \end{bmatrix}$$

(26)

then the systems (25) in vector form are written as

$$\begin{aligned}
 \dot{x}_1 &= f_1(x_1) + F_1(x_1)\hat{\theta}_1 + F_1(x_1)\tilde{\theta}_1 \\
 \dot{x}_2 &= f_2(x_2) + F_2(x_2)\hat{\theta}_2 + F_2(x_2)\tilde{\theta}_2 + u_1 \\
 \dot{x}_3 &= f_3(x_3) + F_3(x_3)\hat{\theta}_3 + F_3(x_3)\tilde{\theta}_3 + u_2,
 \end{aligned} \tag{27}$$

where

$$f_1(x_1) = \begin{bmatrix} d_{11}(x_{31} - x_{11}) \\ x_{11}x_{13} + d_{12}(x_{32} - x_{12}) \\ x_{11}x_{12} + d_{13}(x_{33} - x_{13}) \end{bmatrix},$$

$$F_1(x_1) = \begin{bmatrix} x_{11} & x_{12} & 0 & 0 & 0 \\ 0 & 0 & \hat{c}_1 & \hat{d}_1 & 0 \\ 0 & 0 & 0 & 0 & x_{13} \end{bmatrix},$$

$$\hat{\theta}_1 = \begin{bmatrix} \hat{a}_1 \\ \hat{b}_1 \\ \hat{c}_1 \\ \hat{d}_1 \\ \hat{e}_1 \end{bmatrix},$$

$$\tilde{\theta}_1 = \begin{bmatrix} \tilde{a}_1 \\ \tilde{b}_1 \\ \tilde{c}_1 \\ \tilde{d}_1 \\ \tilde{e}_1 \end{bmatrix},$$

$$f_2(x_2) = \begin{bmatrix} d_{21}(x_{11} - x_{21}) \\ -x_{21}x_{23} + d_{22}(x_{12} - x_{22}) \\ x_{21}x_{22} + d_{23}(x_{13} - x_{23}) \end{bmatrix},$$

$$F_2(x_2) = \begin{bmatrix} x_{21} & x_{22} & 0 & 0 \\ 0 & 0 & x_{22} & 0 \\ 0 & 0 & 0 & x_{23} \end{bmatrix},$$

$$\hat{\theta}_2 = \begin{bmatrix} \hat{a}_2 \\ \hat{b}_2 \\ \hat{c}_2 \\ \hat{e}_2 \end{bmatrix},$$

$$\tilde{\theta}_2 = \begin{bmatrix} \tilde{a}_2 \\ \tilde{b}_2 \\ \tilde{c}_2 \\ \tilde{e}_2 \end{bmatrix},$$

$$f_3(x_3) = \begin{bmatrix} d_{31}(x_{21} - x_{31}) \\ -x_{32} - x_{31}x_{33} + d_{32}(x_{22} - x_{32}) \\ x_{31}x_{32} + d_{33}(x_{23} - x_{33}) \end{bmatrix},$$

$$F_3(x_3) = \begin{bmatrix} x_{31} & x_{32} & 0 & 0 \\ 0 & 0 & x_{31} & 0 \\ 0 & 0 & 0 & x_{33} \end{bmatrix},$$

$$\hat{\theta}_3 = \begin{bmatrix} \hat{a}_3 \\ \hat{b}_3 \\ \hat{c}_3 \\ \hat{e}_3 \end{bmatrix},$$

$$\tilde{\theta}_3 = \begin{bmatrix} \tilde{a}_3 \\ \tilde{b}_3 \\ \tilde{c}_3 \\ \tilde{e}_3 \end{bmatrix}.$$

For the antisynchronization, the errors

$$e_{r1} = \begin{bmatrix} e_{r11} \\ e_{r12} \\ e_{r13} \end{bmatrix},$$

$$e_{r2} = \begin{bmatrix} e_{r21} \\ e_{r22} \\ e_{r23} \end{bmatrix}$$

are

$$\begin{aligned} e_{r1} &= x_2 + x_1, \\ e_{r2} &= x_3 + x_2. \end{aligned} \quad (30)$$

Therefore

$$\begin{aligned} \begin{bmatrix} \dot{e}_{r1} \\ \dot{e}_{r2} \end{bmatrix} &= \begin{bmatrix} f_2(x_2) + F_2(x_2)\hat{\theta}_2 + f_1(x_1) + F_1(x_1)\hat{\theta}_1 \\ F_2(x_2)\hat{\theta}_2 + f_3(x_3) + F_3(x_3)\hat{\theta}_3 \end{bmatrix} \\ &+ \begin{bmatrix} F_1(x_1)\tilde{\theta}_1 + F_2(x_2)\tilde{\theta}_2 \\ F_2(x_2)\tilde{\theta}_2 + F_3(x_3)\tilde{\theta}_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \end{aligned} \quad (31)$$

