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Research Article

An Identity Involving the Integral of the First-Kind Chebyshev Polynomials

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We used the algebraic manipulations and the properties of Chebyshev polynomials to obtain an interesting identity involving the power sums of the integral of the first-kind Chebyshev polynomials and solved an open problem proposed by Wenpeng Zhang and Tingting Wang.

1. Introduction

As we all know, the famous Chebyshev polynomials of the first and second kind $T_n(x)$ and $U_n(x)$ are defined by the second-order linear recursive formulae $T_0(x) = 1$, $T_1(x) = x$, and $T_{n+1} = 2xT_n(x) - T_{n-1}(x)$ for all $n \geq 1$. $U_0(x) = 1$, $U_1(x) = 2x$, and $U_{n+1} = 2xU_n(x) - U_{n-1}(x)$ for all $n \geq 1$.

The general term formulae of $T_n(x)$ and $U_n(x)$ are

$$T_n(x) = \frac{n}{2}\sum_{k=0}^{[n/2]}(-1)^k\frac{(n-k-1)!}{k!(n-2k)!}(2x)^{n-2k} \tag{1}$$

and

$$U_n(x) = \sum_{k=0}^{[n/2]}(-1)^k\frac{(n-k)!}{k!(n-2k)!}(2x)^{n-2k} \tag{2}$$

Many authors had studied the elementary properties of Chebyshev polynomials and obtained a series of interesting conclusions. For example, C. Cesarano [1], C.-L. Lee and K. B. Wong [2], and Wenpeng Zhang and Tingting Wang [3] proved a series of identities involving Chebyshev polynomials. A. H. Bhrawy et al. (see [4–7]) and N. Bircan and C. Pommerenke [8] obtained many important applications of the Chebyshev polynomials. Xiaoxue Li [9] obtained some identities involving power sums of $T_n(x)$ and $U_n(x)$. At the same time, she also proposed the following open problem.

Whether there exists an exact expression for the derivative or integral of the Chebyshev polynomials of the first kind in terms of the Chebyshev polynomials of the first kind (and vice versa)?

Tingting Wang and Han Zhang [10] partly proved this problem. That is, they proved the identities

$$\begin{aligned}\sum_{m=1}^{h}\left(T'_m(x)\right)^{2n} &= \frac{(-1)^n}{4^n(x^2-1)^n}\cdot\frac{(2n)!}{(n!)^2}\cdot\left(\sum_{m=1}^{h}m^{2n}\right)\\ &+\sum_{i=1}^{n+1}\sum_{k=1}^{n}\binom{2n}{n-k}(-1)^{n-k}\\ &\cdot\frac{r(i,h)T_{2ki}(x)+s(i,h)T_{2k(i-1)}(x)}{4^{n+i}(x^2-1)^{n+i}U_{k-1}^{2i}(x)}\\ &+\sum_{i=1}^{n+1}\sum_{k=1}^{n}\binom{2n}{n-k}(-1)^{n-k}\\ &\cdot\frac{p(i,h)T_{2k(h+i)}(x)+q(i,h)T_{2k(h+i-1)}(x)}{4^{n+i}(x^2-1)^{n+i}U_{k-1}^{2i}(x)}\end{aligned} \tag{3}$$

where $r(i,h)$, $s(i,h)$, $p(i,h)$, and $q(i,h)$ are computable constants.

They also gave the exact expressions for all constants $r(i,h)$, $s(i,h)$, $p(i,h)$, and $q(i,h)$ with $1 \leq i \leq n+1$, if n is a small positive integer. If n is large enough, then they only gave an exact computational method for these constants, but the computation is more complex.

For the power sums $\sum_{m=1}^{h}(\int_0^x T_m(y)dy)^n$, they have not given any results in [10]. Wenpeng Zhang and Tingting Wang [3] proved that

$$\sum_{n=1}^{h}\int_0^x T_{2n}(y)\,dy = \frac{T_{2h+1}(x)}{2(2h+1)} - \frac{x}{2},$$
$$\sum_{n=1}^{h}\int_0^x T_n^2(y)\,dy = \frac{1}{4}\left[\frac{T_{2h+1}(x)}{2h+1} + (2h-1)x\right] \tag{4}$$

and

$$\sum_{m=1}^{\infty}\int_0^1 T_{2m}^{2n+1}(y)\,dy = -\frac{1}{2} + \frac{\pi}{4^{n+1}}\sum_{k=0}^{n}\frac{\binom{2n+1}{n-k}}{2k+1}\cdot\cot\left(\frac{\pi}{4k+2}\right) \tag{5}$$

