Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 7476954, 9 pages
https://doi.org/10.1155/2018/7476954

Research Article

Analysis for Irregular Thin Plate Bending Problems on Winkler Foundation by Regular Domain Collocation Method

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Received 14 March 2018; Accepted 20 May 2018; Published 11 June 2018

Academic Editor: Samuel N. Jator

Regular domain collocation method based on barycentric rational interpolation for solving irregular thin plate bending problems on Winkler foundation is presented in this article. Embedding the irregular plate into a regular domain, the barycentric rational interpolation is used to approximate the unknown function. The governing equation and the boundary conditions of thin plate bending problems on Winkler foundation in a rectangular region can be discretized by the differentiation matrices of barycentric rational interpolation. The additional method or the substitute method is used to impose the boundary conditions. The overconstraint equations can be solved by using the least square method. Numerical solutions of bending deflection for the irregular plate bending problems on Winkler foundation are obtained by interpolating the data on rectangular region. Numerical examples illuminate that the proposed method for irregular thin plate bending problems on Winkler foundation has the merits of simple formulations, efficiency, and relative error precision of 10^{-9} orders of magnitude.

1. Introduction

The computational modeling of the thin plate bending problems on Winkler foundation is the boundary value problems of partial differential equations. Linear elastic thin plates bending problems involving complex geometries, loads, and boundary conditions have been widely studied. There are some traditional methods, such as the finite difference method (FDM), the finite element method (FEM), and the boundary element method (BEM) [1]. It is necessary to point out that these traditional methods not only take a large amount of work to divide elements, but also the precision is not very high. In recent years, the mesh-less method also named as element free method (EFM) for solving partial differential equations has been widely concerned. EFM is well adopted to solve boundary value problems with complex boundary conditions, for which only need nodal data and without join nodes into elements [2, 3]. However, the computation of EFM is very large. The approximation function of EFM does not pass through variable values of the nodes, so it is more difficult to satisfy the essential boundary condition and discontinuity condition of the material.

Collocation method without element division and numerical integration is a truly mesh-less method. The formula is simple and the implementation of the program is convenient by using collocation method [4]. Pseudospectral method (PSM) [5–7] and differential quadrature method (DQM) [8] as high-precision collocation methods are commonly used. PSM is a mesh-less method based on spectral function interpolation. Enslaved to the defined interval of the spectral function, PSM is only applicable to some special intervals, such as [-1,1]. DQM uses the weighted sum of the unknown function value in the calculating nodes to approximate the derivative value of the unknown function, and the approximate weight is generally calculated by Lagrange interpolation [9, 10]. Lagrange interpolation has the disadvantages of numerical instability, so that the calculated results tend to be unstable with the increase of the calculating node numbers [11]. In two-dimensional problem, PSM and DQM apply the tensor product to obtain the approximation functions in a rectangular domain. Both methods cannot be directly used to obtain numerical calculation in geometrically complex domain. When PSM is applied to solve the boundary values on irregular regions, the coordinate transformation of the differential equation needs to be carried out. Although PSM can be used in the regular region, the governing equations become complicated, and it is not conducive to discrete the governing equations. When DQM is applied to solve the boundary values on an irregular domain, it is necessary to transform irregular domain into a regular domain.

We can obtain barycentric Lagrange interpolation by transforming the Lagrange interpolation into barycentric form. The barycentric Lagrange interpolation weight is only related to the node form and is very beneficial to the numerical calculation at any interval [11, 12]. Rewriting Lagrange interpolation formula to barycentric form can effectively prevent computer computing spillovers [13, 14]. Berrut introduces the barycentric rational interpolation and its applications in [15], and Güttel [16] confirms the convergence of barycentric rational interpolation. The barycentric weight is the difference between barycentric Lagrange interpolation and barycentric rational interpolation. Barycentric Lagrange interpolation function is unconditionally stable at the Chebyshev points; barycentric rational function interpolation can also effectively overcome the instable situation [11, 17]. Both of barycentric Lagrange interpolation collocation method and barycentric rational interpolation collocation method with the Chebyshev nodes possess excellent accuracy and maintain the numerical stability. Especially, the barycentric rational interpolation method is also stable at the equidistant nodes [17]. Embedding the irregular plate into a regular domain, the boundary value problem of partial differential equations on complex regions is effectively solved by regular domain method [18].

This article is organized as follows: in Section 2, barycentric rational interpolation functions in 1D and 2D are introduced. In Section 3, computational modeling of irregular thin plate bending problems on Winkler foundation by using regular domain collocation method based on barycentric rational interpolation is given in detail. In Section 4, numerical examples are shown to illustrate the advantages of the proposed method. In Section 5, we draw a conclusion.