By choosing

$$\begin{aligned} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \\ &\cdot \left\{ - \begin{bmatrix} f_2(x_2) + F_2(x_2)\hat{\theta}_2 + f_1(x_1) + F_1(x_1)\hat{\theta}_1 \\ f_3(x_3) + F_3(x_3)\hat{\theta}_3 + F_2(x_2)\hat{\theta}_2 \end{bmatrix} \right. \\ &\left. + \begin{bmatrix} e_{r2} \\ \nu \end{bmatrix} \right\}, \end{aligned} \quad (32)$$

where $\nu = [\nu_1 \ \nu_2 \ \nu_3]^T$ is the new input vector, then system (31) becomes

$$\begin{aligned} \dot{e}_{r1} &= e_2 + F_2(x_2)\tilde{\theta}_2 + F_1(x_1)\tilde{\theta}_1 \\ \dot{e}_{r2} &= \nu + F_3(x_3)\tilde{\theta}_3 + F_2(x_2)\tilde{\theta}_2. \end{aligned} \quad (33)$$

The nominal system for (33) is

$$\begin{aligned} \dot{e}_{r1} &= e_2 \\ \dot{e}_{r2} &= \nu_0, \end{aligned} \quad (34)$$

where $\nu_0 = [\nu_{01} \ \nu_{02} \ \nu_{03}]^T$ is the nominal input vector.

The sliding surface for nominal system (34) is $\sigma_0 = e_{r1} + e_{r2}$; that is,

$$\begin{bmatrix} \sigma_{01} \\ \sigma_{02} \\ \sigma_{03} \end{bmatrix} = \begin{bmatrix} e_{r11} + e_{r21} \\ e_{r12} + e_{r22} \\ e_{r13} + e_{r23} \end{bmatrix}. \quad (35)$$

(29) The nominal system (34) is asymptotically stable if $\nu_0 = -e_{r2} - k\sigma_0 - k \text{sign}(\sigma_0)$, $k > 0$. The sliding surface for system (33) is $\sigma = \sigma_0 + z = e_{r1} + e_{r2} + z$, where $z = [z_1 \ z_3 \ z_3]^T$ is some integral term and $z(0)$ is chosen in such a way that $\sigma(0) = 0$. Defining $\nu = \nu_0 + \nu_s$ where $\nu_0 = [\nu_{01} \ \nu_{02} \ \nu_{03}]^T$ is the nominal

input and $v_s = [v_{s1} \ v_{s2} \ v_{s3}]^T$ is compensator, then system (33) can be rewritten as

$$\begin{aligned}\dot{e}_{r11} &= e_{r21} + \tilde{a}_2 x_{21} + \tilde{b}_2 x_{22} + \tilde{a}_1 x_{11} + \tilde{b}_1 x_{12} \\ \dot{e}_{r12} &= e_{r22} + \tilde{c}_2 x_{22} + \tilde{c}_1 x_{11} + \tilde{d}_1 x_{12} \\ \dot{e}_{r13} &= e_{r23} + \tilde{e}_2 x_{23} + \tilde{e}_1 x_{13} \\ \dot{e}_{r21} &= v_1 + \tilde{a}_3 x_{31} + \tilde{b}_3 x_{32} + \tilde{a}_2 x_{21} + \tilde{b}_2 x_{22} \\ \dot{e}_{r22} &= v_2 + \tilde{c}_3 x_{31} + \tilde{c}_2 x_{22} \\ \dot{e}_{r23} &= v_3 + \tilde{e}_3 x_{33} + \tilde{e}_2 x_{23}.\end{aligned}\quad (36)$$

Then $\dot{\sigma} = \dot{e}_{r1} + \dot{e}_{r2} + \dot{z}$ gives

$$\begin{aligned}\dot{\sigma}_1 &= e_{r21} + v_{01} + v_{s1} + \dot{z}_1 + \tilde{a}_1 x_{11} + \tilde{b}_1 x_{12} + 2\tilde{a}_2 x_{21} \\ &\quad + 2\tilde{b}_2 x_{22} + \tilde{a}_3 x_{31} + \tilde{b}_3 x_{32} \\ \dot{\sigma}_2 &= e_{r22} + v_{02} + v_{s2} + \dot{z}_2 + \tilde{c}_1 x_{11} + \tilde{d}_1 x_{12} + 2\tilde{c}_2 x_{22} \\ &\quad + \tilde{c}_3 x_{31} \\ \dot{\sigma}_3 &= e_{r23} + v_{03} + v_{s3} + \dot{z}_3 + 2\tilde{e}_2 x_{23} + \tilde{e}_1 x_{13} + \tilde{e}_3 x_{33}.\end{aligned}\quad (37)$$

By choosing a Lyapunov function

$$\begin{aligned}V &= \frac{1}{2} \{ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \tilde{a}_1^2 + \tilde{a}_2^2 + \tilde{a}_3^2 + \tilde{b}_1^2 + \tilde{b}_2^2 + \tilde{b}_3^2 \\ &\quad + \tilde{c}_1^2 + \tilde{c}_2^2 + \tilde{c}_3^2 + \tilde{e}_1^2 + \tilde{e}_2^2 + \tilde{e}_3^2 + \tilde{d}_1^2 \}\end{aligned}\quad (38)$$

then $\dot{V} < 0$ if the adaptive laws for $\tilde{a}_o, \tilde{a}_o, \tilde{b}_o, \tilde{b}_o, \tilde{c}_o, \tilde{c}_o, \tilde{e}_{ro}, \tilde{e}_{ro}, \tilde{d}_1, \tilde{d}_1$, $o = 1, 2, 3$ and the values of v_{so} , $o = 1, 2, 3$ are chosen as

