Based on the Bayesian principle, the modern uncertainty evaluation methods can fully integrate prior and current sample information, determine the prior distribution according to historical information, and deduce the posterior distribution by integrating prior distribution and the current sample data with the Bayesian model. As such, it is possible to evaluate uncertainty, updating in real time the uncertainty of the measuring instrument according to regular measurement, and timely reflect the latest information on the accuracy of the measurement system. Based on the Bayesian information fusion and statistical inference principle, the model of uncertainty evaluation is established. The maximum entropy principle and the hill-climbing search optimization algorithm are introduced to determine the prior distribution probability density function and the sample information likelihood function. The probability density function of posterior distribution is obtained by the Bayesian formula to achieve the optimization estimation of uncertainty. Three methods of measurement uncertainty evaluation based on Bayesian analysis are introduced: the noninformative prior, the conjugate prior, and the maximum entropy prior distribution. The advantages and limitations of each method are discussed.

1. Introduction

Measurement uncertainty is an important parameter in measurement results. To give scientific and proper evaluation of measurement uncertainty is an important factor to guarantee the development of modern measuring science. It is over two decades since “Guide to the Expression of Uncertainty in Measurement (GUM)” was first released in 1993. Initially, it mainly served in the field of physical science. Now, it is widely used in chemistry, biology, medicine, forensic medicine, astronomy, and other fields. Measurement uncertainty plays an increasingly prominent role in modern science [1–3].

Some of the existing studies on measurement uncertainty evaluation are based on historical experience, experts’ opinion, and prior data and fail to take the real time measurement data into account [4, 5]. Some of the studies are based on the information of measured samples and fail to take the historical information of the measurement system into account [6, 7]. These methods can not reflect the most up-to-date status of the measurement system and thus have negative impact on the reliability and soundness of uncertainty evaluation results.

With the advances in computer technology, new uncertainty evaluation methods have been developed and applied. Using Bayesian method to evaluate measurement uncertainty is an important direction for modern uncertainty evaluation methods to progress. Based on Bayesian information fusion, the uncertainty evaluation method can fully integrate the prior and the current sample information. The prior distribution is determined by the historical information, and the posterior distribution is deduced by integrating prior distribution and the current sample data with the Bayesian model, so as to achieve both the evaluation and updating of uncertainty [8–15].

Prior distribution is an important part of Bayesian statistical model. Properly determining the prior distribution
according to historical information is a key point for evaluating measurement uncertainty with Bayesian methods. Most researches on Bayesian prior distribution merely give a brief introduction to some prior distribution methods but fail to propose in-depth analysis of the advantages, limitations, and application scope of the methods [16]. In addition, few studies have applied the Bayesian principle and its prior distribution to the evaluation and updating of measurement uncertainty. Therefore, this paper makes a comparative analysis of Bayesian evaluation methods based on the noninformative prior, conjugate priors, and maximum entropy prior. As such, it is significant for the scientific and proper evaluation of measurement uncertainty.

2. Bayesian Principle

The Bayesian principle takes measurement parameter \( \theta \) as random variable and identifies its prior distribution according to the historical information of \( \theta \). After obtaining the measurement sample \( X = (x_1, x_2, x_3, \ldots, x_n) \), it integrates prior information and current sample information based on the Bayes formula \( \pi(\theta \mid x) \propto l(x \mid \theta)\pi(\theta) \) to obtain \( \theta \) posterior distribution, thus achieving statistical inference of \( \theta \) [17]. Here, \( \pi(\theta) \) is the prior density function of \( \theta \), \( \pi(\theta \mid x) \) is the posterior density function, and \( l(x \mid \theta) \) is the sample likelihood function.

Probability density function (PDF) of posterior distribution includes all information related to \( \theta \) in the population, sample, and prior information; it is more reliable to carry out a statistical inference of \( \theta \) based on the posterior distribution \( \pi(\theta \mid x) \).

The prior and posterior distributions are important constituents of the Bayesian statistical model. The key points for the Bayesian evaluation method are to identify prior distribution according to historical information and select proper methods to determine the posterior distribution. Based on different prior distributions, this paper studies proper methods to determine the posterior distribution as best estimation value \( \hat{\mu} \) of the Bayesian method for parameter estimation, if no or little prior information is available, it is possible to use the population information and the noninformative prior distribution. Noninformative prior distribution methods include the hypothesis-based Bayesian prior distribution method, position or the scale parameter prior distribution method, and the Jeffreys prior distribution method.

The prior distribution method combining subjective and nonsubjective information determines prior distribution by using prior information, population information, or sample information. It includes the maximum entropy prior distribution, the conjugate prior distribution, the maximum digital information prior distribution, and the multilevel prior distribution. In the maximum entropy prior distribution, the concept of entropy is introduced to measure the overall uncertainty in the prior density \( \pi(\theta) \). In the distribution satisfying the given conditions, the prior distribution that can maximize the entropy is obtained. In addition to measured data, the maximum entropy prior distribution method does not generate any subjective assumptions about the probability distribution of the parameters. It is thus highly scientific and objective.

The prior distribution and posterior distribution of the conjugate prior distribution share the same probability distribution group. When integrating sample information, the prior information has changed the value of the distribution parameter, with the advantage of the conjugate distribution. But in practical application, probability distribution with assumed prior information is required.

