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Research Article

Family of a-Ary Univariate Subdivision Schemes Generated by Laurent Polynomial

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The generalized symbols of the family of a-ary ($a \geq 2$) univariate stationary and nonstationary parametric subdivision schemes have been presented. These schemes are the new version of Lane-Riesenfeld algorithms. Comparison shows that our proposed family has higher continuity and generation degree comparative to the existing subdivision schemes. It is observed that many existing binary and ternary schemes are the special cases of our schemes. The analysis of proposed family of subdivision schemes is also presented in this paper.

1. Introduction

Subdivision schemes are powerful tools in CAGD for generation of smooth curves and surfaces. Subdivision schemes are popular in many practical applications such as multiresolution modeling and character animation.

The curve generation in computer graphics especially by subdivision schemes has applications in signal compression [1]. It has been shown that the subdivision schemes are suitable algorithm for compressing both regular and fractal-like signals. Herley [2] used subdivision schemes to derive necessary and sufficient conditions under which a discrete-time signal can be exactly interpolated from only one out of every M sample and show how designs may be carried out. Taubin [3] described a new tool for interactive free-form fair surface design. By generalizing classical discrete Fourier analysis to two-dimensional discrete surface signals- functions defined on polyhedral surfaces of arbitrary topology.

Initially Lane and Riesenfeld present an algorithm [4] for subdividing uniform B-spline schemes of order l, with $l \in \mathbb{N}$. After that, this algorithm was used in different variants [5]. In stationary context, Deslauriers-Dubuc schemes [6] were characterized by a symbol containing the factor $(1+z)^l/2^{l-1}$. Hormann and Sabin [7] presented a family of subdivision schemes with cubic precision (with $l \in \mathbb{N}$). Ashraf et al. [8] presented a new family of subdivision schemes by using six-point variant on the Lane-Riesenfeld algorithm. In nonstationary context, Conti and Romani [5] presented general affine combination of B-spline subdivision masks and its nonstationary counterparts. Conti and Romani [9] also presented algebraic conditions on nonstationary subdivision schemes for exponential polynomial reproduction. Novara and Romani [10] offered an extension of nonstationary Lane-Riesenfeld algorithm and a nonstationary family of alternating primal/dual subdivision schemes with reproduction of $\{1, x, e^{tx}, e^{-tx}\}$, $t \in [0, \pi) \cup i\mathbb{R}^+$.

During the last decade, much work has been published on parametric subdivision schemes. Parametric subdivision schemes are good to control the shapes in curve and surface design. Siddiqi and Rehan [11] modified 3-point binary and ternary subdivision schemes with a tension parameter which generate a family of C^1 limiting curves for certain range of tension parameter. Using same technique, they also presented an improved four point scheme [12]. Mustafa et al. [13] presented a general formula for odd point ternary approximating subdivision schemes with a shape parameter. Ghaffar et al. [14] proposed a 3-point scheme for any arity. They also presented 4-point, α-ary subdivision schemes with tension parameter [15]. Zheng et al. [16] presented a family

of integer point binary subdivision schemes with tension parameter.

1.1. Motivation. Lane-Riesenfeld algorithm has been used for the generation of a family of binary parametric subdivision schemes [17, 18]. But till now, nobody has used Lane-Riesenfeld algorithms for the generation of higher arity parametric subdivision schemes. The benefits of higher arity subdivision schemes are as follows:

(i) The rate of convergence increases with the increase of arity of the scheme.

(ii) The support of the schemes decreases as arity increases.

The main objective of this work is to present univariate parametric high arity subdivision schemes. The objective has been obtained by using Lane-Riesenfeld algorithm and parametric Laurent polynomials. The highlights of proposed work are as follows:

(i) The generalization of Lane-Riesenfeld algorithm to generate high arity schemes

(ii) A unified way to present families of univariate stationary and nonstationary high arity parametric subdivision schemes

(iii) To propose families of schemes with higher continuity and generation degree comparative to the existing schemes [7, 17–19] (see Tables 3 and 4)

(iv) To present families of schemes so that existing binary and ternary subdivision schemes become special cases (see Section 4.1).

The paper is organized as follows. In Section 2, we present construction of family of a-ary univariate approximating subdivision schemes. Analysis of the proposed family is presented in Section 3. Comparison and special cases of our proposed family are presented in Section 4. Section 5 is for nonstationary version of Section 2. Conclusions are drawn in Section 6.

2. Construction of Algorithm

In this section, a generalized B-spline symbol for a-ary scheme has been presented using the well-known Lane-Riesenfeld algorithm. This algorithm is based on the smoothing operator

$$S_a(z) = \frac{\left(1 + z + z^2 + \cdots + z^{a-1}\right)}{a} \tag{1}$$

and refining factor

$$R_a(z) = S_a(z)\left(1 + z + z^2 + \cdots + z^{a-1}\right). \tag{2}$$

Now by applying the smoothing operator $S_a(z)$, n times, after one application of the refining operator $R_a(z)$, we get

$$A_n^a(z) = \left(S_a(z)\right)^n R_a(z). \tag{3}$$

Its simplest form is

$$A_n^a(z) = \frac{\left(1 + z + z^2 + \cdots + z^{a-1}\right)^{n+2}}{a^{n+1}}, \tag{4}$$

which is the general symbol of $(n+1)$th degree polynomial B-spline. The symbol $A_n^a(z)$ defined in (4) is able to generate $\Pi_{n+1} = \operatorname{span}\{1, x, x^2, \ldots, x^{n+1}\}$.

Laurent polynomial (symbol) for odd arity (i.e., a is odd integer) is defined as

$$\begin{aligned} P_o^a(z) = (1 - aw) + w\Big(&z^{-(a-1)} + z^{-(a-3)} + \cdots + z^{(a-a)} \\ &+ \cdots + z^{(a-3)} + z^{(a-1)}\Big). \end{aligned} \tag{5}$$

Laurent polynomial (symbol) for even arity (i.e a is even integer) is defined as

$$\begin{aligned} &P_e^a(z) \\ &\quad = (1 - aw) \\ &\qquad + w\left(z^{-(a-1)} + z^{-(a-3)} + \cdots + z^{(a-3)} + z^{(a-1)}\right). \end{aligned} \tag{6}$$

The general symbols for odd and even arity schemes can be defined as

$$S_{n,o}^a(z) = A_n^a(z)\, P_o^a(z), \tag{7}$$

$$S_{n,e}^a(z) = A_n^a(z)\, P_e^a(z), \tag{8}$$

respectively. From (7) and (8), we get a-ary subdivision schemes with tension parameter w. The members of the families can be easily obtained corresponding to the values of a and n. Table 1 gives the complexity and mask of the subdivision schemes.