$$\begin{aligned}\dot{z}_1 &= -e_{r21} - v_{01}, \\ v_{s1} &= -k_1 s_1 \\ \dot{z}_2 &= -e_{r22} - v_{02}, \\ v_{s2} &= -k_2 s_2 \\ \dot{z}_3 &= -e_{r23} - v_{03}, \\ v_{s3} &= -k_3 s_3 \\ \dot{\tilde{a}}_1 &= -\sigma_1 x_{11} - k_4 \tilde{a}_1, \\ \dot{\tilde{a}}_1 &= -\tilde{a}_1 \\ \dot{\tilde{a}}_2 &= -2\sigma_1 x_{21} - k_5 \tilde{a}_2, \\ \dot{\tilde{a}}_2 &= -\tilde{a}_2 \\ \dot{\tilde{a}}_3 &= -\sigma_1 x_{31} - k_6 \tilde{a}_3,\end{aligned}$$

$$\begin{aligned}\dot{\tilde{a}}_3 &= -\tilde{a}_3 \\ \dot{\tilde{b}}_1 &= -\sigma_1 x_{12} - k_7 \tilde{b}_1, \\ \dot{\tilde{b}}_1 &= -\tilde{b}_1 \\ \dot{\tilde{b}}_2 &= -2\sigma_1 x_{22} - k_8 \tilde{b}_2, \\ \dot{\tilde{b}}_2 &= -\tilde{b}_2 \\ \dot{\tilde{b}}_3 &= -\sigma_1 x_{32} - k_9 \tilde{b}_3, \\ \dot{\tilde{b}}_3 &= -\tilde{b}_3 \\ \dot{\tilde{c}}_1 &= -\sigma_2 x_{11} - k_{10} \tilde{c}_1, \\ \dot{\tilde{c}}_1 &= -\tilde{c}_1 \\ \dot{\tilde{c}}_2 &= -2\sigma_2 x_{22} - k_{11} \tilde{c}_2, \\ \dot{\tilde{c}}_2 &= -\tilde{c}_2 \\ \dot{\tilde{c}}_3 &= -\sigma_2 x_{31} - k_{12} \tilde{c}_3, \\ \dot{\tilde{c}}_3 &= -\tilde{c}_3 \\ \dot{\tilde{e}}_{r1} &= -\sigma_3 x_{13} - k_{13} \tilde{e}_{r1}, \\ \dot{\tilde{e}}_{r1} &= -\tilde{e}_{r1} \\ \dot{\tilde{e}}_{r2} &= -2\sigma_3 x_{23} - k_{14} \tilde{e}_{r2}, \\ \dot{\tilde{e}}_{r2} &= -\tilde{e}_{r2} \\ \dot{\tilde{e}}_{r3} &= -\sigma_3 x_{33} - k_{15} \tilde{e}_{r3}, \\ \dot{\tilde{e}}_{r3} &= -\tilde{e}_{r3} \\ \dot{\tilde{d}}_1 &= -\sigma_2 x_{12} - k_{16} \tilde{d}_1, \\ \dot{\tilde{d}}_1 &= -\tilde{d}_1,\end{aligned}$$

$$k_o > 0, \quad o = 1, 2, \dots, 16.$$

(39)

We have

$$\begin{aligned}\dot{V} &= -k_2 \sigma_1^2 - k_2 \sigma_2^2 - k_3 \sigma_3^2 - k_4 \tilde{a}_1^2 - k_5 \tilde{a}_2^2 - k_6 \tilde{a}_3^2 \\ &\quad - k_7 \tilde{b}_1^2 - k_8 \tilde{b}_2^2 - k_9 \tilde{b}_3^2 - k_{10} \tilde{c}_1^2 - k_{11} \tilde{c}_2^2 - k_{12} \tilde{c}_3^2 \\ &\quad - k_{13} \tilde{e}_{r1}^2 - k_{14} \tilde{e}_{r2}^2 - k_{15} \tilde{e}_{r3}^2 - k_{16} \tilde{d}_1^2.\end{aligned}\quad (40)$$

From this we conclude that $\sigma_o, \tilde{e}_{ro}, \tilde{d}_1, \tilde{a}_o, \tilde{b}_o, \tilde{c}_o \rightarrow 0$, $o = 1, 2, 3$. Since $\sigma_o \rightarrow 0$, $o = 1, 2, 3$, therefore $e_{r1}, e_{r2} \rightarrow 0$. Therefore, antisynchronization is realized. For complete synchronization, the error is $e'_{r1} = x_3 - x_1$, which can be written as $e'_{r1} = x_3 - x_1 = x_3 + x_2 - x_2 - x_1 = e_{r2} - e_{r1}$. As

$e_{r1}, e_{r2} \rightarrow 0$ therefore $e'_{r1} \rightarrow 0$. Thus complete synchronization is achieved. Results of simulations are depicted in Figures 2–4.

Example 2 ($M = 4$). In this example, only Lorenz systems with unknown parameters are chosen to examine the effectiveness of the proposed method.