2.2. Bayesian Posterior Distribution. According to the Bayesian principle, the calculation formula for the Bayesian posterior distribution can be expressed as

\[
\pi(\theta \mid x) \propto l(x \mid \theta)\pi(\theta) \tag{1}
\]

An appropriate method to identify the prior density \( \pi(\theta) \) has to be selected; then, the sample likelihood function \( l(x \mid \theta) \) is obtained with reference to the sample information. Thus, the posterior distribution \( \pi(\theta \mid x) \) can be obtained by Bayesian formula.

Select proper sufficient statistic for parameter estimation, and supply all information of the parameter included in the sample. After this, posterior distribution can be obtained by sufficient statistic estimation. This method applies to estimating the sufficient statistic of \( \theta \) whose distribution is already known.

3. Bayesian Uncertainty Evaluation Methods

It can be known from GUM [18] that when the measurement result is the average of the \( n \) tests, standard uncertainty can be expressed as the standard deviation of the average. So, formula (1) can be used to create a model for average value \( \mu \), by using the expectation of the density function of its posterior distribution as best estimation value \( \hat{\mu} \) of
measurement result, and the standard deviation can be used to express the standard uncertainty \( u \).

\[
\hat{\mu} = E[\pi(\mu \mid x)] \\
\sigma = \sqrt{D[\pi(\mu \mid x)]}
\]  

(2)

3.1. Bayesian Uncertainty Evaluation Based on Noninformative Prior Distribution

3.1.1. Noninformative Prior Distribution. Suppose \( X = (X_1, X_2, X_3, \ldots, X_n) \) are simple samples drawn from the population \( f(x \mid \theta) \). When there is no available prior information for \( \theta \), the square root of the Fisher information matrix is used as the noninformative prior of \( \theta \). The log-likelihood function of the parameter \( \theta \) is

\[
I(\theta \mid x) = \ln \left[ \prod_{i=1}^{n} f(x_i \mid \theta) \right] = \sum_{i=1}^{n} \ln f(x_i \mid \theta)
\]

(3)

Its Fisher information matrix is

\[
I = \begin{bmatrix}
I_{ij}(\theta)
\end{bmatrix}
\]

(4)

\[
I_{ij}(\theta) = E_{X\theta} \left[ -\frac{\partial^2 I}{\partial \theta_i \partial \theta_j} \right], \quad (i, j = 1, 2, \ldots, p)
\]

So, noninformative prior density of \( \theta \) can be expressed as

\[
\pi(\theta) = \sqrt{\det I(\theta)}
\]

(5)

where \( \det I(\theta) \) denotes the determinant of \( p \times p \) square matrix \( I(\theta) \).

Suppose \( X = (X_1, X_2, X_3, \ldots, X_n) \) are simple samples drawn from normal population \( N(\mu, \sigma^2) \). So, when \( \mu \) and \( \sigma^2 \) are independent of each other,

\[
\pi(\mu, \sigma^2) = \pi(\mu) \pi(\sigma^2) = \frac{1}{\sigma^2}
\]

(6)

3.1.2. The Posterior Distribution and Its Uncertainty of Noninformative Prior. Suppose \( Y \sim N(\theta, \sigma^2) \), where \( \theta \) and \( \sigma^2 \) are unknown; the prior distribution of \( \theta \) and \( \sigma^2 \) is noninformative prior. \( Y = (Y_1, Y_2, Y_3, \ldots, Y_n) \) are simple samples drawn from the population. The joint noninformative prior distribution density function of \( (\theta, \sigma^2) \) is

\[
\pi(\mu, \sigma^2) = \pi_1(\theta) \pi_2(\sigma^2) = \frac{1}{\sigma^2}
\]

(7)

Suppose \( \bar{Y} = (1/n) \sum_{i=1}^{n} Y_i \) and \( S^2 = (1/n) \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \). Let \( T = (\bar{Y}, S^2) \) be the joint sufficient statistics of \( (\theta, \sigma^2) \) and it can be learned that \( Y \sim N(\theta, \sigma^2/n) \), \( \chi^2(\nu) \), and \( \nu = n-1 \). So the likelihood function of \( (\theta, \sigma^2) \) is

\[
l(\theta, \sigma^2 \mid T) = \sqrt{\frac{n}{2\pi \sigma^2}} \exp \left[ \frac{-n(\bar{Y} - \theta)^2}{2\sigma^2} \right] \frac{1}{\sqrt{2^\nu \Gamma(\nu/2)}}
\]

(8)

It can be obtained by the Bayesian formula that the joint posterior PDF of \( \theta \) and \( \sigma^2 \) is

\[
\pi(\theta, \sigma^2 \mid T) = \sqrt{\frac{n}{2\pi \sigma^2}} \cdot \frac{1}{\Gamma(\nu/2)} \cdot \left( \frac{\nu S^2 + n(\bar{Y} - \theta)^2}{2\sigma^2} \right)^{\nu/2-1} \exp \left[ -\frac{\nu S^2 + n(\bar{Y} - \theta)^2}{2\sigma^2} \right]
\]

(9)

