

Research Article

Controllability, Reachability, and Stabilizability of Finite Automata: A Controllability Matrix Method

Yalu Li,¹ Wenhui Dou,¹ Haitao Li ,^{1,2} and Xin Liu¹

¹School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China

²Institute of Data Science and Technology, Shandong Normal University, Jinan 250014, China

Correspondence should be addressed to Haitao Li; haitaoli09@gmail.com

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This paper investigates the controllability, reachability, and stabilizability of finite automata by using the semitensor product of matrices. Firstly, by expressing the states, inputs, and outputs as vector forms, an algebraic form is obtained for finite automata. Secondly, based on the algebraic form, a controllability matrix is constructed for finite automata. Thirdly, some necessary and sufficient conditions are presented for the controllability, reachability, and stabilizability of finite automata by using the controllability matrix. Finally, an illustrative example is given to support the obtained new results.

1. Introduction

In the research field of theoretical computer science, finite automaton is one of the simplest models of computation. Finite automaton is a device whose states take values from a finite set. It receives a discrete sequence of inputs from the outside world and changes its state according to the inputs. The study of finite automata has received many scholars' research interest in the last century [1–5] due to its wide applications in engineering, computer science, and so on.

As we all know, controllability and stabilizability analysis of finite automata are fundamental topics, which are important and necessary to the solvability of many related problems [1, 4, 6]. The concepts of controllability, reachability, and stabilizability of finite automata were defined in [2] by resorting to the classic control theory. The controllability of a deterministic Rabin automaton was studied in [7] by defining the “controllability subset.” Kobayashi et al. [8] investigated the state feedback stabilization of a deterministic finite automaton and presented some new results.

Recently, a new matrix product, namely, the semitensor product (STP) of matrices, has been proposed by Cheng et al. [9]. Up to now, STP has been successfully applied to many research fields related to finite-valued systems like Boolean networks [10–20], multivalued logical networks [21–23],

game theory [24, 25], finite automata [5, 26], and so on [27–35]. The main feature of STP is to convert a finite-valued system into an equivalent algebraic form [22]. Thus, STP provides a convenient way for the construction and analysis of finite automata [5, 26]. Xu and Hong [5] provided a matrix-based algebraic approach for the reachability analysis of finite automata with the help of STP. Yan et al. [26] studied the controllability and stabilizability analysis of finite automata based on STP and presented some novel results. It should be pointed out that although the concepts of controllability, reachability, and stabilizability of finite automata come from classic control theory, there exist fewer results on the construction of controllability matrix for finite automata.

In this paper, we investigate the controllability, reachability, and stabilizability of deterministic finite automata by using STP. The main contribution of this paper is to construct a controllability matrix for finite automata based on the algebraic form. Using the controllability matrix, we present some necessary and sufficient conditions for the controllability, reachability, and stabilizability of finite automata. Compared with the existing results [5, 26], our results are more easily verified via MATLAB.

The rest of this paper is organized as follows. Section 2 contains some necessary preliminaries on the semitensor product of matrices and finite automata. Section 3 studies

the controllability, reachability, and stabilizability of finite automata and presents the main results of this paper. In Section 4, an illustrative example is given to support our new results, which is followed by a brief conclusion in Section 5.

Notations. \mathbb{R} , \mathbb{N} , and \mathbb{Z}_+ denote the set of real numbers, the set of natural numbers, and the set of positive integers, respectively. $\Delta_n := \{\delta_n^k : k = 1, \dots, n\}$, where δ_n^k denotes the k th column of the $n \times n$ identity matrix I_n . An $n \times t$ matrix M is called a logical matrix, if $M = [\delta_n^{i_1} \ \delta_n^{i_2} \ \dots \ \delta_n^{i_t}]$, which is briefly denoted by $M = \delta_n [i_1 \ i_2 \ \dots \ i_t]$. The set of $n \times t$ logical matrices is denoted by $\mathcal{L}_{n \times t}$. Given a real matrix A , $\text{Col}_i(A)$, $\text{Row}_j(A)$, and $(A)_{i,j}$ denote the i th column, the j th row, and the (i, j) th element of A , respectively. $A > 0$ if and only if $(A)_{i,j} > 0$ holds for any i, j . $\text{Blk}_i(A)$ denote the i th block of an $n \times mn$ matrix A .