3. Analysis of $S_{n,e}^a$ and $S_{n,o}^a$ Schemes

In this section, we will present the complete analysis of a family of a-ary subdivision schemes. We present continuity, Hölder regularity, generation degree, and reproduction degree of proposed family of schemes.

3.1. Convergence and Smoothness Analysis. Continuity is an important property of subdivision schemes. Continuity of a subdivision scheme refers to the differentiability of the limit curve produced by subdivision process. A high continuity subdivision scheme gives more smooth limit curve. We use Laurent polynomial method [20] to calculate integer class continuity of the $S_{n,e}^a$ and $S_{n,o}^a$-schemes. Hölder regularity is an extension of convergence and continuity which gives more information about any scheme. Lower and upper bounds on Hölder continuity are calculated by using Riouls method [21]. Moreover, exact upper bounds on Hölder continuity can also be derived by using Floater and Muntingh algorithm [22].

Theorem 1. *The family of a-ary subdivision schemes corresponding to the symbols $S_{n,e}^a$ and $S_{n,o}^a$ is C^{n+1} continuous.*

Table 1: Here we present arity, complexity, and mask of the schemes corresponding to different values of n; here m shows complexity of the schemes (i.e., 2-, 3-, ..., -point schemes), while a, $S^a_{n,e}$, and $S^a_{n,o}$ stand for arity and mask of the schemes, respectively.

n	a	m	Mask
0	2	3	$S^2_{0,e} = \frac{1}{2}[w, 1, -2w+2, 1, w]$
1	2	3	$S^2_{1,e} = \frac{1}{4}[w, 1+w, -2w+3, -2w+3, 1+w, w]$
2	2	4	$S^2_{2,e} = \frac{1}{8}[w, 1+2w, -w+4, -4w+6, -w+4, 1+2w, w]$
3	2	4	$S^2_{3,e} = \frac{1}{16}[w, 1+3w, w+5, -5w+10, -5w+10, w+5, 1+3w, w]$
4	2	5	$S^2_{4,e} = \frac{1}{32}[w, 1+4w, 6+4w, -4w+15, 20-10w, -4w+15, 6+4w, 1+4w, w]$
0	3	3	$S^3_{0,o} = \frac{1}{3}[w, 2w, 1+w, 2-2w, -4w+3, 2-2w, 1+w, 2w, w]$
1	3	4	$S^3_{1,o} = \frac{1}{9}[w, 3w, 1+4w, 3+w, -5w+6, -8w+7, -5w+6, 3+w, 1+4w, 3w, w]$
2	3	5	$S^3_{2,o} = \frac{1}{27}[w, 4w, 1+8w, 4+8w, 10, -12w+16, -18w+19, -12w+16, \ldots, 4w, w]$
3	3	5	$S^3_{3,o} = \frac{1}{81}[w, 5w, 1+13w, 5+20w, 16w+15, -4w+30, -30w+45, -42w+51, -30w+45, \ldots, 5w, w]$
4	3	6	$S^3_{4,o} = \frac{1}{243}[w, 6w, 1+19w, 6+38w, 49w+21, 32w+50, -18w+90, -76w+126, -102w+141, -76w+126, \ldots, 6w, w]$
0	4	4	$S^4_{0,e} = \frac{1}{4}[w, 2w, 4w, 1+2w, 2-w, 3-4w, 1-2w, 3-4w, 2-w, 1+2w, 4w, 2w, w]$
1	4	4	$S^4_{1,e} = \frac{1}{16}[w, 3w, 7w, 9w+1, 7w+3, w+6, -11w+10, -17w+12, -17w+12, -11w+10, \ldots, 3w, w]$
2	4	5	$S^4_{2,e} = \frac{1}{64}[w, 4w, 11w, 20w+1, 26w+4, 24w+10, 6w+20, -20w+31, -44w+40, -56w+44, -44w+40, \ldots, 4w, w]$
3	4	6	$S^4_{3,e} = \frac{1}{256}[w, 5w, 16w, 36w+1, 61w+5, 81w+15, 76w+35, 36w+65, -34w+101, -114w+135, -164w+155, -164w+155, \ldots, 5w, w]$
4	4	6	$S^4_{4,e} = \frac{1}{1024}[w, 6w, 22w, 58w+1, 118w+6, 194w+21, 254w+56, 254w+120, 159w+216, -36w+336, -276w+456, -476w+546, -556w+580, -476w+546, \ldots, 6w, w]$
0	5	4	$S^5_{0,o} = \frac{1}{5}[w, 2w, 4w, 6w, 4w+1, 2, 3-3w, -8w+4, -12w+5, -8w+4, \ldots, 2w, w]$
1	5	5	$S^5_{1,o} = \frac{1}{25}[w, 3w, 7w, 13w, 17w+1, 16w+3, 6+11w, -w+10, -19w+15, -31w+18, -34w+19, -31w+18, \ldots, 3w, w]$
2	5	5	$S^5_{2,o} = \frac{1}{125}[w, 4w, 11w, 24w, 41w+1, 56w+4, 64w+10, 56w+20, 24w+35, -24w+52, -74w+68, -116w+80, -134w+85, -116w+80, \ldots, 4w, w]$
3	5	6	$S^5_{3,o} = \frac{1}{625}[w, 5w, 16w, 40w, 81w+1, 136w+5, 196w+15, 241w+35, 241w+70, 176w+121, 46w+185, -134w+255, -324w+320, -464w+365, -514w+381, -464w+365, \ldots, 5w, w]$
4	5	7	$S^5_{4,o} = \frac{1}{3125}[w, 6w, 22w, 62w, 143w+1, 278w+6, 469w+21, 694w+56, 895w+126, 990w+246, 900w+426, 570w+666, 5w+951, -700w+1246, -1390w+1506, -1900w+1686, -2090w+1751, -1900w+1686, \ldots, 6w, w]$