2. Barycentric Rational Interpolation Functions in 1 D and 2D

2.1. Barycentric Rational Interpolation Function in 1D. Given that a function $u(x)$ is defined on the interval $a = x_1 < x_2 < \cdots < x_n = b$ and the function values on the nodes are $u_j = u(x_j)$, $j = 1, 2, \cdots, n$, the barycentric rational interpolation of the function $u(x)$ is as follows [11]:

$$u(x) = \frac{\sum_{j=1}^{n} \left(w_j / \left(x - x_j\right)\right) u_j}{\sum_{j=1}^{n} \left(w_j / \left(x - x_j\right)\right)}, \quad (1)$$

and the barycentric rational interpolation weight is as follows:

$$w_j = \sum_{i \in J_j} (-1)^i \prod_{k=i,k\neq j}^{i+d} \frac{1}{\left(x_j - x_k\right)}, \quad j = 1, 2, \cdots, n, \quad (2)$$

where $J_j = \{i \in I : j - d \le i \le j\}$, $I = \{1, 2, \ldots, n\}$ are index sets, d is the rational interpolation parameter, and $0 \le d \le n$。

The barycentric rational interpolation of function $u(x)$ can be simplified as

$$u(x) = \sum_{j=1}^{n} L_j(x) u_j, \quad (3)$$

where $L_k(x)$ is a basis function of barycentric rational and $L_k(x) = (w_k/(x - x_k)) / \sum_{j=1}^{n}(w_j/(x - x_j)), k = 1, 2, \cdots, n.$

The mth-order derivative of function $u(x)$ can be written as

$$u^{(m)}(x) = \frac{d^m u(x)}{dx^m} = \sum_{j=1}^{n} L_j^{(m)}(x) u_j. \quad (4)$$

So the mth-order derivative of function $u(x)$ on the nodes $x_1 < x_2 < \cdots < x_n$ can be written as

$$u^{(m)}(x_i) = u_i^{(m)} = \frac{d^m u(x_i)}{dx^m} = \sum_{j=1}^{n} D_{ij}^{(m)} u_j, \quad (5)$$

and formula (5) can be written in the following matrix form [11]:

$$\mathbf{u}^{(m)} = \mathbf{D}^{(m)} \mathbf{u}. \quad (6)$$

In formula (6), $\mathbf{u}^{(m)} = [u_1^{(m)}, u_2^{(m)}, \cdots, u_n^{(m)}]^T$ and $\mathbf{u} = [u_1, u_2, \cdots, u_n]^T$ represent the column vector of mth-order derivative value and the value of the function $u(x)$ on the nodes, respectively. Matrix $\mathbf{D}^{(m)}$ indicates the unknown function mth-order barycentric rational interpolation differential matrix on nodes $x_1, x_2, \cdots, x_n$, which is composed of the elements $D_{ij}^{(m)} = L_j^{(m)}(x_i)$.

2.2. Barycentric Rational Interpolation Functions in 2D. Given a function $u(x, y)$ on a regular domain $\Omega_0 = [a, b] \times [c, d]$, and the function values on the nodes are $u_{ij} = u(x_i, y_j)$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$. The barycentric rational interpolation of the function $u(x, y)$ is

$$u(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} L_i(x) M_j(y) u_{ij}, \quad (7)$$

where $L_i(x)$, $M_j(y)$ are the basis functions of barycentric rational interpolation in 1D.

The $(l + k)$ th partial derivative of formula (7) can be written as

$$\frac{\partial^{l+k} u}{\partial x^l \partial y^k} = \sum_{i=1}^{m} \sum_{j=1}^{n} L_i^{(l)}(x) M_j^{(k)}(y) u_{ij}, \quad (8)$$

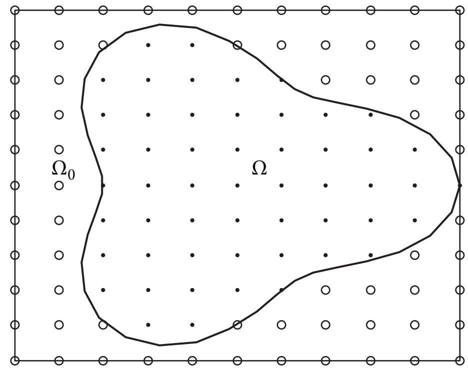


Figure 1: The irregular plate is embedded into a rectangular domain. ("" represents the node inside; "" represents the node outside and inside.)

The $(l+k)$th partial derivative values of function $u(x, y)$ on nodes (x_i, y_j), $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ are also can be written as

$$u_{pq}^{(l,k)} := \frac{\partial^{l+k} u(x_s, y_t)}{\partial x^l \partial y^k} = \sum_{i=1}^{m} \sum_{j=1}^{n} L_i^{(l)}(x_s) M_j^{(k)}(y_t) u_{ij}, \quad s = 1, 2, \ldots, m, \ t = 1, 2, \ldots, n. \tag{9}$$