These can be expressed as

$$\begin{aligned}
 \dot{x}_{11} &= -10x_{11} + 10x_{12} + d'_{11}(x_{41} - x_{11}) \\
 \dot{x}_{12} &= 28x_{11} - x_{12} - x_{11}x_{13} + d'_{12}(x_{42} - x_{12}) \\
 \dot{x}_{13} &= -\frac{8}{3}x_{13} + x_{11}x_{12} + d'_{13}(x_{43} - x_{13}) \\
 \dot{x}_{21} &= -10x_{21} + 10x_{22} + d'_{21}(x_{11} - x_{21}) + u'_{11} \\
 \dot{x}_{22} &= 28x_{21} - x_{22} - x_{21}x_{23} + d'_{22}(x_{12} - x_{22}) + u'_{12} \\
 \dot{x}_{23} &= -\frac{8}{3}x_{23} + x_{21}x_{22} + d'_{23}(x_{13} - x_{23}) + u'_{13} \\
 \dot{x}_{31} &= -10x_{31} + 10x_{32} + d'_{31}(x_{21} - x_{31}) + u'_{21} \\
 \dot{x}_{32} &= 28x_{31} - x_{32} - x_{31}x_{33} + d'_{32}(x_{22} - x_{32}) + u'_{22} \\
 \dot{x}_{33} &= -\frac{8}{3}x_{33} + x_{31}x_{32} + d'_{33}(x_{23} - x_{33}) + u'_{23} \\
 \dot{x}_{41} &= -10x_{41} + 10x_{42} + d'_{41}(x_{31} - x_{41}) + u'_{31} \\
 \dot{x}_{42} &= 28x_{41} - x_{42} - x_{41}x_{43} + d'_{42}(x_{32} - x_{42}) + u'_{32} \\
 \dot{x}_{43} &= -\frac{8}{3}x_{43} + x_{41}x_{42} + d'_{43}(x_{33} - x_{43}) + u'_{33}.
 \end{aligned} \tag{41}$$

With unknown parameters above four systems can be represented as

$$\begin{aligned}
 \dot{x}_{11} &= ax_{11} + bx_{12} + d'_{11}(x_{41} - x_{11}) \\
 \dot{x}_{12} &= cx_{11} - x_{12} - x_{11}x_{13} + d'_{12}(x_{42} - x_{12})
 \end{aligned} \tag{42}$$

$$\begin{aligned}
 \dot{x}_{13} &= dx_{13} + x_{11}x_{12} + d'_{13}(x_{43} - x_{13}) \\
 \dot{x}_{21} &= ax_{21} + bx_{22} + d'_{21}(x_{11} - x_{21}) + u'_{11} \\
 \dot{x}_{22} &= cx_{21} - x_{22} - x_{21}x_{23} + d'_{22}(x_{12} - x_{22}) + u'_{12}
 \end{aligned} \tag{43}$$

$$\begin{aligned}
 \dot{x}_{23} &= dx_{23} + x_{21}x_{22} + d'_{23}(x_{13} - x_{23}) + u'_{13} \\
 \dot{x}_{31} &= ax_{31} + bx_{32} + d'_{31}(x_{21} - x_{31}) + u'_{21} \\
 \dot{x}_{32} &= cx_{31} - x_{32} - x_{31}x_{33} + d'_{32}(x_{22} - x_{32}) + u'_{22}
 \end{aligned} \tag{44}$$

$$\begin{aligned}
 \dot{x}_{33} &= dx_{33} + x_{31}x_{32} + d'_{33}(x_{23} - x_{33}) + u'_{23} \\
 \dot{x}_{41} &= ax_{41} + bx_{42} + d'_{41}(x_{31} - x_{41}) + u'_{31} \\
 \dot{x}_{42} &= cx_{41} - x_{42} - x_{41}x_{43} + d'_{42}(x_{32} - x_{42}) + u'_{32} \\
 \dot{x}_{43} &= dx_{43} + x_{41}x_{42} + d'_{43}(x_{33} - x_{43}) + u'_{33}.
 \end{aligned} \tag{45}$$

Let $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ be estimates of a, b, c, d , respectively, and let $\tilde{a} = a - \hat{a}$, $\tilde{b} = b - \hat{b}$, $\tilde{c} = c - \hat{c}$, $\tilde{d} = d - \hat{d}$ be the errors in estimation of a, b, c, d , respectively.

Defining

$$\begin{aligned}
 x_1 &= \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}, \\
 x_2 &= \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}, \\
 x_3 &= \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix}, \\
 x_4 &= \begin{bmatrix} x_{41} \\ x_{42} \\ x_{43} \end{bmatrix}, \\
 u'_1 &= \begin{bmatrix} u'_{11} \\ u'_{12} \\ u'_{13} \end{bmatrix}, \\
 u'_2 &= \begin{bmatrix} u'_{21} \\ u'_{22} \\ u'_{23} \end{bmatrix}, \\
 u'_3 &= \begin{bmatrix} u'_{31} \\ u'_{32} \\ u'_{33} \end{bmatrix}, \\
 \hat{\theta} &= \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \\ \hat{d} \end{bmatrix}, \\
 \tilde{\theta} &= \begin{bmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{c} \\ \tilde{d} \end{bmatrix},
 \end{aligned} \tag{46}$$