Namely, the joint posterior distributions of \( \theta \) and \( \sigma^2 \) follow

\[
\pi_1(\theta \mid \sigma^2, Y) \sim N(\bar{Y}, \frac{\sigma^2}{n})
\]

(10)

\[
\pi_2(\sigma^2 \mid Y) \sim \Gamma^{-1}(\frac{\nu}{2}, \frac{\nu S^2}{2})
\]

3.2. Bayesian Uncertainty Evaluation Based on Conjugate Prior. When certain prior information and sample distribution are known, uncertainty evaluation can be performed by the conjugate Bayesian method. The conjugate prior distribution can make the form of the prior distribution and the posterior distribution identical and categorize them in the same probability distribution group. The prior information, after integrating sample information, just changes the distribution of parameter values, which is consistent with a subjective judgment; furthermore, the integration of prior information and the sample information can form a prior chain. Namely, posterior distribution obtained after every information integration can serve as the prior distribution for the next calculation. Performed repeatedly, this will form a chain which enables continuous integration and updating of measurement information.

3.2.1. Conjugate Prior Distribution. Suppose that prior data \( (X_{01}, X_{02}, X_{03}, \ldots, X_{0n}) \) follows normal distribution \( N(\mu, \sigma^2) \). It can be learned from mathematical statistics that

\[
\bar{X}_0 = \frac{1}{n} \sum_{i=1}^{n} X_{0i}, \quad \bar{X}_0 \sim N\left(\mu, \frac{\sigma^2}{n_0}\right)
\]

(12)

\[
S_0 = \sum_{i=1}^{n_0} (X_{0i} - \bar{X}_0)^2, \quad \frac{S_0}{\sigma^2} \sim \chi^2_{n_0}
\]

where \( \bar{X}_0 \) and \( S_0 \) are independent of each other.
Conjugate prior distributions of $\mu$ and $\sigma^2$ are

$$
\mu \sim N\left(\overline{X}_0, \frac{\sigma^2}{n_0}\right) \\
\sigma^2 \sim \Gamma^{-1}\left(\frac{n_0-1}{2}, \frac{S_0}{2}\right)
$$

So, the joint prior distribution density of $(\mu, \sigma^2)$ is

$$
\pi(\mu, \sigma^2) = \pi(\mu) \pi(\sigma^2) = \sqrt{\frac{n_0}{2\pi\sigma^2}} \cdot \exp \left[-\frac{S_0 + n_0 (\mu - \overline{X}_0)^2}{2\sigma^2}\right]
$$

3.2.2. The Posterior Distribution and Its Uncertainty of Conjugate Prior. Suppose $(X_{11}, X_{12}, X_{13}, \ldots, X_{1n_1})$ are samples drawn from the normal population $N(\theta, \sigma^2)$. Here, $\theta$ and $\sigma^2$ are unknown. The joint conjugate prior density function of $\theta$ and $\sigma^2$ can be provided with reference to formula (14).

Use $\overline{X}_1 = (1/n_1) \sum_{i=1}^{n_1} x_{1i}$ and $S_1 = \sum_{i=1}^{n_1} (x_{1i} - \overline{X}_1)^2$ as the sufficient statistic of $\theta$ and $\sigma^2$, and its sample likelihood function is

$$
l(\theta, \sigma^2 | X) = \left(\frac{1}{2\pi\sigma^2}\right)^{n_1/2} \exp \left[-\frac{S_1 + n_1 (\theta - \overline{X}_1)^2}{2\sigma^2}\right]
$$

It can be obtained by the Bayesian formula that the joint posterior PDF of $(\theta, \sigma^2)$ is

$$
\pi(\theta, \sigma^2 | X) \propto \left(\frac{\sigma^2}{m}\right)^{-1/2} \exp \left[-\frac{m (\theta - \overline{X})^2}{2\sigma^2}\right]
$$

where

$$
m = n_0 + n_1 \\
\overline{X} = \frac{n_0 \overline{X}_0 + n_1 \overline{X}_1}{n_0 + n_1} \\
S = \frac{n_0 n_1}{n_0 + n_1} \left(\overline{X}_0 - \overline{X}_1\right)^2 + S_0 + S_1
$$

Namely, the posterior distributions of $(\theta, \sigma^2)$ follow

$$
\pi(\sigma^2 | X) \sim \Gamma^{-1}\left(\frac{m-1}{2}, \frac{S}{2}\right) \\
\pi(\theta | \sigma^2, X) \sim N\left(\overline{X}, \frac{\sigma^2}{m}\right)
$$

It can be known from GUM [18] and formula (2) that when $\theta$ and $\sigma^2$ are unknown, the best evaluation value and the standard uncertainty of the posterior distribution based on conjugate prior distribution are

$$
\hat{\theta} = \frac{n_0 \overline{X}_0 + n_1 \overline{X}_1}{n_0 + n_1} \\
u = \sqrt{\frac{S}{m - 3}}
$$

3.2.3. Bayesian Uncertainty Component Updating Method Based on Conjugate Prior. In a practical uncertainty evaluation, some uncertainty components shall be determined by type A evaluation method based on the measurement data. If these uncertainty components are determined by merely one test, the included information will be limited and lack representativeness. The operational status of the apparatus and the work pieces are subject to change with time. The uncertainty identified in just one test can not reflect the most updated information in the evaluation. Thus, the Bayesian information integration is proposed in this paper. The information integration model based on the Bayesian formula is established to realize timely and continuous updating of uncertainty component and enable uncertainty evaluation results to reflect the most updated status of the measurement system and enhance the reliability of the uncertainty evaluation results.