Formula (9) can be written as formula (10) by using matrix tensor product, "⊗", and the symbols of barycentric rational interpolation differential matrix $\mathbf{V}^{(l,k)} = \mathbf{E}^{(l)} \otimes \mathbf{F}^{(k)}$:

$$\mathbf{U}^{(l,k)} := \mathbf{V}^{(l,k)} \mathbf{U} = \left(\mathbf{E}^{(l)} \otimes \mathbf{F}^{(k)}\right) \mathbf{U}, \tag{10}$$

where $\mathbf{U} = [u_{11}, u_{12}, \cdots, u_{1n}, u_{21}, u_{22}, \cdots, u_{mn}, u_{m1}, u_{m2}, \cdots, u_{mn}]^T$ and $\mathbf{E}^{(l)}, \mathbf{F}^{(k)}$ represent lth and kth order barycentric rational differential matrix on nodes $x_1, x_2, \cdots, x_m$ and $y_1, y_2, \cdots, y_n$, respectively [11]. Denote $\mathbf{E}^{(0)} = \mathbf{I}_m$, $\mathbf{F}^{(0)} = \mathbf{I}_n$, $\mathbf{I}_m, \mathbf{I}_n$ are mth and nth order identity matrix, respectively. Matrices $\mathbf{E}^{(l)}, \mathbf{F}^{(k)}$ are composed of the elements $E_{ij}^{(l)} = L_i^{(l)}(x_j)$, $i, j = 1, 2, \ldots, m$ and $F_{ij}^{(k)} = M_i^{(k)}(y_j)$, $i, j = 1, 2, \ldots, n$, respectively. $\mathbf{U}^{(l,k)}$ is $m \times n$-dimensional column vectors which is written as $\mathbf{U}^{(l,k)} = [u_{11}^{(l,k)}, u_{12}^{(l,k)}, \cdots, u_{1n}^{(l,k)}, u_{21}^{(l,k)}, u_{22}^{(l,k)}, \cdots, u_{mn}^{(l,k)}, u_{m1}^{(l,k)}, u_{m2}^{(l,k)}, \cdots, u_{mn}^{(l,k)}]^T$.

3. Computational Modeling and Formulations by Using Regular Domain Collocation Method

3.1. Computational Modeling. According to the basic assumption of the Kirchhoff theory [19], in this paper, for homogeneous, isotropic, elastic plate, the standard governing equation form as modified Helmholtz equation can be obtained as

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + \lambda^4 u(x, y) = \frac{q(x, y)}{D} \quad \text{in } \Omega, \tag{11}$$

where $u = u(x, y)$ is the unknown deflection, $q(x, y)$ is the density of lateral force, $D = Et^3/12(1 - \upsilon^2)$, with E Young's modulus, t is the thickness of thin plates, and ν is Possion's ratio of elasticity. $\lambda^4 = k/D$, where k is the foundation stiffness.

The boundary conditions involve clamped edges, simply supported edges, and free edges, which can be denoted as Γ_1, Γ_2, and Γ_3, respectively, and then the boundary of irregular domain Ω is $\Gamma = \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, the boundary conditions are given as follows [1, 20]:

$$\begin{aligned}
&\Gamma_1 : u = 0, \\
&\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right) u = 0, \\
&\Gamma_2 : u = 0, \\
&\left[\left(l^2 + \nu m^2\right)\frac{\partial^2}{\partial x^2} + \left(\nu l^2 + m^2\right)\frac{\partial^2}{\partial y^2} + (2 - \nu)\, lm \frac{\partial^2}{\partial x \partial y}\right] u = 0, \\
&\Gamma_3 : \left[\left(l^2 + \nu m^2\right)\frac{\partial^2}{\partial x^2} + \left(\nu l^2 + m^2\right)\frac{\partial^2}{\partial y^2} + (2 - \nu)\, lm \frac{\partial^2}{\partial x \partial y}\right] u = 0, \\
&\left\{\begin{matrix} -\left[(1-\nu) m^2 + 1\right] l \dfrac{\partial^3}{\partial x^3} + (1-\nu)\left(2l^2 - m^2 - 1\right) m \dfrac{\partial^3}{\partial x^2 \partial y} \\ + (1-\nu)\left(2m^2 - l^2 - 1\right) l \dfrac{\partial^3}{\partial x \partial y^2} - \left[(1-\nu) l^2 + 1\right] m \dfrac{\partial^3}{\partial y^3} \end{matrix}\right\} u \\
&= 0,
\end{aligned} \tag{12}$$

where $l = \cos(n, x)$, $m = \sin(n, x)$, and n is the outward normal direction.

3.2. Regular Domain Collocation Method for Irregular Thin Plates Bending Problems on Winkler Foundation. As shown in Figure 1, the irregular plate Ω is embedded into a rectangular domain $\Omega_0 = [a, b] \times [c, d]$, and $\Omega \subset \Omega_0$, the boundary of the rectangular domain Ω_0 is denoted as Γ_0,and the boundary of the irregular plate Ω is denoted as Γ. In order to reduce the number of nodes outside the irregular domain, the rectangular domain should be as small as possible.