Wenpeng Zhang and Tingting Wang [3] also proposed the following two open problems. Whether there exists a exact calculation formula for

$$\sum_{k=1}^{h}\left(\int_0^x T_k(y)\,dy\right)^n \text{ or } \sum_{m=0}^{\infty}\int_0^1 T_{2m+1}^{2n+1}(y)\,dy. \tag{6}$$

In this paper, as a note of [3, 9], we shall use the analytic and elementary method to give an interesting computational formula for the second sums of (6). That is, we shall prove the following conclusion.

Theorem 1. *For any positive integer n, one has the identity*

$$\sum_{m=0}^{\infty}\int_0^1 T_{2m+1}^{2n+1}(y)\,dy = \frac{\pi}{4^{n+1}}\sum_{k=0}^{n}(-1)^k\cdot\frac{\binom{2n+1}{n-k}}{2k+1}\cdot\cot\left(\frac{\pi}{4}+\frac{\pi}{4}\cdot\frac{(-1)^k}{2k+1}\right). \tag{7}$$

Taking $n = 1$ *and 2, noting that* $\cot(\pi/2) = 0$, $\cot(\pi/6) = \sqrt{3}$, *and* $\cot(3\pi/10) = \tan(\pi/5)$, *from our theorem we may immediately deduce the following two identities:*

$$\sum_{m=0}^{\infty}\int_0^1 T_{2m+1}^{3}(y)\,dy = -\frac{\pi}{48}\cdot\sqrt{3} \tag{8}$$

and

$$\sum_{m=0}^{\infty}\int_0^1 T_{2m+1}^{5}(y)\,dy = \frac{\pi}{320}\cdot\tan\left(\frac{\pi}{5}\right) - \frac{5\pi}{192}\cdot\sqrt{3}. \tag{9}$$

2. Several Simple Lemmas

In this section, we shall give several simple lemmas, which are necessary in the proof of our theorem. Hereinafter, we shall use a few basic results, including the properties of $\sin x$, for which we refer the reader to the introductory books by Chengdong Pan and Chengbiao Pan [11]. First we have the following.

Lemma 2. *For any integer* $k \geq 0$, *one has the identity*

$$F(k) \equiv \sum_{m=0}^{\infty}\left(\frac{1+(-1)^m}{(2m+1)(4k+1)+1} - \frac{1-(-1)^m}{(2m+1)(4k+1)-1}\right) = \frac{\pi}{2}\cdot\frac{1}{4k+1}\cdot\cot\left(\frac{\pi(2k+1)}{2(4k+1)}\right). \tag{10}$$

Proof. For any real number x, from the infinite product of $\sin(\pi x)$, we have

$$\sin(\pi x) = \pi x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right) \tag{11}$$

Taking the logarithm for (11) and then differentiating it for x we have

$$\pi\cot(\pi x) = \frac{1}{x} - \sum_{n=1}^{\infty}\frac{2x}{n^2-x^2} = \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{n+x}-\frac{1}{n-x}\right). \tag{12}$$

Let N be a positive integer. Then from the properties of infinite series we have

$$F(k) = \lim_{N\to+\infty}\sum_{0\leq m\leq 2N}\left(\frac{1+(-1)^m}{(2m+1)(4k+1)+1} - \frac{1-(-1)^m}{(2m+1)(4k+1)-1}\right). \tag{13}$$

On the other hand, we have

$$\sum_{0\leq m\leq 2N}\left(\frac{1+(-1)^m}{(2m+1)(4k+1)+1} - \frac{1-(-1)^m}{(2m+1)(4k+1)-1}\right) = \frac{1}{2k+1} + \sum_{m\leq N}\frac{2}{(4m+1)(4k+1)+1} - \sum_{m\leq N}\frac{2}{(4m-1)(4k+1)-1} = \frac{1}{2k+1} + \frac{2}{4k+1}\left(\sum_{m\leq N}\frac{1}{4m+1+1/(4k+1)} - \sum_{m\leq N}\frac{1}{4m-1-1/(4k+1)}\right) = \frac{1}{2k+1}$$

$$+\frac{1}{2(4k+1)}\left(\sum_{m\le N}\frac{1}{m+(2k+1)/2(4k+1)}-\sum_{m\le N}\frac{1}{m-(2k+1)/2(4k+1)}\right)=\frac{1}{2k+1}+\frac{1}{2(4k+1)}\sum_{m\le N}\left(\frac{1}{m+(2k+1)/2(4k+1)}-\frac{1}{m-(2k+1)/2(4k+1)}\right). \tag{14}$$