2. Preliminaries

2.1. Semitensor Product of Matrices. In this part, we recall some necessary preliminaries on STP. For details, please refer to [9].

Definition 1. Given two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$, the semitensor product of A and B is defined as

$$A \ltimes B = (A \otimes I_{\alpha/n}) (B \otimes I_{\alpha/p}), \quad (1)$$

where $\alpha = \text{lcm}(n, p)$ is the least common multiple of n and p and \otimes is the Kronecker product of matrices.

Lemma 2. *STP has the following properties:*

(1) Let $X \in \mathbb{R}^{t \times 1}$ be a column vector and $A \in \mathbb{R}^{m \times n}$. Then

$$X \ltimes A = (I_t \otimes A) \ltimes X. \quad (2)$$

(2) Let $X \in \mathbb{R}^{m \times 1}$ and $Y \in \mathbb{R}^{n \times 1}$ be two column vectors. Then

$$Y \ltimes X = W_{[m,n]} \ltimes X \ltimes Y, \quad (3)$$

where $W_{[m,n]} \in \mathcal{L}_{m \times mn}$ is called the swap matrix.

2.2. Finite Automata. In this subsection, we recall some definitions of finite automata.

A finite automaton is a seven-tuple $A = (X, U, Y, f, g, x_0, X_m)$, in which X , U , and Y are finite sets of states, input symbols, and outputs, respectively; x_0 and $X_m \subset X$ are the initial state and the set of accepted states; f and g are transition and output functions, which are defined as $f : X \times U \rightarrow 2^X$ and $g : X \times U \rightarrow 2^Y$, where 2^X and 2^Y denote the power set of X and Y , respectively; that is, $f(x, u) \subset X$, $g(x, u) \subset Y$. U^* represents the finite string set on U , which does not include the empty transition. Given an initial state $x_0 \in X$ and an input symbol $u \in U$, the function f uniquely determines the next subset of states, that is, $f(x_0, u) \subset X$, while the function g uniquely determines the next subset of outputs; that is, $g(x_0, u) \subset Y$.

Throughout this paper, we only consider the deterministic finite automata; that is, $|f(x, u)| \leq 1$ holds for any $x \in X$ and $u \in U$. In addition, we only investigate the controllability, reachability, and stabilizability of deterministic finite automata, and thus we do not use Y and g in the seven-tuple $A = (X, U, Y, f, g, x_0, X_m)$.

In the following, we recall the definitions of controllability, reachability, and stabilizability for deterministic finite automata.

Definition 3. (i) A state $x_p \in X$ is said to be controllable to $x_q \in X$, if there exists a control sequence $u^t \in U^*$ such that $f(x_p, u^t) = x_q$.

(ii) A state $x_p \in X$ is said to be controllable, if $x_p \in X$ is controllable to any state $x_q \in X$.

Definition 4. (i) A state $x_q \in X$ is said to be reachable from $x_p \in X$, if there exists a control sequence $u^t \in U^*$ such that $f(x_p, u^t) = x_q$.

(ii) A state $x_q \in X$ is said to be reachable, if $x_q \in X$ is reachable from any state $x_p \in X$.

Given two nonempty sets $X_1 \subseteq X$ and $X_2 \subseteq X$ satisfying $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \emptyset$, we have the following definitions.

Definition 5. A nonempty set of state $X_1 \subseteq X$ is said to be controllable, if, for any state $x_q \in X_2$, there exist an $x_p \in X_1$ and a control sequence $u^t \in U^*$ such that $f(x_p, u^t) = x_q$.

Definition 6. A nonempty set of state $X_2 \subseteq X$ is said to be reachable, if, for any state $x_p \in X_1$, there exist an $x_q \in X_2$ and a control sequence $u^t \in U^*$ such that $f(x_p, u^t) = x_q$.

Definition 7. A nonempty set of state $X_1 \subseteq X$ is said to be 1-step returnable, if, for any state $x_0 = x_p \in X_1$, there exists an input $u_i \in U$ such that $f(x_p, u_i) = x_s \in X_1$.

Definition 8. A nonempty set of state $X_1 \subseteq X$ is said to be stabilizable, if X_1 is reachable and 1-step returnable.