Now, Ω_0 is called the computational domain, and Ω is called the physical domain. $x_1, x_2, \cdots, x_m$ and $y_1, y_2, \cdots, y_n$ are nodes in x and y coordinate directions, respectively, and can be used to form tensor product grid points, (x_i, y_j), $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

The discrete formulation of (11) on computational domain Ω_0 can be obtained as (13) by using formula (10):

$$\left[\mathbf{V}^{(4,0)} + 2\mathbf{V}^{(2,2)} + \mathbf{V}^{(0,4)} + \lambda^4 \mathbf{I}\right] \mathbf{U} = \frac{\mathbf{q}}{D}. \tag{13}$$

N points (x_k^b, y_k^b), $k = 1, 2, \ldots, N$ are selected on the boundary Γ of Ω. In general, N should be greater than the number of nodes on Ω_0. The discrete formulation of Γ can be obtained by barycentric rational interpolation:

$$\begin{aligned}
\Gamma_1 : & \sum_{i=1}^{m}\sum_{j=1}^{n} L_i\left(x_k^b\right) M_j\left(y_k^b\right) u_{ij} = 0, \\
& \left[\sum_{i=1}^{m}\sum_{j=1}^{n} L_i'\left(x_k^b\right) M_j\left(y_k^b\right) u_{ij}\right] n_x^k \\
& \quad + \left[\sum_{i=1}^{m}\sum_{j=1}^{n} L_i\left(x_k^b\right) M_j'\left(y_k^b\right) u_{ij}\right] n_y^k = 0, \\
& \qquad k = 1, 2, \ldots, N, \\
\Gamma_2 : & \sum_{i=1}^{m}\sum_{j=1}^{n} L_i\left(x_k^b\right) M_j\left(y_k^b\right) u_{ij} = 0, \\
& \left[\sum_{i=1}^{m}\sum_{j=1}^{n} L_i''\left(x_k^b\right) M_j\left(y_k^b\right) u_{ij}\right] \left(\left(n_x^k\right)^2 + \nu\left(n_y^k\right)^2\right) \\
& \quad + \left[\sum_{i=1}^{m}\sum_{j=1}^{n} L_i'\left(x_k^b\right) M_j'\left(y_k^b\right) u_{ij}\right] \left(\left(n_y^k\right)^2 + \nu\left(n_x^k\right)^2\right) \\
& \quad + (2-\nu)\, n_x^k n_y^k \left[\sum_{i=1}^{m}\sum_{j=1}^{n} L_i'\left(x_k^b\right) M_j'\left(y_k^b\right) u_{ij}\right] = 0, \\
& \qquad k = 1, 2, \ldots, N, \\
\Gamma_3 : & \left[\sum_{i=1}^{m}\sum_{j=1}^{n} L_i''\left(x_k^b\right) M_j\left(y_k^b\right) u_{ij}\right] \left(\left(n_x^k\right)^2 + \nu\left(n_y^k\right)^2\right) \\
& \quad + \left[\sum_{i=1}^{m}\sum_{j=1}^{n} L_i'\left(x_k^b\right) M_j'\left(y_k^b\right) u_{ij}\right] \left(\left(n_y^k\right)^2 + \nu\left(n_x^k\right)^2\right) \\
& \quad + (2-\nu)\, n_x^k n_y^k \left[\sum_{i=1}^{m}\sum_{j=1}^{n} L_i'\left(x_k^b\right) M_j'\left(y_k^b\right) u_{ij}\right] = 0, \\
& \qquad k = 1, 2, \ldots, N, \\
& - \left[(1-\nu)\left(n_y^k\right)^2 + 1\right] \\
& \cdot n_x^k \left[\sum_{i=1}^{m}\sum_{j=1}^{n} L_i'''\left(x_k^b\right) M_j\left(y_k^b\right) u_{ij}\right] + (1-\nu) \\
& \cdot \left(2\left(n_x^k\right)^2 - \left(n_y^k\right)^2 - 1\right) \\
& \cdot n_y^k \left[\sum_{i=1}^{m}\sum_{j=1}^{n} L_i''\left(x_k^b\right) M_j'\left(y_k^b\right) u_{ij}\right] + (1-\nu) \\
& \cdot \left(2\left(n_y^k\right)^2 - \left(n_x^k\right)^2 - 1\right)\left(n_x^k\right) \\
& \cdot \left[\sum_{i=1}^{m}\sum_{j=1}^{n} L_i'\left(x_k^b\right) M_j''\left(y_k^b\right) u_{ij}\right] \\
& - \left[(1-\nu)\left(n_x^k\right)^2 + 1\right] \\
& \cdot n_y^k \left[\sum_{i=1}^{m}\sum_{j=1}^{n} L_i\left(x_k^b\right) M_j'''\left(y_k^b\right) u_{ij}\right] = 0, \\
& \qquad k = 1, 2, \ldots, N,
\end{aligned} \tag{14}$$

where (n_y^k, n_y^k) is outward unit normal vector on point (x_k^b, y_k^b), $k = 1, 2, \ldots, N$. Formula (14) can be written as follows:

$$\begin{aligned}
\Gamma_1 : \mathbf{B}_0\mathbf{U} &= 0, \\
\mathbf{B}_1\mathbf{U} &= 0 \\
\Gamma_2 : \mathbf{B}_0\mathbf{U} &= 0, \\
\mathbf{B}_2\mathbf{U} &= 0 \\
\Gamma_3 : \mathbf{B}_2\mathbf{U} &= 0, \\
\mathbf{B}_3\mathbf{U} &= 0
\end{aligned} \tag{15}$$