systems (43)–(45) are represented as

$$\begin{aligned}
 \dot{x}_1 &= f_1(x_1) + F_1(x_1)\hat{\theta} + F_1(x_1)\tilde{\theta} \\
 \dot{x}_2 &= f_2(x_2) + F_2(x_2)\hat{\theta} + F_2(x_2)\tilde{\theta} + u'_1 \\
 \dot{x}_3 &= f_3(x_3) + F_3(x_3)\hat{\theta} + F_3(x_3)\tilde{\theta} + u'_2 \\
 \dot{x}_4 &= f_4(x_4) + F_4(x_4)\hat{\theta} + F_4(x_4)\tilde{\theta} + u'_3,
 \end{aligned} \tag{47}$$

where

$$f_1(x_1) = \begin{bmatrix} d'_{11}(x_{41} - x_{11}) \\ -x_{12} - x_{11}x_{13} + d'_{12}(x_{42} - x_{12}) \\ x_{11}x_{12} + d'_{13}(x_{43} - x_{13}) \end{bmatrix},$$

$$F_1(x_1) = \begin{bmatrix} x_{11} & x_{12} & 0 & 0 \\ 0 & 0 & x_{11} & 0 \\ 0 & 0 & 0 & x_{13} \end{bmatrix}$$

$$f_2(x_2) = \begin{bmatrix} d'_{21}(x_{11} - x_{21}) \\ -x_{22} - x_{21}x_{23} + d'_{22}(x_{12} - x_{22}) \\ x_{21}x_{22} + d'_{23}(x_{13} - x_{23}) \end{bmatrix},$$

$$F_2(x_2) = \begin{bmatrix} x_{21} & x_{22} & 0 & 0 \\ 0 & 0 & x_{21} & 0 \\ 0 & 0 & 0 & x_{23} \end{bmatrix}$$

$$f_3(x_3) = \begin{bmatrix} d'_{31}(x_{21} - x_{31}) \\ -x_{32} - x_{31}x_{33} + d'_{32}(x_{22} - x_{32}) \\ x_{31}x_{32} + d'_{33}(x_{23} - x_{33}) \end{bmatrix},$$

$$F_3(x_3) = \begin{bmatrix} x_{31} & x_{32} & 0 & 0 \\ 0 & 0 & x_{31} & 0 \\ 0 & 0 & 0 & x_{33} \end{bmatrix}$$

$$f_4(x_4) = \begin{bmatrix} d'_{41}(x_{31} - x_{41}) \\ -x_{42} - x_{41}x_{43} + d'_{42}(x_{32} - x_{42}) \\ x_{41}x_{42} + d'_{43}(x_{33} - x_{43}) \end{bmatrix},$$

$$F_4(x_4) = \begin{bmatrix} x_{41} & x_{42} & 0 & 0 \\ 0 & 0 & x_{41} & 0 \\ 0 & 0 & 0 & x_{43} \end{bmatrix}.$$

For the antisynchronization, the error vectors

$$e_{r1} = \begin{bmatrix} e_{r11} \\ e_{r12} \\ e_{r13} \end{bmatrix},$$

$$e_{r2} = \begin{bmatrix} e_{r21} \\ e_{r22} \\ e_{r23} \end{bmatrix},$$

$$e_{r3} = \begin{bmatrix} e_{r31} \\ e_{r32} \\ e_{r33} \end{bmatrix}$$

(48)

are

$$\begin{aligned} e_{r1} &= x_2 + x_1, \\ e_{r2} &= x_3 + x_2, \\ e_{r3} &= x_4 + x_3. \end{aligned} \quad (50)$$

Therefore

$$\begin{aligned} \begin{bmatrix} \dot{e}_{r1} \\ \dot{e}_{r2} \\ \dot{e}_{r3} \end{bmatrix} &= \begin{bmatrix} (F_1(x_1) + F_2(x_2))\hat{\theta} + f_1(x_1) + f_2(x_2) \\ (F_2(x_2) + F_3(x_3))\hat{\theta} + f_2(x_2) + f_3(x_3) \\ (F_3(x_3) + F_4(x_4))\hat{\theta} + f_3(x_3) + f_4(x_4) \end{bmatrix} \\ &+ \begin{bmatrix} F_1(x_1) + F_2(x_2) \\ F_2(x_2) + F_3(x_3) \\ F_3(x_3) + F_4(x_4) \end{bmatrix} \tilde{\theta} \\ &+ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix}. \end{aligned} \quad (51)$$

Choose

$$\begin{aligned} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \\ &\cdot \left\{ - \begin{bmatrix} f_2(x_2) + f_1(x_1) + (F_2(x_2) + F_1(x_1))\hat{\theta} \\ f_3(x_3) + f_2(x_2) + (F_3(x_3) + F_2(x_2))\hat{\theta} \\ f_4(x_4) + f_3(x_3) + (F_4(x_4) + F_3(x_3))\hat{\theta} \end{bmatrix} \right. \\ &\left. + \begin{bmatrix} e_{r2} \\ e_{r3} \\ v \end{bmatrix} \right\}, \end{aligned} \quad (52)$$

where v is the new input vector.