3. Main Results

In this section, we investigate the controllability, reachability, and stabilizability of deterministic finite automata by constructing a controllability matrix.

3.1. Controllability Matrix. For a deterministic finite automaton $A = (X, U, Y, f, g, x_0, X_m)$, where $X = \{x_1, \dots, x_n\}$ and $U = \{u_1, \dots, u_m\}$, we identify x_i as δ_n^i ($i = 1, \dots, n$) and call δ_n^i the vector form of x_i . Then, X can be denoted as Δ_n ; that is, $X = \{\delta_n^1, \dots, \delta_n^n\}$. Similarly, for U , we identify u_j with δ_m^j ($j = 1, \dots, m$) and call δ_m^j the vector form of u_j . Then, $U = \{\delta_m^1, \dots, \delta_m^m\}$.

Using the vector form of elements in X and U , Yan et al. [26] construct the transition structure matrix (TSM) of $A = (X, U, Y, f, g, x_0, X_m)$ as $F = [F_1 \ \dots \ F_m]$. One can see that if

there exists a control $u_j \in U$ which moves state x_p to state x_q , then

$$\delta_n^q = F \times \delta_m^j \times \delta_n^p. \quad (4)$$

In this case, $(F_j)_{q,p} = 1$. Otherwise, $(F_j)_{q,p} = 0$. Thus, setting

$$M = \sum_{j=1}^m F_j \in \mathbb{R}^{n \times n}, \quad (5)$$

then one can use M to judge whether or not state x_p is controllable to state x_q in one step. Precisely, state x_p is controllable to state x_q in one step, if and only if $(M)_{q,p} > 0$.

Now, we show that, for any $t \in \mathbb{Z}_+$, state x_p is controllable to state x_q at the t th step, if and only if $(M^t)_{q,p} > 0$. We prove it by induction. Obviously, when $t = 1$, the conclusion holds. Assume that the conclusion holds for some $t \in \mathbb{Z}_+$. Then, for the case of $t + 1$, state x_p is controllable to state x_q at the $(t + 1)$ th step, if and only if there exists some state $x_r \in X$ such that state x_p is controllable to state x_r at the t th step and state x_r is controllable to state x_q in one step. Hence,

$$(M^{t+1})_{q,p} = \sum_{i=1}^n (M)_{q,i} (M^t)_{i,p} \geq (M)_{q,r} (M^t)_{r,p} > 0. \quad (6)$$

By induction, for any $t \in \mathbb{Z}_+$, state x_p is controllable to state x_q at the t th step, if and only if $(M^t)_{q,p} > 0$. Thus, $\sum_{t=1}^{\infty} M^t$ contains all the controllability information of the finite automata. Noticing that M is an $n \times n$ square matrix, by Cayley-Hamilton theorem, we only need to consider $t \leq n$. Then, we define the controllability matrix for finite automata as follows.

Definition 9. Set $M = \sum_{j=1}^m F_j \in \mathbb{R}^{n \times n}$. The controllability matrix of finite automata is $C = \sum_{t=1}^n M^t$.

Based on the controllability matrix, we have the following result.

Algorithm 10. Consider the finite automata $A = (X, U, Y, f, g, x_0, X_m)$. Then, the controls which force x_p to x_q in the shortest time l can be designed by the following steps:

- (1) Find the smallest integer l such that, for

$$\begin{aligned} \mathcal{F}_0^l &:= M^{l-1}F \\ &= [\text{Blk}_1(\mathcal{F}_0^l) \quad \text{Blk}_2(\mathcal{F}_0^l) \quad \cdots \quad \text{Blk}_m(\mathcal{F}_0^l)], \end{aligned} \quad (7)$$

there exists a block, say, $\text{Blk}_\theta(\mathcal{F}_0^l)$, satisfying $[\text{Blk}_\theta(\mathcal{F}_0^l)]_{q,p} > 0$.