Imposing formula (15) to (13), the numerical solution on computational domain can be obtained. When the boundary of physical domain coincides with the boundary of computational domain, apply the substitute method to impose the boundary conditions; otherwise apply the additional method to impose the boundary conditions. Using formula (7), numerical solutions of bending deflection for the irregular plate bending problems on Winkler foundation are obtained by interpolating the data on a rectangular region.

4. Numerical Examples and Discussion

To demonstrate the high accuracy and efficiency of the regular domain collocation method based on barycentric rational interpolation for solving irregular thin plate bending problems on Winkler foundation, two numerical examples are performed in this section. The method is validated by employing exact solutions with known boundary conditions and evaluating the computational errors. The computational programs of examples are compiled by MATLAB.

The exact solutions of numerical examples are obtained from [1, 20]. The absolute error and relative error of numerical computation are defined, respectively, as $E_a = \|\mathbf{u}^c - \mathbf{u}^e\|_\infty$,

TABLE 1: The absolute errors and relative errors of regular domain collocation method with different number of nodes in Example 1.

m	n	N	E_a	E_r
13	13	48	4.9336×10^{-10}	1.0459×10^{-5}
15	15	56	4.3383×10^{-12}	9.1976×10^{-8}
17	17	64	9.4875×10^{-13}	2.0115×10^{-8}
19	19	72	1.3409×10^{-13}	2.8429×10^{-9}
21	21	80	3.0355×10^{-13}	6.4993×10^{-9}
23	23	88	4.0824×10^{-13}	8.6551×10^{-9}

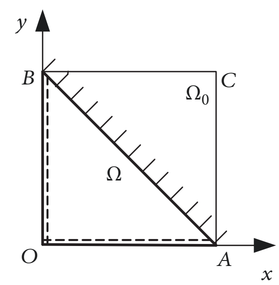


FIGURE 2: An irregular thin plate *OAB* is embedded into a rectangular *OACB*.

$E_r = \|\mathbf{u}^c - \mathbf{u}^e\|_2/\|\mathbf{u}^e\|_2$, where $\mathbf{u}^c, \mathbf{u}^e$ are vectors of numerical computational values and analytical solution, respectively. In the numerical analysis, the second kind Chebyshev points, $x_i = (b+a)/2 + ((b-a)/2)\cos(i\pi/m)$, $i = 0, 1, 2, \ldots, m$, $y_j = (d+c)/2 + ((d-c)/2)\cos(i\pi/n)\, j = 0, 1, 2, \ldots, n$, are used.

Example 1. The following bending problem of the irregular thin plate *OAB* with $\Omega = \{(x, y) : \ x > 0, y > 0, x + y < 1\}$ is embedded into the rectangular domain *OACB* with $\Omega_0 = [0, 1] \times [0, 1]$ in Figure 2. *AO* and *OB* are simple supported, *BA* is clamped, and $A(1, 0)m$, $B(0, 1)m$, $E = 7.0 \times 10^9 P_a$, $t = 0.1m$, $\nu = 0.3$, $D = Et^3/12(1-\nu^2)$, with the load $q(x, y) = q_0(1 + \lambda^4/4\pi^4)\sin\pi x \sin\pi y$, as q_0 is uniform load and $q_0 = 1.0 \times 10^3 N/m$, $\lambda^4 = 0.4\pi^4$. The exact solution is as follows: $u_e(x, y) = (q_0/4\pi^4 D)\sin\pi x \sin\pi y$.

The boundary conditions can be imposed according to (15). Choose $m \times n$ nodes on the computational domain and $N = 2(m+n-2)$ nodes on the edge of the physical boundary. The boundary conditions of *AO* and *OB* can be imposed by applying the substitute method, and the boundary condition of *BA* can be imposed by applying the additional method.

The absolute error and relative error of Example 1 by using regular collocation method based on barycentric rational interpolation with the different number of (m, n) Chebyshev nodes in x and y direction and N points on the boundary of Γ are listed in Table 1. It can be seen from Table 1 the following: ① adopting (19, 19) Chebyshev nodes in x and y direction and 88 points on the boundary of Γ, the absolute error and relative error reach 10^{-13} and 10^{-9} orders of magnitude, respectively. ② The calculation accuracy of the proposed method is very excellent. With the number of computational nodes increasing, the computational errors increase; when computational nodes increase up to a certain number of (19, 19) Chebyshev nodes in x and y direction, the accuracy is not improved because of the round-off error effect.