In this paper, updating of measurement repeatability uncertainty is taken as an example to describe the updating of uncertainty component by the Bayesian method. The distribution of measurement repeatability generally follows the normal distribution. The prior distribution and the posterior distribution share the same form in the information integration. Thus, the conjugate Bayesian method can be used for constantly updating the measurement repeatability uncertainty component.

Suppose the measurement array of the repeatability test is $X = (x_1, x_2, x_3, \ldots, x_n)$, $X \sim N(\theta, \sigma^2)$. The repeatability uncertainty component can be obtained by the Bessel formula:

$$
u = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \overline{X})^2}{n(n-1)}}
$$

If the number of measurements for the first repeatability test is $n_0$ and the measurement result is $X = (x_{01}, x_{02}, \ldots, x_{0n_0})$, it can be known from the conjugate
Bayesian uncertainty evaluation method that the conjugate prior distribution of $\sigma^2$ is

$$\pi (\sigma^2) = \frac{\sqrt{S_0^{-1}}}{\sqrt{2^{n_0-1}} \Gamma((n_0 - 1)/2)} \left( \frac{1}{\sigma^2} \right)^{(n_0+1)/2} \exp \left( - \frac{S_0}{2\sigma^2} \right)$$  \hspace{1cm} (21)$$

Thus, the uncertainty component of the first repeatability test is

$$u_0 = \sqrt{\frac{S_0}{n_0 - 1}}$$  \hspace{1cm} (22)$$

Suppose the number of measurements for the second repeatability test is $n_1$, and the measurement result is $X = (x_{11}, x_{12}, x_{13}, \ldots, x_{1n_1})$. Use the newest repeatability data to update $\sigma^2$ whose likelihood function is expressed as

$$l(\sigma^2 | x) \propto \left( \frac{1}{\sigma^2} \right)^{n/2} \exp \left( - \frac{S_1}{2\sigma^2} \right)$$  \hspace{1cm} (23)$$

It can be obtained from the Bayesian formula that the posterior PDF and distribution of $\sigma^2$ are

$$\pi (\sigma^2 | x) \propto \pi (\sigma^2) l(\sigma^2 | x) \propto \left( \frac{1}{\sigma^2} \right)^{(n_0+n_1-1)/2-1} \exp \left( - \frac{S_0+S_1}{2\sigma^2} \right) (\sigma^2)^{-\frac{(n_0+n_1-1)}{2}}$$  \hspace{1cm} (24)$$

As a result, the repeatability uncertainty component after update is

$$u_1 = \sqrt{\frac{S_0+S_1}{n_0+n_1-3}}$$  \hspace{1cm} (26)$$

The general formula for updating of the repeatability uncertainty component can be expressed as

$$u_1 = \sqrt{\frac{(n_0-1) u_0^2 + \sum_{j=1}^{n_1} (x_{1j} - \bar{x}_1)^2}{n_0+n_1-3}}$$  \hspace{1cm} (27)$$

3.3. Bayesian Uncertainty Evaluation Based on the Maximum Entropy Principle. The probability distribution of random variables is difficult to be determined accurately, and usually only the mean value and variance of the measured results can be obtained. There are many kinds of distribution tallying with the measured values of random variables. Among the distribution, one has the maximum information entropy. Entropy can be used to measure the value of the information tested. Maximum entropy indicates the cases in which the information appears with the greatest probability. Namely, the probability distribution of random variables obtained at the maximum entropy best accords with objective law. This is an effective method and law to identify the probability distribution of random variables, namely, the maximum entropy principle. The maximum entropy principle can be used to identify the PDF of prior distribution and sample information. It can effectively reduce the risk of predicting the probability distribution of random variables and make uncertainty evaluation more objective and reasonable.

3.3.1. Maximum Entropy Prior Distribution. Suppose the random variables $x$. Its only PDF $f(x)$ can be obtained by the maximum entropy function $H(x)$:

$$H (x) = - \int_{-\infty}^{+\infty} f (x) \ln f (x) \, dx \rightarrow \max$$  \hspace{1cm} (28)$$

The constraint condition of $f(x)$ is

$$\int_{-\infty}^{+\infty} x^i f (x) \, dx = m_i \hspace{1cm} (29)$$

where $m_i$ indicates the origin moment of the $i$th rank, $i = 1, 2, 3, \ldots, N$.

The maximum entropy principle is used to solve the problem of PDF of the prior distribution and the sample information. Specifically, it can be converted to seeking the extreme value problem under constraint conditions. This key problem can be solved by introducing the optimization algorithm [19]. As one method for solving the extreme value problems, the mathematical meaning of the optimization algorithm is to find the extreme value of the target function under a group of constraining equalities and inequalities. The Lagrange multiplier method is introduced to carry out in-depth research of the above problem.