We have

$$\begin{bmatrix} \dot{e}_{r1} \\ \dot{e}_{r2} \\ \dot{e}_{r3} \end{bmatrix} = \begin{bmatrix} e_{r2} \\ e_{r3} \\ v \end{bmatrix} + \begin{bmatrix} F_2(x_2) + F_1(x_1) \\ F_3(x_3) + F_2(x_2) \\ F_4(x_4) + F_3(x_3) \end{bmatrix} \tilde{\theta} \quad (53)$$

or

$$\begin{aligned} \dot{e}_{r1} &= e_{r2} + (F_2(x_2) + F_1(x_1))\tilde{\theta} \\ \dot{e}_{r2} &= e_{r3} + (F_3(x_3) + F_2(x_2))\tilde{\theta} \\ \dot{e}_{r3} &= v + (F_4(x_4) + F_3(x_3))\tilde{\theta}. \end{aligned} \quad (54)$$

The nominal system for (54) is

$$\begin{aligned} \dot{e}_{r1} &= e_{r2} \\ \dot{e}_{r2} &= e_{r3} \\ \dot{e}_{r3} &= v_0. \end{aligned} \quad (55)$$

The Hurwitz sliding surface vector for nominal system (55) can be defined as

$$\sigma_0 = e_{r1} + 2e_{r2} + e_{r3}, \quad (56)$$

where

$$\sigma_0 = \begin{bmatrix} \sigma_{01} \\ \sigma_{02} \\ \sigma_{03} \end{bmatrix} = \begin{bmatrix} e_{r11} + 2e_{r21} + e_{r31} \\ e_{r12} + 2e_{r22} + e_{r32} \\ e_{r13} + 2e_{r23} + e_{r33} \end{bmatrix}. \quad (57)$$

Then the derivative of the above system is $\dot{\sigma}_0 = \dot{e}_{r1} + 2\dot{e}_{r2} = e_{r2} + 2e_{r3} + v_0$. By choosing $v_0 = -e_{r2} - 2e_{r3} - k\sigma_0 - k \text{sign}(\sigma_0)$, $k > 0$, we have $\dot{\sigma}_0 = -k \text{sign}(\sigma_0) - k\sigma_0$. So the error dynamics (55) are asymptotically stable.

Sliding surface for the system (55) is described as $\sigma = \sigma_0 + z = e_{r1} + 2e_{r2} + e_{r3} + z$, where $z = [z_1 \ z_2 \ z_3]^T$ is some integral term, by choosing $z(0)$ in such a way that $\sigma(0) = 0$. Select $v = v_0 + v_s$, where $v_0 = [v_{01} \ v_{02} \ v_{03}]^T$ is the nominal input and $v_s = [v_{s1} \ v_{s2} \ v_{s3}]^T$ is compensator term computed later.

The system (54) can be rewritten as

$$\begin{aligned} \dot{e}_{r11} &= e_{r21} + \tilde{a}x_{21} + \tilde{b}x_{22} + \tilde{a}x_{11} + \tilde{b}x_{12} \\ \dot{e}_{r12} &= e_{r22} + \tilde{c}x_{21} + \tilde{c}x_{11} \\ \dot{e}_{r13} &= e_{r23} + \tilde{d}x_{23} + \tilde{d}x_{13} \\ \dot{e}_{r21} &= e_{r31} + \tilde{a}x_{31} + \tilde{b}x_{32} + \tilde{a}x_{21} + \tilde{b}x_{22} \\ \dot{e}_{r22} &= e_{r32} + \tilde{c}x_{31} + \tilde{c}x_{21} \\ \dot{e}_{r23} &= e_{r33} + \tilde{d}x_{33} + \tilde{d}x_{23} \\ \dot{e}_{r31} &= v_{01} + v_{s1} + \tilde{a}x_{41} + \tilde{b}x_{42} + \tilde{a}x_{31} + \tilde{b}x_{32} \\ \dot{e}_{r32} &= v_{02} + v_{s2} + \tilde{c}x_{41} + \tilde{c}x_{31} \\ \dot{e}_{r33} &= v_{03} + v_{s3} + \tilde{d}x_{43} + \tilde{d}x_{33}. \end{aligned} \quad (58)$$

Then $\dot{\sigma} = \dot{e}_{r1} + 2\dot{e}_{r2} + \dot{e}_{r3} + \dot{z}$ gives