Three-dimensional image and contour map of numerical solutions by using regular collocation method based on barycentric rational interpolation with (19,19) Chebyshev nodes in x and y direction and 72 points on the boundary of Γ in Example 1 are shown in Figures 3(a) and 3(b), respectively. Three-dimensional image and contour map of the absolute errors by using regular collocation method based on barycentric rational interpolation with (19,19) Chebyshev nodes in x and y direction and 72 points on the boundary of Γ in Example 1 are shown in Figures 4(a) and 4(b), respectively.

From Figures 3 and 4, we can clearly see that the proposed method is able to obtain a high accuracy with using fewer points.

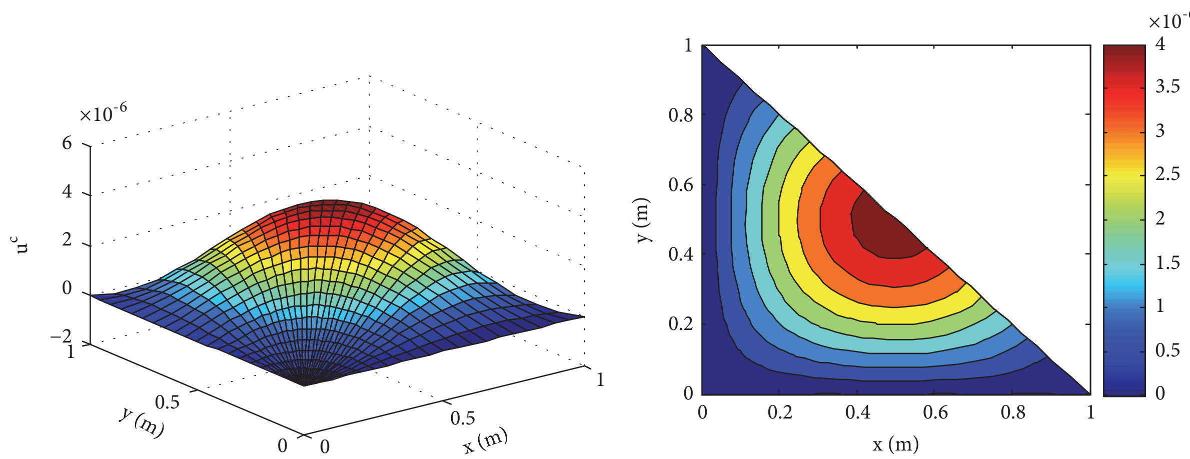
Example 2. Consider the following bending problem of irregular thin plate in [20]. The irregular plate *ABCD* is embedded into the rectangular domain *OBEC* with $\Omega_0 = [0, 1]\times[0, 1]$ in Figure 5. *AB* and *CD* are simple supported, *DA* is clamped and *BC* is free. *BC* is a circinal arc and its center is at the coordinate origin. $A(0.5, 0)m$, $B(1, 0)m$, $C(0, 1)m$, $D(0, 0.5)m$, $E = 7.0\times10^9 P_a$, $t = 0.1m$, $\nu = 0.3$, $D = Et^3/12(1-\nu^2)$, with the load $q(x, y) = q_0(1+\lambda^4/4\pi^4)\sin\pi x \sin\pi y$, as q_0 is uniform load and $q_0 = 1.0\times10^2 N/m$, $\lambda^4 = 0.4\pi^4$. The exact solution is as follows: $u_e(x, y) = (q_0/4\pi^4 D)\sin\pi x \sin\pi y$.

Choose $m \times n$ nodes on the computational domain and $N = 2(m+n-2)$ nodes on the edge of the physical boundary. The boundary conditions can be imposed according to formula (15). The boundary conditions of *AB and CD* are imposed by applying the substitute method; *AD* and *BC* can be imposed by applying the additional method.

The absolute error and relative error of Example 2 with the different number of (m, n) Chebyshev nodes in x and y direction and N points on the boundary of Γ are listed in Table 2.