The Lagrange multiplier $\lambda_i, i = 1, 2, 3, \ldots, n$, is introduced to obtain

$$\bar{H} = H (x) + (\lambda_0 + 1) \left[ \int_{-\infty}^{+\infty} f (x) \, dx - 1 \right] + \sum_{i=1}^{n} \lambda_i \left[ \int_{-\infty}^{+\infty} x^i f (x) \, dx - m_i \right]$$  \hspace{1cm} (30)$$

It can be obtained from maximum entropy extreme condition $d\bar{H}/df(x) = 0$ that

$$f (x) = \exp \left( \lambda_0 + \sum_{i=1}^{n} \lambda_i x^i \right)$$  \hspace{1cm} (31)$$
By considering formula (29), it can be obtained that
\[
\lambda_0 = -\ln \int_{-\infty}^{+\infty} \exp \left( \sum_{i=1}^{n} \lambda_i x_i \right) dx,
\]
\[
m_i = \frac{\int_{-\infty}^{+\infty} x_i \exp \left( \sum_{i=1}^{n} \lambda_i x_i \right) dx}{\int_{-\infty}^{+\infty} \exp \left( \sum_{i=1}^{n} \lambda_i x_i \right) dx}
\]
(32)

Suppose the residual \( v_i \) is
\[
v_i = 1 - \frac{\int_{-\infty}^{+\infty} x_i \exp \left( \sum_{i=1}^{n} \lambda_i x_i \right) dx}{m_i \int_{-\infty}^{+\infty} \exp \left( \sum_{i=1}^{n} \lambda_i x_i \right) dx}
\]
(33)

When the sum of squared residuals is minimum, the optimal solution of \( \lambda_i \) will be obtained, thus obtaining the random variable PDF for the maximum entropy. Solution of the PDF of prior distribution and the sample likelihood function with the maximum entropy principle is ultimately transformed into a parameter optimization problem which aims at
\[
\lambda_i \mid \min \left[ f(\lambda_i) = \sum_{i=1}^{n} v_i^2 \right]
\]
(34)

In this paper, the hill-climbing search optimization algorithm is introduced to work out the optimum solution for parameter \( \lambda_i \) [20]. The basic idea of the hill-climbing search optimization algorithm is as follows: Compare values of neighboring nodes with the current node. If the value of the current node is the greatest one, return the value of the current node; if the value is less than the value of the compared node, replace the current node with the one with greater value. Repeat this process, thus obtaining the optimum node and the maximum value. In the hill-climbing search optimization algorithm, partial nodes can be selected for the solution, thus improving the efficiency. The flow chart for using the hill-climbing search optimization algorithm to optimize the PDF of random variable according to the maximum entropy principle is shown in Figure 1.

According to above analysis, the procedures for using the maximum entropy principle to identify prior distribution random variable PDF and sample likelihood function are as follows:

1. Identify data’s integration interval based on prior data (or sample data).
2. Determine the sample matrix \( m_i \) of prior data (or sample data).
3. Set the initial value of Lagrange multipliers \( \lambda_1, \lambda_2, \lambda_3 \) as \( \lambda_{i0} \).
4. According to the flow chart in Figure 1, use MATLAB software for calculation, and obtain Lagrange multiplier’s optimal solution \( \lambda_i \). Solve \( \lambda_0 \) with reference to formula (29).
5. Introduce the calculated results \( \lambda_i \) and \( \lambda_0 \) into formula (31) to obtain the prior distribution PDF (or sample likelihood function).

3.3.2. Uncertainty Evaluation Model Based on the Maximum Entropy Principle. Suppose the PDF of the random variables of prior distribution are \( f_1(x) \). The PDF of current sample data is \( f_2(x \mid \theta) \). The PDF \( g(x) \) of posterior distribution can be obtained by the calculation formula for the Bayesian posterior distribution:
\[
g(x) = \frac{f_1(x) f_2(x \mid \theta)}{\int_{\Theta} f_1(x) f_2(x \mid \theta) dx}
\]
(35)

where \( \Theta \) is the parameter space.
According to GUM [18], the optimal estimate of the posterior distribution and its standard uncertainty based on the maximum entropy principle are expressed as

\[
\hat{x} = \int_a^b x \hat{g}(x) \, dx
\]

and

\[
u = \left[ \int_a^b (x - \hat{x})^2 \hat{g}(x) \, dx \right]^{1/2}
\]

(36)

The modern uncertainty evaluation method, using the maximum entropy principle to determine the prior distribution probability density function and the likelihood function, can effectively avoid the subjective assumption of data distribution type and render the identification of prior and posterior distributions more reasonable and reliable.

The posterior distribution obtained by formula (35) can be used as prior information for subsequent evaluation. By using formula (36), the most updated measurement information can be included as the measurement goes on, thus achieving continuous updating of the uncertainty evaluation.

4. Simulation Test

Suppose that the random variable \(X\) follows the normal distribution \(N(30, 0.02^2)\). The software MATLAB is used to perform random sampling from \(X\). 16 groups of random numbers are obtained in sequence, as shown in Table 1.