- (2) Set $u(0) = \delta_m^\theta$ and $x_q = \delta_n^q$. If $l = 1$, stop. Otherwise, go to Step (3).
- (3) Find r and η such that $[\text{Blk}_\eta(\mathcal{F}_0^{l-1})]_{q,r} > 0$ and $[\text{Blk}_\theta(\mathcal{F}_0^{l-1})]_{r,p} > 0$, where $\mathcal{F}_0^1 = F$ and $\mathcal{F}_0^{l-1} = M^{l-2}F$. Set $u(l-1) = \delta_m^\eta$ and $x(l-1) = \delta_n^r$.

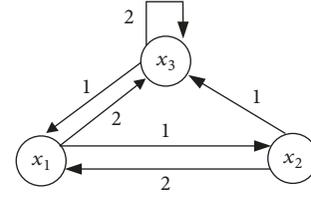


FIGURE 1: A finite automata.

- (4) If $l - 1 = 1$, stop. Otherwise, replace l and q by $l - 1$ and r , respectively, and go to Step (3).

Example 11. Consider a finite automaton $A = (X, U, Y, f, g, x_0, X_m)$ given in Figure 1, where $X = \{x_1, x_2, x_3\}$, $U = \{1, 2\}$. Suppose that $x_0 = x_1$ and $X_m = \{x_3\}$. Then, X can be denoted as $\Delta_3 = \{\delta_3^1, \delta_3^2, \delta_3^3\}$. Similarly, $U = \Delta_2 = \{\delta_2^1, \delta_2^2\}$, $x_0 = \delta_3^1$, $X_m = \{\delta_3^2\}$.

The transition structure matrix of the finite automata A is

$$F = \delta_3 [2 \ 3 \ 1 \ 3 \ 1 \ 3]. \quad (8)$$

Split $F = [F_1 \ F_2]$, where $F_1 = \delta_3 [2 \ 3 \ 1]$ and $F_2 = \delta_3 [3 \ 1 \ 3]$. Then,

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \quad (9)$$

Thus, the controllability matrix is

$$C = \sum_{t=1}^3 M^t = \begin{bmatrix} 4 & 5 & 5 \\ 3 & 2 & 2 \\ 7 & 7 & 7 \end{bmatrix}. \quad (10)$$

By Algorithm 10, one can obtain that $[\text{Blk}_1(\mathcal{F}_0^2)]_{2,3} > 0$. Setting $x_p = \delta_3^3$, $u(0) = \delta_2^1$, and $x_q = \delta_3^2$, one can find $r = 1$ and $\eta = 1$ such that $[\text{Blk}_1(\mathcal{F}_0^1)]_{2,1} > 0$ and $[\text{Blk}_1(\mathcal{F}_0^1)]_{1,3} > 0$. Let $u(1) = \delta_2^1$ and $x_1 = \delta_3^1$. Hence, state x_3 is controllable to state x_2 at the 2nd step.

3.2. Controllability, Reachability, and Stabilizability. In this part, we study the controllability, reachability, and stabilizability of deterministic finite automata based on the controllability matrix.

According to the meaning of controllability matrix, we have the following results.

Theorem 12. The state $x_p \in X$ is controllable, if and only if

$$\text{Col}_p(C) > 0. \quad (11)$$

Proof.

Necessity. Suppose that the state $x_p \in X$ is controllable. By Definition 3, $x_p \in X$ is controllable to any state $x_q \in X$. Based on (4), one can see that there exists a control sequence $\{u_{i_\alpha} : \alpha = 0, \dots, s-1\} \subseteq U$ ($1 \leq s \leq n$) satisfying

$\delta_n^q = F_{i_{s-1}} \cdots F_{i_0} \times \delta_n^p$. Thus, $(F_{i_{s-1}} \cdots F_{i_0})_{q,p} > 0$, which implies that

$$(C)_{q,p} \geq (F_{i_{s-1}} \cdots F_{i_0})_{q,p} > 0. \quad (12)$$

From the arbitrariness of q , we have $\text{Col}_p(C) > 0$.

Sufficiency. Suppose that $\text{Col}_p(C) > 0$ holds. Then, for any state $x_q \in X$, one can find some $(F_{i_{s-1}} \cdots F_{i_0})_{q,p} > 0$. Therefore, under the control sequence $\{u_{i_\alpha} : \alpha = 0, \dots, s-1\} \subseteq U$, the state $x_p \in X$ is controllable to $x_q \in X$. From the arbitrariness of q , the state $x_p \in X$ is controllable. \square

Theorem 13. *The state $x_q \in X$ is reachable, if and only if*

$$\text{Row}_q(C) > 0. \quad (13)$$

Proof.