Proof. From (8), we have

$$S^a_{n,e}(z) = \frac{1}{a^{n+1}}\left(1+z+z^2+\cdots+z^{a-1}\right)^{n+2}\left((1-aw) + w\left(z^{-(a-1)}+z^{-(a-3)}+\cdots+z^{(a-3)}+z^{(a-1)}\right)\right). \tag{9}$$

This implies

$$S^a_{n,e}(z) = \left(\frac{1+z+z^2+\cdots+z^{a-1}}{a}\right)^{n+1} b_e(z), \tag{10}$$

with $b_e(z) = (1+z+z^2+\cdots+z^{a-1})c_e(z)$, where $c_e(z) = (1-aw+w(z^{-(a-1)}+z^{-(a-3)}+\cdots+z^{(a-3)}+z^{(a-1)}))$.

$$\|c_e(z)\|_\infty = \max\{|w|+|w|, |w|+|w|, |w|+|w|, |w| + |w|, \ldots, |1-aw|\}. \tag{11}$$

In binary case, the condition for $\|c_e(z)\|_\infty < 1$ is $0 < w < 1/2$. For all other even arity subdivision schemes the condition for $\|c_e(z)\|_\infty < 1$ is $0 < w < 2/a$. Therefore, by [20], if $\|c_e(z)\|_\infty < 1$, then $c_e(z)$ is contractive and $b_e(z)$ converges. If $b_e(z)$ is convergent then scheme corresponding to $S^a_{n,e}(z)$ is C^{n+1} continuous.

Similarly from (7), we have

$$S_{n,o}^{a}(z) = \frac{1}{a^{n+1}}\left(1+z+z^{2}+\cdots+z^{a-1}\right)^{n+2}\left(1-aw + w\left(z^{-(a-1)}+z^{-(a-3)}+\cdots+z^{(a-a)}+\cdots+z^{(a-3)} + z^{(a-1)}\right)\right). \quad (12)$$

This implies

$$S_{n,o}^{a}(z) = \left(\frac{1+z+z^{2}+\cdots+z^{a-1}}{a}\right)^{n+1} b_{o}(z), \quad (13)$$

with $b_o(z) = (1+z+z^2+\cdots+z^{a-1})c_o(z)$, where $c_o(z) = (1-aw+w(z^{-(a-1)}+z^{-(a-3)}+\cdots+z^{(a-a)}+\cdots+z^{(a-3)}+z^{(a-1)}))$.

$$\|c_o(z)\|_{\infty} = \max\{|w|+|w|, |w|+|w|, |w|+|w|, |w| + |w|, \ldots, |aw|\}. \quad (14)$$

The condition for $\|c_o(z)\|_\infty < 1$ is $-1/a < w < 1/a$. Therefore, by [20], if $\|c_o(z)\|_\infty < 1$, then $c_o(z)$ is contractive and $b_o(z)$ converges. If $b_o(z)$ is convergent then scheme corresponding to $S_{n,o}^a(z)$ is C^{n+1} continuous. □

Theorem 2. *The lower bound of Hölder regularity of a family of a-ary subdivision schemes corresponding to the symbol $S_{n,e}^a(z)$ is computed as $r \geq n+2-\log_a(\mu)$, where μ is defined as $\mu = a - a^2w$ and $\mu = aw$. The value of μ depends on w.*

Proof. From (9), we have

$$b_i = \begin{cases} a - a^2 w & \text{when } i = \dfrac{a}{2} \\ aw & \text{when } i = 0, 1, \ldots, a,\ i \neq \dfrac{a}{2}, \end{cases} \quad (15)$$

$k = n+2$, $q = 0, 1, 2, \ldots, a$, and $B_0, B_1, \ldots, B_a$ are the matrices with elements $(B_q)_{i,j} = b_{a-1+i-aj+q}$. By [21] the lower bound of Hölder regularity is given by $r \geq k - \log_a(\mu)$, where μ is the joint spectral radius of the matrices $B_0, B_1, \ldots, B_a$; that is, $\mu = \rho(B_0, B_1, \ldots, B_a)$. For bounds on Hölder regularity we calculate $\max\{\rho(B_0), \rho(B_1), \ldots, \rho(B_a)\} \leq \mu \leq \max\{\|(B_0)\|_\infty, \|(B_1)\|_\infty, \ldots, \|(B_a)\|_\infty\}$. Since μ is bounded from below by the spectral radii and from above by the norm of the metrics $B_0, B_1, \ldots, B_a$, then $\mu = \max\{aw, aw, \ldots, a - a^2w\}$. This implies that μ depends on the value of w. The lower bound Hölder regularity of the scheme corresponding to the symbol $S_{n,e}^a(z)$ is $r \geq n+2-\log_a(\mu)$. This completes the proof. □

Remark 3. The upper bound of Hölder regularity of the families of binary and quaternary schemes corresponding to symbols $S_{n,e}^2(z)$ and $S_{n,e}^4(z)$ is $r \leq n+2-\log_2(2-2w)$ and $r \leq n+2-\log_4(\mu)$, respectively, where μ is defined as

$$\mu = 4 - 16w \quad \text{if } 0 < w \leq 0.2$$
$$\mu = 8w \quad \text{if } 0.2 < w < 0.5. \quad (16)$$

Remark 4. Similarly, we can easily compute the lower and upper bounds of a family of a-ary subdivision schemes corresponding to the symbol $S_{n,o}^a(z)$.

3.2. Generation and Reproduction Analysis. The subdivision scheme with symbol $S_{n,e}^a(z)$ reproduces polynomials of degree d with respect to the parameterizations $\tau = (1/a)(d/dz)S_{n,e}^a(z)|_{z=1}$ if and only if

$$\left(S_{n,e}^{a}\right)^{(k)}(-1) = 0,$$
$$\left(S_{n,e}^{a}\right)^{(k)}(1) = a\prod_{j=0}^{k-1}(\tau - j), \quad (17)$$
$$k = 0, 1, \ldots, d.$$

Polynomial reproduction of degree d requires polynomial generation of degree d [20].