$$\begin{aligned} \dot{\sigma}_1 &= \dot{e}_{r11} + 2\dot{e}_{r21} + \dot{e}_{r31} + \dot{z}_1 \\ &= e_{r21} + \tilde{a}x_{21} + \tilde{b}x_{22} + \tilde{a}x_{11} + \tilde{b}x_{12} + 2e_{r31} \\ &\quad + 2\tilde{a}x_{31} + 2\tilde{b}x_{32} + 2\tilde{a}x_{21} + 2\tilde{b}x_{22} + v_{01} + v_{s1} \\ &\quad + \tilde{a}x_{41} + \tilde{b}x_{42} + \tilde{a}x_{31} + \tilde{b}x_{32} + \dot{z}_1 \\ \dot{\sigma}_2 &= \dot{e}_{r12} + 2\dot{e}_{r22} + \dot{e}_{r32} + \dot{z}_2 \\ &= e_{r22} + \tilde{c}x_{21} + \tilde{c}x_{11} + 2e_{r32} + 2\tilde{c}x_{31} + 2\tilde{c}x_{21} \\ &\quad + v_{02} + v_{s2} + \tilde{c}x_{41} + \tilde{c}x_{31} + \dot{z}_2 \\ \dot{\sigma}_3 &= \dot{e}_{r13} + 2\dot{e}_{r23} + \dot{e}_{r33} + \dot{z}_3 \\ &= e_{r23} + \tilde{d}x_{23} + \tilde{d}x_{13} + 2e_{r33} + 2\tilde{d}x_{33} + 2\tilde{d}x_{23} \\ &\quad + v_{03} + v_{s3} + \tilde{d}x_{43} + \tilde{d}x_{33} + \dot{z}_3 \end{aligned} \quad (59)$$

or

$$\begin{aligned} \dot{\sigma}_1 &= e_{r21} + 2e_{r31} + \tilde{a}\{x_{11} + 3x_{21} + 3x_{31} + x_{41}\} \\ &\quad + \tilde{b}\{x_{12} + 3x_{22} + 3x_{32} + x_{42}\} + v_{01} + v_{s1} \\ &\quad + \dot{z}_1 \\ \dot{\sigma}_2 &= e_{r22} + 2e_{r32} + \tilde{c}\{x_{11} + 3x_{21} + 3x_{31} + x_{41}\} + v_{02} \\ &\quad + v_{s2} + \dot{z}_2 \\ \dot{\sigma}_3 &= e_{r23} + 2e_{r33} + \tilde{d}\{x_{13} + 3x_{23} + 3x_{33} + x_{43}\} + v_{03} \\ &\quad + v_{s3} + \dot{z}_3. \end{aligned} \quad (60)$$

Choose the Lyapunov function: $V = (1/2)\{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2\}$.

Then $\dot{V} < 0$ if the adaptive laws for $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \hat{a}, \hat{b}, \hat{c}, \hat{d}$ and v_{so} , $o = 1, 2, 3$ are chosen as

$$\begin{aligned} \dot{z}_1 &= -e_{r21} - 2e_{r31} - v_{01}, \\ v_{s1} &= -k_1 s_1 \\ \dot{z}_2 &= -e_{r22} - 2e_{r32} - v_{02}, \\ v_{s2} &= -k_2 s_2 \\ \dot{z}_3 &= -e_{r23} - 2e_{r33} - v_{03}, \\ v_{s3} &= -k_3 s_3 \\ \dot{\tilde{a}} &= -\sigma_1 (x_{11} + 3x_{21} + 3x_{31} + x_{41}) - k_4 \tilde{a}, \\ \dot{\tilde{a}} &= -\dot{\tilde{a}} \\ \dot{\tilde{b}} &= -\sigma_1 (x_{12} + 3x_{22} + 3x_{32} + x_{42}) - k_5 \tilde{b}, \\ \dot{\tilde{b}} &= -\dot{\tilde{b}} \\ \dot{\tilde{c}} &= -\sigma_2 (x_{11} + 3x_{21} + 3x_{31} + x_{41}) - k_6 \tilde{c}, \\ \dot{\tilde{c}} &= -\dot{\tilde{c}} \\ \dot{\tilde{d}} &= -\sigma_3 (x_{13} + 3x_{23} + 3x_{33} + x_{43}) - k_7 \tilde{d}, \\ \dot{\tilde{d}} &= -\dot{\tilde{d}}, \end{aligned} \quad (61)$$

$$k_o > 0, \quad o = 1, \dots, 7.$$

From this we conclude that $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \sigma_1, \sigma_2, \sigma_3, \tilde{d} \rightarrow 0$. Since $\sigma_o \rightarrow 0$, $o = 1, 2, 3$, therefore $e_{r1}, e_{r2}, e_{r3} \rightarrow 0$. Thus antisynchronization is achieved.

For complete synchronization, the errors are $e'_{r1} = x_3 - x_1$, $e'_{r2} = x_4 - x_2$ which can be written as

$$\begin{aligned} e'_{r1} &= x_3 - x_1 = x_3 + x_2 - x_2 - x_1 = e_{r2} - e_{r1} \\ e'_{r2} &= x_4 - x_2 = x_4 + x_3 - x_3 - x_2 = e_{r3} - e_{r2}. \end{aligned} \quad (62)$$

As $e_{r1}, e_{r2}, e_{r3} \rightarrow 0$ therefore $e'_{r1}, e'_{r2} \rightarrow 0$; thus complete synchronization is achieved. Simulation results are shown in Figures 5–7.