Necessity. Suppose that the state $x_q \in X$ is reachable. By Definition 4, $x_q \in X$ is reachable to any state $x_p \in X$. One can obtain from (4) that there exists a control sequence $\{u_{i_\alpha} : \alpha = 0, \dots, s-1\} \subseteq U$ ($1 \leq s \leq n$) satisfying $\delta_n^q = F_{i_{s-1}} \cdots F_{i_0} \times \delta_n^p$. Thus, $(F_{i_{s-1}} \cdots F_{i_0})_{q,p} > 0$, which shows that

$$(C)_{q,p} \geq (F_{i_{s-1}} \cdots F_{i_0})_{q,p} > 0. \quad (14)$$

From the arbitrariness of p , one can conclude that $\text{Row}_q(C) > 0$.

Sufficiency. Suppose that $\text{Row}_q(C) > 0$. Then, for any state $x_p \in X$, there exists some $(F_{i_{s-1}} \cdots F_{i_0})_{q,p} > 0$. Hence, under the control sequence $\{u_{i_\alpha} : \alpha = 0, \dots, s-1\} \subseteq U$, the state $x_q \in X$ is reachable to $x_p \in X$. By Definition 4, the state $x_q \in X$ is reachable. \square

Given two nonempty sets $X_1 = \{\delta_n^{p_1}, \dots, \delta_n^{p_s}\}$ and $X_2 = \{\delta_n^{p_{s+1}}, \dots, \delta_n^{p_n}\}$, where $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \emptyset$, define

$$C_{X_1} = \left[\sum_{\beta=1}^s (C)_{p_{s+1}, p_\beta} \cdots \sum_{\beta=1}^s (C)_{p_n, p_\beta} \right], \quad (15)$$

$$R_{X_2} = \left[\sum_{\gamma=s+1}^n (C)_{p_\gamma, p_1} \cdots \sum_{\gamma=s+1}^n (C)_{p_\gamma, p_s} \right].$$

Based on Theorems 12 and 13, we have the following result.

Theorem 14. (i) *The nonempty set $X_1 \subseteq X$ is controllable, if and only if $C_{X_1} > 0$.*

(ii) *The nonempty set $X_2 \subseteq X$ is reachable, if and only if $R_{X_2} > 0$.*

Proof.

(i) *Necessity.* Suppose that the nonempty set $X_1 \subseteq X$ is controllable. By Definition 5, for any state $\delta_n^{p_\gamma} \in X_2$, there exist a $\delta_n^{p_\beta} \in X_1$ and a control sequence $u^t \in U^*$ such that $f(\delta_n^{p_\beta}, u^t) = \delta_n^{p_\gamma}$. Based on Theorems 12 and 13, for a fixed $\delta_n^{p_\gamma} \in X_2$, $\gamma = s+1, s+2, \dots, n$, at least one of the following

cases is true: $(C)_{p_\gamma, p_1} > 0, (C)_{p_\gamma, p_2} > 0, \dots, (C)_{p_\gamma, p_s} > 0$.

Therefore, for a fixed $\delta_n^{p_\gamma} \in X_2$, one can conclude that $\sum_{\beta=1}^s (C)_{p_\gamma, p_\beta} > 0$. From the arbitrariness of p_γ , one can see that

$$C_{X_1} = \left[\sum_{\beta=1}^s (C)_{p_{s+1}, p_\beta} \cdots \sum_{\beta=1}^s (C)_{p_n, p_\beta} \right] > 0. \quad (16)$$

Sufficiency. Suppose that $C_{X_1} = [\sum_{\beta=1}^s (C)_{p_{s+1}, p_\beta} \cdots \sum_{\beta=1}^s (C)_{p_n, p_\beta}] > 0$. Then, for any $\gamma = s+1, \dots, n$, we have $\sum_{\beta=1}^s (C)_{p_\gamma, p_\beta} > 0$. It means that, for any state $\delta_n^{p_\gamma} \in X_2$, there exist a $\delta_n^{p_\beta} \in X_1$ and a control sequence $u^t \in U^*$ such that $f(\delta_n^{p_\beta}, u^t) = \delta_n^{p_\gamma}$. By Definition 5, the nonempty set $X_1 \subseteq X$ is controllable.