4.1. Bayesian Uncertainty Evaluation Based on Noninformative Prior. Calculate the variance of the numbers in Group 1:

\[
S_0^2 = \frac{1}{n} \sum_{j=1}^{n} (x_{1j} - \bar{x}_1)^2 = 0.000135
\]

(37)

According to formula (11), the posterior distribution standard uncertainty of the measurement information in Group 1 is calculated:

\[
u_0 = 0.0132
\]

(38)

In the same way, the variance and the posterior distribution standard uncertainty of numbers in Group 2 are calculated:

\[
S_1^2 = 0.000334
\]

\[
u_1 = 0.0207
\]

(39)

Repeat above calculation to obtain the simulation result (Table 2) of standard uncertainty evaluated by the Bayesian method in noninformative prior.

4.2. Bayesian Uncertainty Evaluation Based on Conjugate Prior. Take the data in Group \(i\) as prior information and data in Group \(i + 1\) as sample information.

Calculate the average, standard deviation, \(S_0\), and standard uncertainty of the data in Group 1 (prior data):

\[
\mu_0 = \bar{x}_1 = \frac{1}{10} \sum_{j=1}^{10} x_{1j} = 30.00015
\]

\[
u_0 = \sigma_0 = \sqrt{\frac{\sum_{j=1}^{10} (x_{1j} - \bar{x}_1)^2}{(10 - 1) 10}} = 0.0132
\]

(40)

\[
S_0 = \sum_{j=1}^{10} (x_{1j} - \bar{x}_1)^2 = 0.00135
\]

It can be obtained from the data in Group 2 (sample data) that

\[
S_1 = \sum_{j=1}^{10} (x_{2j} - \bar{x}_2)^2 = 0.00334
\]

(41)

According to formula (19), the standard uncertainty of the measurement information of the prior data and the sample data in Group 1 is

\[
u_1 = 0.0167
\]

(42)

With the posterior distribution after the first information integration as the prior information, the following conclusion from the data in Group 3 (the new sample data) is obtained:

\[
S_2 = \sum_{j=1}^{10} (x_{3j} - \bar{x}_3)^2 = 0.00325
\]

(43)

The standard uncertainty obtained from the integrated data in the first three groups:

\[
u_2 = 0.0197
\]

(44)

The posterior distribution obtained this time is used as the prior information of next evaluation. The above information integration is repeated; thus the Bayesian uncertainty evaluation and updating results are obtained based on conjugate prior, as shown in Table 3.

4.3. Bayesian Uncertainty Evaluation Based on The Maximum Entropy Prior. The prior data integration interval of the data in Group 1 is \([29.9794, 30.0190]\). The third-order moment is used as the constraining condition for calculation, and the first three-order sample moment of the prior data is obtained:

\[
m_i = [30.0141, 900.8416, 27038.1091]
\]

(45)

In MATLAB, set the initial values of \(\lambda_1, \lambda_2, \lambda_3\) as \(\lambda_0 = [-20, 1, 0]\).

According to the chart in Figure 1, MATLAB can be programmed to obtain the optimal solution \(\hat{\lambda}_1 = [-19.3154, 0.7, -0.01]\) and achieve \(\lambda_0 = 221.9628\), thus getting the PDF of prior distribution:

\[
f_1(x) = \exp\left(221.9628 - 19.3154x + 0.7x^2 - 0.01x^3\right)
\]

(46)
Table 1: MATLAB simulation random sampling data.