Theorem 5. *Generation degree of family of a-ary subdivision schemes corresponding to the symbols $S_{n,e}^a(z)$ and $S_{n,o}^a(z)$ is $n+1$.*

Proof. The Laurent polynomial of family of even ary subdivision schemes defined in (9) can be written as

$$S_{n,e}^{a}(z) = \left(\frac{1+z+z^{2}+\cdots+z^{a-1}}{a}\right)^{(n+1)+1} b_{e}(z), \quad (18)$$

where $b_e(z) = a(1 - aw + w(z^{-(a-1)} + z^{-(a-3)} + \cdots + z^{(a-3)} + z^{(a-1)}))$.

Similarly the Laurent polynomial of family of odd ary subdivision schemes defined in (12) can be written as

$$S_{n,o}^{a}(z) = \left(\frac{1+z+z^{2}+\cdots+z^{a-1}}{a}\right)^{(n+1)+1} b_{o}(z), \quad (19)$$

where $b_o(z) = a(1 - aw + w(z^{-(a-1)} + z^{-(a-3)} + \cdots + z^{(a-a)} + \cdots + z^{(a-3)} + z^{(a-1)}))$.

Hence generation degree is $n+1$. □

Theorem 6. *The family of binary schemes corresponding to the symbol $S_{n,e}^2(z)$ reproduces polynomial of degree 1 with respect to the dual parameterizations for odd values of n and primal parametrization for even values of n.*

Proof. By taking the first derivative of (9) and putting $z = 1$, we get $(S_{n,e}^2)'(1) = n+2$. This implies that $\tau = (n+2)/2$, so the scheme corresponding to the symbol $S_{n,e}^2(z)$ has dual parametrization for odd values of n and primal parametrization for even values of n. We can easily verify that the first and second derivatives of $S_{n,e}^2(z)$ at $z = -1$ are equal to 0. Further we can also verify (17) for $k = 0$ and 1. This completes the proof. □

Remark 7. The family of ternary schemes corresponding to the symbol $S_{n,o}^3(z)$ reproduces polynomial of degree 1 with respect to the dual parameterizations with $\tau = n+2$.

Remark 8. The family of quaternary schemes corresponding to the symbol $S_{n,e}^4(z)$ reproduces polynomial of degree 1 with respect to the dual parameterizations for odd values of n and primal parametrization for even values of n with $\tau = (6n+2)/4$.

Table 2: Here C, G_d, R_d, τ, and P are continuity, generation degree, reproduction degree, shift parameter, and parametrization, respectively.

Scheme	C	G_d	R_d	τ	P
$S^2_{n,e}$	C^{n+1} when $w \in (0, 1/2)$	$n+1$	1	$\tau = \dfrac{n+2}{2}$	Primal for even n
	C^{n+2} when $w = 1/4$	$n+3$			Dual for odd n
$S^3_{n,o}$	C^{n+1} when $w \in (-1/3, 1/3)$	$n+1$	1	$\tau = n+2$	Dual $\forall n$
$S^4_{n,e}$	C^{n+1} when $w \in (0, 1/2)$	$n+1$	1	$\tau = \dfrac{6n+2}{4}$	Primal for even n
					Dual for odd n
$S^5_{n,o}$	C^{n+1} when $w \in (-1/5, 1/5)$	$n+1$	1	$\tau = 2n+4$	Dual $\forall n$

Remark 9. The family of quinary schemes corresponding to the symbol $S^5_{n,o}(z)$ reproduces polynomial of degree 1 with respect to the dual parameterizations with $\tau = 2n + 4$.

Similarly, we can compute the degree of reproduction of a family of a-ary subdivision schemes corresponding to the symbols $S^a_{n,e}(z)$ and $S^a_{n,o}(z)$. In Table 2, we briefly present the above analysis of the schemes $S^a_{n,e}$ and $S^a_{n,o}$.

4. Comparison and Special Cases of $S^a_{n,e}$ and $S^a_{n,o}$

In this section, we will present the special cases, comparison, and applications of the schemes.

4.1. Special Cases of the Schemes $S^a_{n,e}$ and $S^a_{n,o}$. It is observed that many existing binary and ternary subdivision schemes are the special cases of our proposed $S^a_{n,e}$ and $S^a_{n,o}$ schemes.

In *binary case* (i.e., $S^2_{n,e}$), the following existing schemes are the special cases:

(i) If $w = 0$, we get B-spline of $(n+1)$th degree [4].

(ii) If we take $w = -N/8$, it gives Hormann and Sabin's family [7].

(iii) If we take $n = 2m - 1$ and $w = 1/4 + 2w$, we get the general formula of integer point binary approximating subdivision schemes [16].

(iv) For $n = 1$ and $w = 0$, we have the mask of well-known 2-point Chaikin's scheme described in [23].

(v) If we take $n = 1$ and $w = -3/8$, we get a 3-point binary approximate subdivision scheme described in [11].

(vi) For $n = 1$ and $w = -3/8 + 4\mu$, we get a 3-point binary scheme [11].

(vii) By setting $n = 1$ and $w = \mu_0 - 1/8$, we get a 3-point binary scheme [14].

(viii) For $n = 1$ and $w = 1/8$, we get a 3-point binary scheme [24].

(ix) For $n = 1$ and $w = 4\mu$, we get a 3-point binary scheme [25].

(x) For $n = 1$ and $w = 1/4$, we get a 3-point binary scheme [26].

(xi) By setting $n = 1$ and $w = 1/6 + \mu$, we get a 3-point binary scheme [27].

(xii) For $n = 1$ and $w = -12w$, we get a 3-point binary scheme [28].

(xiii) By setting $n = 1$ and $w = w/4$, we get a 3-point binary scheme [29].

(xiv) For $n = 3$ and $w = 16\mu$, we get a 4-point binary scheme [12].

(xv) If we take $n = 3$, we get a 4-point binary scheme [15].