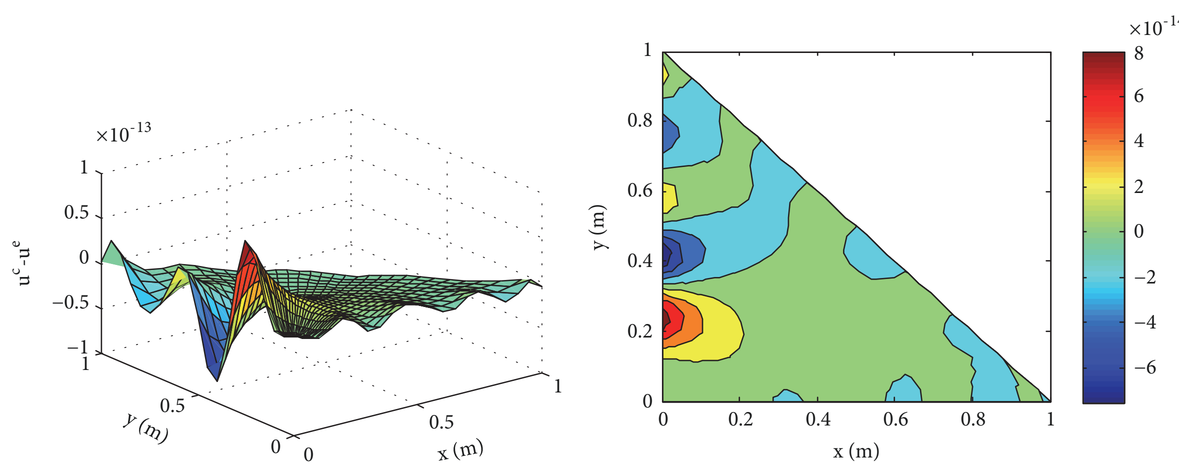
Table 2: The absolute errors and relative errors of regular domain collocation method with different number of nodes in Example 2.

m	n	N	E_a	E_r
14	14	52	7.5541×10^{-10}	9.9326×10^{-6}
15	15	56	3.6219×10^{-12}	5.9213×10^{-7}
16	16	60	1.3531×10^{-12}	2.2121×10^{-7}
17	17	64	1.5618×10^{-13}	2.5533×10^{-8}
18	18	68	1.0369×10^{-13}	1.6952×10^{-8}
19	19	72	1.3445×10^{-14}	2.1981×10^{-9}
20	20	76	8.0780×10^{-15}	1.3206×10^{-9}
21	21	80	29391×10^{-14}	4.8051×10^{-9}
22	22	84	3.0511×10^{-14}	4.9882×10^{-9}
23	23	88	2.1492×10^{-14}	3.51367×10^{-9}



(a) Three-dimensional image of the deflection (b) Contour map of the deflection

Figure 3: Numerical solutions of the deflection using irregular domain collocation method in Example 1.



(a) Three-dimensional image of the deflection (b) Contour map of the deflection

Figure 4: The absolute errors using regular domain collocation method in Example 1.

It can be seen from Table 2 the following: ① adopting (20, 20) Chebyshev nodes in x and y direction and 88 points on the boundary of Γ, the absolute error and relative error reach 10^{-15} and 10^{-9} orders of magnitude, respectively. ② The calculation accuracy of the proposed method is very excellent. With the number of computational nodes increasing, the computational errors increase; when computational nodes increase up to a certain number of (20, 20) Chebyshev nodes in x and y direction, the accuracy is not improved because of the round-off error effect.

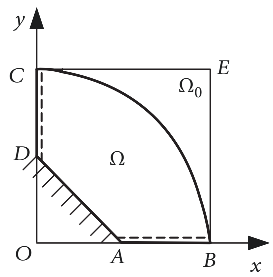
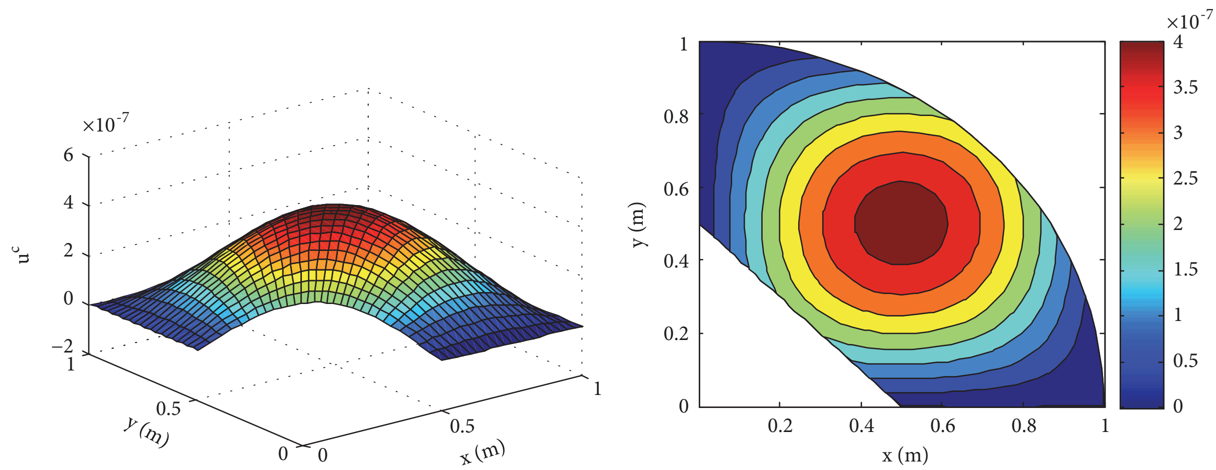