<table>
<thead>
<tr>
<th></th>
<th>x_11</th>
<th>x_12</th>
<th>x_13</th>
<th>x_14</th>
<th>x_15</th>
<th>x_16</th>
<th>x_17</th>
<th>x_18</th>
<th>x_19</th>
<th>x_20</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_1</td>
<td>30.0037</td>
<td>29.9794</td>
<td>30.0190</td>
<td>30.0061</td>
<td>30.0027</td>
<td>30.0103</td>
<td>30.0052</td>
<td>29.9812</td>
<td>29.9968</td>
<td>29.9971</td>
</tr>
<tr>
<td>x_2</td>
<td>29.9894</td>
<td>30.0306</td>
<td>29.9945</td>
<td>29.9962</td>
<td>29.9765</td>
<td>29.9858</td>
<td>29.9903</td>
<td>29.9825</td>
<td>30.0336</td>
<td>29.995</td>
</tr>
<tr>
<td>x_3</td>
<td>29.9787</td>
<td>30.0089</td>
<td>29.9948</td>
<td>30.0055</td>
<td>30.0000</td>
<td>29.9911</td>
<td>29.9699</td>
<td>29.9954</td>
<td>30.0247</td>
<td>30.0321</td>
</tr>
<tr>
<td>x_4</td>
<td>30.0087</td>
<td>29.9795</td>
<td>29.9898</td>
<td>30.0248</td>
<td>29.9909</td>
<td>29.9394</td>
<td>30.0002</td>
<td>29.9396</td>
<td>29.9852</td>
<td>29.981</td>
</tr>
<tr>
<td>x_5</td>
<td>29.9787</td>
<td>29.9687</td>
<td>30.0008</td>
<td>30.0020</td>
<td>30.0321</td>
<td>29.9983</td>
<td>30.0036</td>
<td>29.9994</td>
<td>30.007</td>
<td>30.0187</td>
</tr>
<tr>
<td>x_6</td>
<td>30.0065</td>
<td>30.0217</td>
<td>30.0201</td>
<td>29.9870</td>
<td>30.0051</td>
<td>29.9811</td>
<td>30.0000</td>
<td>30.0185</td>
<td>30.000</td>
<td>29.9989</td>
</tr>
<tr>
<td>x_7</td>
<td>30.0108</td>
<td>30.0317</td>
<td>30.0201</td>
<td>29.9870</td>
<td>30.0051</td>
<td>29.9811</td>
<td>30.0000</td>
<td>30.0185</td>
<td>30.000</td>
<td>29.9989</td>
</tr>
<tr>
<td>x_8</td>
<td>30.0108</td>
<td>30.0317</td>
<td>30.0201</td>
<td>29.9870</td>
<td>30.0051</td>
<td>29.9811</td>
<td>30.0000</td>
<td>30.0185</td>
<td>30.000</td>
<td>29.9989</td>
</tr>
<tr>
<td>x_9</td>
<td>30.0108</td>
<td>30.0317</td>
<td>30.0201</td>
<td>29.9870</td>
<td>30.0051</td>
<td>29.9811</td>
<td>30.0000</td>
<td>30.0185</td>
<td>30.000</td>
<td>29.9989</td>
</tr>
<tr>
<td>x_10</td>
<td>30.0108</td>
<td>30.0317</td>
<td>30.0201</td>
<td>29.9870</td>
<td>30.0051</td>
<td>29.9811</td>
<td>30.0000</td>
<td>30.0185</td>
<td>30.000</td>
<td>29.9989</td>
</tr>
<tr>
<td>x_11</td>
<td>30.0108</td>
<td>30.0317</td>
<td>30.0201</td>
<td>29.9870</td>
<td>30.0051</td>
<td>29.9811</td>
<td>30.0000</td>
<td>30.0185</td>
<td>30.000</td>
<td>29.9989</td>
</tr>
<tr>
<td>x_12</td>
<td>30.0108</td>
<td>30.0317</td>
<td>30.0201</td>
<td>29.9870</td>
<td>30.0051</td>
<td>29.9811</td>
<td>30.0000</td>
<td>30.0185</td>
<td>30.000</td>
<td>29.9989</td>
</tr>
<tr>
<td>x_13</td>
<td>30.0108</td>
<td>30.0317</td>
<td>30.0201</td>
<td>29.9870</td>
<td>30.0051</td>
<td>29.9811</td>
<td>30.0000</td>
<td>30.0185</td>
<td>30.000</td>
<td>29.9989</td>
</tr>
<tr>
<td>x_14</td>
<td>30.0108</td>
<td>30.0317</td>
<td>30.0201</td>
<td>29.9870</td>
<td>30.0051</td>
<td>29.9811</td>
<td>30.0000</td>
<td>30.0185</td>
<td>30.000</td>
<td>29.9989</td>
</tr>
<tr>
<td>x_15</td>
<td>30.0108</td>
<td>30.0317</td>
<td>30.0201</td>
<td>29.9870</td>
<td>30.0051</td>
<td>29.9811</td>
<td>30.0000</td>
<td>30.0185</td>
<td>30.000</td>
<td>29.9989</td>
</tr>
<tr>
<td>x_16</td>
<td>30.0108</td>
<td>30.0317</td>
<td>30.0201</td>
<td>29.9870</td>
<td>30.0051</td>
<td>29.9811</td>
<td>30.0000</td>
<td>30.0185</td>
<td>30.000</td>
<td>29.9989</td>
</tr>
</tbody>
</table>

Table 2: Simulation results of Bayesian uncertainty evaluation based on noninformative prior.

<table>
<thead>
<tr>
<th></th>
<th>u_0</th>
<th>u_1</th>
<th>u_2</th>
<th>u_3</th>
<th>u_4</th>
<th>u_5</th>
<th>u_6</th>
<th>u_7</th>
</tr>
</thead>
<tbody>
<tr>
<td>u_0</td>
<td>0.0132</td>
<td>0.0207</td>
<td>0.0204</td>
<td>0.0241</td>
<td>0.0192</td>
<td>0.0278</td>
<td>0.0209</td>
<td>0.0179</td>
</tr>
<tr>
<td>u_4</td>
<td>0.0324</td>
<td>0.0245</td>
<td>0.0172</td>
<td>0.0284</td>
<td>0.0124</td>
<td>0.0185</td>
<td>0.0252</td>
<td>0.0162</td>
</tr>
</tbody>
</table>

Table 3: Simulation results of Bayesian uncertainty evaluation based on conjugate prior.

<table>
<thead>
<tr>
<th></th>
<th>u_0</th>
<th>u_1</th>
<th>u_2</th>
<th>u_3</th>
<th>u_4</th>
<th>u_5</th>
<th>u_6</th>
<th>u_7</th>
</tr>
</thead>
<tbody>
<tr>
<td>u_0</td>
<td>0.0132</td>
<td>0.0167</td>
<td>0.0197</td>
<td>0.0222</td>
<td>0.0219</td>
<td>0.0220</td>
<td>0.0215</td>
<td>0.0190</td>
</tr>
<tr>
<td>u_4</td>
<td>0.0158</td>
<td>0.0179</td>
<td>0.0214</td>
<td>0.0238</td>
<td>0.0175</td>
<td>0.0184</td>
<td>0.0219</td>
<td>0.0203</td>
</tr>
</tbody>
</table>

According to formula (36), the standard uncertainty of the prior data is \( u_0 = 0.0185 \).