(xvi) If we take $n = 3$ and $w = -16\mu$, we get a 4-point binary scheme [30].

(xvii) For $n = 3$ and $w = -5/8$, we get a 4-point binary scheme [31].

(xviii) If we take $n = 3$ and $w = 1/24$, we get a 4-point binary scheme [19].

(xix) For $n = 4$ and $w = 32\mu$, we get a 5-point binary relaxation scheme [32].

(xx) By setting $n = 4$ and $w = \mu/4$, we get a 5-point binary relaxation scheme [33].

(xxi) After substituting $a = 2$ into (8), we obtain a general symbol for stationary quasi-splines [34].

In *Ternary case* (i.e., $S^3_{n,o}$), the following existing schemes are the special cases:

(i) By setting $n = 3n + 1$ and $w = 1/12 + w$, we get odd point ternary approximating subdivision scheme described in [13].

(ii) For $n = 0$ and $w = 1/24+3\mu$, we get a modified ternary 3-point scheme described in [11].

(iii) If we take $n = 0$ and $w = 3\mu - 1$, we get a 3-point ternary scheme [35].

(iv) By setting $n = 0$ and $w = \mu_0/12$, we get a 3-point ternary scheme [14].

4.2. Application and Comparison. Aim of this subsection is to present the comparison of $S^a_{n,e}$, $S^a_{n,o}$, and existing subdivision schemes. In Table 3, we present the comparison of proposed family of binary $S^2_{n,e}$ schemes with existing schemes having the same number of entries in mask and same support size. In this table, the comparison of continuity, generation degree, support, and approximation order of the subdivision schemes is presented. We see that continuity and generation degree

Table 3: Here we the present comparison of the family of binary $S^2_{n,e}$ schemes. Here N.E, C, G_d, S and OA are number of entries in mask, continuity analysis, generation degree, support and approximation order of subdivision schemes.

Schemes	N.E	C	G_d	S	OA
$S^2_{1,e}$-scheme	6	C^2 for $w \in (0, 1/2)$	Quadratic	5	2
		C^3 for $w = 1/4$	Quartic for $w = 1/4$		
Scheme [7]	6	C^2	Quadratic	5	3
Scheme [17]	6	C^1	Linear	5	2
Scheme [24]	6	C^2	Quadratic	5	2
$S^2_{2,e}$-scheme	7	C^3 for $w \in (0, 1/2)$	Cubic	6	2
		C^4 for $w = 1/4$	Quintic for $w = 1/4$		
Scheme [7]	7	C^1	Quartic	6	4
Scheme [17]	7	C^1	Linear	6	2
Scheme [18]	7	C^1 for $w \in (0, 0.1545)$	Linear	6	2
Scheme [36]	7	C^1 for $\mu \in (-1/16, 1/16)$	Linear	6	2
		C^4 for $\mu = 0$	Quintic for $\mu = 0$		
$S^2_{3,e}$-scheme	8	C^4 for $w \in (0, 1/2)$	Quartic	7	2
		C^5 for $w = 1/4$	Sextic for $w = 1/4$		
Scheme [7]	8	C^2	Quintic	7	4
Scheme [17]	8	C^2	Quadratic	7	2
Scheme [18]	8	C^2 for $w \in (0, 0.1545)$	Quadratic	7	2
Scheme [36]	8	C^2 for $\mu \in (-1/16, 1/16)$	Quadratic	7	2
		C^5 for $\mu = 0$	Sextic for $\mu = 0$		
$S^2_{4,e}$-scheme	9	C^5 for $w \in (0, 1/2)$	Quintic	8	2
		C^6 for $w = 1/4$	Septic for $w = 1/4$		
Scheme [7]	9	C^3	Quintic	8	4
Scheme [17]	9	C^1	Linear	8	2
Scheme [18]	9	C^3 for $w \in (-3/16, -2/16)$	Cubic	8	2
Scheme [36]	9	C^3 for $\mu \in (-1/16, 1/16)$	Cubic	8	2
		C^6 for $\mu = 0$	Septic for $\mu = 0$		
$S^2_{5,e}$-scheme	10	C^6 for $w \in (0, 1/2)$	Sextic	9	2
		C^7 for $w = 1/4$	Octic for $w = 1/4$		
Scheme [7]	10	C^7	Nonic	9	4
Scheme [17]	10	C^2	Quadratic	9	2
Scheme [18]	10	C^4 for $w \in (-7/50, -1/24)$	Quartic	9	2
Scheme [36]	10	C^4 for $\mu \in (-1/16, 1/16)$	Quartic	9	2
		C^7 for $\mu = 0$	Octic for $\mu = 0$		

of our proposed family of binary $S^2_{n,e}$ schemes are higher comparative to the existing subdivision schemes. Similarly comparison of ternary $S^3_{n,o}$ subdivision schemes is presented in Table 4. Visual performances of our proposed family members at different values of w are presented in Figure 1. We also observed that the maximum smoothness is obtained at $w = 1/4$. In Figure 2, we show that our proposed scheme gives better smoothness comparative to existing schemes $a_{2,\mu}$ of [18] and $a_{2,w}$ of [36].

5. Nonstationary Algorithm

Let us define

$$v^k = \frac{1}{2}\left(e^{i(t/2^{k+1})} + e^{-i(t/2^{k+1})}\right) = \cos\left(\frac{t}{2^{k+1}}\right), \quad \text{with } t \in [0, \pi) \cup i\mathbb{R}^+, \tag{20}$$

where

$$v^0 = \cos\left(\frac{t}{2}\right) = \begin{cases} \cos\left(\frac{\beta}{2}\right) \in (0,1) & \text{if } t = \beta,\ \beta \in (0, \pi), \\ 1 & \text{if } t = 0, \\ \cosh\left(\frac{\beta}{2}\right) \in (1, +\infty) & \text{if } t = i\beta,\ \beta \in \mathbb{R}^+. \end{cases} \tag{21}$$

We also define

$$v^{k+1} = \left(\frac{1 + v^k}{2}\right)^{1/2}. \tag{22}$$

Table 4: Here we present the comparison of the family of ternary $S^3_{n,o}$ schemes. Here m, C, S, and G_d are complexity, continuity analysis, support size, and generation degree of subdivision schemes, respectively.