(ii) *Necessity.* Suppose that the nonempty set $X_2 \subseteq X$ is reachable. By Definition 6, for any state $\delta_n^{p_\beta} \in X_1$, there exist a $\delta_n^{p_\gamma} \in X_2$ and a control sequence $u^t \in U^*$ such that $f(\delta_n^{p_\beta}, u^t) = \delta_n^{p_\gamma}$. Based on Theorems 12 and 13, for a fixed $\delta_n^{p_\beta} \in X_1$, $\beta = 1, 2, \dots, s$, at least one of the following cases is true: $(C)_{p_{s+1}, p_\beta} > 0, (C)_{p_{s+2}, p_\beta} > 0, \dots, (C)_{p_n, p_\beta} > 0$. Therefore, for a fixed $\delta_n^{p_\beta} \in X_1$, one can see that $\sum_{\gamma=s+1}^n (C)_{p_\gamma, p_\beta} > 0$. From the arbitrariness of p_β , we have

$$\left[\sum_{\gamma=s+1}^n (C)_{p_\gamma, p_1} \cdots \sum_{\gamma=s+1}^n (C)_{p_\gamma, p_s} \right] > 0. \quad (17)$$

Sufficiency. Suppose that $[\sum_{\gamma=s+1}^n (C)_{p_\gamma, p_1} \cdots \sum_{\gamma=s+1}^n (C)_{p_\gamma, p_s}] > 0$. Then, for any $\beta = 1, 2, \dots, s$, we have $\sum_{\gamma=s+1}^n (C)_{p_\gamma, p_\beta} > 0$.

It means that, for any state $\delta_n^{p_\beta} \in X_1$, there exist a $\delta_n^{p_\gamma} \in X_2$ and a control sequence $u^t \in U^*$ such that $f(\delta_n^{p_\beta}, u^t) = \delta_n^{p_\gamma}$. By Definition 6, the nonempty set $X_2 \subseteq X$ is reachable. \square

Finally, we study the stabilizability of deterministic finite automata.

For $X_1 = \{\delta_n^{p_1}, \dots, \delta_n^{p_s}\}$ and $X_2 = \{\delta_n^{p_{s+1}}, \dots, \delta_n^{p_n}\}$, define

$$M_{X_1} = \left[\sum_{i=1}^s (M)_{p_i, p_1} \cdots \sum_{i=1}^s (M)_{p_i, p_s} \right]. \quad (18)$$

Theorem 15. *The nonempty set $X_1 \subseteq X$ is 1-step returnable, if and only if $M_{X_1} > 0$.*

Proof. By Definition 7, one can see that the nonempty set $X_1 \subseteq X$ is 1-step returnable, if and only if, for any state $\delta_n^{p_\beta} \in X_1$, there exist an input $u_i \in U$ and some $\delta_n^{p_i} \in X_1$ such that $f(\delta_n^{p_\beta}, u_i) = \delta_n^{p_i}$, that is, for a fixed $\delta_n^{p_\beta} \in X_1$, $\beta = 1, 2, \dots, s$, at least one of the following cases is true: $(M)_{p_1, p_\beta} > 0, (M)_{p_2, p_\beta} > 0, \dots, (M)_{p_s, p_\beta} > 0$. Hence, $\sum_{i=1}^s (M)_{p_i, p_\beta} > 0, \forall \beta = 1, \dots, s$. From the arbitrariness of p_β , one can obtain that $M_{X_1} > 0$. \square

Based on Theorems 14 and 15, we have the following result.

Corollary 16. *The nonempty set $X_1 \subseteq X$ is stabilizable, if and only if $M_{X_1} > 0$ and $R_{X_1} > 0$.*

Proof. By Definition 8, $X_1 \subseteq X$ is stabilizable, if and only if X_1 is reachable and 1-step returnable. Based on Theorems 14 and 15, the conclusion follows. \square

Remark 17. Compared with the existing results on the controllability and stabilizability of deterministic finite automata [5, 26], the main advantage of our results is to propose a unified tool, that is, controllability matrix, for the study of deterministic finite automata. The new conditions are more easily verified via MATLAB.