Taking $N\to+\infty$ in (14) and then combining (12) and (13) we may immediately deduce the identity

$$F(k)=\sum_{m=0}^{\infty}\left(\frac{1+(-1)^m}{(2m+1)(4k+1)+1}-\frac{1-(-1)^m}{(2m+1)(4k+1)-1}\right)=\frac{1}{2k+1}+\frac{1}{2(4k+1)}\cdot\sum_{m=1}^{\infty}\left(\frac{1}{m+(2k+1)/2(4k+1)}-\frac{1}{m-(2k+1)/2(4k+1)}\right)=\frac{\pi}{2}\cdot\frac{1}{4k+1}\cdot\cot\left(\frac{\pi(2k+1)}{2(4k+1)}\right). \tag{15}$$

This proves Lemma 2. □

Lemma 3. *For any positive integer k, one has the identity*

$$\sum_{m=0}^{\infty}\left(\frac{1-(-1)^m}{(2m+1)(4k-1)+1}-\frac{1+(-1)^m}{(2m+1)(4k-1)-1}\right)=-\frac{\pi}{2}\cdot\frac{1}{4k-1}\cot\left(\frac{\pi}{2}\cdot\frac{2k-1}{4k-1}\right). \tag{16}$$

Proof. For any positive integer N, we have the identity

$$\sum_{0\le m\le 2N}\left(\frac{1-(-1)^m}{(2m+1)(4k-1)+1}-\frac{1+(-1)^m}{(2m+1)(4k-1)-1}\right)=\frac{-1}{2k-1}+\frac{2}{4k-1}\cdot\sum_{m\le N}\frac{1}{4m-1+1/(4k-1)}-\frac{2}{4k-1}\sum_{m\le N}\frac{1}{4m+1-1/(4k-1)}=\frac{-1}{2k-1}+\frac{1}{2(4k-1)}\sum_{0\le m\le N}\left(\frac{1}{4m-(2k-1)/2(4k-1)}-\frac{1}{4m+(2k-1)/2(4k-1)}\right). \tag{17}$$

Taking $N\to+\infty$ in (17), from the method of proving Lemma 2, we have

$$\sum_{m=0}^{\infty}\left(\frac{1-(-1)^m}{(2m+1)(4k-1)+1}-\frac{1+(-1)^m}{(2m+1)(4k-1)-1}\right)=\frac{-1}{2k-1}+\frac{1}{2(4k-1)}\cdot\sum_{m=1}^{\infty}\left(\frac{1}{4m-(2k-1)/2(4k-1)}-\frac{1}{4m+(2k-1)/2(4k-1)}\right)=-\frac{\pi}{2}\cdot\frac{1}{4k-1}\cdot\cot\left(\frac{\pi}{2}\cdot\frac{2k-1}{4k-1}\right). \tag{18}$$

This proves Lemma 3. □

Lemma 4. *For any positive integer n, one has the identity*

$$\int_0^x T_{2n+1}(y)\,dy=\frac{1}{2}\left[\frac{T_{2n+2}(x)}{2n+2}+\frac{(-1)^n(2n+1)}{2n(n+1)}-\frac{T_{2n}(x)}{2n}\right]. \tag{19}$$

Proof. See Lemma 1 in [3]. □

Lemma 5. *For any nonnegative integer n, one has the expressions of x^n in the following form:*

$$x^{2n+1}=\frac{(2n+1)!}{4^n}\sum_{k=0}^{n}\frac{1}{(n-k)!(n+k+1)!}T_{2k+1}(x) \tag{20}$$