m	Schemes	Type	C	S	G_d
3	$S^3_{0,o}$-scheme	Approx.	C^1 for $w \in (0, 1/2)$	4	Linear
3	Scheme [37]	Int.	C^1	4	Linear
3	Scheme [37]	Approx.	C^2	4	Quadratic
3	Scheme [38]	Approx.	C^1	4	Linear
5	$S^3_{3,o}$-scheme	Approx.	C^4 for $w \in (0, 1/2)$	7	Quartic
5	Scheme [37]	Int.	C^1	7	Linear
5	Scheme [35]	Approx.	C^3	7	Cubic
7	$S^3_{6,o}$-scheme	Approx.	C^7 for $w \in (0, 1/2)$	10	Septic
7	Scheme [35]	Approx.	C^3	10	Cubic

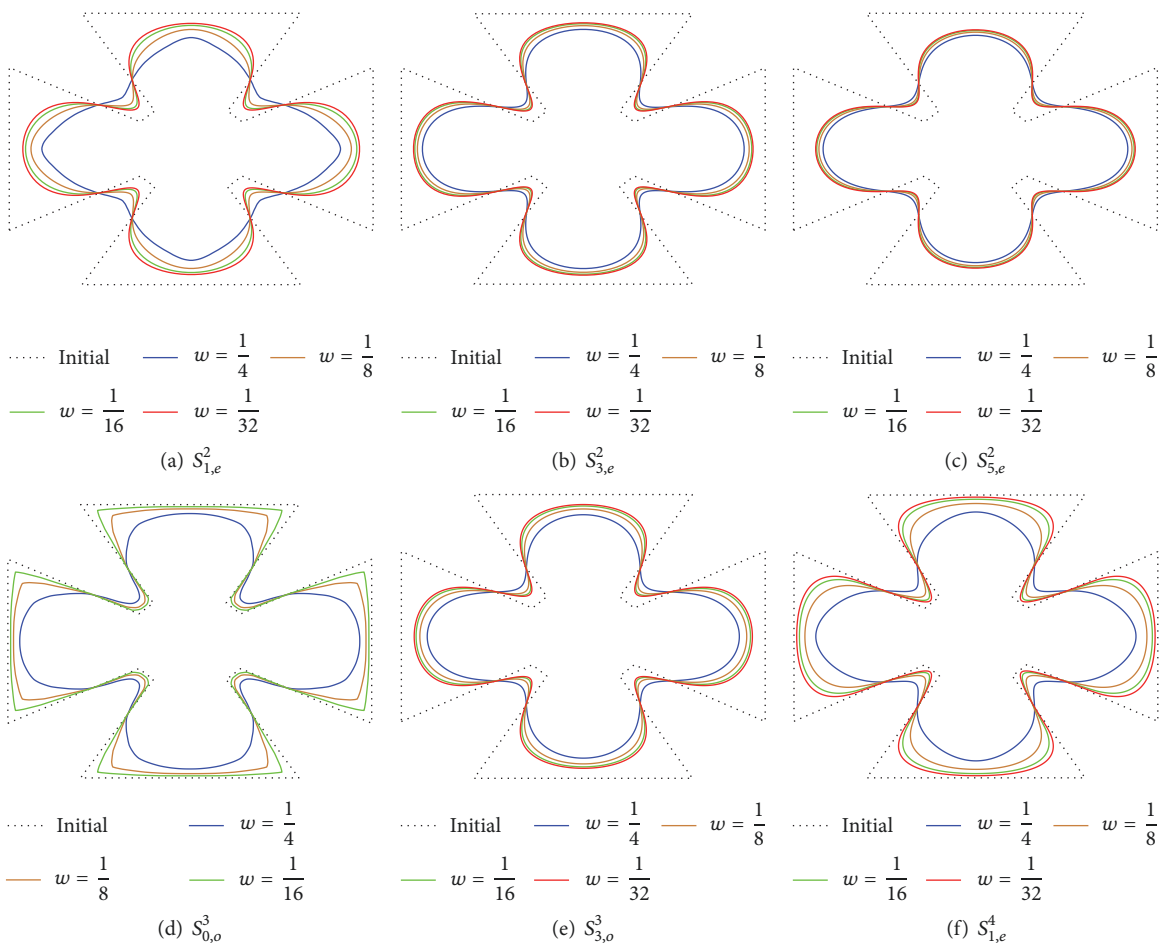

Figure 1: Limiting close curves generated by different members of the family of a-ary schemes at $w = 1/4$, $1/8$, $1/16$, and $1/32$.

Remember that if $t = 0$ then $v^k = v^{k+1} = 1$; similarly we have

$$\lim_{k\to+\infty} v^k = \lim_{k\to+\infty} v^{k+1} = 1. \tag{23}$$

In nonstationary context, we define the Laurent polynomial for odd arity as

$$P^a_{o,k}(z) = \left(1 - aw^k\right) + w^k\left(z^{-(a-1)} + z^{-(a-3)} + \cdots + z^{(a-a)} + \cdots + z^{(a-3)} + z^{(a-1)}\right) \tag{24}$$

and Laurent polynomial for even arity as

$$P^a_{e,k}(z) = \left(1 - aw^k\right) + w^k\left(z^{-(a-1)} + z^{-(a-3)} + \cdots + z^{(a-3)} + z^{(a-1)}\right), \tag{25}$$