4. An Illustrative Example

Consider the finite automata $A = (X, U, Y, f, g, x^0, X^m)$ given in Figure 2, where $X = \{x_1, x_2, x_3, x_4\}$ and $U = \{1, 2\}$.

From Figure 2, we can see that $x_1 \xrightarrow{1} x_2 \xrightarrow{1} x_3 \xrightarrow{2} x_4$ and $x_1 \xrightarrow{1} x_2 \xrightarrow{2} x_1$. Therefore, by Definition 3, one can obtain that x_1 is controllable. Similarly, by Definition 3, we conclude that x_2, x_3 , and x_4 are also controllable. By Definition 4, we can also find that all the states are reachable.

Assume that $X_1 = \{x_1, x_2\}$ and $X_2 = \{x_3, x_4\}$. Since $x_1 \in X_1 \xrightarrow{1} x_2 \xrightarrow{1} x_3 \in X_2$ and $x_1 \in X_1 \xrightarrow{2} x_4 \in X_2$, by Definition 5, one can see that X_1 is controllable. Similarly, we also obtain that X_2 is controllable. Since $x_3 \xrightarrow{2} x_4 \xrightarrow{1} x_2 \xrightarrow{2} x_1$ and $x_1 \xrightarrow{1} x_2 \xrightarrow{1} x_3 \xrightarrow{2} x_4$, by Definition 6, we can obtain that X_1 and X_2 are reachable. From Figure 2, we can see that $x_1 \xrightarrow{1} x_2, x_2 \xrightarrow{2} x_1, x_3 \xrightarrow{2} x_4$, and $x_4 \xrightarrow{2} x_3$. Hence, by Definition 7, X_1 and X_2 are 1-step returnable. By Definition 8, the sets X_1 and X_2 are stabilizable.

Now, we check the above properties based on the controllability matrix.

The transition structure matrix of the finite automata A is

$$F = \delta_4 [2 \ 3 \ 3 \ 2 \ 4 \ 1 \ 4 \ 3]. \quad (19)$$

Split $F = [F_1 \ F_2]$, where $F_1 = \delta_4 [2 \ 3 \ 3 \ 2]$ and $F_2 = \delta_4 [4 \ 1 \ 4 \ 3]$. Then,

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}. \quad (20)$$

Thus, the controllability matrix is

$$C = \sum_{t=1}^4 M^t = \begin{bmatrix} 3 & 4 & 2 & 3 \\ 7 & 5 & 5 & 6 \\ 12 & 13 & 14 & 14 \\ 8 & 8 & 9 & 7 \end{bmatrix}. \quad (21)$$

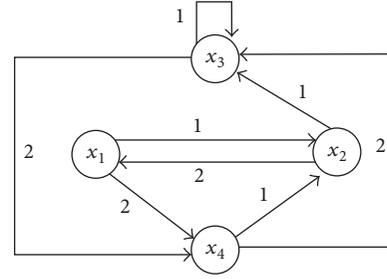


FIGURE 2: A finite automata.

Since all rows and columns of C are positive, by Theorems 12 and 13, any state $x_i \in X$ is controllable and reachable, $i = 1, 2, 3, 4$.

A simple calculation gives $C_{X_1} = [25 \ 16]$, $R_{X_1} = [7 \ 9]$, $C_{X_2} = [5 \ 11]$, $R_{X_2} = [20 \ 21]$, $M_{X_1} = [10 \ 9]$, and $M_{X_2} = [23 \ 21]$. By Theorems 14 and 15 and Corollary 16, X_1 and X_2 are controllable, reachable, 1-step returnable, and stabilizable, respectively.

5. Conclusion

In this paper, we have investigated the controllability, reachability, and stabilizability of deterministic finite automata by using the semitensor product of matrices. We have obtained the algebraic form of finite automata by expressing the states, inputs, and outputs as vector forms. Based on the algebraic form, we have defined the controllability matrix for deterministic finite automata. In addition, using the controllability matrix, we have presented several necessary and sufficient conditions for the controllability, reachability, and stabilizability of finite automata. The study of an illustrative example has shown that the obtained new results are effective.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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