Similarly, the likelihood function of the current sample is

\[
 f_2(x) = \exp \left(-164.4860 - 21.4307x + 0.9x^2\right) \quad (47)
\]

It can be obtained by the Bayesian formula that the PDF of posterior distribution is

\[
 g(x) = \exp \left(57.4768 - 40.7461x + 1.6x^2 - 0.01x^3\right) \quad (48)
\]

According to formula (36), the standard uncertainty of the posterior distribution is \( u_1 = 0.0226 \).

The posterior distribution obtained this time is used as the prior information of next evaluation. The above information integration is repeated; thus the Bayesian uncertainty evaluation and updating results are obtained based on the maximum entropy, as shown in Table 4.

4.4. Result Analysis. The simulation results of the three Bayesian uncertainty evaluation methods are compared, and the comparison results are shown in Figure 2.

Figure 2 shows that the simulations of the Bayesian uncertainty evaluation based on noninformative prior and conjugate prior show significant fluctuation.

The method based on noninformative prior is suitable for the cases with no or little prior information available. It obtains the uncertainty of each group of the measured data by Bayesian statistical inference, without integrating the measured data in different groups.

By repeated information integration, the obtained value of the method based on conjugate prior is close to the theoretical value of the standard uncertainty. The method can use historical data and current data as prior information; i.e., the posterior distribution can be used as the prior information for further test. The obtained newer posterior distribution and the prior distribution are categorized in the same distribution type, and they can provide sound basis for subsequent evaluation of uncertainty. However, the conjugate prior method requires a specific distribution of known prior information. As to measurement information, we must assume that it follows a certain distribution, and this is more or less subjective. Therefore, the conjugate prior
Table 4: Simulation results of Bayesian uncertainty evaluation based on maximum entropy prior.

<table>
<thead>
<tr>
<th>( u_0 )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
<th>( u_5 )</th>
<th>( u_6 )</th>
<th>( u_7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0185</td>
<td>0.0226</td>
<td>0.0253</td>
<td>0.0172</td>
<td>0.0222</td>
<td>0.0216</td>
<td>0.0207</td>
<td>0.0203</td>
</tr>
<tr>
<td>( u_8 )</td>
<td>( u_9 )</td>
<td>( u_{10} )</td>
<td>( u_{11} )</td>
<td>( u_{12} )</td>
<td>( u_{13} )</td>
<td>( u_{14} )</td>
<td>( u_{15} )</td>
</tr>
<tr>
<td>0.0198</td>
<td>0.0189</td>
<td>0.0206</td>
<td>0.0213</td>
<td>0.0208</td>
<td>0.0213</td>
<td>0.0201</td>
<td>0.0195</td>
</tr>
</tbody>
</table>

The results of Bayesian uncertainty evaluation based on non-informative prior, conjugate prior, and maximum entropy prior are compared. The advantages and the scope of their application are introduced, with the aim of providing scientific basis for the application of Bayesian uncertainty evaluation methods. Among them, the noninformative prior Bayesian uncertainty evaluation method does not integrate measured data. There is a greater fluctuation with the uncertainty. The uncertainty showed in the two dynamic evaluation models of Bayesian methods based on the conjugate prior and the maximum entropy prior tends to be closer to the theoretical value as a result of repeated data integration. The uncertainty evaluation and updated result showed in these two methods are more objective and reasonable.

5. Conclusions

The proposed Bayesian uncertainty evaluation method can fully integrate historical information and real time sample information in the measurement system. It renders the uncertainty evaluation more consistent with the latest state of the measurement system, thus providing good foundation for the development of modern uncertainty evaluation methods and their application in dynamic uncertainty evaluation.

Based on analyzing noninformative prior and the conjugate Bayesian methods, the present paper proposes the maximum entropy Bayesian uncertainty evaluation method. Prior distribution and the sample’s likelihood function are determined by the maximum entropy principle, thus avoiding the risk of subjective assumptions; optimized uncertainty evaluation is achieved by introducing hill-climbing searching algorithm and programming-based calculation. The proposed Bayesian uncertainty evaluation model can be used to achieve the updating of uncertainty components. The mathematical model is deduced based on conjugate prior to achieve real time and continuous updating of uncertainty components.

By simulating the real cases, Bayesian evaluation methods based on the noninformative prior, conjugate priors, and maximum entropy prior are compared. The advantages and the scope of their application are introduced, with the aim of providing scientific basis for the application of Bayesian uncertainty evaluation methods. Among them, the noninformative prior Bayesian uncertainty evaluation method does not integrate measured data. There is a greater fluctuation with the uncertainty. The uncertainty showed in the two dynamic evaluation models of Bayesian methods based on the conjugate prior and the maximum entropy prior tends to be closer to the theoretical value as a result of repeated data integration. The uncertainty evaluation and updated result showed in these two methods are more objective and reasonable.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

**Acknowledgments**

This research is supported by National Key Research and Development Plan of China (no. 2016YFF0203801).

**References**