where

$$w^k = \frac{2w}{v^{k+1}\left(v^{k+1} + 1\right)}. \tag{26}$$

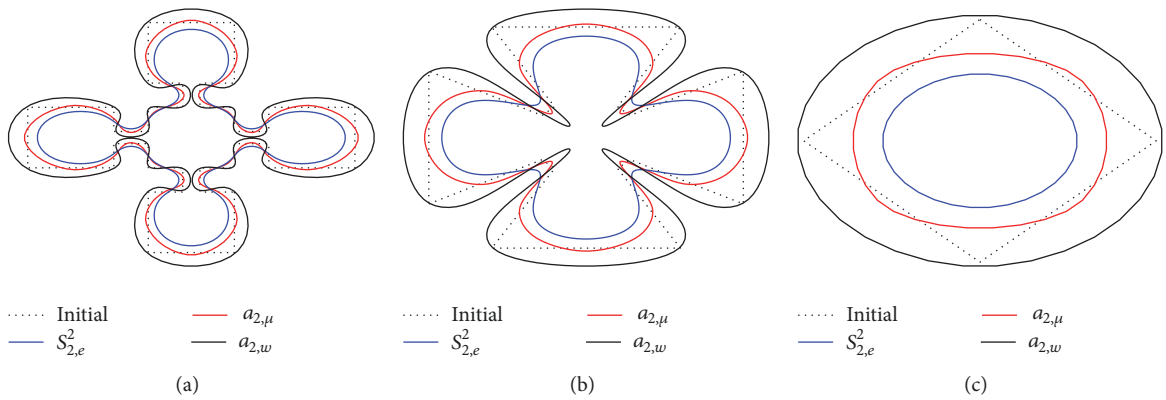

FIGURE 2: (a), (b), and (c) present comparison between limit curves for close polygons produced by schemes corresponding to $S^2_{3,e}$, $a_{2,\mu}$ of [18] and $a_{2,w}$ of [36] at $w = \mu = 1/10$.

Then general symbols for odd arity and even arity subdivision schemes can be defined as

$$S^a_{n,o,k}(z) = A^a_n(z) P^a_{o,k}(z), \tag{27}$$

$$S^a_{n,e,k}(z) = A^a_n(z) P^a_{e,k}(z), \tag{28}$$

respectively.

(27) and (28) are the general symbols of family of a-ary subdivision schemes with tension parameter. By substituting the different values of a and n, we get the symbol of family members of a-ary nonstationary subdivision schemes.

5.1. Family of Binary Nonstationary Schemes. In this subsection, we will present the nonstationary family of binary subdivision schemes. After substituting $a = 2$ in (28), we get family of binary nonstationary subdivision schemes. The general symbol of binary nonstationary subdivision schemes is given by

$$\begin{aligned} S^2_{n,e,k}(z) &= \frac{(1+z)^{n+2}}{2^{n+1}v^{k+1}(v^{k+1}+1)}\left(\left(v^{k+1}\left(v^{k+1}+1\right)-4w\right)\right. \\ &\quad \left.-2w\left(z^{-1}+z\right)\right). \end{aligned} \tag{29}$$

We get the family members of a family of nonstationary subdivision schemes at different values of n.

For $n = 1$, we get a nonstationary dual 3-point scheme

$$\begin{aligned} f^{k+1}_{2i} &= \frac{1}{4v^{k+1}(v^{k+1}+1)}\left(\left(v^{k+1}\left(v^{k+1}+1\right)+2w\right)f^k_{i-1}\right. \\ &\quad \left.+\left(3v^{k+1}\left(v^{k+1}+1\right)-4w\right)f^k_i+2wf^k_{i+1}\right), \\ f^{k+1}_{2i+1} &= \frac{1}{4v^{k+1}(v^{k+1}+1)}\left(2wf^k_{i-1}\right. \\ &\quad +\left(3v^{k+1}\left(v^{k+1}+1\right)-4w\right)f^k_i \\ &\quad \left.+\left(v^{k+1}\left(v^{k+1}+1\right)+2w\right)f^k_{i+1}\right). \end{aligned} \tag{30}$$

For $n = 2$, we get a relaxed 3-point nonstationary primal scheme

$$\begin{aligned} f^{k+1}_{2i} &= \frac{1}{8v^{k+1}(v^{k+1}+1)}\left(\left(v^{k+1}\left(v^{k+1}+1\right)+4w\right)f^k_{i-1}\right. \\ &\quad +\left(6v^{k+1}\left(v^{k+1}+1\right)-8w\right)f^k_i \\ &\quad \left.+\left(v^{k+1}\left(v^{k+1}+1\right)+4w\right)f^k_{i+1}\right), \\ f^{k+1}_{2i+1} &= \frac{1}{8v^{k+1}(v^{k+1}+1)}\left(2wf^k_{i-1}\right. \\ &\quad +\left(4v^{k+1}\left(v^{k+1}+1\right)-2w\right)f^k_i \\ &\quad \left.+\left(4v^{k+1}\left(v^{k+1}+1\right)-2w\right)f^k_{i+1}+2wf^k_{i+2}\right). \end{aligned} \tag{31}$$

For $n = 3$, we get a 4-point dual nonstationary scheme

$$\begin{aligned} f^{k+1}_{2i} &= \frac{1}{16v^{k+1}(v^{k+1}+1)}\left(\left(v^{k+1}\left(v^{k+1}+1\right)+6w\right)f^k_{i-1}\right. \\ &\quad +\left(5v^{k+1}\left(v^{k+1}+1\right)+2w\right)f^k_i \\ &\quad \left.+\left(10v^{k+1}\left(v^{k+1}+1\right)-10w\right)f^k_{i+1}+2wf^k_{i+2}\right), \\ f^{k+1}_{2i+1} &= \frac{1}{16v^{k+1}(v^{k+1}+1)}\left(2wf^k_{i-1}\right. \\ &\quad +\left(10v^{k+1}\left(v^{k+1}+1\right)-10w\right)f^k_i \\ &\quad +\left(5v^{k+1}\left(v^{k+1}+1\right)+2w\right)f^k_{i+1} \\ &\quad \left.+\left(v^{k+1}\left(v^{k+1}+1\right)+6w\right)f^k_{i+2}\right). \end{aligned} \tag{32}$$

Similarly, we can generate family of ternary, quaternary, and a-ary subdivision schemes. Figure 3 represents the application of nonstationary scheme $S^2_{1,e,k}$ at different values of w.