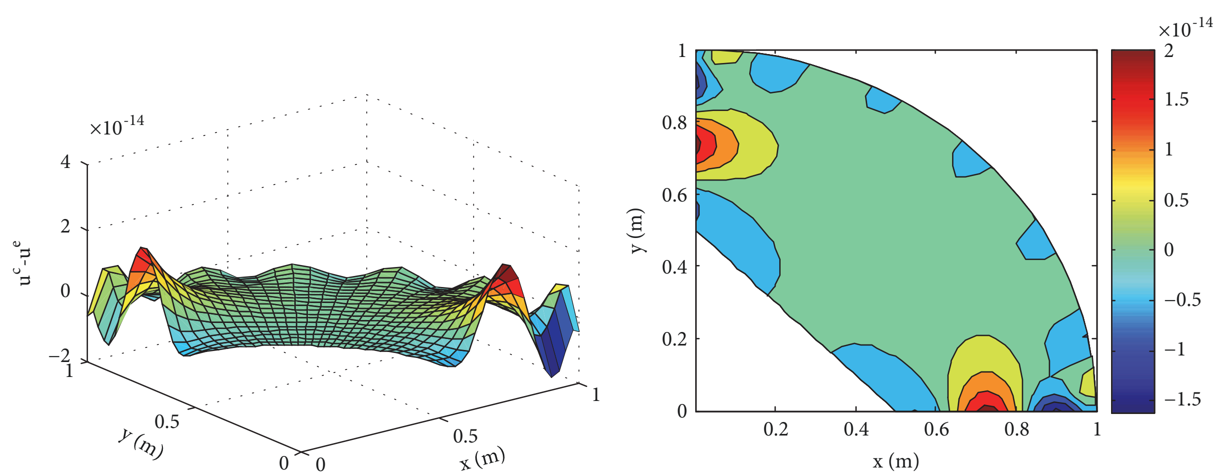
FIGURE 5: An irregular thin plate *ABCD* is embedded into a rectangular *OBCE* in Example 2.



(a) Three-dimensional image of the deflection

(b) Contour map of the deflection

FIGURE 6: The image of numerical solutions on irregular domain in Example 2.



(a) Three-dimensional image of the deflection

(b) Contour map of the deflection

FIGURE 7: The absolute error using regular domain collocation method in Example 2.

Three-dimensional image and contour map of numerical solutions with (19,19) Chebyshev nodes in x and y direction and 72 points on the boundary of Γ in Example 2 are shown in Figures 6(a) and 6(b), respectively. Three-dimensional image and contour map of the absolute errors by using regular collocation method based on barycentric rational interpolation with (19,19) Chebyshev nodes in x and y direction and 72 points on the boundary of Γ in Example 2 are shown in Figures 7(a) and 7(b), respectively.

From Figures 6 and 7, we can clearly see that the proposed method is able to obtain a high accuracy with using fewer points.

The relative error of u is 10^{-7} orders of magnitude by using differential quadrature Trefftz method in [20].

In another highly accurate numerical method, Fourier differential quadrature method for irregular thin plate bending problems on Winkler foundation in [1], the best absolute error of FDQM is in the range of 10^{-9}-10^{-10} and the best relative error of FDQM is 10^{-9}. Thus, the numerical precision of the proposed method in this paper is the same as that of the highly accurate FDQM and higher than differential quadrature Trefftz method. However, compared with differential quadrature Trefftz method and FDQM, the formulations of regular domain collocation method based on barycentric rational interpolation in this article are more simple and the amount of calculation by using regular domain collocation method is smaller.

Numerical Examples 1 and 2 demonstrate that the proposed method in this article can be effective for irregular domains and arbitrary complex boundary conditions.

5. Conclusions

A highly accurate regular domain collocation method based on barycentric rational interpolation is efficient for solving irregular thin plate bending problems with complex boundaries and arbitrary loads on Winkler foundation in this article. Using barycentric rational interpolation method, a stabilized, high-precision interpolation method, we can accurately and conveniently discretized boundary conditions on irregular boundary. The calculation of precision of relative error reaches 10^{-9} orders of magnitude, and the numerical stability of the method in this article is very excellent.

In two-dimensional problem, PSM and DQM apply the tensor product to obtain the approximation functions in a rectangular domain. Both methods cannot be directly used to obtain numerical calculation in geometrically complex domain. Compared with PSM and DQM, the proposed method in this article does not need to process coordinate transformation or transform an irregular domain into a regular domain. It can be directly applied to numerical computation in complex regions. The calculation formula is simple and the calculation precision is high. The calculation program compiled by MATLAB is effective, reliable, convenient, and also modularized and of general purpose, which can be used by the engineering designers.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research has been supported by the National Natural Science Foundation of China under Grant no. 51778135 and Aeronautical Science Foundation of China under Grant no. 20130969010. This research also has been supported by the Fundamental Research Funds for the Central Universities and Postgraduate Research & Practice Innovation Program of Jiangsu Province under Grant no. KYCX18_0113.

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