Proof. In fact this is Lemma 4 of [12]. □

3. Proof of the Theorem

Now we shall complete the proof of our theorem. Taking x as $T_m(x)$ in Lemma 5 and noting that $T_{2k+1}(T_m(x))=T_{(2k+1)m}(x)$ (see Lemma 3 of [13]) we have

$$T_m^{2n+1}(x)=\frac{(2n+1)!}{4^n}\sum_{k=0}^{n}\frac{1}{(n-k)!(n+k+1)!}T_{(2k+1)m}(x) \tag{21}$$

Applying (1) with $x = \pm 1$ we have $T_n(\pm 1) = (\pm 1)^n$, and then from (21), Lemma 2, Lemma 3, and Lemma 4 we can deduce the identity

$$\sum_{m=0}^{\infty} \int_0^1 T_{2m+1}^{2n+1}(y)\,dy = \frac{(2n+1)!}{4^n} \cdot \sum_{k=0}^{n} \frac{1}{(n-k)!\,(n+k+1)!} \sum_{m=0}^{\infty} \int_0^1 T_{(2k+1)(2m+1)}(y)\,dy = \frac{(2n+1)!}{2\cdot 4^n} \cdot \sum_{k=0}^{n} \frac{1}{(n-k)!\,(n+k+1)!} \sum_{m=0}^{\infty} \left(\frac{T_{4km+2m+2k+2}(1)}{4km+2m+2k+2} - \frac{T_{4km+2m+2k}(1)}{4km+2m+2k} \right) + \frac{(2n+1)!}{2\cdot 4^n} \cdot \sum_{k=0}^{n} \frac{1}{(n-k)!\,(n+k+1)!} \sum_{m=0}^{\infty} \left(\frac{(-1)^{m+k}}{4km+2m+2k+2} + \frac{(-1)^{m+k}}{4km+2m+2k} \right) = \frac{2}{4^{n+1}} \sum_{k=0}^{n} \frac{(2n+1)!}{(n-k)!\,(n+k+1)!} \cdot \sum_{m=0}^{\infty} \left(\frac{1+(-1)^{k+m}}{4km+2m+2k+2} - \frac{1-(-1)^{m+k}}{4km+2m+2k} \right) = \frac{2}{4^{n+1}} \sum_{k=0}^{[n/2]} \frac{(2n+1)!}{(n-2k)!\,(n+2k+1)!} \cdot \sum_{m=0}^{\infty} \left(\frac{1+(-1)^m}{(4k+1)(2m+1)+1} - \frac{1-(-1)^m}{(4k+1)(2m+1)-1} \right) + \frac{2}{4^{n+1}} \cdot \sum_{k=0}^{[(n+1)/2]} \frac{(2n+1)!}{(n-2k+1)!\,(n+2k)!} \cdot \sum_{m=0}^{\infty} \left(\frac{1+(-1)^m}{(4k-1)(2m+1)+1} - \frac{1-(-1)^m}{(4k-1)(2m+1)-1} \right) = \frac{\pi}{4^{n+1}} \cdot \sum_{k=0}^{[n/2]} \frac{(2n+1)!}{(n-2k)!\,(n+2k+1)!\,(4k+1)} \cot\left(\frac{\pi}{2} \cdot \frac{2k+1}{4k+1} \right) - \frac{\pi}{4^{n+1}} \cdot \sum_{k=1}^{[(n+1)/2]} \frac{(2n+1)!}{(n-2k+1)!\,(n+2k)!\,(4k-1)} \cot\left(\frac{\pi}{2} \cdot \frac{2k-1}{4k-1} \right) = \frac{\pi}{4^{n+1}} \sum_{k=0}^{[n/2]} \frac{\binom{2n+1}{n-2k}}{4k+1} \cot\left(\frac{\pi}{2} \cdot \frac{2k+1}{4k+1} \right) - \frac{\pi}{4^{n+1}} \sum_{k=1}^{[(n+1)/2]} \frac{\binom{2n+1}{n+2k}}{4k-1} \cot\left(\frac{\pi}{2} \cdot \frac{2k-1}{4k-1} \right) = \frac{\pi}{4^{n+1}} \cdot \sum_{k=0}^{n} (-1)^k \frac{\binom{2n+1}{n-k}}{2k+1} \cot\left(\frac{\pi}{4} + \frac{\pi}{4} \cdot \frac{(-1)^k}{2k+1} \right). \tag{22}$$

This completes the proof of our theorem.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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