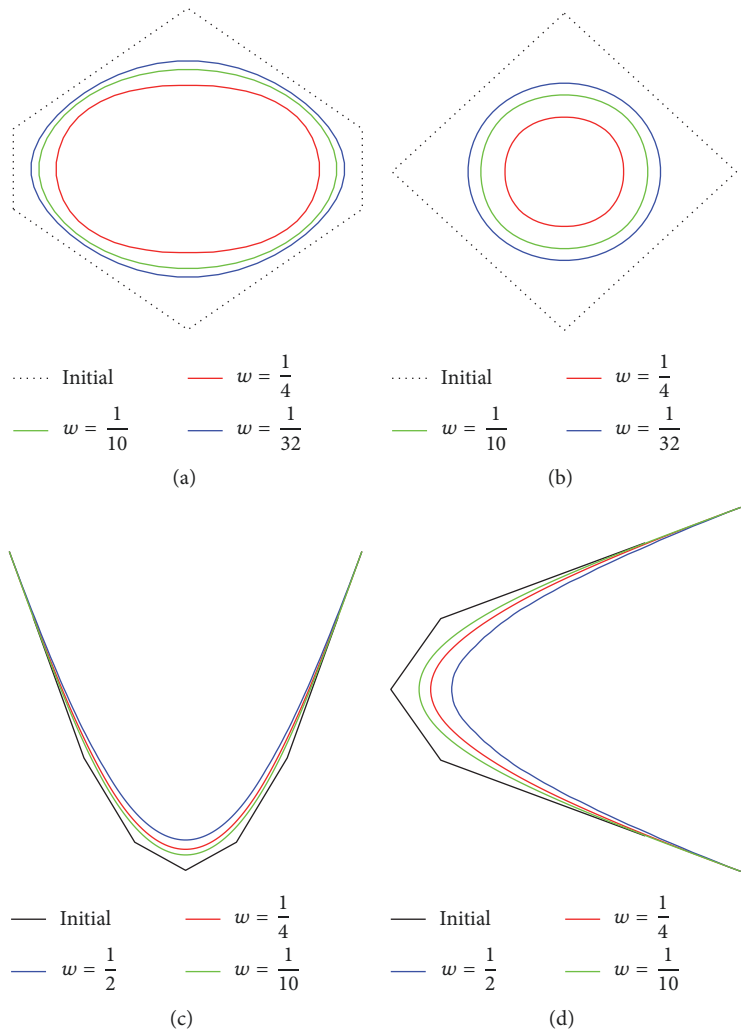

FIGURE 3: Reproduction of conics by scheme corresponding to $S^2_{1,e,k}$.

Lemma 10. *The 3-point nonstationary scheme defined in (30) is asymptotically equivalent to the scheme*

$$\begin{aligned} f_{2i}^{k+1} &= \frac{1}{4}\left((1+w)\, f_{i-1}^{k} + (3-2w)\, f_{i}^{k} + w f_{i+1}^{k}\right), \\ f_{2i+1}^{k+1} &= \frac{1}{4}\left(w f_{i-1}^{k} + (3-2w)\, f_{i}^{k} + (1+w)\, f_{i+1}^{k}\right). \end{aligned} \tag{33}$$

Proof. We can easily verify the above result using (23). □

Similarly, we have the following lemmas.

Lemma 11. *The relaxed 3-point nonstationary scheme defined in (31) is asymptotically equivalent to the scheme*

$$\begin{aligned} & f_{2i}^{k+1} \\ &\quad = \frac{1}{8}\left((1+2w)\, f_{i-1}^{k} + (6-4w)\, f_{i}^{k} + (1+2w)\, f_{i+1}^{k}\right), \\ & f_{2i+1}^{k+1} \\ &\quad = \frac{1}{8}\left(w f_{i-1}^{k} + (4-w)\, f_{i}^{k} + (4-w)\, f_{i+1}^{k} + w f_{i+2}^{k}\right). \end{aligned} \tag{34}$$

Lemma 12. *The 4-point nonstationary scheme defined in (32) is asymptotically equivalent to the scheme*

$$\begin{aligned} f_{2i}^{k+1} &= \frac{1}{16}\left((1+3w)\, f_{i-1}^{k} + (10-5w)\, f_{i}^{k}\right. \\ &\quad \left. + (w+5)\, f_{i+1}^{k} + w f_{i+2}^{k}\right), \\ f_{2i+1}^{k+1} &= \frac{1}{16}\left(w f_{i-1}^{k} + (w+5)\, f_{i}^{k} + (10-5w)\, f_{i+1}^{k}\right. \\ &\quad \left. + (1+3w)\, f_{i+2}^{k}\right). \end{aligned} \tag{35}$$

Lemma 13. *The Laurent polynomials of even and odd ary nonstationary schemes are asymptotically equivalent to the*

Laurent polynomials of even and odd ary stationary schemes, respectively.

$$\lim_{k\to+\infty} P_{e,k}^{a}(z) = P_{e}^{a}(z),$$
$$\lim_{k\to+\infty} P_{o,k}^{a}(z) = P_{o}^{a}(z). \tag{36}$$

Proof. By (23) and (26), $\lim_{k\to+\infty} w^k = w$; hence the proof is completed. □

Lemma 14. *The family of a-ary nonstationary subdivision schemes corresponding to the nonstationary symbols $S_{n,e,k}^{a}(z)$ and $S_{n,o,k}^{a}(z)$ are asymptotically equivalent to the family of a-ary stationary subdivision schemes corresponding to the symbols $S_{n,e}^{a}(z)$ and $S_{n,o}^{a}(z)$, respectively.*

$$\lim_{k\to+\infty} S_{n,e,k}^{a}(z) = S_{n,e}^{a}(z),$$
$$\lim_{k\to+\infty} S_{n,o,k}^{a}(z) = S_{n,o}^{a}(z). \tag{37}$$

Proof. We can easily verify the above result using (27), (28), and Lemma 13. □

6. Conclusion

In this paper, we have presented a simplest way to construct the a-ary ($a \geq 2$) univariate stationary and nonstationary parametric subdivision schemes. Our proposed families of schemes have good properties comparative to the exiting subdivision schemes. It is also observed that many existing binary and ternary subdivision schemes are the special cases of our proposed families. We also present the analysis of families of subdivision schemes. Purposed algorithms are the extension of the well-known Lane-Riesenfeld algorithm in both stationary and nonstationary context.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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