FIGURE 2

FIGURE 3

5. Simulation Results and Discussion

Figures 2–4 show simulation results for Example 1 with $M = 3$. The initial (starting point) conditions are chosen as $(x_{11}(0) = 10, x_{12}(0) = 20, x_{13}(0) = 30), (x_{21}(0) = -5.8, x_{22}(0) = 8, x_{23}(0) = 10),$ and $(x_{31}(0) = 11, x_{32}(0) = 15, x_{33}(0) = 26)$. The coupling parameters are chosen as $d_{11} = d_{21} = d_{13} = d_{23} = d_{31} = d_{33} = 0, d_{12} = 10, d_{22} = 11$ and $d_{32} = 1$. Figure 2(a) shows that the errors $e_{r11}, e_{r12},$

$e_{r13}, e_{r21}, e_{r22},$ and e_{r23} asymptotically go to zero. Figure 2(b) displays the errors $e'_{r31}, e'_{r32},$ and $e'_{r33},$ converging to origin. Figure 3(a) shows system states $x_{11}, x_{21}, x_{31},$ Figure 3(b) shows system states $x_{12}, x_{22},$ and $x_{32},$ and Figure 3(c) shows system states $x_{13}, x_{23},$ and $x_{33}.$ From these figures we can see that systems $x_1(t)$ and $x_2(t),$ and systems $x_2(t)$ and $x_3(t)$ achieve the antisynchronization, and systems $x_1(t)$ and $x_3(t)$ attain complete synchronization and therefore milestone is attained, that is, hybrid synchronization. Figure 4 shows the

(a) Parameter estimation a_1, b_1, c_1, d_1 , and e_{r1} (b) Parameter estimation a_2, b_2, c_2 , and e_{r2} (c) Parameter estimation a_3, b_3, c_3 , and e_{r3}

FIGURE 4

adaptive estimation of parameters a, b, c, d , and e_r for the three systems. Figure 4(a) shows the estimation of a_1, b_1, c_1, d_1 , and e_{r1} , the parameters of the first system, which converge to their true values of $-35, 35, -7, 28$, and -3 , respectively. Figure 4(b) shows the estimation of a_2, b_2, c_2 , and e_{r2} , and the parameters of the second system, which converges to their true values of $-36, 36, 20$, and 3 , respectively. Figure 4(c) shows the estimation of a_3, b_3, c_3 , and e_{r3} , the parameters of the third system, which converges to their true values $-10, 10, 28$, and $-8/3$, respectively.

Figures 5–7 show simulation results for Example 2 with $M = 4$. The initial (starting point) conditions are chosen as

(a) Convergence of error system $e_{r11}, e_{r12}, e_{r13}, e_{r21}, e_{r22}, e_{r23}, e_{r31}, e_{r32}$, and e_{r33} (b) Convergence of error system e'_{r11}, e'_{r12} , and e'_{r13} (c) Convergence of error system e''_{r21}, e''_{r22} , and e''_{r23}

FIGURE 5

$(x_{11}(0) = 4, x_{12}(0) = 5, x_{13}(0) = -3), (x_{21}(0) = 5, x_{22}(0) = 2, x_{23}(0) = -5), (x_{31}(0) = 11, x_{32}(0) = 15, x_{33}(0) = 10)$, and $(x_{41}(0) = 4, x_{42}(0) = 5, x_{43}(0) = -3)$. The coupling parameters are chosen as $d_{12} = d_{22} = d_{32} = d_{13} = d_{23} = d_{33} = d_{42} = d_{43} = 0, d_{11} = d_{21} = 1$ and $d_{31} = d_{41} = -1$. Figure 5(a) shows the errors $e_{r11}, e_{r12}, e_{r13}, e_{r21}, e_{r22}, e_{r23}, e_{r31}, e_{r32}$, and e_{r33} asymptotically converge to zero. Figure 5(b) shows the errors e'_{r11}, e'_{r12} , and e'_{r13} asymptotically converge to zero. Figure 5(c) shows the errors e''_{r21}, e''_{r22} , and e''_{r23}

FIGURE 6

FIGURE 7: Parameter estimation a , b , c , and d .

asymptotically converge to zero. Figure 6(a) shows system states x_{11} , x_{21} , x_{31} , and x_{41} , Figure 6(b) shows system states x_{12} , x_{22} , x_{32} , and x_{42} , and Figure 6(c) shows system states x_{13} , x_{23} , x_{33} , and x_{43} . From these figures we can see that systems $x_1(t)$ and $x_2(t)$, systems $x_2(t)$ and $x_3(t)$, and systems $x_3(t)$ and $x_4(t)$ achieve the antisynchronization, and systems $x_1(t)$ and $x_3(t)$ and systems $x_2(t)$ and $x_4(t)$ attained the complete synchronization and hence hybrid synchronization is attained. Figure 7 shows an adaptive estimation of parameters a , b , c ,

and d for four systems which converge to their true values of -10, 10, 28, and $-8/3$ respectively.

6. Conclusion

In this article, we presented control design method for parameter identification and synchronization/antisynchronization of many attached chaotic systems connected in the ring topology. The methodology is created using adaptive integral SMC. The error system was transformed into a special structure containing nominal part and some unknown terms. The unknown terms were computed adaptively. Then the error system was stabilized using integral sliding mode control. The stabilizing controller for the error system is created that contains the nominal control and the compensator control. Simulation results show that antisynchronization and complete synchronization were achieved with the proposed control laws and that the uncertain parameters converge to their actual values.

Conflicts of Interest

The authors do not have any conflicts of interest regarding the publication of this paper.

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